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Generalised Cauchy-Riemann Lightlike Submanifolds of Indefinite Kenmotsu Manifolds

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Abstract. In this paper we introduce the notion of generalised Cauchy-Riemann (GCR) lightlike submanifolds of indefinite Kenmotsu manifold which includes invariant, contact CR, contact screen Cauchy-Riemann (contact SCR) lightlike subclasses [12]. A condition has been discussed for GCR-lightlike submanifold of an indefinite Kenmotsu manifold to be minimal. We have also studied totally contact umbilical GCR-lightlike submanifolds. Examples of GCR-lightlike submanifold of an indefinite Kenmotsu manifold have also been given.

Keywords: Degenerate metric, Kenmotsu manifold, CR-submanifold.

MSC 2000 classification: primary 53C40; secondary 53C15, 53C50, 53D15

Introduction

In the theory of submanifolds of semi-Riemannian manifolds it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of normal vector bundle and the tangent bundle is non-trivial making it more interesting and remarkably different from the study of non-degenerate submanifolds. The geometry of lightlike submanifolds of indefinite Kaehler manifolds was studied by Duggal and Bejancu [6]. They have also studied possible lightlike submanifolds of indefinite Kaehler manifolds.

On the other hand, a general notion of lightlike submanifolds of indefinite Sasakian manifolds was introduced by Duggal and Sahin [8]. Recently we defined the lightlike submanifolds of indefinite Kenmotsu manifolds [12] and have studied Cauchy-Riemann and screen Cauchy-Riemann lightlike submanifolds. Moreover, we obtained that there do not exist inclu-

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sion relation between these two classes. The objective of this paper is to define a generalised Cauchy-Riemann lightlike submanifold of indefinite Kenmotsu manifolds, which includes invariant, screen real, contact CR lightlike subcases and real hypersurfaces.

In section 1, we have collected the formulae and information which are useful in subsequent sections. In section 2, we have studied GCR-lightlike submanifolds of an indefinite Kenmotsu manifold. In section 3, we have obtained the existence and non-existence conditions for GCR-lightlike submanifolds of indefinite Kenmotsu manifolds and have given an example of GCR-lightlike submanifold of R_4^{13}. In section 4, we have studied minimal GCR-lightlike submanifolds of indefinite Kenmotsu manifolds and have given an example of minimal GCR lightlike submanifold in R_4^{15}.

1 Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, V, \eta, \overline{g}\}$, where ϕ is a (1,1) tensor field, V a vector field, η a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$\left\{ \begin{array}{l} \phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1 \\ \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y), \quad \overline{g}(X, V) = \eta(X) \end{array} \right. \tag{1}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite Kenmotsu manifold if [5],

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(\phi X, Y)V + \eta(Y)\phi X, \quad \text{and} \quad \overline{\nabla}_X V = -X + \eta(X)V \tag{2}$$

for any $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denote the Levi-Civita connection on $\overline{M}$.

A submanifold M^m immersed in a semi-Riemannian manifold $\{\overline{M}^{m+k}, \overline{g}\}$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\overline{g}$ whose radical distribution $\mathrm{Rad}(TM)$ is of rank r, where $1 \leq r \leq m$. Now, $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \bigcup_{x \in M} \{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_x M\} \tag{3}$$

Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian comlementary distribution of $\mathrm{Rad}(TM)$ in TM, that is, $TM = \mathrm{Rad}(TM) \perp S(TM)$.

We consider a *screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $\mathrm{Rad}(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}$($cf.$[6], $page$144).

Let $\mathrm{tr}(TM)$ be the complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$\left\{ \begin{array}{l} tr(TM) = ltr(TM) \perp S(TM^{\perp}) \\ T\overline{M}|_M = S(TM) \perp [\mathrm{Rad}(TM) \bigoplus ltr(TM)] \perp S(TM^{\perp}). \end{array} \right. \tag{4}$$

A submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is said to be

(i) r-lightlike if $r < \min\{m, k\}$;

(ii) coisotropic if $r = k < m$, $S(TM^{\perp}) = \{0\}$;

(iii) isotropic if $r = m < k$, $S(TM) = \{0\}$;

(iv) totally lightlike if $r = m = k$, $S(TM) = \{0\} = S(TM^{\perp})$.

