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Classifications and Isolation Phenomena of Bi-Harmonic Maps and Bi-Yang-Mills Fields

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Abstract. Classifications of all biharmonic isoparametric hypersurfaces in the unit sphere, and all biharmonic homogeneous real hypersurfaces in the complex or quaternionic projective spaces are shown. Answers in case of bounded geometry to Chen's conjecture or Caddeo, Montaldo and Piu's one on biharmonic maps into a space of non positive curvature are given. Gauge field analogue is shown, indeed, the isolation phenomena of bi-Yang-Mills fields are obtained.

Keywords: biharmonic maps, harmonic maps, Yang-Mills fields

MSC 2000 classification: primary 53C43; secondary 58E20, 53C07

Introduction

Theory of harmonic maps plays a central roll in variational problems, which are, for smooth maps between Riemannian manifolds $\varphi : M \to N$, critical maps of the energy functional $E(\varphi) = \frac{1}{2}\int_M \|d\varphi\|^2 \, v_g$. By extending the notion of harmonic maps, in 1983, J. Eells and L. Lemaire [7] proposed the problem to consider the k-harmonic maps which are critical maps of the functional

$$E_k(\varphi) = \frac{1}{2}\int_M \|(d+\delta)^k\varphi\|^2 \, v_g, \quad (k = 1, 2, \cdots).$$

After G.Y. Jiang [14] studied the first and second variation formulas of E_k for $k = 2$, whose critical maps are called biharmonic maps, there have been extensive studies in this area (for instance, see [4], [17], [18], [22], [20], [11], [13], [24], etc.). Harmonic maps are always biharmonic maps by definition. One of main central problems is to classify the biharmonic maps, or to ask whether or not the converse to the above is true when the target Riemannian manifold (N, h) has non positive curvature (B. Y. Chen's conjecture [5] or Caddeo, Montaldo and Piu's one [4]). The main results of this paper related to biharmonic submanifolds are : (i) the classification of all biharmonic isoparametric hypersurfaces in the unit sphere, i.e., those with constant principal curvatures, see §3, 4; (ii) the construction of the first examples and classification of all biharmonic homogeneous real hypersurfaces in the complex or quaternionic projective spaces, see §5, 6, 7. Next, we give answers to Chen's conjecture and Caddeo, Montaldo and Piu's one in §8. Indeed, we show that all biharmonic maps or biharmonic submanifolds of bounded geometry into the target space which has non positive curvature, must be harmonic. Here, that biharmonic maps are of bounded geometry means that the curvature of the domain manifold is bounded, and the norms of the tension field and its covariant derivative are L^2.

Recently, the notion of gauge field analogue of biharmonic maps, i.e., bi-Yang-Mills fields was proposed ([1]). In this paper, we show the isolation phenomena of bi-Yang-Mills fields like the one for Yang-Mills fields (cf. Bourguignon-Lawson [3]), i.e., all bi-Yang-Mills fields over compact Riemanian manifolds with Ricci curvature bounded below by a positive constant k, and the pointwise norm of curvature tensor are bounded above by $k/2$, must be Yang-Mills fields. We also show the L^2-isolation phenopmena which are similar as Min-Oo's result ([19]) for Yang-Mills fields. These interesting phenomena suggest the existence of a unified theory between biharmonic maps and Yang-Mills fields.

1 Preliminaries

In this section, we prepare materials for the first variation formula for the bi-energy functional and bi-harmonic maps. Let us recall the definition of a harmonic map $\varphi : (M, g) \to (N, h)$, of a comoact Riemannian manifold (M, g) into another Riemannian manifold (N, h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi)\, v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for all variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\frac{d}{dt}\bigg|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \tag{1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$ $(x \in M)$, and the *tension field* of φ is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined frame field on (M, g). The second fundamental form $B(\varphi)$ of φ is defined by

$$\begin{aligned} B(\varphi)(X, Y) &= (\widetilde{\nabla} d\varphi)(X, Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y) \\ &= \nabla^N_{d\varphi(X)} d\varphi(Y) - d\varphi(\nabla_X Y), \end{aligned} \tag{2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Furthermore, ∇, and ∇^N, are connections on TM, TN of (M, g), (N, h), respectively, and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced one on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2.1), φ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula of the energy functional is also well known which is given as follows. Assume that φ is harmonic. Then,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g, \tag{3}$$

where J is an ellptic differential operator, called *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \tag{4}$$

where $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V$ is the *rough Laplacian* and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}V = \sum_{i=1}^m R^N(V, d\varphi(e_i))d\varphi(e_i)$, and R^N is the curvature tensor of (N, h) given by $R^N(U, V) = \nabla^N{}_U \nabla^N{}_V - \nabla^N{}_V \nabla^N{}_U - \nabla^N{}_{[U,V]}$ for U, $V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire proposed ([7]) polyharmonic (k-harmonic) maps and Jiang studied ([14]) the first and second variation formulas of bi-harmonic maps. Let us consider the *bi-energy functional* defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g, \tag{5}$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$. Then, the first variation formula is given as follows.

Theorem 1. (*the first variation formula*)

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g, \tag{6}$$

where

$$\tau_2(\varphi) = J(\tau(\varphi)) = \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)), \tag{7}$$

J is given in (2.4).

For the second variational formula, see [14] or [12].

Definition 1. A smooth map φ of M into N is called to be *bi-harmonic* if $\tau_2(\varphi) = 0$.

For later use, we need the following three lemmas.

Lemma 1. (*Jiang*) *Let $\varphi : (M^m, g) \to (N^n, h)$ be an isometric immersion of which mean curvuture vector field $\mathbb{H} = \frac{1}{m}\tau(\varphi)$ is parallel, i.e., $\nabla^\perp \mathbb{H} = 0$, where $\nabla^\perp$ is the induced connection of the normal bundle $T^\perp M$ by φ. Then,*

$$\begin{aligned}\overline{\Delta}\tau(\varphi) = &\sum_{i=1}^m h(\overline{\Delta}\tau(\varphi), d\varphi(e_i)) d\varphi(e_i) \\ &- \sum_{i,j=1}^m h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j))(\widetilde{\nabla}_{e_i} d\varphi)(e_j),\end{aligned} \tag{8}$$

where $\{e_i\}$ is a locally defined orthonormal frame field of (M, g).

Proof. Let us recall the definition of $\nabla^\perp$: For any section $\xi \in \Gamma(T^\perp M)$, we decompose $\overline{\nabla}_X \xi$ according to $TN|_M = TM \oplus T^\perp M$ as follows.

$$\overline{\nabla}_X \xi = \nabla^N_{\varphi_* X} \xi = \nabla^T_{\varphi_* X} \xi + \nabla^\perp_{\varphi_* X} \xi.$$

By the assumption $\nabla^\perp \mathbb{H} = 0$, i.e., $\nabla^\perp_{\varphi_* X} \tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$, we have

$$\overline{\nabla}_X \tau(\varphi) = \nabla^T_{\varphi_* X} \tau(\varphi) \in \Gamma(\varphi_* TM). \tag{9}$$

Thus, for all $i = 1, \cdots, m$,

$$\overline{\nabla}_{e_i}\tau(\varphi) = \sum_{j=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j))d\varphi(e_j) \tag{10}$$

because $\{d\varphi(e_j)_x\}_{j=1}^m$ is an orthonormal basis with respect to h, of $\varphi_* T_x M$ $(x \in M)$.

Now let us calculate

$$\overline{\nabla}^*\overline{\nabla}\tau(\varphi) = -\sum_{i=1}^{m}\{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\tau(\varphi) - \overline{\nabla}_{\nabla_{e_i}e_i}\tau(\varphi)\}. \tag{11}$$

Indeed, we have

$$\begin{aligned}\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\tau(\varphi) = &\sum_{j=1}^{m}\{h(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\tau(\varphi)) + h(\overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i}d\varphi(e_j))\}d\varphi(e_j)\\ &+\sum_{j=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j))\overline{\nabla}_{e_i}d\varphi(e_i),\end{aligned} \tag{12}$$

and

$$\overline{\nabla}_{\nabla_{e_i}e_i}\tau(\varphi) = \sum_{j=1}^{m} h(\overline{\nabla}_{\nabla_{e_i}e_i}, d\varphi(e_j))d\varphi(e_j), \tag{13}$$

so that we have

$$\begin{aligned}\overline{\nabla}^*\overline{\nabla}\tau(\varphi) = &\sum_{j=1}^{m} h(\overline{\nabla}^*\overline{\nabla}\tau(\varphi), d\varphi(e_j))d\varphi(e_j)\\ &-\sum_{i,j=1}^{m}\{h(\overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i}d\varphi(e_j))\}d\varphi(e_j)\\ &\qquad + h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j))\overline{\nabla}_{e_i}d\varphi(e_i)\}.\end{aligned} \tag{14}$$

Denoting $\nabla_{e_i}e_j = \sum_{k=1}^m \Gamma_{ij}^k e_k$, we have $\Gamma_{ij}^k + \Gamma_{ik}^j = 0$. Since $(\widetilde{\nabla}_{e_i}d\varphi)(e_j)$ $= \overline{\nabla}_{e_i}(d\varphi(e_j)) - d\varphi(\nabla_{e_i}e_j)$ is a local section of $T^\perp M$, we have for the the second term of the RHS of (2.14), for each fixed $i = 1, \cdots, m$,

$$\begin{aligned}&\sum_{j=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi),\overline{\nabla}_{e_i}d\varphi(e_j))d\varphi(e_j)\\ &\qquad= \sum_{j=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi), (\widetilde{\nabla}_{e_i}d\varphi)(e_j) + d\varphi(\nabla_{e_i}e_j))d\varphi(e_j)\\ &\qquad= \sum_{j=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(\nabla_{e_i}e_j))d\varphi(e_j)\\ &\qquad= \sum_{j,k=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_k))\, d\varphi(\Gamma_{ij}^k e_j)\\ &\qquad= -\sum_{j,k=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_k))\, d\varphi(\Gamma_{ik}^j e_j)\\ &\qquad= -\sum_{k=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_k))\, d\varphi(\nabla_{e_i}e_k).\end{aligned} \tag{15}$$

Substituting (2.15) into (2.14), we have the desired (2.8). $\boxed{QED}$

Lemma 2. (*Jiang*) *Under the same assumption as Lemma 2.1, we have*

$$\overline{\Delta}\tau(\varphi) = -\sum_{j,k=1}^{m} h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k))d\varphi(e_j) + \sum_{i,j=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j))(\widetilde{\nabla}_{e_i} d\varphi)(e_j). \tag{16}$$

Proof. Since $h(\tau(\varphi), d\varphi(e_j)) = 0$, differentiating it by e_i, we have

$$\begin{aligned} h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j)) &= -h(\tau(\varphi), \overline{\nabla}_{e_i} d\varphi(e_j)) \\ &= -h(\tau(\varphi), \overline{\nabla}_{e_i} d\varphi(e_j) - d\varphi(\nabla_{e_i} e_j)) \\ &= -h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j)). \end{aligned} \tag{17}$$

For the first term of (2.8), we have for each $j = 1, \cdots, m$,

$$h(\overline{\Delta}\tau(\varphi), d\varphi(e_j)) - 2\sum_{i=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i} d\varphi(e_j)) + h(\tau(\varphi), \overline{\Delta} d\varphi(e_j)) = 0, \tag{18}$$

which follows by the expression (2.11) of $\overline{\Delta}\tau(\varphi)$, differentiating the first equation of (2.17) by e_i, and doing $h(\tau(\varphi), d\varphi(e_j)) = 0$ by $\nabla_{e_i} e_i$.

