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Vector space partitions and designs Part II – constructions

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Abstract. This article is the second part and companion article to Part I on the basic theory of what are called focal-spreads; partitions of finite vector spaces of dimension $t+k$ by one subspace of dimension t (the 'focus') and the remaining subspaces of dimension k, a 'focal-spread of type (t,k)'. In Part I, additive focal-spreads are shown to be equivalent to additive partial spreads. Focal-spreads of type $(k+1,k)$ also produce $2-(q^{k+1},q,1)$-designs, and various other double and triple-spreads. Also, in Part I, there are two different methods given to construct focal-spreads, one of which is due to Beutelspacher, the other method being a coordinate method similar to the theory available for translation planes. Here, we shall give a new construction that we term "going up," which also allows a specification of certain subplanes of the focal-spread.

Keywords: vector space partition, designs, focal-spread, going up

MSC 2000 classification: 51E23 (primary), 51A40.

1 Introduction

Let V_{t+k} be a vector space of dimension $t+k$ over $GF(q)$ that is covered by one subspace of dimension t and a set of q^t mutually disjoint k-subspaces. If $t=k$, we have a t-spread. If $t\neq k$, we say that the partition is a 'focal-spread' and the unique t-dimensional subspace in the cover is said to be the 'focus'. The k-subspaces in the cover are said to be k-components and the set of k-subspaces is the associated partial Sperner k-spread.

In Part I, the authors (Jha and Johnson [2]) have generalized a construction of Buetelspacher [1], that constructs 'focal-spreads'. Furthermore, when $t=k+1$, there are various associated $2-(q^{k+1},q,1)$-designs, admitting a set of $(q^{k+1}-1)/(q-1)$ parallel classes of q^{k+1} lines each. Indeed, the construction of these designs provide new partitions of vector spaces of dimension $2k$ over $GF(q)$ admitting exactly $q+1$ k-subspaces and $q^{k+1}-q$ $(k-1)$-subspaces. Other applications of the theory of focal-spreads involve construction of new maximal partial spreads, and a spread-theoretic characterization of the dualization of semifields.

In this part, we give a new construction of focal-spreads, by a procedure that we term 'going up'. This process allows us to specify subspaces of order q^k, in a focal-spread of type $(k(s-1),k)$, where the focus has dimension $k(s-1)=t$ and we have an associated partial

Sperner k-spread of q^t k-dimensional subspaces. Furthermore, if a focal-spread is defined more generally as a partition of a vector space of dimension $t+k$ over $GF(q)$ by one subspace of dimension t' and the remaining subspaces of dimension k, we say the focal-spread is of type (t,t',k), when $t'<t$. The going up process allows constructions of this more general type of focal-spreads.

For convenience of the reader, we repeat the construction of Beutelspacher.

1.1 Beutelspacher's Construction

Let V_{t+k} be a vector space of dimension $t+k$ over $GF(q)$ for $t>k$ and let L be a subspace of dimension t. Let V_{2t} be a vector space of dimension $2t$ containing V_{t+k} ($t>k$ required here) and let S_t be a t-spread containing L. There are always at least Desarguesian t-spreads with this property. Let M_t be a component of the spread S_t not equal L. Then $M_t \cap V_{t+k}$ is a subspace of V_{t+k} of dimension at least k. But, since M_t is disjoint from L, the dimension is precisely k and we then obtain a focal-spread with focus L. This construction may be found in Beutelspacher [1], Lemma 2, p. 205. The corresponding focal-spread shall be called a 'k-cut of the t-spread' and we shall use the notation $F=S_t \backslash V_{t+k}$ for the focal-spread F.

The formal definition is

Definition 1. A partition of a finite-dimensional vector space by a partial Sperner k-spread and a subspace of dimension $t>k$, shall be called a 'focal-spread of type (t,k)'. The unique subspace of dimension t of the partition shall be called the 'focus' of the focal-spread.

We define a 'planar extension' of a focal-spread as a t-spread such that the focal-spread is of type (t,k) and arises from the t-spread as a k-cut.

2 Going Up

In this section, we provide a new method, called 'going up' for the construction of focal-spreads. The idea arose from the authors' construction (Jha, Johnson [3]) of k-spreads in tk-dimensional vector spaces over $GF(q)$.