Let $\overline{\nabla}$, ∇ and ∇^t denote the linear connections on $\overline{M}$, M and vector bundle tr(TM), respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X, Y \in \Gamma(TM), \tag{5}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \quad \forall U \in \Gamma(tr(TM)), \tag{6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively and A_U is the shape operator of M with respect to U. Moreover, according to the decomposition (4), h^l, h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^\perp))$-valued *lightlike second fundamental form* and *screen second fundamental form* of M, respectively, then

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \forall X, Y \in \Gamma(TM), \tag{7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X,N), N \in \Gamma(ltr(TM)), \tag{8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X,W), W \in \Gamma(S(TM^\perp)), \tag{9}$$

where $D^l(X,W)$ and $D^s(X,N)$ are the projections of ∇^t on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^\perp))$, respectively and ∇^l, ∇^s are linear connections on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^\perp))$, respectively. We call ∇^l, ∇^s the lightlike and screen transversal connections on M, and A_N, A_W are shape operators on M with respect to N and W, respectively. Using (5) and (7)$\sim$(9), we obtain

$$\overline{g}(h^s(X,Y), W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{10}$$

$$\overline{g}(D^s(X,N), W) = \overline{g}(N, A_W X). \tag{11}$$

Let $\overline{P}$ denote the projection of TM on $S(TM)$ and let ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and Rad(TM), respectively. Then from the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_X \overline{P} Y = \nabla^*_X \overline{P} Y + h^*(X, \overline{P} Y), \tag{12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{13}$$

for X, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\, TM)$, where h^*, A^* are the second fundamental form and shape operator of distributions $S(TM)$ and Rad(TM), respectively. From (12) and (13), we get

$$\overline{g}(h^l(X, \overline{P} Y), \xi) = g(A^*_\xi X, \overline{P} Y), \tag{14}$$

$$\overline{g}(h^*(X, \overline{P} Y), N) = g(A_N X, \overline{P} Y), \tag{15}$$

$$\overline{g}(h^l(X,\xi), \xi) = 0, \qquad A^*_\xi \xi = 0. \tag{16}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, by using (7), we obtain

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y), Z) + \overline{g}(h^l(X,Z), Y). \tag{17}$$

However, it is important to note that ∇^*, ∇^{*t} are metric connections on $S(TM)$ and Rad(TM), respectively.

A plane section Π in $T_x\overline{M}$ of an indefinite Kenmotsu manifold $\overline{M}$ is called a ϕ-section if it is spanned by a unit vector X orthogonal to V and ϕX, where X is non-null vector field on $\overline{M}$. The sectional curvature $K(\Pi)$ with respect to Π determined by X is called a ϕ-sectional curvature. If $\overline{M}$ has a ϕ-sectional curvature c which does not depend on the ϕ-section at each point, then c is constant in $\overline{M}$. Then, $\overline{M}$ is called an indefinite Kenmotsu space form and is denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of $\overline{M}(c)$ is given by [5]

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c-3}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ + \overline{g}(X,Z)\eta(Y)V - \overline{g}(Y,Z)\eta(X)V + \overline{g}(\phi Y,Z)\phi X \\ + \overline{g}(\phi Z,X)\phi Y - 2\overline{g}(\phi X,Y)\phi Z\} \quad (18)\end{aligned}$$

for any X, Y and Z vector fields on $\overline{M}$.

Definition 1. A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is totally umbilical in M if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that for all $X, Y \in \Gamma(TM)$,

$$h(X,Y) = Hg(X,Y) \quad (19)$$

Using (7) and (19), it is easy to see that M is totally umbilical if and only if on each coordinate neighbourhood $\ddot{U}$ there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(ltr(TM^\perp))$ such that

$$\begin{cases} h^l(X,Y) = H^l g(X,Y) & D^l(X,W) = 0 \\ h^s(X,Y) = H^s g(X,Y) & \forall X,Y \in \Gamma(TM), \quad W \in \Gamma(S(TM^\perp)). \end{cases} \quad (20)$$

Similar to the concept of contact totally umbilical submanifold of Sasakian manifold introduced in the book of Yano and Kon (cf.[10], page 374), we define:

Definition 2. If the second fundamental form h of a submanifold M, tangent to the structure vector field V, of an indefinite Kenmotsu manifold $\overline{M}$, be of the form

$$h(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha \quad (21)$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M. Then M is called contact totally umbilical submanifold of $\overline{M}$. Further if $\alpha = 0$, then it is called totally geodesic.

The above Definition also holds for a lightlike submanifold M. For a contact totally umbilical submanifold M, we have

$$\begin{cases} h^l(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_l \\ h^s(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_s \end{cases} \quad (22)$$

where $\alpha_s \in \Gamma(S(TM^\perp))$ and $\alpha_l \in \Gamma(ltr(TM))$.

We have the following definition by Bejancu and Duggal [3].

Definition 3. A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is minimal if

(i) $h^s = 0$ on $\mathrm{Rad}(TM)$;

(ii) trace $h = 0$, where trace is written with respect to g restricted to $S(TM)$.