For the second term of (2.8), we have by (2.9) and (2.17),

$$\begin{aligned} h(\overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i} d\varphi(e_j)) &= h(\overline{\nabla}_{e_i}\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j) + d\varphi(\nabla_{e_i} e_j)) \\ &= h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(\nabla_{e_i} e_j)) \\ &= -h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(\nabla_{e_i} e_j)). \end{aligned} \tag{19}$$

For the third term $h(\tau(\varphi), \overline{\Delta} d\varphi(e_j))$ of (2.18), we have

$$h(\tau(\varphi), \overline{\Delta} d\varphi(e_j)) = \sum_{k=1}^{m} h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k)) - 2\sum_{k=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_k} d\varphi)(\nabla_{e_k} e_j)). \tag{20}$$

Because, by making use of $(\widetilde{\nabla}_X d\varphi)(Y) = \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y)$ and $h(\tau(\varphi), d\varphi(X)) = 0$ $(X, Y \in \mathfrak{X}(M))$, the LHS of (2.20) coincides with

$$\begin{aligned} & h(\tau(\varphi), -\sum_{k=1}^{m}\{\overline{\nabla}_{e_k}\overline{\nabla}_{e_k} - \overline{\nabla}_{\nabla_{e_k} e_k}\} d\varphi(e_j)) \\ &= h(\tau(\varphi), -\sum_{k=1}^{m}\{(\widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_k} d\varphi)(e_j) + 2(\widetilde{\nabla}_{e_k} d\varphi)(\nabla_{e_k} e_j) \\ &\qquad - (\widetilde{\nabla}_{\nabla_{e_k} e_k} d\varphi)(e_j)\}) \\ &= h(\tau(\varphi), (\widetilde{\nabla}^*\widetilde{\nabla} d\varphi)(e_j)) - 2h(\tau(\varphi), (\widetilde{\nabla}_{e_k} d\varphi)(\nabla_{e_k} e_j)) \\ &= h(\tau(\varphi), \Delta d\varphi(e_j) - S d\varphi(e_j)) - 2h(\tau(\varphi), (\widetilde{\nabla}_{e_k} d\varphi)(\nabla_{e_k} e_j)), \end{aligned} \tag{21}$$

where the last equality follows from the Weitzenböck formula for the Laplacian $\Delta = d\delta + \delta d$ acting on 1-forms on (M, g):

$$\Delta d\varphi = \widetilde{\nabla}^* \widetilde{\nabla} d\varphi + S d\varphi. \tag{22}$$

Here, we have

$$\begin{aligned} S d\varphi(e_j) &:= \sum_{k=1}^{m} (\widetilde{R}(e_k, e_j) d\varphi)(e_k) \\ &= \sum_{k=1}^{m} \{R^N(d\varphi(e_k), d\varphi(e_j)) d\varphi(e_k) - d\varphi(R^M(e_k, e_j)e_k)\}, \end{aligned} \tag{23}$$

and

$$\Delta d\varphi(e_j) = d\delta d\varphi(e_j) = -d\tau(\varphi)(e_j) = -\overline{\nabla}_{e_j}\tau(\varphi). \tag{24}$$

Substituting these into (2.24), and using $h(\tau(\varphi), d\varphi(X)) = 0$ for all $X \in \mathfrak{X}(M)$, (2.24) coincides with

$$\sum_{k=1}^{m} \{h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k)) d\varphi(e_k)) - 2h(\tau(\varphi), (\widetilde{\nabla}_{e_k} d\varphi)(\nabla_{e_k} e_j))\},$$

which implies (2.20).

Substituting (2.19) and (2.20) into (2.18), we have

$$\begin{aligned} h(\overline{\Delta}\tau(\varphi), d\varphi(e_j)) &= -2\sum_{i=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(\nabla_{e_i} e_j)) \\ &\quad - \sum_{k=1}^{m} h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k)) d\varphi(e_k)) \\ &\quad + 2\sum_{k=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_k} d\varphi)(\nabla_{e_k} e_j)) \\ &= \sum_{k=1}^{m} h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k)) d\varphi(e_k)). \end{aligned} \tag{25}$$

Substituting (2.19) and (2.25) into (2.8), we have (2.16). QED

Lemma 3. *Let $\varphi : (M^m, g) \to (N^{m+1}, h)$ be an isometric immersion which is not harmonic. Then, the condition that $\|\tau(\varphi)\|$ is constant is equivalent to the one that*

$$\overline{\nabla}_X \tau(\varphi) \in \Gamma(\varphi_* TM), \quad \forall X \in \mathfrak{X}(M), \tag{26}$$

that is, the mean curvature tensor is parallel with respect to $\nabla^{\perp}$.

PROOF. Assume that φ is not harmonic. Then, if $\|\tau(\varphi)\|$ is constant,

$$Xh(\tau(\varphi), \tau(\varphi)) = 2h(\overline{\nabla}_X \tau(\varphi), \tau(\varphi)) = 0 \tag{27}$$

for all $X \in \mathfrak{X}(M)$, so we have $\overline{\nabla}_X \tau(\varphi) \in \Gamma(\varphi_* TM)$ because $\dim M = \dim N - 1$ and $\tau(\varphi) \neq 0$ everywhere on M. The converse is true from the above equality (2.27). QED

2 Biharmonic maps into the unit sphere

In this section, we give the classification of all the biharmonic isometrically immersed hypersurfaces of the unit sphere with constant principal curvatures. In order to show it, we need the following theorem.

Theorem 2. *(cf. Jiang [14]) Let $\varphi : (M^m, g) \to S^{m+1}\left(\frac{1}{\sqrt{c}}\right)$ be an isometric immersion of an m-dimensional compact Riemannian manifold (M^m, g) into the $(m+1)$-dimensional sphere with constant sectional curvature $c > 0$. Assume that the mean curvature of φ is nonzero constant. Then, φ is biharmonic if and only if square of the pointwise norm of $B(\varphi)$ is constant and $\|B(\varphi)\|^2 = cm$.*

Proof. For completeness, we give a brief proof, here. By Lemma 2.3, the condition (2.9) holds under the condition that the mean curvature of φ is constant. So, we may apply Lemmas 2.1 and 2.2.

Since the curvature tensor R^N of $S^{m+1}\left(\frac{1}{\sqrt{c}}\right)$ is given by

$$R^N(U, V)W = c\{h(V, W)U - h(W, U)V\}, \quad U, V, W \in \mathfrak{X}(N),$$

$R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k)$ is tangent to $\varphi_* TM$. By (2.16) of Lemma 2.2,

$$\overline{\Delta}\tau(\varphi) = \sum_{i,j=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j))(\widetilde{\nabla}_{e_i} d\varphi)(e_j). \tag{28}$$

Furthermore, we have

$$\begin{aligned}\mathcal{R}(\tau(\varphi)) &= \sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) \\ &= c\sum_{i=1}^{m}\{h(d\varphi(e_i), d\varphi(e_i))\tau(\varphi) - h(d\varphi(e_i), \tau(\varphi))d\varphi(e_i)\} \\ &= cm\tau(\varphi).\end{aligned} \tag{29}$$

Then, $\varphi : (M, g) \to S^{m+1}(\frac{1}{\sqrt{c}})$ is biharmonic if and only if

$$\begin{aligned}\tau_2(\varphi) &= \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)) \\ &= \sum_{i,j=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j))(\widetilde{\nabla}_{e_i} d\varphi)(e_j) - cm\tau(\varphi) \\ &= 0.\end{aligned} \tag{30}$$

If we denote by ξ, the unit normal vector field to $\varphi(M)$, the second fundamental form $B(\varphi)$ is of the form $B(\varphi)(e_i, e_j) = (\widetilde{\nabla}_{e_i} d\varphi)(e_j) = h_{ij}\xi$. Then, we have $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) = \sum_{i=1}^m h_{ii}\,\xi$ and $\|B(\varphi)\|^2 = \sum_{i,j=1}^m h_{ij}h_{ij}$. Substituting these into (3.3), we have

$$\tau_2(\varphi) = \sum_{k=1}^{m} h_{kk}\left(\sum_{i,j=1}^{m} h_{ij}h_{ij} - cm\right)\xi = 0, \tag{31}$$

That is, $\|B(\varphi)\|^2 = cm$ since $\sum_{k=1}^m h_{kk} \neq 0$. *QED*

Next, we prepare the necessary materials on isoparametric hypersurfaces M in the unit sphere $S^n(1)$ following Münzner ([21]) or Ozeki and Takeuchi ([23]).

Let $\varphi : (M, g) \to S^n(1)$ be an isometric immersion of (M, g) into the unit sphere $S^n(1)$ and denote by (N, h), the unit sphere $S^n(1)$ with the canonical metric. Assume that $\dim M = n-1$. The shape operator A_ξ is a linear operator of T_xM into itself defined by

$$g(A_\xi X, Y) = h(\varphi_*(\nabla_X Y), \xi),\ X, Y \in \mathfrak{X}(M),$$

where ξ is the unit normal vector field along M. The eigenvalues of A_ξ are called the *principal curvatures*. M is called *isoparametric* if all the principal curvatures are constant in $x \in M$. It is known that there exists a homogeneous polynomial F on $\mathbb{R}^{n+1}$ of degree g whose restriction to $S^n(1)$, denoted by f, called *isoparametric function*, M is given by $M = f^{-1}(t)$ for some $t \in I = (-1, 1)$. For each $t \in I$, $\xi_t = \frac{\nabla f}{\sqrt{g(\nabla f, \nabla f)}}$ is a smooth unit normal vector field along $M_t = f^{-1}(t)$, and all the distinct principal curvatures of M_t with respect to ξ_t are given as

$$k_1(t) > k_2(t) > \cdots > k_{g(t)}(t)$$

with their multiplicities $m_j(t)$ $(j = 1, \cdots, g(t))$. And $g = g(t)$ is constant in t, and is should be $g = 1, 2, 3, 4,$ or 6. Furthermore, it holds that

$$\begin{aligned} m_1(t) &= m_3(t) = \cdots = m_1, \\ m_2(t) &= m_4(t) = \cdots = m_2, \\ k_j(t) &= \cot\left(\frac{(j-1)\pi + \cos^{-1} t}{g}\right) \quad (j = 1, \cdots, g). \end{aligned} \tag{32}$$

where m_1 and m_2 are constant in $t \in I$. We also have

$$\|B(\varphi)\|^2 = \|A_x\|^2 = \sum_{j=1}^{g(t)} m_j(t) k_j(t)^2. \tag{33}$$

Indeed, if we denote by λ_i $(i = 1, \cdots, m$ $(m = \dim M)$, all the principal curvature counted with their multiplicities, we may choose orthonomal eigenvectors $\{X_i\}_{i=1}^m$ of T_xM in such a way that $A_\xi X_i = \lambda_i X_i$ $(i = 1, \cdots, m)$. Then, we have $h(B(X_i, X_j), \xi) = g(A_\xi(X_i), X_j) = \lambda_i \delta_{ij}$, and $\|B(X_i, X_j)\|^2 = {\lambda_i}^2 \delta_{ij}$. Thus, we have

Proposition 1. *Let $\varphi : (M, g) \to S^n(1)$ be an isoparametric hypersurface in the unit sphere $S^n(1)$, $\dim M = n - 1$. Then,*

$$\|B(\varphi)\|^2 = \sum_{j=1}^m {\lambda_j}^2. \tag{34}$$

PROOF. Indeed, we have

$$\|B(\varphi)\|^2 = \|A_\xi\|^2 = \sum_{i,j=1}^m \|B(X_i, X_j)\|^2 = \sum_{j=1}^m {\lambda_j}^2,$$

which is (3.7). QED

3 Biharmonic isoparametric hypersurfaces

Now, our main theorem in this section is

Theorem 3. *Let $\varphi : (M, g) \to S^n(1)$ be an isometric immersion* $(\dim M = n - 1)$ *which is isoparametric. Then, (M, g) is biharmonic if and only if (M, g) is one of the following:*

(i) $M = S^{n-1}\left(\frac{1}{\sqrt{2}}\right) \subset S^n(1)$, (*a small sphere*)

(ii) $M = S^{n-p}\left(\frac{1}{\sqrt{2}}\right) \times S^{p-1}\left(\frac{1}{\sqrt{2}}\right) \subset S^n(1)$, *with $n - p \neq p - 1$* (*the Clifford torus*), *or*

(iii) *$\varphi : (M, g) \to S^n(1)$ is harmonic, i.e., minimal.*

PROOF. The proof is divided into the cases $g = 1, 2, 3, 4, \text{or}\, 6$. It is known that for the cases $g = 1, 2$, all the (M, g) are homogeneous, and are classified into two cases. For $g = 3, 4$ or 6, we will show there are no nonharmonic biharmonic isoparametric hypersurfaces in the unit sphere.

Case 1: $g = 1$. In this case, $m_1 = m_2 = n - 1$ and $k_1(t) = \cot x$, $x = \cos^{-1} t$ with $0 < x < \pi$, $-1 < t < 1$. Then, we have immediately:

minimal $\Longleftrightarrow \cot x = 0 \Longleftrightarrow t = 0$ (a great sphere).

Furthermore, we have:

biharmonic and nonminimal $\Longleftrightarrow (n-1)\cot^2 x = n - 1 \Longleftrightarrow t = \pm\frac{1}{\sqrt{2}}$ (a small sphere).