We remind the reader that any translation plane of order q^k with kernel containing $GF(q)$ has a spread of the form

$$x=0, y=0, y=xM, \text{ where } M \in S,$$

where S is a set of non-singular matrices, whose differences are also non-singular, and where x and y are k-vectors. So, we have a vector space V_{2k} of dimension $2k$ over $GF(q)$ such that V_{2k} is partitioned into the following mutually disjoint sets: $V_{2k}=\{(0,0)\} \cup \{(x,0); x \text{ is a non-zero } k\text{-vector}\} \cup \{(0,y); y \text{ is a non-zero } k\text{-vector}\} \cup \{(x,xM); M \in S; x \text{ a non-zero } k\text{-vector}\}$. We seek a similar partition of sk-dimensional vector spaces over $GF(q)$.

So, that it may be easily conceptualized in general, we show how to construct a k-spread of a $3k$-dimensional vector space over $GF(q)$. Choose five translation planes π_i, for with matrix spreads S_i, $i=1,2,...,5$, in a sequence. Choose a $3k$-dimensional vector space V_{3k} over $GF(q)$, represent vectors as (x_1,x_2,x_3), where x_i are k-vectors for $i=1,2,3$ and partition V_{3k} into mutually disjoint sets as follows:

$$\begin{aligned} V_{3k}=\{(0,0,0)\} &\cup \{(x_1,0,0); x_1 \text{ is a non-zero } k \text{ -vector}\} \cup \\ &\cup \{(0,x_2,0); x_2 \text{ is a non-zero } k\text{-vector}\} \cup \\ &\cup \{(0,0,x_3); x_3 \text{ is a non-zero } k\text{-vector}\} \cup \\ &\cup \{(x_1,x_1M_1,0); x_1 \text{ is a non-zero } k\text{-vector; } M_1 \in S_1\} \cup \end{aligned}$$

$$\cup\{(x_1,0,x_1M_2);x_1 \text{ is a non-zero } k\text{-vector; } M_2\in S_2\}\cup$$
$$\cup\{(0,x_1,x_1M_3);x_1 \text{ is a non-zero } k\text{-vector; } M_3\in S_3\}\cup$$
$$\cup\{(x_1,x_1M_4,x_1M_5);x_1 \text{ is a non-zero } k \text{ -vector; } M_4\in S_4,\ M_5\in S_5\}$$

The sets into which V_{3k} is partitioned are called '$j = 0$-sets', for $j = 0, 1, 2, 3$, if there are exactly j zeros appearing in one of sets of the partition and the remaining vectors are non-zero k-vectors. So, there are exactly $\begin{pmatrix} 3 \\ j \end{pmatrix}$ $j = 0$-sets of the same type. One important note is that, by this construction, there are translation subplanes of order q^k of as many as five mutually non-isomorphic types in the associated Sperner space k-spread.

If we begin with a vector space V_{tk} of dimension tk-over $GF(q)$, we may construct a k-spread in V_{tk} by a choice of any sequence of $\sum_{j=0}^{t-2}\binom{t}{j}(t-j-1) = N_t$ translation planes of order q^k and associated spreads S_i, for $i = 1, 2, ..., N_t$. Of course, when $t = 3$, we have $\sum_{j=0}^{3-2}\binom{t}{j}(3-j-1) = 2+\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 5$, so we may choose at least five mutually non-isomorphic translation subplanes of order q^k in the Sperner space given by the k-spread. Certainly, as t grows, so does the number of possible mutually non-isomorphic translation subplanes of order q^k that may be chosen in the Sperner k-spread. For example, even for $t = 4$, we have $\sum_{j=0}^{t-2}\binom{t}{j}(t-j-1) = 3+\begin{pmatrix} 4 \\ 1 \end{pmatrix}2+\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 3+8+6 = 17$.

2.1 The Focal-Spread Construction for $4k$-dimensional Spaces

So, we begin with a vector space V_{tk} of dimension tk over $GF(q)$. Represent a vector in the form $(x_1, x_2, ..., x_t)$, where each x_i, for $i = 1, ..., t$ are, in turn, k-vectors over $GF(q)$.

Now using this idea, we may also construct focal-spreads. Suppose we do this in a vector space V_{4k} of dimension $4k$ over $GF(q)$. With vectors written in the form (x_1, x_2, x_3, x_4), choose the $3k$-dimensional subspace given by equation $x_4 = 0$. If we were to construct a k-spread in a $4k$-space, we would use 17 translation planes of order q^k, as above. If we were to construct a k-spread in a $3k$-space, we would require 5 translation planes of order q^k. If we ignore the k-spread construction of x_4, then the remaining vectors can be covered using $17 - 5 = 12$ translation planes of order q^k.

