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On L_1-Convergence of the r-th Derivative of Cosine Series with r-quasi convex coefficients

N. L. Braha
Department of Mathematics and Computer Sciences, Avenue "Mother Theresa " 5, Prishtinë, 10000, Kosova
nbraha@yahoo.com

Xh. Z. Krasniqi
Department of Mathematics and Computer Sciences, Avenue "Mother Theresa " 5, Prishtinë, 10000, Kosova
xheki00@hotmail.com

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Abstract. We study L_1-convergence of $r-th$ derivative of modified cosine sums introduced in [2]. Exactly it is proved the L_1- convergence of $r-th$ derivative of modified cosine sums with r-quasi convex coefficients.

Keywords: cosine sums, L_1-convergence, r-quasi convex null sequences.

MSC 2000 classification: 42A20, 42A32.

Introduction

It is well known that if a trigonometric series converges in L_1-metric to a function $f \in L_1$, then it is the Fourier series of the function f. Riesz [4] gave a counter example showing that in a metric space L_1 we cannot expect the converse of the above said result to hold true. This motivated the various authors to study L_1-convergence of the trigonometric series. During their investigations some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in L_1-metric to the sum of the trigonometric series whereas the classical series itself may not. In this context we introduced in [2], new modified cosine series given by relation

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2},$$

and for this modified cosine series we will prove L_1-Convergence of the $r-th$ Derivative of Cosine Series with r-quasi convex coefficients.

In the sequel we will briefly describe the notations and definitions which are used throughout the paper. Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1}$$

be cosine trigonometric series with its partial sums denoted by

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx,$$

and let $g(x) = \lim_{n\to\infty} S_n(x)$.

Definition 1.1 A sequence of scalars (a_n) is said to be quasi semi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} - \Delta^2 a_n| < \infty, (a_0 = 0), \tag{2}$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}, \Delta a_n = a_n - a_{n+1}$.

Definition 1.2 A sequence of scalars (a_n) is said to be r-quasi convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n^{r+1}|\Delta a_{n-1} - \Delta a_n| < \infty, (a_0 = 0, r \geq 0). \tag{3}$$

Note: If we replace $r = 0$, in definition 1.2, we get the concept of the 0-quasi convex sequences.

As usually with $D_n(x)$ and $\tilde{D}_n(x)$ we shall denote the Dirichlet and its conjugate kernels defined by

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx, \quad \tilde{D}_n(x) = \sum_{k=1}^{n} \sin kx.$$

Everywhere in this paper the constants in the O-expression denote positive constants and they may be different in different relations. All other notations are like as in [5], [4].

1 Preliminaries

To prove the main results we need the following lemmas:

Lemma 2.1 If (a_n) is 0-quasi convex null sequence, then it is a quasi semi-convex sequence too.

Proof. From the definition 1.2 it follows that the following relation

$$\sum_{n=1}^{\infty} n|\Delta a_{n-1} - \Delta a_n| < \infty,$$

holds. Now we have:

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} - \Delta^2 a_n| = \sum_{n=1}^{\infty} n|(\Delta a_{n-1} - \Delta a_n) - (\Delta a_n - \Delta a_{n+1})| \leq$$

$$\sum_{n=1}^{\infty} n|\Delta a_{n-1} - \Delta a_n| + \sum_{n=1}^{\infty} n|\Delta a_n - \Delta a_{n+1}| < \infty.$$

QED

In [2] were introduced new modified cosine sums as follows:

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}, \tag{4}$$

and is studied the L_1-convergence of these modified cosine sum with quasi semi-convex coefficients proving the following theorem.

Theorem 2.2[2] Let (a_n) be a quasi semi-convex null sequence, then $N_n(x)$ converges to $f(x)$ in L_1-norm.

Based in lemma 2.1 and Theorem 2.2 we obtain the following:

Corollary 2.3 Let (a_n) be a 0-quasi convex null sequence, then $N_n(x)$ converges to $f(x)$ in L_1-norm.

The main goal of the present work is to study the L_1-convergence of $r-th$ derivative of these new modified cosine sums with r-quasi convex null coefficients. We point out here that a lot of authors investigated the L_1-convergence of the series (1), see for example [1], [3].

Lemma 2.4 If $x \in [\epsilon, \pi], \epsilon > 0$ and $m \in N$, then the following estimate holds

$$\left|\left(\frac{\tilde{D}_m(x)}{2\sin^2\frac{x}{2}}\right)^{(r)}\right| = O_{r,\epsilon}\left(m^{r+1}\right), \quad (r = 0, 1, 2\dots)$$

where $O_{r,\epsilon}$ depends only on r and ϵ.