Case 2: $g = 2$. In this case, $m_1 = p - 1$, $m_2 = n - p$ with $(2 \leq p \leq \left[\frac{n+1}{2}\right])$. Then, we have immediately,

$$\begin{aligned}\text{minimal} &\Longleftrightarrow (p-1)\cot\left(\frac{x}{2}\right) + (n-p)\cot\left(\frac{\pi + x}{2}\right) = 0\\ &\Longleftrightarrow \cos^2\left(\frac{x}{2}\right) = \frac{n-p}{n-1}\\ &\Longleftrightarrow t = \frac{n+1-2p}{n-1}\end{aligned}$$

with $x = \cos^{-1} t$ $(t \in (-1, 1))$. On the other hand, by Proposition 3.1,

$$\begin{aligned}\text{biharmonic} &\Longleftrightarrow (p-1)\cot^2\left(\frac{x}{2}\right) + (n-p)\cot^2\left(\frac{\pi + x}{2} = n - 1\right)\\ &\Longleftrightarrow t = 0,\ \frac{n+1-2p}{n-1}\end{aligned}$$

with $x = \cos^{-1} t$ $(t \in (-1, 1))$. Thus,

biharmonic and nonminimal $\Longleftrightarrow t = 0$, $k_1(0) = 1$ $(m_1 = p - 1)$, $k_2(0) = -1$ $(m_2 = n - p)$ $p - 1 \neq n - p$.

Case 3: $g = 3$. In this case, all the isoparametric hypersurfaces are classified into four cases, and $m_1 = m_2$ are $1, 2, 4$ or, 8, and $\dim M$ is $3, 6, 12$ or 24, respectively. By Proposition 3.1, it suffices to show in the case $\dim M = 3$,

$$\cot^2\left(\frac{x}{3}\right) + \cot^2\left(\frac{\pi + x}{3}\right) + \cot^2\left(\frac{2\pi + x}{3}\right) \geq 6 > 3 \quad (0 < x < \pi). \tag{35}$$

To prove (4.3), we only see the LHS of (4.3) coincides with

$$\cot^2\left(\frac{x}{3}\right) + \left(\frac{\cot\frac{x}{3} - \sqrt{3}}{\sqrt{3}\cot\frac{x}{3} + 1}\right)^2 + \left(\frac{\cot\frac{x}{3} + \sqrt{3}}{-\sqrt{3}\cot\frac{x}{3} + 1}\right)^2, \tag{36}$$

which is bigger than or equal to 6 when $0 < x < \pi$. Remark that $0 < \cot\frac{x}{3} < \frac{1}{\sqrt{3}}$ $(0 < x < \pi)$. And the arguments go the same way as $\dim M = 6, 12, 24$. Thus, due to Proposition 3.1 and Theorem 3.1, there are no nonminimal biharmonic hypersurfaces in this case.

Case 4: $g = 4$. In this case, we have

$$\begin{aligned}\|B(\varphi)\|^2 &= m_1(t)\cot^2\left(\frac{x}{4}\right) + m_2(t)\cot^2\left(\frac{\pi + x}{4}\right)\\ &\quad + m_1(t)\cot^2\left(\frac{2\pi + x}{4}\right) + m_2(t)\cot^2\left(\frac{3\pi + x}{4}\right)\\ &= m_1(t)\left\{\cot^2\left(\frac{x}{4}\right) + \frac{1}{\cot^2\left(\frac{x}{4}\right)}\right\}\end{aligned}$$

$$+ m_2(t)\left\{\left(\frac{\cot\left(\frac{x}{4}\right)-1}{\cot\left(\frac{x}{4}\right)+1}\right)^2+\left(\frac{\cot\left(\frac{x}{4}\right)+1}{\cot\left(\frac{x}{4}\right)-1}\right)^2\right\}$$
$$\geq 2m_1(t)+2m_2(t)=\dim M, \tag{37}$$

and equality holds if and only if

$$\begin{cases}\cot^2\left(\dfrac{x}{4}\right)=\dfrac{1}{\cot^2\left(\frac{x}{4}\right)},\\ \left(\dfrac{\cot\left(\frac{x}{4}\right)-1}{\cot\left(\frac{x}{4}\right)+1}\right)^2=\left(\dfrac{\cot\left(\frac{x}{4}\right)+1}{\cot\left(\frac{x}{4}\right)-1}\right)^2,\end{cases} \tag{38}$$

because, for all $a>0$ and $b>0$, $\frac{a+b}{2}\geq\sqrt{ab}$ and equality holds if and only if $a=b$. But, it is impossible that (4.6) holds. Thus, we have $\|B(\varphi)\|^2>\dim M$. In this case, due to Proposition 3.1 and Theorem 3.1, there are no nonharmonic biharmonic immersions φ.

Case 5: $g=6$. In this case, we have

$$\|B(\varphi)\|^2=m_1(t)\cot^2\left(\frac{x}{6}\right)+m_2(t)\cot^2\left(\frac{\pi+x}{6}\right)$$
$$+m_1(t)\cot^2\left(\frac{2\pi+x}{6}\right)+m_2(t)\cot^2\left(\frac{3\pi+x}{6}\right)$$
$$+m_1(t)\cot^2\left(\frac{4\pi+x}{6}\right)+m_2(t)\cot^2\left(\frac{5\pi+x}{6}\right)$$
$$=m_1(t)\left\{\cot^2\left(\frac{x}{6}\right)+\left(\frac{\cot\frac{x}{6}-\sqrt{3}}{\sqrt{3}\cot\frac{x}{6}+1}\right)^2+\left(\frac{\cot\frac{x}{6}+\sqrt{3}}{-\sqrt{3}\cot\frac{x}{6}+1}\right)^2\right\}$$
$$+m_2(t)\left\{\frac{1}{\cot^2\left(\frac{x}{6}\right)}+\left(\frac{\sqrt{3}\cot\frac{x}{6}-1}{\cot\frac{x}{6}+\sqrt{3}}\right)^2+\left(\frac{\sqrt{3}\cot\frac{x}{6}+1}{-\cot\frac{x}{6}+\sqrt{3}}\right)^2\right\}. \tag{39}$$

Here, we denote by $f(y)$, the bracket of the first term of the RHS of (4.7), where $y=\cot\frac{x}{6}>\sqrt{3}$ $(0<x<\pi)$. Then, we have $\frac{df}{dy}>0$ and $\lim_{y\to\sqrt{3}}f(y)=6$. And we denote by $g(y)$, the bracket of the second term of the RHS of (4.7), where $y=\cot\frac{x}{6}>\sqrt{3}$ $(0<x<\pi)$. Then, we have $\frac{dg}{dy}<0$ and $\lim_{y\to\infty}g(y)=6$. Therefore, we have

$$\|B(\varphi)\|^2\geq 6(m_1(t)+m_2(t))>3(m_1(t)+m_2(t))=\dim M. \tag{40}$$

Thus, due to Proposition 3.1 and Theorem 3.1, there are also no nonharmonic biharmonic immersions φ in this case. QED

4 Biharmonic maps into the complex projective space

In the following two sections, we show classification of all homogeneous real hypersurfaces in the complex n-dimensional projective space $\mathbb{C}P^n(c)$ with positive constant holomorphic sectional curvature $c>0$ which are *biharmonic*. To do it, we need first the following theorem analogue to Theorem 3.1 which charcterizes the biharmonic maps.

Theorem 4. *Let (M,g) be a real $(2n-1)$-dimensional compact Riemannian manifold, and $\varphi:\ (M,g)\to\mathbb{C}P^n(c)$ be an isometric immersion with non-zero constant mean curvature. Then, the necessary and sufficient condition for φ to be biharmonic is*

$$\|B(\varphi)\|^2=\frac{n+1}{2}c. \tag{41}$$

PROOF. By Lemma 2.3, the mean curvature vector of φ is parallel with respect to $\nabla^{\perp}$, so we may apply Lemmas 2.1 and 2.2 in this case. Let us recall the fact that the curvature tensor of $(N,h)=\mathbb{C}P^n(c)$ is given by

$$R^N(U,V)W=\frac{c}{4}\big\{h(V,W)U-h(U,W)V\\ +h(JV,W)JU-h(JU,W)JV+2h(U,JV)JW\big\},$$

where J is the adapted almost complex tensor, and U,V and W are vector fields on $\mathbb{C}P^n(c)$. Then, we have

$$R^N(d\varphi(e_j),d\varphi(e_k))d\varphi(e_k)=\frac{c}{4}\big\{d\varphi(e_j)-\delta_{jk}\,d\varphi(e_k)\\ +3h(d\varphi(e_j),Jd\varphi(e_k))\,Jd\varphi(e_k)\big\}. \tag{42}$$

Then, we have

$$\sum_{j,k=1}^m h\big(\tau(\varphi),R^N(d\varphi(e_j),d\varphi(e_k))d\varphi(e_k)\big)\,d\varphi(e_j)=0. \tag{43}$$

Because the LHS of (5.3) coincides with

$$\begin{aligned}\frac{3c}{4}&\sum_{j,k=1}^m h\big(d\varphi(e_j),Jd\varphi(e_k)\big)\,h\big(\tau(\varphi),Jd\varphi(e_k)\big)\,d\varphi(e_j)\\ &=\frac{3c}{4}\sum_{j,k=1}^m h\big(Jd\varphi(e_j),d\varphi(e_k)\big)\,h\big(J\tau(\varphi),d\varphi(e_k)\big)\,d\varphi(e_j)\\ &=\frac{3c}{4}\sum_{j=1}^m h\Big(Jd\varphi(e_j),\sum_{k=1}^m h\big(J\tau(\varphi),d\varphi(e_k)\big)d\varphi(e_k)\Big)\,d\varphi(e_j)\\ &=\frac{3c}{4}\sum_{j=1}^m h\big(Jd\varphi(e_j),J\tau(\varphi)\big)\,d\varphi(e_j)\\ &=\frac{3c}{4}\sum_{j=1}^m h(d\varphi(e_j),\tau(\varphi))\,d\varphi(e_j)=0.\end{aligned} \tag{44}$$

Here the third equality follows from that $J\tau(\varphi)\in\Gamma(\varphi_*TM)$ which is due to $h(J\tau(\varphi),\tau(\varphi))=0$, $0\neq\tau(\varphi)\in T^{\perp}M$ and $\dim M=2n-1$. Since $\{d\varphi(e_k)\}_{k=1}^m$ is an orthonormal basis of $\varphi_*(T_xM)$ at each $x\in M$, $J\tau(\varphi)=\sum_{k=1}^m h(J\tau(\varphi),d\varphi(e_k))d\varphi(e_k)$.

By (2.16) in Lemma 2.2, we have

$$\overline{\Delta}\tau(\varphi)=\sum_{i,j=1}^m h\big(\tau(\varphi),(\widetilde{\nabla}_{e_i}d\varphi)(e_j)\big)\,(\widetilde{\nabla}_{e_i}d\varphi)(e_j). \tag{45}$$

Furthermore, we have

$$\mathcal{R}(\tau(\varphi))=\frac{c}{4}(m+3)\tau(\varphi). \tag{46}$$

Because the LHS of (5.6) is equal to

$$\sum_{k=1}^m R^N(\tau(\varphi),d\varphi(e_k))d\varphi(e_k)=\frac{c}{4}\big\{m\tau(\varphi)\\ -3\sum_{k=1}^m h(J\tau(\varphi),d\varphi(e_k))\,Jd\varphi(e_k)\big\}$$

$$= \frac{c}{4}\{m\tau(\varphi) - 3J(J\tau(\varphi))\}$$
$$= \frac{c}{4}(m+3)\tau(\varphi). \tag{47}$$

Now the sufficient and necessary condition for φ to be biharmonic is that

$$\tau_2(\varphi) = \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)) = 0 \tag{48}$$

which is equivalent to

$$\sum_{i,j=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j))(\widetilde{\nabla}_{e_i} d\varphi)(e_j) - \frac{c}{4}(m+3)\tau(\varphi) = 0. \tag{49}$$

Here, we may denote as

$$B(\varphi)(e_i, e_j) = (\widetilde{\nabla}_{e_i} d\varphi)(e_j) = h_{ij}\,\xi$$
$$\tau(\varphi) = \sum_{k=1}^{m} (\widetilde{\nabla}_{e_k} d\varphi)(e_k) = \sum_{k=1}^{m} h_{kk}\xi, \tag{50}$$

where ξ is the unit normal vector field along $\varphi(M)$. Thus, the LHS of (5.9) coincides with

$$\sum_{i,j,k=1}^{m} h_{kk}h_{ij}h_{ij} - \frac{c}{4}(m+3)\sum_{k=1}^{m} h_{kk}$$
$$= \left(\sum_{k=1}^{m} h_{kk}\right)\left\{\sum_{i,j=1}^{m} h_{ij}h_{ij} - \frac{c}{4}(m+3)\right\}$$
$$= \|\tau(\varphi)\|^2 \left\{\|B(\varphi)\|^2 - \frac{c}{2}(n+1)\right\}, \tag{51}$$

which yields the desired (5.1) due to the assumption that $\|\tau(\varphi)\|$ is a non-zero constant. QED

5 Biharmonic Homogeneous real hypersurfaces in the complex projective space

In this section, we classify all the *biharmonic* homogeneous real hypersurfaces in the complex projective space $\mathbb{C}P^n(c)$.