We obtain a focal-spread of type $(3k, k)$ as follows: Let $x_4 = 0$, denote the focus, a subspace of dimension $3k$. There are $(q^{4k} - 1)/(q^k - 1)$ k-subspaces in what would be a k-spread in the $4k$-dimensional vector space and there would be $(q^{3k} - 1)/(q^k - 1)$ k-subspaces in the k-spread for the $3k$-dimensional vector space $x_4 = 0$. Hence, there are

$$(q^{4k} - 1)/(q^k - 1) - (q^{3k} - 1)/(q^k - 1) = q^{3k}$$

k-subspaces in a partial spread distinct from $x_4 = 0$.

Now we generalize this example. In the following, we define a planar k-spread to be a k-spread of a $2k$-dimensional vector space over $GF(q)$, thus equivalent to a translation plane of order q^k with kernel containing $GF(q)$. We are working with vector spaces V of dimension tk over $GF(q)$, with vectors represented in the form $(x_1, x_2, ..., x_t)$, where the x_i are k-vectors, for $i = 1, 2, ..., t$. We partition the set of vectors into the '$j = 0$-sets', which are the sets of vectors with a set of j k-vectors equal to 0 in fixed locations, where the remaining entries are non-zero. For example, $j = (t-1)$-zero sets have the general form $(x_1, 0, 0, ..., 0)$, where x_1 is a non-zero k-vector, and the $j = 1$-zero sets are sets of vectors with exactly one entry equal to 0, such as $\{(0, x_2, ..., x_t);\ x_i \text{ are non-zero } k\text{-vectors},\ i = 2, ..., t\}$.

The following is the corresponding construction of k-spreads in tk-dimensional vector spaces over $GF(q)$.

Theorem 1. *(Jha and Johnson [3]) Let V be a tk-dimensional $GF(q)$-subspace and represent vectors in the form $(x_1, x_2, ..., x_t)$, where the x_i are k-vectors, for $i = 1, 2, ..., t$. Let $\mathcal{S}_t$ be a sequence of $\sum_{j=0}^{t-2} \binom{t}{j}(t-j-1) = N_t$ planar k-spreads over $GF(q)$. We may represent each of the planar k-spreads as follows:*

We identify two common components $x = 0, y = 0$, and where x and y are k-vectors. Then if $\mathcal{M}_i$ is a planar k-spread, there is a set of $q^k - 1$ nonsingular matrices $M_{i,z}$, for $z = 1, 2, ..., q^k - 1$, $i = 1, 2, ..., N_t$, whose differences are also non-singular.

Hence, we represent the planar k-spread by $y = xM_{iz}$, for $z = 1, 2, ..., q^k - 1$. Consider the $j = 0$-sets for $j = 0, 1, 2, ..., t-2$, and assume an ordering for the N_t planar k-spreads. For each such $j = 0$-set, for $j > 2$, we eliminate the zero elements and write vectors in the form $(x_1, x_2, ..., x_{t-j})$, where x_w, for $w = 1, 2, ..., t-j$, are all non-zero k-vectors. We partition this set by

$$(x_1, x_1M_{1,z_1}, x_1M_{2,z_2}, x_1M_{3,z_3}, ..., x_1M_{t-j,z_{t-j}})$$

where M_{i,z_i} varies over $\mathcal{M}_i$, and where the indices z_i are independent of each other. By adjoining the zero vector, we then may consider

$$y = (x_1M_{1,z_1}, x_1M_{2,z_2}, x_1M_{3,z_3}, ..., x_1M_{t-j,z_{t-j}})$$

as a k-subspace for fixed M_{i,z_i}, for each $i = 1, 2, ..., t-j$. Then, together with the $j = t-1$ zero sets adjoined by the zero vector, we obtain a spread of k-spaces of a tk-dimensional vector space over $GF(q)$ as a union of $j = 0$-sets (with the zero vector adjoined to each).

2.2 The 'Going Up' Construction of Focal-spreads

We are, of course, interested in constructing focal-spreads of type (t, k), defined over a $t+k$-dimensional $GF(q)$ vector spaces partitioned by a subspace of dimension t and a partial Sperner k-spread. However, it is certainly possible to define a focal-spread over a finite dimensional vector space as a partition of the vectors into one subspace of dimension t', the focus, and the remaining subspaces of dimension k, where the dimension is $t + k$, for $t' \leq t$. In this case, this type of focal-spread would not arise as a k-cut, at least directly. We call such partitions focal-spreads of type (t, t', k). In this setting, $q^{t+k} - q^{t'} = q^{t'}(q^{t-t'+k} - 1)$ must be divisible by $q^k - 1$ so that $t - t'$ is divisible by k.