Proof. By Leibniz formula we have

$$\begin{aligned}
\left(\frac{\tilde{D}_m(x)}{2\sin^2\frac{x}{2}}\right)^{(r)} &= \sum_{i=0}^{r}\binom{r}{i}\left(\frac{1}{2\sin^2\frac{x}{2}}\right)^{(r-i)}\left(\tilde{D}_m(x)\right)^{(i)} \\
&= \sum_{i=0}^{r}\binom{r}{i}\left(\frac{1}{2\sin^2\frac{x}{2}}\right)^{(r-i)}\sum_{j=1}^{m} j^i \sin\left(jx + \frac{i\pi}{2}\right) \\
&= O(1)m^{r+1}\sum_{i=0}^{r}\binom{r}{i}\left(\frac{1}{2\sin^2\frac{x}{2}}\right)^{(r-i)}. \qquad (5)
\end{aligned}$$

We shall prove by mathematical induction the equality $\left(\frac{1}{2\sin^2\frac{x}{2}}\right)^{(r)} = \frac{P_r(\cos\frac{x}{2})}{\sin^{r+2}\frac{x}{2}}$, where P_r is a cosine polynomial of degree r.

Namely, we have $\left(\frac{1}{2\sin^2\frac{x}{2}}\right)' = -\frac{1}{2}\frac{\left(\cos\frac{x}{2}\right)}{\sin^3\frac{x}{2}} = \frac{P_1(\cos\frac{x}{2})}{\sin^3\frac{x}{2}}$, so that for $r = 1$ the above equality is true.

Assume that the equality $F(x) := \left(\frac{1}{2\sin^2\frac{x}{2}}\right)^{(r)} = \frac{P_r(\cos\frac{x}{2})}{\sin^{r+2}\frac{x}{2}}$ holds. For the $(r+1)-th$ derivative of $\frac{1}{2\sin^2\frac{x}{2}}$ we get

$$\begin{aligned}
&F'(x) := \\
&= \frac{P_r'(\cos\frac{x}{2})\left(-\frac{1}{2}\sin^{r+3}\frac{x}{2}\right) - P_r(\cos\frac{x}{2})(r+2)\sin^{r+1}\frac{x}{2}\cdot\frac{1}{2}\cdot\cos\frac{x}{2}}{\sin^{2r+4}\frac{x}{2}} \\
&= \frac{-\frac{1}{2}\sin^{r+1}\frac{x}{2}\left[P_r'(\cos\frac{x}{2})\sin^2\frac{x}{2} + P_r(\cos\frac{x}{2})(r+2)\cos\frac{x}{2}\right]}{\sin^{r+1}\frac{x}{2}\cdot\sin^{r+3}\frac{x}{2}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\frac{-P_r'(\cos\frac{x}{2})\frac{1}{2}(1-\cos x)-P_r(\cos\frac{x}{2})(r+2)\cos\frac{x}{2}}{\sin^{r+3}\frac{x}{2}}\\
&= \frac{1}{2}\frac{-\frac{1}{2}P_r'(\cos\frac{x}{2})+\cos x\cdot\frac{1}{2}P_r'(\cos\frac{x}{2})-P_r(\cos\frac{x}{2})(r+2)\cos\frac{x}{2}}{\sin^{r+3}\frac{x}{2}}\\
&= \frac{1}{2}\frac{(-1/2)H_{r-1}(\cos\frac{x}{2})+(1/2)H_{r-1}(\cos\frac{x}{2})\cos x-(r+2)P_r(\cos\frac{x}{2})\cos\frac{x}{2}}{\sin^{r+3}\frac{x}{2}}\\
&= \frac{Q_{r+1}(\cos\frac{x}{2})-(r+2)R_{r+1}(\cos\frac{x}{2})}{\sin^{r+3}\frac{x}{2}}=\frac{T_{r+1}(\cos\frac{x}{2})}{\sin^{r+3}\frac{x}{2}}, \qquad (6)
\end{aligned}$$

where H_{r-1}, Q_{r+1}, R_{r+1}, T_{r+1} are cosine polynomials of degree $r-1$ and $r+1$ respectively. Therefore for $x\in[\epsilon,\pi]$, $\epsilon>0$, from (5) and (6) we obtain

$$\left|\left(\frac{\tilde{D}_m(x)}{2\sin^2\frac{x}{2}}\right)^{(r)}\right| = O(1)m^{r+1}\sum_{i=0}^{r}\binom{r}{i}\frac{\left|P_{r-i}(\cos\frac{x}{2})\right|}{\sin^{r-i+2}\frac{x}{2}}=O_{r,\epsilon}\left(m^{r+1}\right).$$

QED

Lemma 2.5 If $x\in[\epsilon,\pi],\epsilon>0$ and $m\in N$, then the following estimate holds

$$\left|\left(\frac{D_m(x)}{2\sin^2\frac{x}{2}}\right)^{(r)}\right|=O_{r,\epsilon}\left(m^{r+1}\right),\quad (r=0,1,2\dots)$$

where $O_{r,\epsilon}$ depends only on r and ϵ.