First, let us recall the classification theorem of all the homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$ due to R. Takagi (cf. [26]) based on a work by W.Y. Hsiang and H.B. Lawson ([10]). Let U/K be a symmetric space of rank two of compact type, and $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decomposition of the Lie algebra $\mathfrak{u}$ of U, and the Lie subalgebra $\mathfrak{k}$ corresponding to K. Let $\langle X, Y\rangle = -B(X,Y)$ $(X, Y \in \mathfrak{p})$ be the inner product on $\mathfrak{p}$, $\|X\|^2 = \langle X, X\rangle$, and $S := \{X \in \mathfrak{p};\, \|X\| = 1\}$, the unit sphere in the Euclidean space $(\mathfrak{p}, \langle\,,\,\rangle)$, where B is the Killing form of $\mathfrak{u}$. Consider the adjoint action of K on $\mathfrak{p}$. Then, the orbit $\hat{M} = \mathrm{Ad}(K)A$ through any regular element $A \in \mathfrak{p}$ with $\|A\| = 1$ gives a homogeneous hypersurface in the unit sphere S. Conversely, any homogeneous hypersurface in S can be obtained in this way ([10]).

Let us take as U/K, a *Hermitian* symmetric space of compact type of rank two of complex dimension $(n+1)$, and identify $\mathfrak{p}$ with $\mathbb{C}^{n+1}$. Then, the adjoint orbit $\hat{M} = \mathrm{Ad}(K)A$ of K through any regular element A in $\mathfrak{p}$ is again a homogeneous hypersurface in the unit sphere S. Let π :

$\mathbb{C}^{n+1} - \{\mathbf{0}\} = \mathfrak{p} - \{\mathbf{0}\} \to \mathbb{C}P^n$ be the natural projection. Then, the projection induces the Hopf fibration of S onto $\mathbb{C}P^n$, denoted also by π, and $\varphi : M := \pi(\hat{M}) \hookrightarrow \mathbb{C}P^n$ gives a homogeneous real hypersurface in the complex projective space $\mathbb{C}P^n(4)$ with constant holomorphic sectional curvature 4. Conversely, any homogeneous real hypersurface M in $\mathbb{C}P^n(4)$ is given in this way ([26]). Furthermore, all such hypersurfaces are classified into the following five types:

(1) A-type: $\mathfrak{u} = \mathfrak{su}(p+2) \oplus \mathfrak{su}(q+2)$, $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(p+1) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(q+1) + \mathfrak{u}(1))$, where $0 \le p \le q$, $0 < q$, $p + q = n - 1$, and $\dim M = 2n - 1$.

(2) B-type: $\mathfrak{u} = \mathfrak{o}(m+2)$, $\mathfrak{k} = \mathfrak{o}(m) \oplus \mathbb{R}$, where $3 \le m$, $\dim M = 2m - 3$.

(3) C-type: $\mathfrak{u} = \mathfrak{su}(m+2)$, $\mathfrak{k} = \mathfrak{s}(\mathfrak{o}(m) + \mathfrak{o}(2))$, where $3 \le m$, and $\dim M = 4m - 3$.

(4) D-type: $\mathfrak{o}(10)$, $\mathfrak{u}(5)$, and $\dim M = 17$.

(5) E-type: $\mathfrak{u} = \mathfrak{e}_6$, $\mathfrak{k} = \mathfrak{o}(10) \oplus \mathbb{R}$, and $\dim M = 29$.

He also gave ([27], [28]) lists of the principal curvatures and their multiplicities of these M as follows:

(1) A-type: Assume that

$$U/K = \frac{SU(p+2) \times SU(q+2)}{S(U(p+1) \times U(1)) \times S(U(q+1) \times U(1)},$$

then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by the Riemannian product of two odd dimensional spheres,

$$\hat{M} = \hat{M}_{p,q} = S^{2p+1}(\cos u) \times S^{2q+1}(\sin u) \subset S^{2n+1}, \tag{52}$$

where $0 < u < \frac{\pi}{2}$. The projection $M_{p,q}(u) := \pi(\hat{M}_{p,q}(u))$ is a homogeneous real hypersurface of $\mathbb{C}P^n(4)$. The principal curvatures of $M_{p,q}$ with $0 \le p \le q$, $0 < q$, are given as

$$\begin{cases} \lambda_1 = -\tan u \quad (\text{with multiplicity } m_1 = 2p), \\ \qquad\qquad (m_1 = 0 \text{ if } p = 0), \\ \lambda_2 = \cot u \quad (\text{with multiplicity } m_2 = 2q), \\ \lambda_3 = 2\cot(2u) \quad (\text{with multiplicity } m_3 = 1). \end{cases} \tag{53}$$

Thus, the mean curvature H of $M_{p,q}(u)$ is given by

$$\begin{aligned} H &= \frac{1}{2n-1}\{2q \cot u - 2p \tan u + 2\cot(2u)\} \\ &= \frac{1}{2n-1}\{(2q+1)\cot u - (2p+1)\tan u\}. \end{aligned} \tag{54}$$

The constant $\|B(\varphi)\|^2$ which is the sum of all the square of principal curvatures with their multiplicities, is given by

$$\begin{aligned} \|B(\varphi)\|^2 &= 2q \cot^2 u + 2p \tan^2 u + 4\cot^2(2u) \\ &= (2q+1)\cot^2 u + (2p+1)\tan^2 u - 2. \end{aligned} \tag{55}$$

(2) B-type: Assume that $U/K = SO(m+2)/(SO(m) \times SO(2))$, $(m := n+1)$, and then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by

$$\hat{M} = \{SO(n+1) \times SO(2)\}/\{SO(n-1) \times \mathbb{Z}_2\} \subset S^{2n+1}.$$

The real hypersurface $\varphi : M \hookrightarrow \mathbb{C}P^n$ is a tube over a complex quadric with radius $\frac{\pi}{4} - u$ $(0 < u < \frac{\pi}{4})$ or a tube over a totally geodesic real projective space $\mathbb{R}P^n$ with radius u

$(0 < u < \frac{\pi}{4})$. The principal curvatures of M are given as

$$\begin{cases} \lambda_1 = -\cot u \quad (\text{with multiplicity } m_1 = n-1), \\ \lambda_2 = \tan u \quad (\text{with multiplicity } m_2 = n-1), \\ \lambda_3 = 2\tan(2u) \quad (\text{with multiplicity } m_3 = 1). \end{cases} \tag{56}$$

Thus, the mean curvature of M is given by

$$\begin{aligned} H &= \frac{1}{2n-1}\{-(n-1)\cot u + (n-1)\tan u + 2\cot(2u)\} \\ &= -\frac{1}{2n-1} \cdot \frac{(n-1)t^4 - 2(n+1)t^2 + n - 1}{t(t^2-1)}, \end{aligned} \tag{57}$$

where $t = \cot u$. The constant $\|B(\varphi)\|^2$ is given by

$$\begin{aligned} \|B(\varphi)\|^2 &= (n-1)\cot^2 u + n - 1\tan^2 u + 4\tan^2(2u) \\ &= (n-1)t^2 + \frac{n-1}{t^2} + \frac{16t^2}{(t^2-1)^2} \\ &= \frac{(n-1)(X-1)^2(X^2+1) + 16X^2}{X(X-1)^2}, \end{aligned} \tag{58}$$

where $X := t^2$.

(3) C-type: Assume that $U/K = SU(m+2)/S(U(m)\times U(2))$, $(n = 2m+1)$, and then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by

$$\hat{M} = S(U(m)\times U(2))/(T^2 \times SU(m-2)) \subset S^{2n+1}.$$

The real hypersurface $\varphi :\ M \hookrightarrow \mathbb{C}P^n$ is a tube over the Segre imbeding of $\mathbb{C}^1 \times \mathbb{C}P^m$ with radius u $(0 < u < \frac{\pi}{4})$. The principal curvatures of M are given by

$$\begin{cases} \lambda_1 = -\cot u \quad (\text{with multiplicity } m_1 = n-3), \\ \lambda_2 = \cot\left(\frac{\pi}{4} - u\right) \quad (\text{with multiplicity } m_2 = 2), \\ \lambda_3 = \cot\left(\frac{\pi}{2} - u\right) \quad (\text{with multiplicity } m_3 = n-3), \\ \lambda_4 = \cot\left(\frac{3\pi}{4} - u\right) \quad (\text{with multiplicity } m_4 = 2), \\ \lambda_5 = -2\tan(2u) \quad (\text{with multiplicity } m_5 = 1). \end{cases} \tag{59}$$

Then,

$$\lambda_1 = -t,\ \lambda_2 = \frac{t+1}{t-1},\ \lambda_3 = \frac{1}{t},\ \lambda_4 = -\frac{t-1}{t+1},\ \lambda_5 = -t + \frac{1}{t},$$

where $t = \cot u$. The mean curvature of M is given by

$$\begin{aligned} H &= \frac{1}{2n-1}\left\{(n-3)(-t) + 2\frac{t+1}{t-1} + (n-3)\frac{1}{t} - 2\frac{t-1}{t+1} - t + \frac{1}{t}\right\} \\ &= -\frac{(n-2)t^4 - 2(n+2)t^2 + n - 2}{t(t^2-1)}. \end{aligned} \tag{60}$$

The constant $\|B(\varphi)\|^2$ is given by

$$\begin{aligned}\|B(\varphi)\|^2 &= (n-3)t^2 + 2\left(\frac{t+1}{t-1}\right)^2 + (n-3)\frac{1}{t^2} \\ &\quad + 2\left(\frac{t-1}{t+1}\right)^2 + \left(-t+\frac{1}{t}\right)^2 \\ &= \frac{C(X)}{X(X-1)^2}, \end{aligned} \tag{61}$$

where

$$\begin{aligned} C(X) &:= (n-2)X^2(X-1)^2 + (n-2)(X-1)^2 \\ &\quad + 4X(X^2+6X+1) - 2X(X-1)^2, \end{aligned} \tag{62}$$

and $X := t^2$.

(4) D-type: Assume that $U/K = O(10)/U(5)$, and then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by

$$\hat{M} = U(5)/(SU(2)\times SU(2)\times U(1)) \subset S^{19}.$$

The real hypersurface $\varphi : M \hookrightarrow \mathbb{C}P^9$ is a tube over the Plücker imbeding of $\mathrm{Gr}_2(\mathbb{C}^5)$ with radius u $(0 < u < \frac{\pi}{4})$. The principal curvatures of M are given by

$$\begin{cases} \lambda_1 = -\cot u \quad \text{(with multiplicity } m_1 = 4), \\ \lambda_2 = \cot\left(\frac{\pi}{4} - u\right) \quad \text{(with multiplicity } m_2 = 4), \\ \lambda_3 = \cot\left(\frac{\pi}{2} - u\right) \quad \text{(with multiplicity } m_3 = 4), \\ \lambda_4 = \cot\left(\frac{3\pi}{4} - u\right) \quad \text{(with multiplicity } m_4 = 4), \\ \lambda_5 = -2\tan(2u) \quad \text{(with multiplicity } m_5 = 1). \end{cases} \tag{63}$$

Then,

$$\lambda_1 = -t,\ \lambda_2 = \frac{t+1}{t-1},\ \lambda_3 = \frac{1}{t},\ \lambda_4 = -\frac{t-1}{t+1},\ \lambda_5 = -t + \frac{1}{t},$$

where $t = \cot u$. The mean curvature of M is given by

$$\begin{aligned} H &= \frac{1}{17}\left\{4(-t) + 4\frac{t+1}{t-1} + 4\,\frac{1}{t} - 4\frac{t-1}{t+1} - t + \frac{1}{t}\right\} \\ &= -\frac{5t^4 - 26t^2 + 5}{17t(t^2-1)} = -\frac{(5t^2-1)(t^2-5)}{17t(t^2-1)}. \end{aligned} \tag{64}$$

The constant $\|B(\varphi)\|^2$ is given by

$$\begin{aligned}\|B(\varphi)\|^2 &= 4t^2 + 4\left(\frac{t+1}{t-1}\right)^2 + 4\frac{1}{t^2} \\ &\quad + 4\left(\frac{t-1}{t+1}\right)^2 + \left(-t+\frac{1}{t}\right)^2 \\ &= \frac{D(X)}{X(X-1)^2}, \end{aligned} \tag{65}$$

where

$$D(X) := 11X^3 + 63X^2 + X + 5, \tag{66}$$

and $X := t^2$.

