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Two Modular Equations for Squares of the Cubic-functions with Applications

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Abstract. In this paper, we derive two modular identities for cubic functions and are shown to be connected to the Ramanujan cubic continued fraction $G(q)$. Also we have derived many theta function identities which play an important role in proving Ramanujan's modular equations of degree 3.

Keywords: Cubic function, Ramanujan cubic continued fraction, Theta function identities related to modular equations of degree 3.

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1 Introduction

In the sequel, we always assume that $|q| < 1$. For any complex number a, we employ the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

The Ramanujan's general theta function [2] is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,$$

where $|ab| < 1$. Evidently, $f(a,b) = f(b,a)$. Certain special cases of $f(a,b)$ are defined by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

[i]DST, New Delhi for awarding research project[No. SR/S4/MS:517/08] under which this work has been done.

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty$$

and

$$\chi(-q) := (q;q^2)_\infty.$$

For convenience, denote $f(-q^n)$ by f_n for positive integer n. It is easy to see that

$$\varphi(q) = \frac{{f_2}^5}{{f_1}^2{f_4}^2}, \qquad \varphi(-q) = \frac{{f_1}^2}{f_2}, \qquad \psi(q) = \frac{{f_2}^2}{f_1},$$

$$f(q) = \frac{{f_2}^3}{f_1 f_4}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4} \qquad \text{and} \qquad \chi(-q) = \frac{f_1}{f_2}. \tag{1.1}$$

The Rogers-Ramanujan functions are defined as

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{1.2}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.3}$$

where the two equalities on the right of (1.2) and (1.3) are the celebrated Rogers-Ramanujan identities. Fourty modular relations for $G(q)$ and $H(q)$ were found by Ramanujan [12] and are known as Ramanujan's fourty identities. For a history of the forty identities as well as many proofs, see the excellent monograph [5] of B. C. Berndt, et. al. In his Ph. D thesis [13], S. Robins used a computer and the theory of modular forms to discover and prove following new relations for $G(q)$ and $H(q)$, which are analogus to Ramanujan forty identities:

$$G^2(q)H(q^2) - H^2(q)G(q^2) = 2qH(q)H^2(q^2)\frac{f_{10}^2}{f_5^2} \tag{1.4}$$

and

$$G^2(q)H(q^2) + H^2(q)G(q^2) = 2G(q)G^2(q^2)\frac{f_{10}^2}{f_5^2}. \tag{1.5}$$

B. Gordon and R. J. McIntosh [8] have proved (1.4) and (1.5) by employing the following identity due to D. Hickerson [10]:

$$j(-x,q)j(y,q) - j(x,q)j(-y,q) = 2xj(x^{-1}y,q^2)j(qxy,q^2), \tag{1.6}$$

where

$$j(x,q) = (x;q)_\infty (q/x;q)_\infty (q;q)_\infty.$$

The identity (1.6) is the generalization to following identities of Ramanujan [11], [4, Entry 29, p. 45]

Theorem 1.1. If $ab = cd$, then

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = f(ac,bd)f(ad,bc) \tag{1.7}$$

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, ac^2d\right) f\left(\frac{b}{d}, acd^2\right). \tag{1.8}$$

Recently, C. Gugg [9], has obtained an alternating proof of (1.4) and (1.5) and established many interesting results from them related to the following famous Rogers-Ramanujan continued fraction:

$$R(q) := \frac{q^{1/5}}{1}\,\frac{q}{+1}\,\frac{q^2}{+1}\,\frac{q^3}{+1}+\cdots. \tag{1.9}$$

And also, C. Gugg [9] employed (1.4) and (1.5) to give a new proof of five identities of Ramanujan's forty identities and obtained four new identities which are are analogous to Ramanujan's forty identities.

On page 366 of his Lost Notebook [11], Ramanujan investigated another continued fraction

$$G(q) := \frac{q^{1/3}}{1}\,\frac{q+q^2}{+\;1}\,\frac{q^2+q^4}{+\;1}\,\frac{q^3+q^6}{+\;1}+\cdots \tag{1.10}$$

and claimed that there are many results of $G(q)$ which are analogous to those of (1.9). The continued fraction (1.10) is now famous as Ramanujan's cubic continued fraction. Motivated by Ramanujan's claim, H. H. Chan [6] established many identities for $G(q)$.