Proof is similar to the lemma 2.4.

2 Results

In what follows we prove the main result of the paper:

Theorem 3.1 Let (a_n) be a r-quasi convex null sequence, then $N_n^{(r)}(x)$ converges to $g^{(r)}(x)$ in L_1-norm.

Proof. From definition of $S_n(x)$ we have:

$$\begin{aligned}
S_n(x) &= \frac{a_0}{2}+\sum_{k=1}^{n}a_k\cdot\cos kx\\
&= \frac{1}{\left(2\sin\frac{x}{2}\right)^2}\cdot\sum_{k=1}^{n}a_k\cdot\cos kx\cdot\left(2\sin\frac{x}{2}\right)^2\\
&= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\cdot\sum_{k=1}^{n}a_k[\cdot\cos(k+1)x-2\cos kx+cos(k-1)x]\\
&= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\cdot\sum_{k=1}^{n}(a_{k-1}-2a_k+a_{k+1})\cdot\cos kx-\frac{a_0\cos x}{\left(2\sin\frac{x}{2}\right)^2}+\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2}\\
&\quad+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}\\
&\Rightarrow
\end{aligned}$$

$$S_n(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\cdot\sum_{k=1}^{n}\Delta^2a_{k-1}\cos kx-\frac{a_0\cos x}{\left(2\sin\frac{x}{2}\right)^2}+\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2}+$$

$$\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.$$

Applying Abel's transformation, we get

$$S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x) + \frac{\Delta^2 a_{n-1}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_0\cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.$$

Therefore

$$S_n^{(r)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k^{(r)}(x) + \Delta^2 a_{n-1}\cdot\left(\frac{\widetilde{D}_n(x)}{2\sin^2\frac{x}{2}}\right)^{(r)} + a_n\cdot\left(\frac{\cos(n+1)(x)}{2\sin^2\frac{x}{2}}\right)^{(r)} + a_1\cdot\left(\frac{1}{2\sin^2\frac{x}{2}}\right)^{(r)} - a_{n+1}\cdot\left(\frac{\cos nx}{2\sin^2\frac{x}{2}}\right)^{(r)}. \quad (7)$$

On the other hand we have:

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}. \quad (8)$$

Applying Abel's transformation to the relation (8), we get:

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n-1}(\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x) + \frac{\Delta^2 a_{n-1}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} + \frac{\Delta^2 a_n\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}, \quad (9)$$

and the r-derivative of the relation (9) is the following relation:

$$N_n^{(r)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n-1}(\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k^{(r)}(x) + \Delta^2 a_{n-1}\cdot\left(\frac{\widetilde{D}_n(x)}{2\sin^2\frac{x}{2}}\right)^{(r)} + \Delta^2 a_n\cdot\left(\frac{\widetilde{D}_n(x)}{2\sin^2\frac{x}{2}}\right)^{(r)} + a_1\left(\frac{1}{2\sin^2\frac{x}{2}}\right)^{(r)}. \quad (10)$$

By lemma 2.4 and since (a_n) is r-quasi convex null sequence, we have

$$\begin{aligned}
\left|(\Delta^2 a_n)\left(\frac{\tilde{D}_n(x)}{2\sin^2\frac{x}{2}}\right)^{(r)}\right| &= O_{r,\epsilon}\left(\left|n^r\left(\Delta^2 a_n\right)\right|\right)\\
&= O_{r,\epsilon}\left(\left|n^r\sum_{k=n}^{\infty}\left(\Delta^2 a_k - \Delta^2 a_{k+1}\right)\right|\right)\\
&= O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^{r+1}|\Delta^2 a_k - \Delta^2 a_{k+1}|\right)\\
&= O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^{r+1}|\Delta a_k - \Delta a_{k+1}|\right)\\
&\quad + O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^{r+1}|\Delta a_{k+1} - \Delta a_{k+2}|\right) = o(1), n\to\infty. \quad (11)
\end{aligned}$$