Definition 4. A lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$ is a screen real submanifold if $\mathrm{Rad}(TM)$ and $S(TM)$ are, respectively, invariant and anti-invariant with respect to ϕ.

The above definition is the lightlike version (cf. [7]) of totally real submanifold of an almost Hermitian (or contact) manifold [10].

The following result is important for our work.

Proposition 1. *[6] The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM)$, $S(TM^\perp)$ and $ltr(TM)$.*

2 Generalised Cauchy-Riemann (GCR) Lightlike Submanifolds

We have the following definition of a *GCR-lightlike submanifold*:

Definition 5. Let $(M, g, S(TM), S(TM^{\perp}))$ be a real lightlike submanifold, tangent to structure vector field V, immersed in an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$. Then M is called a generalised Cauchy-Riemann lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(A) There exist two subbundles D_1 and D_2 of $\mathrm{Rad}(TM)$ on M such that

$$\mathrm{Rad}\, TM = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM) \tag{23}$$

(B) There exist two vector subbundles D_0 and D' of $S(TM)$ such that over M

$$\begin{cases} S(TM) = \{\phi(D_2) \oplus D'\} \perp D_0 \perp \{V\}, \\ \phi D_0 = D_0, \quad \phi(D') = L_1 \perp L_2 \end{cases} \tag{24}$$

where D_0 is nondegenerate and L_1 and L_2 are vector subbundles of $S(TM^{\perp})$ and $ltr(TM)$, respectively.

Thus we have the following decomposition:

$$TM = D \oplus D' \perp \{V\}, \quad D = \mathrm{Rad}\, TM \perp \phi(D_2) \perp D_0 \tag{25}$$

A contact GCR-lightlike submanifold is said to be proper if $D_0 \neq \{0\}$, $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $L_1 \neq \{0\}$. Thus, from Definition 5, we have
(a) Condition (A) implies that $\dim(\mathrm{Rad}TM) \geq 3$.
(b) Condition (B) implies that $\dim(D) \geq 2s \geq 6$ and $\dim(D_2) = \dim(L_2)$. Thus $\dim(M) \geq 9$ and $\dim(\overline{M}) \geq 13$.
(c) Any proper 9-dimensional contact GCR-lightlike submanifold is 3-lightlike.
(d) (a) and contact distribution $(\eta = 0)$ imply that $\mathrm{index}(\overline{M}) \geq 4$.

Proposition 2. *A GCR-lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$ is a contact CR (respectively, contact SCR lightlike submanifold) if and only if $D_1 = \{0\}$ (respectively, $D_2 = \{0\}$).*

Proof. Let M be a contact CR-lightlike submanifold. Then $\phi\mathrm{Rad}TM$ is a distribution on M such that $\mathrm{Rad}TM \cap \phi\mathrm{Rad}TM = \{0\}$ imply that $ltr(TM) \cap \phi(ltr(TM)) = \{0\}$. Thus it follows that $\phi(ltr(TM)) \subset S(TM)$.

Conversely, suppose M is a GCR-lightlike submanifold of an indefinite Kenmotsu manifold such that $D_1 = \{0\}$. Then from (23), we have $D_2 = \mathrm{Rad}TM$. Therefore, $\mathrm{Rad}TM \cap \phi(\mathrm{Rad}TM) = \{0\}$, implying that M is a contact CR-lightlike submanifold of an indefinite Kenmotsu manifold.

Similarly, it can be proved that GCR-lightlike submanifold of an indefinite Kenmotsu manifold is a contact SCR lightlike submanifold if and only if $D_2 = \{0\}$. The following follows:

Proposition 3. *There exists no coisotropic, isotropic or totally lightlike proper GCR-lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$.*

Proof. If M is isotropic or totally lightlike, then $S(TM) = \{0\}$ and if M is coisotropic then $S(TM^{\perp})$. Hence, conditions (A) and (B) of definition 5 are not satisfied.

It is easy to see that any contact CR-lightlike three-dimensional submanifold is 1-lightlike real hypersurface [12]. Moreover, it is proved in the same paper that contact SCR-lightlike submanifolds have invariant and screen real lightlike subcases. Thus, from Proposition 2 it follows that GCR-lightlike submanifold is an umbrella of real hypersurfaces, invariant, screen real and contact CR-lightlike submanifolds.