(5) E-type: Assume that $U/K = E_6/(\mathrm{Spin}(10) \times U(1)$, and then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by

$$\hat{M} = (\mathrm{Spin}(10) \times U(1))/(SU(4) \times U(1)) \subset S^{31}.$$

The real hypersurface $\varphi : M \hookrightarrow \mathbb{C}P^{15}$ is a tube over the canonical imbeding of $SO(10)/U(5) \subset \mathbb{C}P^{15}$ with radius u $(0 < u < \frac{\pi}{4})$. The principal curvatures of M are given by

$$\begin{cases} \lambda_1 = -\cot u \quad \text{(with multiplicity } m_1 = 8), \\ \lambda_2 = \cot\left(\dfrac{\pi}{4} - u\right) \quad \text{(with multiplicity } m_2 = 6), \\ \lambda_3 = \cot\left(\dfrac{\pi}{2} - u\right) \quad \text{(with multiplicity } m_3 = 8), \\ \lambda_4 = \cot\left(\dfrac{3\pi}{4} - u\right) \quad \text{(with multiplicity } m_4 = 6), \\ \lambda_5 = -2\tan(2u) \quad \text{(with multiplicity } m_5 = 1). \end{cases} \tag{67}$$

Then,

$$\lambda_1 = -t,\ \lambda_2 = \frac{t+1}{t-1},\ \lambda_3 = \frac{1}{t},\ \lambda_4 = -\frac{t-1}{t+1},\ \lambda_5 = -t + \frac{1}{t},$$

where $t = \cot u$. The mean curvature of M is given by

$$\begin{aligned} H &= \frac{1}{29}\left\{8(-t) + 6\frac{t+1}{t-1} + 8\,\frac{1}{t} - 6\frac{t-1}{t+1} - t + \frac{1}{t}\right\} \\ &= -\frac{9t^4 - 42t^2 + 9}{29t(t^2-1)}. \end{aligned} \tag{68}$$

The constant $\|B(\varphi)\|^2$ is given by

$$\begin{aligned} \|B(\varphi)\|^2 &= 8t^2 + 6\left(\frac{t+1}{t-1}\right)^2 + 8\frac{1}{t^2} \\ &\quad + 6\left(\frac{t-1}{t+1}\right)^2 + \left(-t + \frac{1}{t}\right)^2 \\ &= \frac{E(X)}{X(X-1)^2} - 2, \end{aligned} \tag{69}$$

where

$$E(X) := 21X^3 + 99X^2 - 9X + 9, \tag{70}$$

and $X := t^2$.

Now we want to show the following:

Theorem 5. *Let M be any homogeneous real hypersurface in $\mathbb{C}P^n(4)$, so that M is a tube of $A \sim E$ type.*

(I) *Then, for each type, there is a unique u with $0 < u < \frac{\pi}{4}$ in such a way that M is a tube of radius u and is minimal.*

(II) *Assume that M is a biharmonic but not minimal. Then, M is one of type A, D or E. More precisely,*

(1) *in the case of A-type, M is a tube $M_{p,q}(u)$ of $\mathbb{C}P^p \subset \mathbb{C}P^n$ ($p \geq 0$ and $q = (n-1) - p$) of radius u with $0 < u < \frac{\pi}{2}$ of which $t = \cot u$ is a solution of the equation*

$$\cot u = \left\{\frac{p + q + 3 \pm \sqrt{(p-q)^2 + 4(p+q+2)}}{1 + 2q}\right\}^{1/2}. \tag{71}$$

(2) *In the case of D-type, M is a tube of the Plücker imbedding* $\mathrm{Gr}_2(\mathbb{C}^5) \subset \mathbb{C}P^9$ *of radius u with $0 < u < \frac{\pi}{4}$ of which $t = \cot u$ is a unique solution of the equation*

$$41t^6 + 43t^4 + 41t^2 - 15 = 0. \tag{72}$$

I.e., $u = 1.0917\cdots$.

(3) *In the case of E-type, M is a tube of the imbedding $SO(10)/U(5) \subset \mathbb{C}P^{15}$ of radius u with $0 < u < \frac{\pi}{4}$ of which $t = \cot u$ is a unique solution of the equation*

$$13t^6 - 107t^4 + 43t^2 - 9 = 0. \tag{73}$$

I.e., $u = 0.343448\cdots$.

Proof. We give a proof case by case.

Case (1) A-Type: By (6.3), $\varphi : M_{p,q}(u) \hookrightarrow \mathbb{C}P^n(4)$ is harmonic if and only if

$$t := \cot u = \left\{ \frac{2p+1}{2q+1} \right\}^{1/2}. \tag{74}$$

On the other hand, by Theorem 5.1 and (6.4), $\varphi : M_{p,q} \hookrightarrow \mathbb{C}P^n(4)$ is non-harmonic and biharmonic if and only if $t = \cot u$ must satisfy

$$(2q+1)\cot^4 u - 2(p+q+3)\cot^2 u + 2p + 1 = 0, \tag{75}$$

so that

$$t = \cot u = \left\{ \frac{p+q+3 \pm \sqrt{(p-q)^2 + 4(p+q+2)}}{2q+1} \right\}^{1/2} \tag{76}$$

since $p+q+3 \pm \sqrt{(p-q)^2 + 4(p+q+2)}$ is positive but does never be $2p+1$.

Case (2) B-Type: By (6.6), $\varphi : M \hookrightarrow \mathbb{C}P^n(4)$ is harmonic if and only if $t = \cot u$ ($0 < u < \frac{\pi}{4}$) must satisfy

$$(n-1)t^4 - 2(n+1)t^2 + n - 1 = 0, \tag{77}$$

which is equivalent to that

$$t = \cot u = \left\{ \frac{n+1 \pm 2\sqrt{n}}{n-1} \right\}^{1/2} = \frac{\sqrt{n} \pm 1}{\sqrt{n-1}}. \tag{78}$$

On the other hand, by Theorem 5.1 and (6.7), $\varphi : M \hookrightarrow \mathbb{C}P^n(4)$ is non-harmonic but biharmonic if and only if

$$\begin{aligned} f(X) :&= (n-1)(X-1)^2(X^2+1) + 16X^2 - 2(n+1)X(X-1)^2 \\ &= 0, \end{aligned} \tag{79}$$

where $X := t^2$. But, $f(X) > 0$ for all $0 < X < \infty$. Indeed, (1) we have

$$f(X) = (n-1)(X-1)^2 \left\{ X^2 - 2\frac{n+1}{n-1}X + 1 \right\} + 16X^2,$$

which is positive when either $X \geq 4$ and $n \geq 3$ or $X \leq 0.2679$ and $n \geq 3$. Furthermore, (2) we have

$$f(X) = (n-1)(X-1)^4 + 4X\left(4X - (X-1)^2\right),$$

and $4X - (X-1)^2 > 0$ if $0.171573 = 3 - 2\sqrt{2} < X < 3 + 2\sqrt{2} = 5.82843$. So we have, $f(X) > 0$ when $0.172 < X < 5.82$, Thus, by (1) and (2), $f(X) > 0\,(0 < X < \infty)$ when $n \geq 3$. In the case $n = 2$, $f(X) = X^4 - 8X^3 + 30X^2 - 8X + 1 > 0$ on $(0, \infty)$. Thus, (6.28) has no solution for all $n \geq 2$. . Therefore, φ is biharmonic if and only if harmonic in this case.

Case (3) C-Type: By (6.9), $\varphi : M \hookrightarrow \mathbb{C}P^n(4)$ is harmonic if and only if $t = \cot u$ $(0 < u < \frac{\pi}{4})$ must satisfy

$$(n-2)t^4 - 2(n+2)t^2 + n - 2 = 0, \tag{80}$$

which is equivalent to that

$$t = \cot u = \left\{\frac{n + 2 \pm 2\sqrt{2n}}{n-2}\right\}^{1/2} = \frac{\sqrt{n} \pm \sqrt{2}}{\sqrt{n-2}}. \tag{81}$$

On the other hand, by Theorem 5.1 and (6.10), $\varphi : M \hookrightarrow \mathbb{C}P^n(4)$ is non-harmonic but biharmonic if and only if

$$\begin{aligned} g(X) : &= (n-2)X^2(X-1)^2 + (n-2)(X-1)^2 + 4X(X^2 + 6X + 1) \\ &- 2X(X-1)^2 - 2(n+1)X(X-1)^2 = 0, \end{aligned} \tag{82}$$

where $X := t^2$. But, $g(X) > 0$ for all $0 < X < \infty$ and $n \geq 3$. Indeed, (1) we have

$$g(X) = (n-2)(X-1)^2 \left\{X^2 - 2\frac{n+2}{n-2}X + 1\right\} + 4X(X^2 + 6X + 1),$$

which is positive when either $X > 5 + 2\sqrt{6}$ or $0 < X < 5 - 2\sqrt{6}$ if $n \geq 3$. Furthermore, (2) we have

$$g(X) = (n-2)(X-1)^4 + 4X\left(-X^2 + 10X - 1\right),$$

and $-X^2 + 10X - 1 > 0$ if $5 - 2\sqrt{6} < X < 5 + 2\sqrt{6}$. Finally, (3) we have $g(5 \pm 2\sqrt{6}) = (4 \pm 2\sqrt{6})^4 > 0$. Thus, by (1), (2) and (3), $g(X) > 0$ on $(0, \infty)$ when $n \geq 3$. Thus, (6.31) has no solution for all $n \geq 3$. . Therefore, φ is biharmonic if and only if harmonic in this case.

Case (4) D-type. By (6.13), $\varphi : M \hookrightarrow \mathbb{C}P^9$ is harmonic if and only if $t = \cot u = \frac{1}{5}$, and by (6.14), is biharmonic but not harmonic if and only if $t = \cot u$ is a solution of the equation

$$11X^3 + 63X^2 + X + 5 - 20X(X-1)^2 = 0 \tag{83}$$

which is equivalent to

$$h(X) := 11X^3 + 43X^2 + 41X - 15 = 0. \tag{84}$$

This has a solution because $h(0) = -15 < 0$, $h(X) > 0$ for a large X, and the mean value theorem. Indeed, The solution X of (6.33) is 0.278629, and the corresponding $t = \cot u$ is 0.527853, and u is 1.08512.

Case (5) E-type. By (6.17), $\varphi : M \hookrightarrow \mathbb{C}P^{15}$ is harmonic if and only if $t = \cot u = \frac{\sqrt{15} \pm \sqrt{6}}{3}$ if and only if u is 0.443039 or 1.12776. By (6.18), is biharmonic but not harmonic if and only if $t = \cot u$ is a solution of the equation

$$21X^3 + 99X^2 - 9X + 9 - 2X(X-1)^2 = 0 \tag{85}$$

which is equivalent to

$$k(X) := 13X^3 - 107X^2 + 43X - 9 = 0. \tag{86}$$

This has a solution because $k(0) = -9 < 0$, $k(X) > 0$ for a large X, and the mean value theorem. Indeed, The solution X of (6.35) is 7.81906, and the corresponding $t = \cot u$ is 2.79626, and $u = 0.343448$. QED

6 Biharmonic homogeneous real hypersurfaces in the quarternionic projective space

In this section, we show classification of all the real hypersurfaces curvature adapted in the quarternionic projective space $\mathbb{H}P^n(4)$ which are *biharmonic*.