Also after some elementary calculations and by virtue of lemma 2.5 we obtain

$$
\begin{aligned}
a_n \cdot \left(\frac{\cos{(n+1)(x)}}{2\sin^2\frac{x}{2}}\right)^{(r)} - a_{n+1}\cdot\left(\frac{\cos nx}{2\sin^2\frac{x}{2}}\right)^{(r)} = \\
&= a_n\left[\left(\frac{D_{n+1}(x)}{2\sin^2\frac{x}{2}}\right)^{(r)} - \left(\frac{D_n(x)}{2\sin^2\frac{x}{2}}\right)^{(r)}\right] - \qquad (12)\\
&\quad a_{n+1}\left[\left(\frac{D_n(x)}{2\sin^2\frac{x}{2}}\right)^{(r)} - \left(\frac{D_{n-1}(x)}{2\sin^2\frac{x}{2}}\right)^{(r)}\right]\\
&= a_n\cdot O_{r,\epsilon}\left((n+1)^{r+1}-n^{r+1}\right) - a_{n+1}\cdot O_{r,\epsilon}\left(n^{r+1}-(n-1)^{r+1}\right)\\
&= O_{r,\epsilon}\left(n^r\left(a_n-a_{n+1}\right)\right) = O_{r,\epsilon}\left(n^r\sum_{k=n}^{\infty}(\Delta a_k-\Delta a_{k+1})\right)\\
&= O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^r|\Delta a_k-\Delta a_{k+1}|\right) = o(1), n\to\infty, \qquad (13)
\end{aligned}
$$

respectively

$$
\begin{aligned}
g^{(r)}(x) = \lim_{n\to\infty}N_n^{(r)}(x) = \lim_{n\to\infty}S_n^{(r)}(x) = \\
&- \frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{\infty}(\Delta^2 a_{k-1}-\Delta^2 a_k)\widetilde{D}_k^{(r)}(x) + a_1\left(\frac{1}{2\sin^2\frac{x}{2}}\right)^{(r)}.
\end{aligned}
$$

Using lemma 2.4, relations (11) and (12) we get the following:

$$
\begin{aligned}
\int_{-\pi}^{\pi}\left|g^{(r)}(x)-N_n^{(r)}(x)\right|dx &= 2\int_0^{\pi}\sum_{k=n}^{\infty}\left|\Delta^2 a_k-\Delta^2 a_{k+1}\right|\left|\left(\frac{\tilde{D}_k(x)}{2\sin^2\frac{x}{2}}\right)^{(r)}\right|dx\\
&= O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^{r+1}\left|\Delta^2 a_k-\Delta^2 a_{k+1}\right|\right) = o(1), n\to\infty,
\end{aligned}
$$

which proves the theorem. QED

Corollary 3.2 Let (a_n) be a r-quasi convex null sequence, then the necessary and sufficient condition for L_1-convergence of the $r-th$ derivative of the series (1) is $n^{r+1}|a_n| = o(1)$, as $n\to\infty$.

Proof. We have

$$
\begin{aligned}
\left\|g^{(r)}(x)-S_n^{(r)}(x)\right\| &\le \left\|g^{(r)}(x)-N_n^{(r)}(x)\right\| + \left\|N_n^{(r)}(x)-S_n^{(r)}(x)\right\|\\
&= o(1) + \left\|a_n\left(\frac{\cos{(n+1)x}}{2\sin^2\frac{x}{2}}\right)^{(r)}\right\| + \left\|a_{n+1}\left(\frac{\cos nx}{2\sin^2\frac{x}{2}}\right)^{(r)}\right\|\\
&\quad + \left|(\Delta^2 a_n)\left(\frac{\tilde{D}_n(x)}{2\sin^2\frac{x}{2}}\right)^{(r)}\right| \qquad \text{(by relation (11))}\\
&\le o(1) + \left\|a_n\left(\frac{\cos{(n+1)x}}{2\sin^2\frac{x}{2}}\right)^{(r)}\right\| + \left\|a_{n+1}\left(\frac{\cos nx}{2\sin^2\frac{x}{2}}\right)^{(r)}\right\|\\
&\qquad \text{(by lemma 2.5)}\\
&= o(1) + O\left((n+1)^{r+1}|a_n|\right) + O\left((n+1)^{r+1}|a_{n+1}|\right) = o(1).
\end{aligned}
$$

QED

From Theorem 3.1 and corollary 3.2 we deduce the following corollary ($r = 0$):
Corollary 3.3 If (a_n) is 0-quasi convex null sequence of scalars, then the necessary and sufficient condition for L_1-convergence of the cosine series (1) is $\lim_{n\to\infty} n \cdot |a_n| = 0$.

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