Hereafter, $(R_q^{2m+1}, \phi_0, V, \eta, \overline{g})$ will denote the manifold R_q^{2m+1} with its usual Kenmotsu structure given by

$$\begin{cases} \eta = dz, \qquad\qquad V = \partial z, \\ \overline{g} = \eta \otimes \eta + e^{2z}(-\sum_{i=1}^{q/2}(dx^i \otimes dx^i + dy^i \otimes dy^i) \\ \quad + \sum_{i=q+1}^{m}(dx^i \otimes dx^i + dy^i \otimes dy^i), \\ \phi_0(\sum_{i=1}^{m}(X_i \partial x^i + Y_i \partial y^i) + Z\partial z) = \sum_{i=1}^{m}(Y_i \partial x^i - X_i \partial y^i) \end{cases} \tag{26}$$

where (x^i, y^i, z) are the Cartesian coordinates.

Example 1. Let $\overline{M} = (R_4^{13}, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature (-, -, +, +, +, +, -, -, +, +, +, +, +) with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}. \tag{27}$$

Consider a submanifold M of R_4^{13}, defined by

$$\begin{cases} x^4 = x^1 \cos\theta - y^1 \sin\theta, \qquad y^4 = x^1 \sin\theta + y^1 \cos\theta, \\ x^2 = y^3, \qquad x^5 = \sqrt{1 + (y^5)^2}, \qquad y^5 \neq \pm 1 \end{cases} \tag{28}$$

Then a local frame of TM is given by

$$\begin{cases} Z_1 = e^{-z}(\partial x_1 + \cos\theta \partial x_4 + \sin\theta \partial y_4), \\ Z_2 = e^{-z}(-\sin\theta \partial x_4 + \partial y_1 + \cos\theta \partial y_4), \\ Z_3 = e^{-z}(\partial x_2 + \partial y_3), \qquad Z_4 = e^{-z}(\partial x_3 - \partial y_2), \\ Z_5 = e^{-z}\partial x_6, \qquad Z_6 = e^{-z}\partial y_6, \\ Z_7 = e^{-z}(y^5 \partial x_5 + x^5 \partial y_5), \qquad Z_8 = e^{-z}(\partial x_3 + \partial y_2), \\ Z_9 = V = \partial z, \end{cases} \tag{29}$$

Hence, $\mathrm{Rad}TM = \mathrm{span}\{Z_1, Z_2, Z_3\}$. Moreover $\phi_0 Z_1 = -Z_2$ and $\phi_0 Z_3 = Z_4 \in \Gamma S(TM)$. Thus $D_1 = \mathrm{span}\ \{Z_1, Z_2\}$, $D_2 = \mathrm{span}\ \{Z_3\}$. Hence, (A) holds. Next, $\phi_0 Z_5 = -Z_6$, which implies that $D_0 = \{Z_5, Z_6\}$ is invariant with respect to ϕ_0. By direct calculations, we get

$$S(TM^\perp) = \mathrm{span}\ \{W = e^{-z}(x^5 \partial x_5 - y^5 \partial y_5)\}$$

such that $\phi_0(W) = -Z_7$. Hence $L_1 = S(TM^\perp)$ and

$$ltr(TM) = \mathrm{span} \begin{cases} N_1 = e^{-z}(-\partial x_1 + \cos\theta \partial x_4 + \sin\theta \partial y_4), \\ N_2 = e^{-z}(-\sin\theta \partial x_4 - \partial y_1 + \cos\theta \partial y_4), \\ N_3 = e^{-z}(-\partial x_2 + \partial y_3), \end{cases}$$

such that $\phi_0(N_1) = -N_2$ and $\phi_0(N_3) = Z_8$. Hence, $L_2 = \mathrm{span}\{N_3\}$ and $D' = \mathrm{span}\{\phi_0(N_3), \phi_0 W\}$. Thus M is a contact GCR-lightlike submanifold of R_4^{13}.

3 Existence and Non-existence Theorems

We prove an existence Theorem for GCR-lightlike submanifolds in an indefinite Kenmotsu space form:

Theorem 1. *Let M be a lightlike submanifold with structure vector field tangent to M of an indefinite Kenmotsu space form $\overline{M}(c)$ with $c \neq -1$. Then, M is a GCR-lightlike submanifold of $\overline{M}(c)$ if and only if* *(i) The maximal invariant subspaces of T_pM, for $p \in M$, define a distribution*

$$D = D_1 \perp D_2 \perp \phi(D_2) \perp D_0$$

where $RadTM = D_1 \oplus D_2$, and D_0 is non-degenerate invariant distribution.

(ii) There exists a lightlike transversal vector bundle $ltr(TM)$ such that

$$\overline{g}(\overline{R}(X,Y)\xi, N) = 0, \quad \forall X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), \xi \in \Gamma(\mathrm{Rad}\,TM).$$

(iii) There exists a vector subbundle $\overline{D}$ on M such that

$$\overline{g}(\overline{R}(X,Y)W_1, W_2) = 0,\ \forall W_1, W_2 \in \Gamma(\overline{D}),$$

where $\overline{D}$ is orthogonal to D and $\overline{R}$ is the curvature tensor of $\overline{M}(c)$.