We now use the ideas of the k-spread construction to provide focal-spreads of type $(k(t-w), k)$. Choose the $k(t-w)$-dimensional subspace F given by the equation $x_1 = x_2 = ... = x_w = 0$. To construct a k-spread requires $N_{t-w} = \sum_{j=0}^{t-(w+2)} \binom{t-1}{j}(t-j-(w+1))$ spreads. This means that the remaining $q^{k(t-1)}(q^{kw} - 1)/(q^k - 1)$ k-components are obtained using a set of $N_t - N_{t-w}$ spreads as in the previous theorem. Ignoring the set of N_{t-w} spreads covering F, we obtain a focal-spread of type $(kt(t-1), k(t-w), k)$.

We obtain then the following result (see Jha and Johnson [3] for a result similar to part (1), where only $t-1$ spreads were used in a construction). For part (2), the reader is directed to Jha and Johnson [3] for the analogous result for the constructed k-spreads in kt-dimensional vector spaces.

Theorem 2. *Given a vector space of dimension kt over a field $GF(q)$. Choose any $k(t-w)$-dimensional subspace F. Now choose a set of $N_t - N_{t-w}$ planar k-spreads (spreads corresponding to translation planes of order q^k).*

(1) Then, using the method introduced in the previous theorem, we have constructed a focal-spread of type $(k(t-1), k(t-w), k)$.

(2) If, by identifying the components $x = 0, y = 0$, each k-spread admits a collineation group with elements of the general form $(x, y) \to (xA, yA)$, where A is a $k \times k$ matrix, then the focal-spread also admits this group.

The following corollary shows that either there are a large variety of focal-spreads that cannot be obtained as k-cuts, or there are translation planes of order $q^{k(t-1)}$ with widely variable translation subplanes of order q^k.

Corollary 1. *Assume that all focal-spreads of type (s,k) may be obtained as k-cuts from planar s-spreads (corresponding to translation planes of order q^s). Let $s = k(t-1)$. Now specify any given set of*

$$\sum_{j=0}^{t-2} \binom{t}{j} (t-j-1) - \sum_{j=0}^{t-3} \binom{t-1}{j} (t-j-2)$$

translation planes of order q^k. Then there exists a translation plane of order $q^{k(t-1)}$ containing the given set as a set of affine subplanes of order q^k.

Proof. For an example, consider a focal-spread of type $(2k,k)$ obtained from the above process. Choose vectors (x_1, x_2, x_3), where x_i are k-vectors over $GF(q)$. Let $x_1 = 0$, denote the focus, a $2k$-dimensional subspace. Then, the partial Sperner k-spread is given by

$$\{(x_1, x_1M_1, 0)\},\ \{(x_1, 0, x_1M_2)\},\ \{(x_1, x_1M_3, x_1M_4)\},\ \{(x_1, 0, 0)\ \},$$

for $M_i \in \mathcal{M}_i$, k-spreads, $i = 1,2,3,4$. We claim that $\{(x_1, x_1M_1, 0)\}$ extends to a matrix k-spread and provides a subplane of order q^k in the associated translation plane of order q^{2k}. If we consider x_1 as $(z_1, z_2, ..., z_k)$, for $z_i \in GF(q)$ then, in the extension, we take $x = (x_1, w_1, ..., w_k)$, for $w_i \in GF(q)$. Considering $(x_1, x_1M_1, 0)$ as $y = x_1[M_1, 0]$, the latter matrix is a $k \times 2k$ matrix. We extend to $y = x[\begin{matrix} M_1 & 0 \\ M_1^- & S_1 \end{matrix}]$, a $2k \times 2k$ matrix of rank $2k$. Since M_1 has rank k, it follows that S_1 has rank k. Then, consider

$$y = (x_1, 0, 0, ..., 0)[\begin{matrix} M_1 & 0 \\ M_1^- & S_1 \end{matrix}], \text{ for all } M_1 \in \mathcal{M}_1, \text{ and } x_1 = 0,$$

then provides a subplane of order q^k. More generally, in the focal-spread of type $(k(t-w), k)$, the M_1 would still be $k \times k$ but the S_1 would be $k(t-w) \times k(t-w)$. The more general proof is completely analogous and shall be left to the reader. QED

Basically, the following question is completely open: **Given any translation plane π_o of order q^k, choose any integer s. Is there a translation plane Σ_s of order q^{ks} containing π_o as a subplane?** Since it seems likely that the answer to this question is either no or can never be answered, it certainly appears that there are many focal-spreads of type (t,k) that cannot arise from k-cuts.

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