Let $(N,h) = \mathbb{H}P^n(c)$ be the quaternionic projective space with quarternionic sectional curvature $c > 0$. Then, the Riemannian curvature tensor is given by

$$R(U,V)W = \frac{c}{4}\Big\{h(V,W)U - h(U,W)V + \sum_{\alpha=1}^{3}\big(h(J_\alpha V,W)J_\alpha U - h(J_\alpha U,W)J_\alpha V + 2h(U,J_\alpha V)J_\alpha W\big)\Big\},$$

for vector fields U, V and W on $\mathbb{H}P^n(c)$. Here, J_α $(\alpha = 1,2,3)$ are the locally defined adapted three almost complex tensors on $\mathbb{H}P^n(c)$ which satisfy $J_1J_2 = -J_2J_1 = J_3$. Then, we have the following theorem which we omit its proof since one can prove it by the same manner as Theorem 5.1 whose proof is omitted.

Theorem 6. *Let (M,g) be a real $(4n-1)$-dimensional compact Riemannian manifold, and $\varphi : (M,g) \to \mathbb{H}P^n(c)$ be an isometric immersion with constant non-zero mean curvature $(n \geq 2)$. Then, the necessary and sufficient condition for φ to be biharmonic is*

$$\|B(\varphi)\|^2 = (n+2)c. \tag{87}$$

Now, let us recall Berndt's classification ([2]) of all the real hypersurfaces (M,g) in the quarternionic projective space $\mathbb{H}P^n(4)$ which are *curvature adapted*, i.e., $J_\alpha\xi$ is a direction of the principal curvature for all $\alpha = 1,2,3$, where ξ is the unit normal vector field along M.

Theorem 7. (*Berndt* [2]) (I) *All the curvature adapted real hypersurfaces in $\mathbb{H}P^n(4)$ are one of the following:*

(1) *a geodesic sphere $M(u)$ of radius u $(0 < u < \frac{\pi}{2})$,*

(2) *a tube $M(u)$ of radius u $(0 < u < \frac{\pi}{4})$ of the complex projective space $\mathbb{C}P^n \subset \mathbb{H}P^n(4)$, and*

(3) *tubes $M_k(u)$ of radii u $(0 < u < \frac{\pi}{4})$ of the quaternionic projective subspaces $\mathbb{H}P^k \subset \mathbb{H}P^n(4)$ with $1 \leq k \leq n-1$.*

(II) *Furthermore, their principal curvatures are given as follows.*

(1) *The geodesic sphere $M(u)$:*

$$\begin{cases} \lambda_1 = \cot u \text{ (with multiplicity } m_1 = 4(n-1)), \\ \lambda_2 = 2\cot(2u) \text{ (with multiplicity } m_2 = 3). \end{cases} \tag{88}$$

(2) *The tube $M(u)$ of the complex projective space:*

$$\begin{cases} \lambda_1 = \cot u \text{ (with multiplicity } m_1 = 2(n-1)), \\ \lambda_2 = -\tan u \text{ (with multiplicity } m_2 = 2(n-1)), \\ \lambda_3 = 2\cot(2u) \text{ (with multiplicity } m_3 = 1), \\ \lambda_4 = -2\tan(2u) \text{ (with multiplicity } m_4 = 2). \end{cases} \tag{89}$$

(3) *The tubes $M_k(u)$ of the quarternionic projective spaces:*

$$\begin{cases} \lambda_1 = \cot u \text{ (with multiplicity } m_1 = 4(n-k-1)), \\ \lambda_2 = -\tan u \text{ (with multiplicity } m_2 = 4k), \\ \lambda_3 = 2\cot(2u) \text{ (with multiplicity } m_3 = 3). \end{cases} \tag{90}$$

Then, we obtain the following theorem.

Theorem 8. *For all the three classes* (1), (2) *and* (3) *of Theorem 7.2, harmonic (i.e, minimal), and biharmonic but not harmonic real hypersurfaces* $M(u)$ *or* $M_k(u)$ *in* $\mathbb{H}P^n(4)$ *with radii* u *are given as follows:*

(1) *The geodesic sphere* $M(u)$: *The necessary and sufficient condition for* $M(u)$ *is to be harmonic (i.e., minimal) is that* $t = \cot u$ $(0 < u < \frac{\pi}{2})$ *satisfies*

$$t = \sqrt{\frac{3}{4n-1}}, \tag{91}$$

and to be biharmonic but not harmonic is that $t = \cot u$ $(0 < u < \frac{\pi}{2})$ *satisfies*

$$(4n-1)t^4 - 2(2n+7)t^2 + 3 = 0. \tag{92}$$

Both the (7.5) *and* (7.6) *have always solutions.*

(2) *The tube* $M(u)$ *of radius* u $(0 < u < \frac{\pi}{4})$ *of the complex projective space: The necessary and sufficient condition for* $M(u)$ *is to be harmonic (i.e., minimal) is that*

$$(2n-1)t^4 - (4n+5)t^2 + 2(n-1) = 0, \tag{93}$$

and to be biharmonic but not harmonic is that

$$(2n-1)t^8 - 8(n+1)t^6 - (6n+11)t^4 - 2(2n-1)t^2 - 12 = 0. \tag{94}$$

Both the (7.7) *and* (7.8) *have always solutions.*

(3) *The tubes* $M_k(u)$ *of radii* u $(0 < u < \frac{\pi}{4})$ *of the quarternioinic projective subspaces: The necessary and sufficient conditions for* $M_k(u)$ *to be harmonic (i.e., minimal) is that*

$$t = \sqrt{\frac{4k+3}{4n-4k-1}}, \tag{95}$$

and to be biharmonic but not harmonic is that

$$(4n-4k-1)t^4 - 2(2n+4)t^2 + 4k + 3 = 0. \tag{96}$$

Both the (7.9) *and* (7.10) *have always solutions.*

Proof. Case (1): The geodesic sphere $M(u)$. In this case, the mean curvature H of $M(u)$ is given by

$$\begin{aligned} H &= \frac{1}{4n-1}\{4(n-1)\cot u + 3\,2\cot(2u)\} \\ &= 4(n-1)t + 3\left(t - \frac{1}{t}\right), \end{aligned} \tag{97}$$

where $t = \cot u$, so that $M(u)$ is harmonic, i.e., minimal if and only if

$$4(n-1)t^2 + 3t - 3 = 0 \quad \Longleftrightarrow \quad t = \sqrt{\frac{3}{4n-1}}. \tag{98}$$

The square of the second fundamental form $\|B(\varphi)\|^2$ is given by

$$\|B(\varphi)\|^2 = 4(n-1)t^2 + 3\left(t - \frac{1}{t}\right)^2 = (4n-1)t^2 + \frac{3}{t^2} - 6, \tag{99}$$

which yields by Theorem 7.1, that $M(u)$ is biharmonic, but not harmonic if and only if

$$\begin{aligned}&(4n-1)t^2+\frac{3}{t^2}-2(2n+7)=0\\ \iff\quad & t^2=\frac{2n+7\pm\sqrt{n^2+4n+13}}{4n-1},\end{aligned} \tag{100}$$

which has always solutions.

Case (2): The tube $M(u)$ of $\mathbb{C}P^n\subset\mathbb{H}P^n(4)$. In this case, the mean curvature $(4n-1)H$ of $M(u)$ coincides with

$$\begin{aligned}&2(n-1)\cot u+2(n-1)(-\tan u)+2\cot(2u)+2(-2\cot(2u))\\ &=2(n-1)+2(n-1)\left(-\frac{1}{t}\right)+\left(t-\frac{1}{t}\right)+2\left(\frac{-4t}{t^2-1}\right)\\ &=\frac{(2n-1)t^4-(4n+5)t^2+2(n-1)}{t(t^2-1)},\end{aligned} \tag{101}$$

where $t=\cot u$, so that $M(u)$ is harmonic, i.e., minimal if and only if

$$\begin{aligned}&2(n-1)t^4-(4n+5)t^2+2(n-1)=0\\ \iff\quad & t^2=\frac{4n+5\pm\sqrt{3(n+2)(2n+9)}}{2(n-1)},\end{aligned} \tag{102}$$

which has always solutions. On the other hand, $\|B(\varphi)\|^2$ coincides with

$$\begin{aligned}&2(n-1)t^2+\frac{2(n-1)}{t^2}+t^2-2+\frac{1}{t^2}+\frac{32}{(t^2-1)^2}\\ &=\frac{(2n-1)X^2(X-1)^2+(2n-1)(X-1)^2-2X(X-1)^2+32X^2}{X(X-1)^2}\end{aligned} \tag{103}$$

where $X=t^2$. Hence, $M(u)$ is biharmonic, but not harmonic if and only if

$$(2n-1)X^4-8(n+1)X^3-(6n+11)X^2-2(2n-1)X-12=0, \tag{104}$$

with $X=t^2$, $t=\cot u$ with $0<u<\frac{\pi}{4}$. Denoting by $f(t)$ the LHS,

$$f(0)=-12<0,\quad f(t)>0$$

for large t. Thus, by the mean value theorem, (7.18) has always solutions X, so t, but not solutions of (7.16).

Case (3): The tubes of $\mathbb{H}P^k\subset\mathbb{H}P^n(4)$. In this case, the mean curvature H of $M(u)$ is given by

$$\begin{aligned}H&=4(n-k-1)\cot u+4k(-\tan u)+6\cot(2u)\\ &=4(n-k-1)t+4k\left(-\frac{1}{t}\right)+3\left(t-\frac{1}{t}\right),\end{aligned} \tag{105}$$

with $t=\cot u$, so that $M(u)$ is harmonic, i.e., minimal if and only if

$$t=\sqrt{\frac{4k+3}{4n-4k-1}}. \tag{106}$$

On the other hand, $\|B(\varphi)\|^2$ is given by

$$\begin{aligned} \|B(\varphi)\|^2 &= 4(n-k-1)\cot^2 u + 4k\tan^2 u + 12\cot^2(2u) \\ &= (4n-4k-1)t^2 + \frac{4k}{t^2} + 3\left(t - \frac{1}{t}\right)^2 \\ &= (4n-4k-1)t^2 + \frac{4k+3}{t^2} - 6, \end{aligned} \tag{107}$$

so that $M(u)$ is biharmonic, but not harmonic if and only if

$$(4n-4k-1)t^4 - 2(2n+4)t^2 + 4k + 3 = 0, \tag{108}$$

which has always solutions. QED

7 Biharmonic maps into a manifold of nonpositive curvature

In this section, we show answers in case of bounded geometry, to the following conjectures proposed by B.Y. Chen ([5]), and R. Caddeo, S. Montaldo and P. Piu ([4]):

B.Y. Chen's Conjecture. *Any biharmonic submanifold of the Euclidean space is harmonic.*

or more generally,

R. Caddeo, S. Montaldo and P. Piu's conjecture. *The only biharmonic submanifolds of a complete Riemanian manifold whose curvature is nonpositive are the minimal ones.*

Example 1. Let $\varphi : (\mathbb{R}^m, g_0) \ni x = (x_1, \ldots, x_m) \mapsto (\varphi_1, \ldots, \varphi_n) \in (\mathbb{R}^n, h_0)$ be a smooth mapping given by

$$\varphi_i(x) = \sum_{j=1}^m {x_j}^4 - m\,{x_i}^4 \quad (i = 1, \cdots, m),$$

and $\varphi_j(x)$ $(j = m+1, \ldots, n)$ are at most linear, where $(\mathbb{R}^m, g_0)$ and $(\mathbb{R}^n, h_0)$ are the standard Euclidean spaces, respectively. Then, we have

$$\begin{cases} \tau(\varphi) = \Delta\varphi = (\Delta\varphi_1, \ldots, \Delta\varphi_n), \\ \tau_2(\varphi) = \Delta(\Delta\varphi) = 0, \end{cases}$$

where

$$\Delta\varphi_i = 12\left(\sum_{j=1}^m {x_j}^2 - m\,{x_i}^2\right) \quad (i = 1, \ldots, m).$$

Furthermore, we have

$$\begin{aligned} \|\tau(\varphi)\|^2 &= 12^2\, m\left(m\sum_{j=1}^m {x_j}^4 - \left(\sum_{j=1}^m {x_j}^2\right)^2\right) \geq 0, \\ \|\overline{\nabla}\tau(\varphi)\|^2 &= 24^2\, m(m-1)\left(\sum_{j=1}^m {x_j}^2\right)^2. \end{aligned}$$

However, we show

Theorem 9. *Let $\varphi:(M,g)\to(N,h)$ be a biharmonic map from a complete Riemannian manifold (M,g) of bounded sectional curvature $|\mathrm{Riem}^M|\leq C$ into a Riemannian manifold (N,h) of nonpositive curvature, i.e.,* $\mathrm{Riem}^N\leq 0$. *Assume that*

$$\|\tau(\varphi)\|\in L^2(M),\ \ \textit{and}\quad \|\overline{\nabla}\tau(\varphi)\|\in L^2(M). \tag{109}$$

Then, $\varphi:(M,g)\to(N,h)$ is harmonic.