Proof. Suppose that M is a GCR-lightlike submanifold of $\overline{M}(c)$ with $c \neq$ - 1. Then $D = D_1 \perp D_2 \perp \phi(D_2) \perp D_0$ is a maximal invariant subspace. From (18), we have

$$\overline{g}(\overline{R}(X,Y)\xi, N) = -\frac{c+1}{2}\{g(\phi X, Y)\overline{g}(\phi\xi, N)\}$$

$\forall X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), \xi \in \Gamma(D_2)$. Since $c \neq -1$, $g(\phi X, Y) \neq 0$ and $\overline{g}(\phi\xi, N) = 0$, and theerefore, we get

$$\overline{g}(\overline{R}(X,Y)\xi, N) = 0.$$

Similarly we have

$$\overline{g}(\overline{R}(X,Y)W_1, W_2) = -\frac{c+1}{2}\{g(\phi X, Y)\overline{g}(\phi W_1, W_2)\} = 0,$$

$\forall X, Y \in \Gamma(D_0)$, and $W_1, W_2) \in \Gamma\phi(L_1)$.

Conversely, assume that (i), (ii) and (iii) are satisfied. Then, (i) implies that D is a invariant. From (ii) and (18), we have

$$\overline{g}(\phi\xi, N) = 0 \tag{30}$$

which implies that $\phi\xi \in \Gamma(S(TM))$.

From (30), we get that $\overline{g}(\xi, \phi N) = 0$. Hence, a part of $\phi ltr(TM)$ also belongs to $S(TM)$. Similarly from (iii) and (18), we get

$$\overline{g}(\phi W_1, W_2) = 0 \tag{31}$$

which implies that $\phi(\overline{D})$ is orthogonal to $\overline{D}$. Since $\overline{D}$ is non-degenerate,

$$\overline{g}(\phi W_1, \phi W_2) = g(W_1, W_2) \neq 0$$

Also, we have $\overline{g}(\phi\xi, W) = -\overline{g}(\xi, \phi W) = 0$ implies that $\phi(\overline{D})$ is orthogonal to $\mathrm{Rad}(TM)$. This also implies that $\phi(\overline{D})$ does not belong to $ltr(TM)$. On the other hand, invariant and non-invariant D_0 imply $g(\phi W, X) = 0$ for $X \in \Gamma(D_0)$. Thus, $\overline{D} \perp D_0$ and $\phi(\overline{D}) \perp D_0$. Moreover, from a result in [3], we know that the structure vector field V belongs to $S(TM)$. Then summing up the above arguments, we conclude that

$$S(TM) = \{\phi D_2 \oplus M_1\} \perp \overline{D} \perp D_0 \perp \{V\}$$

where $\phi(M_1) \subset ltr(TM)$, which completes the proof.

For any $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX \tag{32}$$

where PX and FX are the tangential and transversal parts of ϕX. Similarly,

$$\phi W = BW + CW, \qquad W \in \Gamma(ltr(TM)) \tag{33}$$

where BW and CW are sections of TM and $tr(TM)$, respectively.

Following are the two non-existence Theorems for GCR-lightlike submanifolds.

Theorem 2. *There exists an induced metric connection on a proper GCR-lightlike submanifold M of an indefinite Kenmotsu manifold with structure vector field tangent to M if and only if for $X \in \Gamma(TM)$, the following hold*

$$P(A^*_{\phi\xi}X - \nabla^{*t}_X \phi\xi) \in \Gamma(\mathrm{Rad}\, TM), \quad \xi \in \Gamma(D_1)$$
$$P(h^*(X, \phi\xi) - \nabla^*_X \phi\xi) \in \Gamma(\mathrm{Rad}\, TM), \quad \xi \in \Gamma(D_2),$$

and $Bh(X, \phi\xi) = 0$, $\xi \in \Gamma(RadTM)$.

Proof. Assume that M admits a metric connection ∇. Then we show that the radical distribution is parallel with respect to ∇ (cf. [6], Theorem 2.4, p.161). From (2), we get

$$\overline{\nabla}_X \phi\xi = \phi\overline{\nabla}_X \xi - \overline{g}(\phi X, \xi)V$$

or,

$$\phi\overline{\nabla}_X \phi\xi = -\overline{\nabla}_X \xi \tag{34}$$

for $X \in \Gamma(TM)$, and $\xi \in \Gamma(\mathrm{Rad}TM)$.