In this paper, we consider the following two analogous functions of the Rogers-Ramanujan functions (1.2) and (1.3) :

$$L(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{f(-q,-q^5)}{\psi(-q)} \tag{1.11}$$

and

$$M(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{f(-q^3,-q^3)}{\psi(-q)}. \tag{1.12}$$

The two inequalities on the right of (1.11) and (1.12) are the cubic identities due to G. E. Andrews [1] and L. J. Slater [15] respectively. Andrews [1] shown that

$$G(q) := q^{1/3}\frac{L(q)}{M(q)} = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}. \tag{1.13}$$

In this paper, we derive following two modular relation for $L(q)$ and $M(q)$ due to W. Chu [7], which are analogous to (1.4) and (1.5):

$$L(q^2)M^2(q) - L^2(q)M(q^2) = 2q\frac{f_3f_2^2f_{12}^5}{f_1f_6^3f_4^3f_8} \tag{1.14}$$

and

$$L(q^2)M^2(q) + L^2(q)M(q^2) = 2\frac{f_3f_4f_{12}}{f_1f_6f_8}. \tag{1.15}$$

We derive four modular relations for $L(q)$ and $M(q)$ by employing (1.14) and (1.15). Also, we derive many theta function identities and also certain identities for $G(q)$. We close this section by noting the following:

$$L(q) := \frac{f_6^2}{f_4f_3} \qquad \text{and} \qquad M(q) := \frac{f_3^2f_2}{f_1f_4f_6}. \tag{1.16}$$

2 Main Theorem

Lemma 2.1. We have

$$L(q^2) = \frac{\varphi(-q^4)}{\varphi(-q^6)}L(q)L(-q) \tag{2.1}$$

and

$$M(q^2) = \frac{\varphi(-q^4)}{\varphi(-q^6)} M(q)M(-q). \tag{2.2}$$

Proof. From Entry 25(iii) of Chapter 16 [4, p. 40], we have

$$\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2) \tag{2.3}$$

and

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2). \tag{2.4}$$

Also from Entry 24(i) of Chapter 16 [4, p. 39], we have

$$\frac{\psi(q)}{\psi(-q)} = \sqrt{\frac{\varphi(q)}{\varphi(-q)}}. \tag{2.5}$$

By definition of $L(q)$, we have

$$L(q^2) = \frac{f(-q^2, -q^{10})}{\psi(-q^2)}.$$

Now using the definition $f(a, b)$ and using the simple fact

$$(q^{2n}; q^{2m})_\infty = (-q^n; q^m)_\infty (q^n; q^m)_\infty$$

in the above, we find that

$$L(q^2) = \frac{\psi(-q)\psi(q)L(q)L(-q)}{\psi(-q^2)}.$$

Using (2.3), we have

$$\begin{aligned} L(q^2) &= \frac{\psi(q^2)\varphi(-q^2)}{\psi(-q^2)\varphi(-q^6)} L(q)L(-q) \\ &= \sqrt{\frac{\varphi(q^2)}{\varphi(-q^2)}} \frac{\varphi(-q^2)}{\varphi(-q^6)} L(q)L(-q) \\ &= \frac{\varphi(-q^4)}{\varphi(-q^6)} L(q)L(-q), \end{aligned}$$

where we have successively applied (2.3), (2.5) and (2.4). This proves (2.1). The proof of (2.2) is similar.

Lemma 2.2. We have

$$\text{(i)} \;\; L(-q)M(q) - L(q)M(-q) = 2q \frac{f_2 f_{12}^4}{f_4^3 f_6^2} \tag{2.6}$$

and

$$\text{(ii)} \;\; L(-q)M(q) + L(q)M(-q) = 2\frac{f_4}{f_2}. \tag{2.7}$$

Proof. Setting $a = q$, $b = q^5$, $c = -q^3$, $d = -q^3$ in (1.8), we find that

$$f(q, q^5)f(-q^3, -q^3) - f(-q, -q^5)f(q^3, q^3) = 2qf^2(-q^2, -q^{10}). \tag{2.8}$$

Now, using (1.11) and (1.12), in the above , we have

$$L(-q)M(q) - L(q)M(-q) = 2q \frac{\psi^2(-q)L^2(-q^2)}{\psi(q)\psi(-q)}. \tag{2.9}$$

Employing (1.1) and (1.16) in the right of the above, we obtain (2.6). Similarly by setting $a = q$, $b = q^5$, $c = -q^3$ and $d = -q^3$ in (1.7) and then using (1.1) and (1.16), we obtain (2.7).