Corollary 1. *Let $\varphi:(M,g)\to(N,h)$ be a biharmonic isometric immersion from a complete Riemannian manifold (M,g) of bounded sectional curvature $|\mathrm{Riem}^M|\leq C$ into a Riemannian manifold (N,h) of nonpositive curvature, i.e.,* $\mathrm{Riem}^N\leq 0$. *Assume that the second fundamental form $\tau(\varphi)$ satisfies that*

$$\|\tau(\varphi)\|\in L^2(M),\ \ \textit{and}\quad \|\overline{\nabla}\tau(\varphi)\|\in L^2(M). \tag{110}$$

Then $\varphi:(M,g)\to(N,h)$ is harmonic.

Before going to prove Theorem 8.1, we prepare a cut off function λ_R $(0<R<\infty)$ on a complete Riemannian manifold (M,g) as follows ([6]). Let μ be a real valued C^∞ function on $\mathbb{R}$ satistying the following conditions:

$$\begin{cases} 0\leq\mu(t)\leq 1 \quad (t\in\mathbb{R}),\\ \mu(t)=1 \qquad (t\leq 1),\\ \mu(t)=0 \qquad (t\geq 2),\\ |\mu'|\leq C,\ \text{and}\ |\mu''|\leq C, \end{cases} \tag{111}$$

where $\mu'(t)$ and $\mu''(t)$ stand for the derivations of the first and second order of $\mu(t)$ with respect to t, respectively. Then, for all $R>0$, the function defined by

$$\lambda_R(x)=\mu\left(\frac{r(x)}{R}\right),\quad (x\in M)$$

is said to be a *cut off function* on (M,g), where

$$r(x)=d(x_0,x),\quad (x\in M)$$

for some fixed point x_0 in M and $d(x,y)$, $(x,y\in M)$ is the Riemannian distance function of (M,g). Then, it is known ([6]) that

Lemma 4. (i) *λ_R is a Lipshitz function on M, and differentiable a.e. on M,*
(ii) $\mathrm{supp}(\lambda_R)\subset B_{2R}(x_0)$,
(iii) $0\leq\lambda_R(x)\leq 1,\quad (x\in M)$,
(iv) $\lambda_R(x)=1,\quad (x\in B_R(x_0))$,
(v) $|\nabla\lambda_R|\leq\frac{C}{R}$, *(a.e. on M),*
(vi) *and if the Ricci curvature of (M,g) is bounded below by a constant $(m-1)(-k)$ for some* $k>0$ $(m=\dim M)$, *then,*

$$|\Delta\lambda_R|\leq\frac{C}{R^2}+\frac{CC'}{R}\quad (\textit{a.e. on } M). \tag{112}$$

Here C' is a positive constant depending only on m and k, $\mathrm{supp}(\lambda_R)$ *stands for the support of λ_R, and $B_r(x):=\{y\in M;\,d(x,y)<r\}$ is the Riemannian disc in (M,g) around x with radius* $r>0$.

Proof. From (i) to (v), see [6], for instance. For (vi), let us recall the estimation of Δr in terms of the lower bound of the Ricci curvature (see [15] for instance): If the Ricci curvature of (M, g) is bounded below by a constant $(m-1)(-k)$ for some $k>0$ ($m = \dim M$), then,

$$\Delta r \leq (m-1)\frac{{f_k}'}{f_k} = \frac{m-1}{\sqrt{k}}\,\frac{\cosh(\sqrt{k}r)}{\sinh(\sqrt{k}r)}, \tag{113}$$

where where $f_k(t) = \frac{\sinh(\sqrt{k}t)}{\sqrt{k}}$ is the unique solution of the initial value problem

$${f_k}'' + (-k)f_k = 0, \quad f_k(0) = 0,\ f_k'(0) = 1.$$

Thus, outside of $B_R(x_0)$, it holds that

$$|\Delta r| \leq \frac{m-1}{\sqrt{k}}\,\frac{\cosh(\sqrt{k}R)}{\sinh(\sqrt{k}R)}. \tag{114}$$

Since $\nabla\lambda_R = \frac{1}{R}\,\mu'\left(\frac{r}{R}\right)\,\nabla r$ (see [15], p. 108), we have, a.e. on M,

$$\Delta\lambda_R = \frac{1}{R^2}\,\mu''\left(\frac{r}{R}\right) + \frac{1}{R}\,\mu'\left(\frac{r}{R}\right)\,\Delta r. \tag{115}$$

Then, together with (8.3), (8.6) and (8.7), we have (8.4). QED

Now let us begin a proof of Theorem 8.1. Let us recall the definition of $e_2(\varphi) = \frac{1}{2}\|\tau(\varphi)\|^2$. We will estimate $\Delta(\lambda_R\, e_2(\varphi))$ as follows.

$$\Delta(\lambda_R\, e_2(\varphi)) = (\Delta\lambda_R)\, e_2(\varphi) + 2g(\nabla\lambda_R, \nabla e_2(\varphi)) + \lambda_R\, \Delta e_2(\varphi). \tag{116}$$

For the LHS of (8.8), we have $\Delta(\lambda_R\, e_2(\varphi)) = \mathrm{div}X$, where $X := \nabla(\lambda_R\, e_2(\varphi))$ which is a C^∞ vector field on M with compact support. Due to Green's theorem,

$$\int_M \Delta(\lambda_R\, e_2(\varphi))v_g = \int_M \mathrm{div}(X)v_g = 0. \tag{117}$$

Furthermore, we have

$$\lim_{R\to\infty}\int_M (\Delta\lambda_R)e_2(\varphi)v_g = 0, \tag{118}$$

$$\lim_{R\to\infty}\int_M g(\nabla\lambda_R, \nabla e_2(\varphi))v_g = 0. \tag{119}$$

Indeed, for (8.10), by (8.4) in Lemma 8.1,

$$\begin{aligned}\left|\int_M (\Delta\lambda_R)e_2(\varphi)v_g\right| &\leq \int_M |\Delta\lambda_R|\, e_2(\varphi)v_g \\ &\leq \int_M \left(\frac{C}{R^2} + \frac{CC'}{R}\right) e_2(\varphi)v_g \\ &= \left(\frac{C}{R^2} + \frac{CC'}{R}\right)\int_M e_2(\varphi)v_g,\end{aligned} \tag{120}$$

where the RHS goes to 0 if $R\to\infty$, since $e_2(\varphi) = \frac{1}{2}\|\tau(\varphi)\|^2 \in L^1(M)$ by the assumptions (8.1). For (8.11), due to (v) in Lemma 8.1,

$$\begin{aligned}\left|\int_M g(\nabla\lambda_R, \nabla e_2(\varphi))v_g\right| &\leq \int_M |g(\nabla\lambda_R, \nabla e_2(\varphi))|v_g \\ &\leq \int_M \|\nabla\lambda_R\|\ \|\nabla e_2(\varphi)\|v_g \\ &\leq \frac{C}{R}\int_M \|\nabla e_2(\varphi)\|v_g,\end{aligned} \tag{121}$$

where the RHS goes to 0 if $R \to \infty$, since

$$\begin{aligned}\int_M \|\nabla e_2(\varphi)\| v_g &= \int_M |g(\overline{\nabla}\tau(\varphi), \tau(\varphi))| v_g \\ &\leq \int_M \|\overline{\nabla}\tau(\varphi)\| \, \|\tau(\varphi)\| v_g \\ &\leq \|\overline{\nabla}\tau(\varphi)\|_{L^2(M)} \|\tau(\varphi)\|_{L^2(M)} < \infty\end{aligned}$$

by the assumptions (8.1).

Thus, due to ((8.8), (8.9), (8.10), (8.11), we obtain

$$\lim_{R\to\infty} \int_M \lambda_R \, \Delta e_2(\varphi) v_g = 0. \tag{122}$$

Now, by the computation (4.1) in [14] in which Jiang used only the assumption that $\varphi : (M, g) \to (N, h)$ is biharmonic, we have

$$\begin{aligned}\Delta e_2(\varphi) &= \sum_{k=1}^{m} g(\overline{\nabla}_{e_k}\tau(\varphi), \overline{\nabla}_{e_k}\tau(\varphi)) + g(-\overline{\nabla}^*\overline{\nabla}\tau(\varphi), \tau(\varphi)) \\ &= \sum_{k=1}^{m} g(\overline{\nabla}_{e_k}\tau(\varphi), \overline{\nabla}_{e_k}\tau(\varphi)) \\ &\quad - \sum_{k=1}^{m} h(R^N(\tau(\varphi), d\varphi(e_k))d\varphi(e_k), \tau(\varphi)).\end{aligned} \tag{123}$$

Then, we have

$$\begin{aligned}\int_M \lambda_R \, \Delta e_2(\varphi) v_g &= \int_M \lambda_R \left(\sum_{k=1}^{m} g(\overline{\nabla}_{e_k}\tau(\varphi), \overline{\nabla}_{e_k}\tau(\varphi)) \right) v_g \\ &+ \int_M \lambda_R \left(- \sum_{k=1}^{m} h(R^N(\tau(\varphi), d\varphi(e_k))d\varphi(e_k), \tau(\varphi)) \right) v_g.\end{aligned} \tag{124}$$

Here, the first term of the RHS of (8.16) goes to 0 when $R \to \infty$, i.e.,

$$\lim_{R\to\infty} \int_M \lambda_R \left(\sum_{k=1}^{m} g(\overline{\nabla}_{e_k}\tau(\varphi), \overline{\nabla}_{e_k}\tau(\varphi)) \right) v_g = 0. \tag{125}$$

Because, both the integrand of (8.17) is nonnegative, and by the curvature assumption of (N, h), $\mathrm{Riem}^N \leq 0$, the integrand of the second term of RHS of (8.16) is nonnegative. Thus, (8.14) implies the desired (8.17).

Notice here, that (8.17) implies

$$\int_M \sum_{k=1}^{m} g(\overline{\nabla}_{e_k}\tau(\varphi), \overline{\nabla}_{e_k}\tau(\varphi)) v_g = 0, \tag{126}$$

which yields that $\overline{\nabla}_X \tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$.

Finally, if we consider a C^∞ vector field X_φ on M defined by

$$X_\varphi := \sum_{k=1}^{m} h(d\varphi(e_k), \tau(\varphi)) e_k,$$

the divergence of X_φ satisfies that

$$\begin{aligned}\operatorname{div}(X_\varphi) &= h(\tau(\varphi), \tau(\varphi)) + \sum_{k=1}^{m} h(d\varphi(e_k), \overline{\nabla}_{e_k}\tau(\varphi)) \\ &= h(\tau(\varphi), \tau(\varphi)) \in L^1(M),\end{aligned} \tag{127}$$

by the above and the assumptions (8.1). Therefore, due to the Green's theorem on a complete Riemannian manifolds (M, g) (see [9] for instance), we obtain

$$\int_M h(\tau(\varphi), \tau(\varphi)) v_g = \int_M \operatorname{div}(X_\varphi) v_g = 0, \tag{128}$$

which yields $\tau(\varphi) = 0$. QED

8 The first variational formula for bi-Yang-Mills fields

From this section, we begin to prepare fundamental materials to state interesting phenomena on bi-Yang-Mills fields which are closely related to biharmonic maps. We will recall the Yang-Mills setting ([3]) and the definition of bi-Yang-Mills fields following Bejan and Urakawa ([1]), and show the isolation phenomena.