Using (7) in (34) we obtain

$$\phi(\nabla_X \phi\xi + h(X, \phi\xi)) = -\nabla_X \xi - h(X, \xi) \tag{35}$$

Considering the tangential part of the above equation for $\xi \in \Gamma(D_1)$ and using (13), (32) and (33), we get

$$\nabla_X \xi = PA^*_{\phi\xi}X - P\nabla^{*t}_X \phi\xi - Bh(X, \phi\xi) \tag{36}$$

Similarly, for $\xi \in \Gamma(D_2)$ and using (12), (32), (33) and (35), we get

$$\nabla_X \xi = Ph^*(X, \phi\xi) - P\nabla^*_X \phi\xi - Bh(X, \phi\xi) \tag{37}$$

Thus, from (36), $\nabla_X \xi \in \Gamma(\mathrm{Rad}TM)$ if and only if

$$P(A^*_{\phi\xi}X - \nabla^{*t}_X \phi\xi) \in \Gamma(\mathrm{Rad}\, TM) \quad \text{and} \quad Bh(X, \phi\xi) = 0 \tag{38}$$

for $X \in \Gamma(TM)$, and $\xi \in \Gamma(D_1)$.

Similarly, from (37), $\nabla_X \xi \in \Gamma(\mathrm{Rad}TM)$ if and only if

$$P(h^*(X, \phi\xi) - \nabla^*_X \phi\xi) \in \Gamma(\mathrm{Rad}\, TM) \quad \text{and} \quad Bh(X, \phi\xi) = 0 \tag{39}$$

for $X \in \Gamma(TM)$ and $\xi \in \Gamma(D_2)$.

Then, the proof follows from (38) and (39).

Theorem 3. *There exists no contact totally umbilical proper GCR-lightlike submanifold M with structure vector field tangent to M of an indefinite Kenmotsu space form $\overline{M}(c)$ with $c \neq$- 1.*

Proof: The proof is similar to the proof of Theorem 4.11 [12].

4 Minimal GCR-lightlike submanifolds

In this section, we study minimal GCR-lightlike submanifold of an indefinite Kenmotsu manifold.

Example 2. Let $\overline{M} = (R^{15}_4, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature (-, -, +, +, +, +, +, -, -, +, +, +, +, +, +) with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial z\} \tag{40}$$

Suppose M is a submanifold of R_4^{15}, given by

$$
\left\{\begin{array}{ll}
x^1 = u^1 \cosh\beta, & y^1 = -u^2 \cosh\beta \\
x^2 = u^3, & y^2 = u^8 \\
x^3 = u^1 \sinh\beta + u^2, & y^3 = -u^2 \sinh\beta + u^1 \\
x^4 = u^3, & y^4 = u^9 \\
x^5 = \cos u^4 \cosh u^5, & y^5 = \sin u^4 \sinh u^5 \\
x^6 = \cos u^6 \cosh u^7, & y^6 = \cos u^6 \sinh u^7 \\
x^7 = \sin u^6 \cosh u^7, & y^7 = \sin u^6 \sinh u^7 \\
\multicolumn{2}{c}{z = u^{10}}
\end{array}\right. \tag{41}
$$

Then it is easy to see that a local frame of TM is given by

$$
\left\{\begin{array}{c}
Z_1 = e^{-z}(\cosh\beta\partial x_1 + \sinh\beta\partial x_3 + \partial y_3) \\
Z_2 = e^{-z}(\partial x_3 - \cosh\beta\partial y_1 - \sinh\beta\partial y_3) \\
Z_3 = e^{-z}(\partial x_2 + \partial x_4) \\
Z_4 = e^{-z}(-\sin u^4 \cosh u^5\partial x_5 + \cos u^4 \sinh u^5\partial y_5) \\
Z_5 = e^{-z}(\cos u^4 \sinh u^5\partial x_5 + \sin u^4 \cosh u^5\partial y_5) \\
Z_6 = e^{-z}(-\sin u^6 \cosh u^7\partial x_6 + \cos u^6 \cosh u^7\partial x_7 \\
-\sin u^6 \sinh u^7\partial y_6 + \cos u^6 \sinh u^7\partial y_7) \\
Z_7 = e^{-z}(\cos u^6 \sinh u^7\partial x_6 + \sin u^6 \sinh u^7\partial x_7 \\
+\cos u^6 \cosh u^7\partial y_6 + \sin u^6 \cosh u^7\partial y_7) \\
Z_8 = e^{-z}\partial y_2, \quad Z_9 = e^{-z}\partial y_4, \quad Z_{10} = \partial z = V
\end{array}\right. \tag{42}
$$