Theorem 2.3. The identities (1.14) and (1.15) hold.

Proof. Using (2.1) and (2.2) in (2.6) and then using (1.1) and (1.16), we obtain (1.14). Using (2.1) and (2.2) in (2.7) and then using (1.1) and (1.16), we deduce (1.15).

3 Applications of (1.14) and (1.15).

Let

$$u := G(q), \qquad v := G(q^2), \qquad w := G(-q) \quad \textit{and} \quad k = \frac{u^2}{v}.$$

Theorem 3.1. (Chan[6]). We have

$$(i) \qquad u + w + 2u^2w^2 = 0 \tag{3.1}$$

and

$$(ii) \qquad u^2 - v + 2v^2u = 0. \tag{3.2}$$

Proof. From (1.13), we have

$$u = q^{1/3}\frac{L(q)}{M(q)} \qquad \text{and} \qquad w = -q^{1/3}\frac{L(-q)}{M(-q)}.$$

Dividing (2.6) throughout by $q^{-1/3}M(q)M(-q)$, and using the above, we find that

$$u + w = 2q^{4/3}\frac{f_2 f_{12}^4}{f_4^3 f_6^2 M(q)M(-q)}. \tag{3.3}$$

From (1.16) and (1.1), we deduce that

$$\frac{f_2 f_{12}^4}{f_4^2 f_6^2} = \frac{L^2(q)L^2(-q)}{M(q)M(-q)}.$$

Using this in the right of (3.3), we deduce (3.1). It is easy to see from (1.16) that

$$\frac{f_3 f_2^2 f_{12}^5}{f_1 f_6^3 f_4^3 f_8} = \frac{L(q)L^2(q)M(q)}{M(q^2)}.$$

Dividing (1.14) throughout by $q^{-2/3}M^2(q)M(q^2)$, and employing the above, we obtain (3.2).

Theorem 3.2. We have

$$\frac{1+k}{1-k} = \frac{\psi^2(q^2)}{q\psi^2(q^6)}. \tag{3.4}$$

Proof. From (1.14) and (1.15), we find that

$$\frac{1+\frac{L^2(q)M(q^2)}{M^2(q)L(q^2)}}{1-\frac{L^2(q)M(q^2)}{M^2(q)L(q^2)}} = \frac{f_4^4 f_6^2}{q f_2^2 f_{12}^4}.$$

Using (1.1) in the right of the above identity and by definition of k, we obtain

$$\frac{1+k}{1-k} = \frac{\psi^2(q^2)}{q\psi^2(q^6)}.$$

This proves (3.4).

Theorem 3.3. We have

$$1+\frac{1}{u^3} = \frac{\psi^4(q)}{q\psi^4(q^3)}, \tag{3.5}$$

$$v^3 = \frac{(1-k)^2}{4k}, \tag{3.6}$$

$$u^3 = \frac{k(1-k)}{2}, \tag{3.7}$$

$$1-8u^3 = \frac{\varphi^4(-q)}{\varphi^4(-q^3)}, \tag{3.8}$$

and

$$2k-1 = \frac{\varphi^2(-q)}{\varphi^2(-q^3)}. \tag{3.9}$$

The identity (3.5) is recorded by Ramanujan in his notebook [11, p. 24] and is proved by Berndt [4, p. 346]. The identity (3.8) is due to Berndt [4, p. 347].

Proof of (3.5). Squaring both sides of (3.4) and then subtracting by -1 on both sides, we find that

$$\frac{\psi^4(q^2)}{q^2\psi^4(q^6)} - 1 = \frac{4vu^2}{(v-u^2)^2}.$$

Using (3.2) in the denominator of right of the above identity, we obtain

$$\frac{\psi^4(q^2)}{q^2\psi^4(q^6)} - 1 = \frac{1}{v^3}. \tag{3.10}$$

Changing q to $q^{1/2}$ in the above, we obtain (3.5).

Proof of (3.6). From (3.10) and (3.4), we have

$$1+\frac{1}{v^3} = \left(\frac{1+k}{1-k}\right)^2$$

which implies

$$v^3 = \frac{(1-k)^2}{4k}.$$

Proof of (3.7). By the definition of k, $v = u^2/k$. Using this in (3.6), then taking square root on both sides, we obtain (3.7).

Proof of (3.8). From (1.1), we see that

$$\frac{\varphi^4(q)}{\varphi^4(q^3)} = \frac{\psi^8(q)}{\psi^8(q^3)} \cdot \frac{\psi^4(q^6)}{\psi^4(q^2)}.$$

Using (3.5) and (3.10) in the right of the above identity

$$\frac{\varphi^4(q)}{\varphi^4(q^3)} = \frac{v^3}{u^6}\left[\frac{(1+u^3)^2}{1+v^3}\right].$$

Now using (3.6) and (3.7), we find that

$$\begin{aligned}\frac{\varphi^4(q)}{\varphi^4(q^3)} &= \left(1-\frac{2}{k}\right)^2 \\ &= 1-\frac{4}{k^2}(1-k) \\ &= 1-8\frac{u^3}{k^3} \\ &= 1-8\frac{v^3}{u^3} \\ &= 1+8w^3,\end{aligned}$$

where, we have successively applied (3.7), definition of k, $v=-uw$. Changing q to $-q$, we obtain (3.8).

Proof of (3.9). Using (3.7) in (3.8), we obtain

$$\frac{\varphi^4(-q)}{\varphi^4(-q^3)} = (2k-1)^2,$$

which is equivalent to (3.9).

Theorem 3.4. We have

$$(i) \qquad M(q)M(q^2) - 2qL(q)L(q^2) = \frac{L^2(q)M^2(q^2)}{L(q^2)M(q)} \tag{3.11}$$

and

$$(ii) \qquad M(q)M(q^2) + 2qL(q)L(q^2) = \frac{\varphi^2(q)}{\varphi^2(q^3)}\,\frac{L^2(q)M^2(q^2)}{M(q)L(q^2)}. \tag{3.12}$$

Proof of (i). The identity (3.11) is equivalent to (3.2).

Proof of (ii). From (3.9) and definition of k, we have

$$2u^2 - v = v\frac{\varphi^2(-q)}{\varphi^2(-q^3)}.$$

Changing q to $-q$ in the above, we deduce that

$$2w^2 - v = v\,\frac{\varphi^2(q)}{\varphi^2(q^3)}.$$

Using $v = -uw$, we see that

$$2\ \frac{v^2}{u^2} - v = v\ \frac{\varphi^2(q)}{\varphi^2(q^3)},$$

which is equivalent to

$$2v - u^2 = u^2\ \frac{\varphi^2(q)}{\varphi^2(q^3)}.$$

Using (3.2) in the above, we get

$$1 + 2uv = \frac{u^2}{v}\ \frac{\varphi^2(q)}{\varphi^2(q^3)}$$

which is readily equivalent to (3.12).

Theorem 3.5. We have

$$\text{(i)} \qquad \varphi^2(q) + \varphi^2(q^3) = 2\varphi^2(q^3)\frac{v}{u^2}, \tag{3.13}$$

$$\text{(ii)} \qquad \varphi^2(q) - \varphi^2(q^3) = 4\varphi^2(q^3)\frac{v^2}{u} \tag{3.14}$$

and

$$\text{(iii)} \qquad \psi^2(q^2) + q\psi^2(q^6) = q\psi^2(q^6)\frac{1}{uv}. \tag{3.15}$$

The identities (3.13) and (3.14) are due to Shen [14]. For an alternative proof of (3.13) and (3.14) one may refer Baruah [3] and Baruah [2] respectively. The identity (3.15) is due to Baruah [3].

Proof of (i). Dividing (3.12) by (3.11) and using the definitions of u and v, we get

$$\frac{1 + 2uv}{1 - 2uv} = \frac{\varphi^2(q)}{\varphi^2(q^3)}. \tag{3.16}$$

Adding 1 on both sides of the above identity, we obtain

$$\varphi^2(q) + \varphi^2(q^3) = \frac{2}{1 - 2uv}\varphi^2(q^3).$$

Now using (3.11) and (1.13) in the denominator of the right hand side of the above, we deduce the required result.