We see that M is a 3-lightlike submanifold with $\mathrm{Rad}TM = span\{Z_1, Z_2, Z_3\}$ and $\phi_0 Z_1 = Z_2$ and $\phi_0 Z_3 = -Z_8 - Z_9 \in \Gamma(S(TM))$. Thus $D_1 = span\{Z_1, Z_2\}$ and $D_2 = span\{Z_3\}$. On the other hand, $\phi_0 Z_4 = Z_5$ and $D_0 = span\{Z_4, Z_5\}$ is invariant. Moreover, since $\phi_0 Z_6$ and $\phi_0 Z_7$ are perpendicular to TM and they are nonnull, we can choose

$$
S(TM^{\perp}) = \mathrm{span}\{\phi_0 Z_6, \phi_0 Z_7\}
$$

Furthermore, the lightlike transversal vector bundle $ltr(TM)$ spanned by

$$
\left\{\begin{array}{c}
N_1 = e^{-z}(-\cosh\beta\partial x_1 - \sinh\beta\partial x_3 + \partial y_3) \\
N_2 = e^{-z}(\partial x_3 + \cosh\beta\partial y_1 + \sinh\beta\partial y_3) \\
N_3 = e^{-z}(-\partial x_2 + \partial x_4)
\end{array}\right. \tag{43}
$$

and $\phi_0 N_1 = N_2$, $\phi_0 N_3 = Z_8 - Z_9 \in \Gamma(S(TM))$. Thus, we have $\phi_0 D' = span\{\phi_0 Z_6, \phi_0 Z_7, \phi_0 N_3\}$. Hence, we conclude that M is a contact GCR-lightlike submanifold of R_4^{15}.

Then a quasi orthonormal basis of $\overline{M}$ along M is given by

$$
\left\{\begin{array}{lll}
\xi_1 = Z_1, & \xi_2 = Z_2, & \xi_3 = Z_3, \\
\phi_0\xi_3 = -e^{-z}(\partial y_2 + \partial y_4), & & \phi_0 N_3 = e^{-z}(\partial y_2 - \partial y_4) \\
e_1 = \frac{1}{\sqrt{\cosh^2 u^5 - \cos^2 u^4}} Z_4, & & e_2 = \frac{1}{\sqrt{\cosh^2 u^5 - \cos^2 u^4}} Z_5 \\
e_3 = \frac{1}{\sqrt{\cosh^2 u^7 + \sinh^2 u^7}} Z_6, & & e_4 = \frac{1}{\sqrt{\cosh^2 u^7 + \sinh^2 u^7}} Z_7 \\
 & V = Z_{10} & \\
W_1 = \frac{1}{\sqrt{cosh^2 u^7 + sinh^2 u^7}} \phi_0 Z_6, & & W_2 = \frac{1}{\sqrt{cosh^2 u^7 + sinh^2 u^7}} \phi_0 Z_7 \\
normal\ \ N_1, & N_2, & N_3
\end{array}\right. \tag{44}
$$

where $\varepsilon_1 = g(e_1, e_1) = 1$, $\varepsilon_2 = g(e_2, e_2) = 1$, $\varepsilon_3 = g(e_3, e_3) = 1$ and $\varepsilon_4 = g(e_4, e_4) = 1$.

By direct calculation and using Gauss formula (7), we get

$$\left\{\begin{array}{c} h(\xi_1,\xi_1)=h(\xi_2,\xi_2)=h(\xi_3,\xi_3)=h(e_1,e_1)=h(e_2,e_2)=0,\\ h(\phi_0\xi_3,\phi_0\xi_3)=h(\phi_0N_3,\phi_0N_3)=h^l(e_3,e_3)=h^l(e_4,e_4)=0,\\ h^s(e_3,e_3)=-\frac{e^{-z}}{(cosh^2u^7+sinh^2u^7)^{\frac{3}{2}}}W_2, \qquad h^s(e_4,e_4)=\frac{e^{-z}}{(cosh^2u^7+sinh^2u^7)^{\frac{3}{2}}}W_2 \end{array}\right. \tag{45}$$

Therefore,

$$\text{trace}h_{g|S(TM)}=\varepsilon_1h^s(e_3,e_3)+\varepsilon_2h^s(e_4,e_4)=h^s(e_3,e_3)+h^s(e_4,e_4)=0 \tag{46}$$

Thus M is a minimal proper contact GCR-lightlike submanifold of R^{15}_4.