Proof of (ii). Subtracting 1 from both sides of (3.16) and using (3.11) and (1.13) on the righthand side, we obtain (3.14).

Proof of (iii). Adding 1 to both sides of (3.4) and then simplifying , we see that

$$\psi^2(q^2) + q\psi^2(q^6) = 2q\psi^2(q^6)\frac{1}{1 - k}.$$

Now using (3.7) in the denominator of the right hand side of the above, we deduce the required result.

Theorem 3.6. We have

$$\text{(i)} \qquad \varphi^2(q) + 3\varphi^2(q^3) = 4\frac{\varphi(q^3)\psi^3(q)}{\varphi(q)\psi(q^3)},$$

$$\text{(ii)} \qquad \varphi^2(q) - 3\varphi^2(q^3) = -2\frac{\varphi(q^3)\varphi^3(-q^2)}{\varphi(q)\varphi(-q^6)}$$

and

$$\text{(iii)} \qquad \psi^2(q^2) - 3q\psi^2(q^6) = \psi^2(q^6)\frac{\varphi^2(-q)\chi^3(-q^3)\chi^3(-q^6)}{\varphi^2(-q^3)\chi(-q)\chi(-q^2)}.$$

Proof of (i). Adding 3 on both sides of (3.16), we see that

$$\frac{4(1-uv)}{1-2uv} = \frac{\varphi^2(q)}{\varphi^2(q^3)} + 3. \tag{3.17}$$

Also note that (3.2) can be rewritten as

$$1 - uv = \frac{(v+u^2)}{2v}.$$

Now using (1.13) in the right hand side of the above and then employing (1.15) and (1.16), we obtain

$$1 - uv = \frac{f_1 f_4^3 f_6^2}{f_2^2 f_3^3 f_{12}}. \tag{3.18}$$

Next, using (3.11), (1.13) and (1.16) along with (3.18) in the left hand side of (3.17), we see that

$$\frac{\varphi^2(q)}{\varphi^2(q^3)} + 3 = \frac{f_2 f_3^3 f_4^2 f_{12}^2}{f_1 f_6^7} \tag{3.19}$$

which is equivalent to the required result upon using (1.1).

Proof of (ii). Adding −9 on both sides of (3.8), we see that

$$-8(1+u^3) = \frac{\varphi^4(-q)}{\varphi^4(-q^3)} - 9.$$

Using (3.5) in the left hand side of the above and then changing q to $-q$, we obtain

$$8\frac{\psi^4(-q)}{q\psi^4(-q^3)}w^3 = \frac{\varphi^4(q)}{\varphi^4(q^3)} - 9.$$

Now substituting $w = -\dfrac{v}{u}$ and using (1.13) in the left hand side of the above, we see that

$$-8\frac{\psi^4(-q)}{\psi^4(-q^3)}\frac{L^2(q^2)}{M^2(q^2)}\frac{M^3(q)}{L^3(q)} = \frac{\varphi^4(q)}{\varphi^4(q^3)} - 9,$$

which can be rewritten as

$$\frac{\varphi^4(q)}{\varphi^4(q^3)} - 9 = -8\frac{f_1 f_2^2 f_3^5 f_4 f_{12}^5}{f_6^{14}} \tag{3.20}$$

on using (1.1) and (1.16). Dividing (3.20) by (3.19), we obtain

$$\frac{\varphi^2(q)}{\varphi^2(q^3)} - 3 = -2\frac{f_1^2 f_2 f_3^2 f_{12}^3}{f_6^7 f_4},$$

which on using (1.1) can be rewritten as

$$\frac{\varphi^2(q)}{\varphi^2(q^3)} - 3 = -2\frac{\varphi^3(-q^2)}{\varphi(q^3)\varphi(q)\varphi(-q^6)}.$$

The above expression on simplification yields the required result.

Proof of (iii). Subtracting 3 from both sides of (3.4), we get

$$\frac{2(2k-1)}{1-k} = \frac{\psi^2(q^2)}{q\psi^2(q^6)} - 3.$$

Now employing (3.9) and (3.7) in the above, we see that

$$\frac{\varphi^2(-q)}{\varphi^2(-q^3)}\frac{1}{uv} = \frac{\psi^2(q^2)}{q\psi^2(q^6)} - 3.$$

Simplifying the above and on using (1.13), we obtain the required result.

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