Now, we prove characterisation results for minimal proper contact GCR-lightlike submanifold. We use the quasi orthonormal frame given by

$$\{\xi_1,...,\xi_q,e_1,...,e_m,V,W_1,...,W_n,N_1,...,N_q\}$$

where $\{\xi_1,...,\xi_q,e_1,...,e_m,V\}\in\Gamma(TM)$ such that $\{\xi_1,...,\xi_{2p}\}$, $\{\xi_{2p+1},...,\xi_q\}$ and $\{e_1,...,e_{2l}\}$ form a basis of D_1, D_2 and D_0 respectively. Moreover, take $\{W_1,...,W_k\}$ a basis of L_1 and $\{N_{2p+1},...,N_q\}$ a basis of L_2. Thus, we have quasi orthonormal basis of M as follows

$$\{\xi_1,..,\xi_{2p},\xi_{2p+1},..,\xi_q,\phi\xi_{2p+1},..,\phi\xi_q,e_1,..,e_l,\phi e_1,..,\phi e_l,\phi W_1,..,\phi W_k,\phi N_{2p+1},..,\phi N_q\}.$$

Theorem 4. *Let M be a proper contact GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then M is minimal if and only if*

$$traceA_{W_j|S(TM)}=0, \qquad traceA^*_{\xi_k|S(TM)}=0, \quad \overline{g}(Y,D^l(X,W))=0 \tag{47}$$

for $X,Y\in\Gamma(RadTM)$ and $W\in\Gamma(S(TM^\perp)$.

Proof. We know that $h^l=0$ on Rad(TM) [3]. Hence, from Definition 3, a GCR-lightlike submanifold is minimal if and only if

$$\sum_{i=1}^{2l}\varepsilon_ih(e_i,e_i)+\sum_{j=2p+1}^{q}h(\phi\xi_j,\phi\xi_j)+\sum_{j=2p+1}^{q}h(\phi N_j,\phi N_j)+\sum_{l=1}^{k}\varepsilon_lh(\phi W_l,\phi W_l)=0,$$

and $h^s=0$ on RadTM.

Now from (10), we have $h^s=0$ on RadTM if and only if $\overline{g}(Y,D^l(X,W))=0$, for $X,Y\in\Gamma(\text{Rad}TM)$, and $W\in\Gamma(S(TM^\perp))$.

On the other hand

$$\left\{\begin{array}{l} \text{trace}h|_{S(TM)}=\frac{1}{q}\sum_{a=1}^{q}\sum_{j=2p+1}^{q}\overline{g}(h^l(\phi\xi_j,\phi\xi_j),\xi_a)N_a+\overline{g}(h^l(\phi N_j,\phi N_j),\xi_a)N_a\\ +\frac{1}{n}\sum_{j=2p+1}^{q}\sum_{b=1}^{n}\varepsilon_b\{\overline{g}(h^s(\phi\xi_j,\phi\xi_j),W_b)W_b+\overline{g}(h^s(\phi N_j,\phi N_j),W_b)W_b\}\\ +\sum_{b=1}^{n}\varepsilon_b\frac{1}{n}\{\sum_{i=1}^{2l}\overline{g}(h^s(e_i,e_i),W_b)W_b+\sum_{l=1}^{k}\overline{g}(h^s(\phi W_l,\phi W_l),W_b)W_b\}\\ +\sum_{c=1}^{q}\frac{1}{q}\{\sum_{i=1}^{2l}\overline{g}(h^l(e_i,e_i),\xi_c)N_c+\sum_{l=1}^{k}\overline{g}(h^l(\phi W_l,\phi W_l),\xi_c)N_c\} \end{array}\right. \tag{48}$$

Using (10) and (14), we get

$$\left\{\begin{array}{l} \text{trace}h|_{S(TM)}=\frac{1}{q}\sum_{a=1}^{q}\sum_{j=2p+1}^{q}g(A^*_{\xi_a}\phi\xi_j,\phi\xi_j)N_a+g(A^*_{\xi_a}\phi N_j,\phi N_j)N_a\\ +\frac{1}{n}\sum_{j=2p+1}^{q}\sum_{b=1}^{n}\varepsilon_b\{g(A_{W_b}\phi\xi_j,\phi\xi_j)W_b+g(A_{W_b}\phi N_j,\phi N_j)W_b\}\\ +\sum_{b=1}^{n}\varepsilon_b\frac{1}{n}\{\sum_{i=1}^{2l}g(A_{W_b}e_i,e_i)W_b+\sum_{l=1}^{k}g(A_{W_b}\phi W_l,\phi W_l)W_b\}\\ +\sum_{c=1}^{q}\frac{1}{q}\{\sum_{i=1}^{2l}g(A_{\xi_c}e_i,e_i)N_c+\sum_{l=1}^{k}g(A^*_{\xi_c}\phi W_l,\phi W_l)N_c\} \end{array}\right. \tag{49}$$

Thus our assertion follows from the above equation.

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