Let us start with the Yang-Mills setting following [3]. Let (E, h) be a real vector bundle of rank r with an inner product h over an m-dimensional compact Riemannian manifold (M, g). Let $\mathcal{C}(E, h)$ be the space of all C^∞-connections of E satisfying the compatibility condition:

$$Xh(s, t) = h(\nabla_X s, t) + h(s, \nabla_X t), \quad s,\, t \in \Gamma(E),$$

for all $X \in \mathfrak{X}(M)$, where $\Gamma(E)$ stands for the space of all C^∞-sections of E. For $\nabla \in \mathcal{C}(E, h)$, let R^∇ be its curvature tensor defined by

$$R^\nabla(X, Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s,$$

for all $X, Y \in \mathfrak{X}(M)$, $s \in \Gamma(E)$. Let $F = \operatorname{End}(E, h)$ be the bundle of endmorphisms of E which are skew symmetric with respect to the inner product h on E. We define the inner product $\langle\,,\,\rangle$ on F by

$$\langle \varphi, \psi \rangle = \sum_{i=1}^{r} h(\varphi u_i, \psi u_i), \quad \varphi, \psi \in F_x,$$

where $\{u_i\}_{i=1}^r$ is an orthonormal basis of E_x with respect to h ($x \in M$). Let us also consider the space of F-valued k-forms on M, denoted by $\Omega^k(F) = \Gamma(\wedge^k T^*M) \otimes F)$, which admits a global inner product $(\,,\,)$ given by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle v_g,$$

where the pointwise inner product $\langle \alpha, \beta \rangle$ is given by

$$\langle \alpha, \beta \rangle = \sum_{i_1 < \cdots < i_k} \langle \alpha(e_{i_1}, \dots, e_{i_k}), \beta(e_{i_1}, \dots, e_{i_k}) \rangle$$

and $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g).

For every $\nabla \in \mathcal{C}(E, h)$, let $d^\nabla : \Omega^k(F) \to \Omega^{k+1}(F)$ be the exterior differentiation with respect to ∇ (cf. [3]), and the adjoint operator $\delta^\nabla : \Omega^{k+1}(F) \to \Omega^k(F)$ given by

$$\delta^\nabla \alpha = (-1)^{k+1} * d^\nabla * \alpha, \quad \alpha \in \Omega^{k+1} F),$$

where $* : \Omega^p(F) \to \Omega^{m-p}(F)$ is the extension of the usual Hodge star operator on (M, g). Then, it holds that

$$(d^\nabla \alpha, \beta) = (\alpha, \delta^\nabla \beta), \quad \alpha \in \Omega^k(F),\, \beta \in \Omega^{k+1}(F).$$

Now let us recall the bi-Yang-Mills functional (see [1]) and Yang-Mills one (see [3]):

Definition 2.

$$\mathcal{Y}M_2(\nabla) = \frac{1}{2}\int_M \|\delta^\nabla R^\nabla\|^2 v_g, \quad \nabla \in \mathcal{C}(E,h), \tag{129}$$

$$\mathcal{Y}M(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2 v_g, \quad \nabla \in \mathcal{C}(E,h), \tag{130}$$

where $\|\delta^\nabla R^\nabla\|$, (resp. $\|R^\nabla\|$) is the norm of $\delta^\nabla R^\nabla \in \Omega^1(F)$ (resp. $R^\nabla \in \Omega^2(F)$) relative to each $\langle\, ,\, \rangle$.

Then, the bi-Yang-Mills fields and the Yang-Mills ones are critical points of the above functionals as follows.

Definition 3. For each $\nabla \in \mathcal{C}(E,h)$, it is a *bi-Yang-Mills field* (resp. *Yang-Mills field*) if for any smooth one-parameter family ∇^t $(|t| < \epsilon)$ with $\nabla^0 = \nabla$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{Y}M_2(\nabla^t) = 0, \quad \left(\textit{resp. } \left.\frac{d}{dt}\right|_{t=0} \mathcal{Y}M(\nabla^t) = 0\right). \tag{131}$$

Then, the first variation formulas are given as

Theorem 10. *([1], [3]) Let* $\alpha = \frac{d}{dt}|_{t=0}\nabla^t \in \Omega^1(F)$. *Then, we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{Y}M_2(\nabla^t) = \int_M \langle(\delta^\nabla d^\nabla + \mathcal{R}^\nabla)(\delta^\nabla R^\nabla), \alpha\rangle v_g, \tag{132}$$

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{Y}M(\nabla^t) = \int_M \langle\delta^\nabla R^\nabla, \alpha\rangle v_g, \tag{133}$$

respectively. Here, $\mathcal{R}^\nabla(\beta) \in \Omega^1(F)$ $(\beta \in \Omega^1(F))$ *is defined by*

$$\mathcal{R}^\nabla(\beta)(X) = \sum_{j=1}^m [R^\nabla(e_j, X), \beta(e_j)], \quad X \in \mathfrak{X}(M). \tag{134}$$

Thus, ∇ *is a bi-Yang-Mills field (resp. Yang-Mills one) if and only if*

$$(\delta^\nabla d^\nabla + \mathcal{R}^\nabla)(\delta^\nabla R^\nabla) = 0 \quad (\textit{resp. } \delta^\nabla R^\nabla = 0). \tag{135}$$

Thus, by this theorem, we have immediately

Corollary 2. *If* ∇ *is a Yang-Mills field, then it is also a bi-Yang-Mills one.*

Lemma 5. *For all* β_1, $\beta_2 \in \Omega^1(F)$, *and* $\varphi \in \Omega^2(F)$, *we have*

$$\langle\varphi, [\beta_1 \wedge \beta_2]\rangle = \langle\mathcal{R}(\varphi)(\beta_2), \beta_1\rangle = \langle\beta_2, \mathcal{R}(\varphi)(\beta_1)\rangle. \tag{136}$$

Proof. For the first equality, we have

$$
\begin{aligned}
\langle \varphi, [\beta_1 \wedge \beta_2] \rangle &= \sum_{i<j} \langle \varphi(e_i, e_j), [\beta_1 \wedge \beta_2](e_i, e_j) \rangle \\
&= \sum_{i<j} \varphi(e_i, e_j), [\beta_1(e_i), \beta_2(e_j)] - [\beta_1((e_j), \beta_2(e_i)] \rangle \\
&= \sum_{i,j=1}^{m} \langle \varphi(e_i, e_j), [\beta_1(e_i), \beta_2(e_j)] \rangle \\
&= \sum_{i=1}^{m} \left\langle \sum_{j=1}^{m} [\varphi(e_j, e_i), \beta_2(e_j)], \beta_1(e_i) \right\rangle \\
&= \sum_{i=1}^{m} \langle \mathcal{R}(\varphi)(\beta_2)(e_i), \beta_1(e_i) \rangle \\
&= \langle \mathcal{R}(\varphi)(\beta_2), \beta_1 \rangle,
\end{aligned}
$$

since $\langle [\eta, \psi], \xi \rangle + \langle \psi, [\eta, \xi] \rangle = 0$ for all endomorphisms η, ψ, and ξ of E_x $(x \in M)$. By the same reason, for the second equality, we have

$$
\begin{aligned}
\langle \mathcal{R}(\varphi)(\beta_2), \beta_1 \rangle &= \sum_{i=1}^{m} \left\langle \sum_{j=1}^{m} [\varphi(e_j, e_i), \beta_2(e_j)], \beta_1(e_i) \right\rangle \\
&= - \sum_{i,j=1}^{m} \langle \beta_2(e_j), [\varphi(e_j, e_i), \beta_1(e_i)] \rangle \\
&= \sum_{j=1} \langle \beta_2(e_j), \sum_{i=1}^{m} [\varphi(e_i, e_j), \beta_1(e_i)] \rangle \\
&= \sum_{j=1}^{m} \langle \beta_2(e_j), \mathcal{R}(\varphi)(\beta_1)(e_j) \rangle \\
&= \langle \beta_2, \mathcal{R}(\varphi)(\beta_1) \rangle,
\end{aligned}
$$

thus, we obtain (9.8). QED

9 Isolation phenomena for bi-Yang-Mills fields

In this section, we finally show very interesting phenomena which assert that Yang-Mills fields are isolated among the space of all bi-Yang-Mills fields over compact Riemannian manifolds with positive Ricci curvature.

Theorem 11. (*bounded isolation phenomena*) *Let (M, g) a compact Riemannian of which Ricci curvature is bounded below by a positive constant $k > 0$, i.e.,* $\mathrm{Ric} \geq k\,\mathrm{Id}$. *Assume that $\nabla \in \mathcal{C}(E, h)$ is a bi-Yang-Mills field with $\|R^{\nabla}\| < \frac{k}{2}$ pointwisely everywhere on M. Then, ∇ is a Yang-Mills field.*

Theorem 12. (*L^2-isolation phenomena*) *Let (M, g) be a four dimensional compact Riemannian manifold of which Ricci curvature is bounded below by a positive constant $k > 0$, i.e.,* $\mathrm{Ric} \geq k\,\mathrm{Id}$. *Assume that $\nabla \in \mathcal{C}(E, h)$ is a bi-Yang-Mills field satisfying that*

$$
\|R^{\nabla}\|_{L^2} < \frac{1}{2} \min \left\{ \frac{\sqrt{c_1}}{18},\ \frac{k}{2} \operatorname{Vol}(M, g)^{1/2} \right\}. \tag{137}
$$

Then, ∇ is a Yang-Mills field. Here, c_1 is the isoperimetric constant of (M, g) given by

$$c_1 = \inf_{W\subset M} \frac{\mathrm{Vol}_3(W)^4}{(\min\{\mathrm{Vol}(M_1),\, \mathrm{Vol}(M_2)\})^3}, \tag{138}$$

where $W \subset M$ runs over all the hypersurfaces in M, and $\mathrm{Vol}_3(W)$ is the three dimensional volume of W with respect to the Riemannian metric on W induced from g, and the complement of W in M has a disjoint union of M_1 and M_2.

To prove Theorem 10.1, we need the following Weitzenböck formula.

Lemma 6. *Assume that $\nabla \in \mathcal{C}(E, h)$ is a bi-Yang-Mills field. Then,*

$$\begin{aligned}\frac{1}{2}\Delta\|\delta^\nabla R^\nabla\|^2 &= \langle 2\mathcal{R}^\nabla(\delta^\nabla R^\nabla) + \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla\rangle \\ &\quad + \sum_{i=1}^m \|\nabla_{e_i}(\delta^\nabla R^\nabla)\|^2.\end{aligned} \tag{139}$$

Here, $\Delta f = \sum_{i=1}^m ({e_i}^2 - \nabla_{e_i} e_i) f$ is the Laplacian acting on smooth functions f on M, and, for all $\alpha \in \Omega^1(F)$,

$$(\alpha \circ \mathrm{Ric})(X) := \alpha(\mathrm{Ric}(X)),\ X \in \mathfrak{X}(M), \tag{140}$$

where Ric *is the Ricci transform of (M, g).*

Proof. Indeed, for the LHS of (10.3), we have

$$\frac{1}{2}\Delta\|\delta^\nabla R^\nabla\|^2 = \langle -\nabla^*\nabla(\delta^\nabla R^\nabla), \delta^\nabla R^\nabla\rangle + \sum_{i=1}^m \langle \nabla_{e_i}(\delta^\nabla R^\nabla), \nabla_{e_i}(\delta^\nabla R^\nabla)\rangle. \tag{141}$$

Let us recall the Weitzenböck formula (cf. [3], p.199, Theorem (3.2)) that

$$\begin{aligned}\Delta^\nabla\alpha &= (d^\nabla\delta^\nabla + \delta^\nabla d^\nabla)\alpha \\ &= \nabla^*\nabla\alpha + \alpha\circ\mathrm{Ric} + \mathcal{R}^\nabla(\alpha), \quad \alpha \in \Omega^1(F).\end{aligned} \tag{142}$$

It holds that

$$\delta^\nabla(\delta^\nabla R^\nabla) = 0. \tag{143}$$

Because for all $\varphi \in \Gamma(F)$,

$$(\delta^\nabla(\delta^\nabla R^\nabla), \varphi) = \int_M \langle R^\nabla, d^\nabla(d^\nabla\varphi)\rangle v_g.$$

But, by using the formula (2.9) in [3], p. 194, the integrand of the RHS coincides with

$$\begin{aligned}\langle R^\nabla, d^\nabla(d^\nabla\varphi)\rangle &= \sum_{i<j}\sum_{s=1}^r \langle R^\nabla(e_i, e_j)u_s, (R^\nabla(e_i, e_j)\varphi)(u_s)\rangle \\ &= \sum_{i<j}\sum_{s=1}^r \langle R^\nabla(e_i, e_j)u_s, (R^\nabla(e_i, e_j)(\varphi(u_s)) - \varphi(R^\nabla(e_i, e_j)u_s)\rangle \\ &= \sum_{i<j}\langle R^\nabla(e_i, e_j), [R(e_i, e_j), \varphi]\rangle \\ &= -\sum_{i<j}\langle [R^\nabla(e_i, e_j), R^\nabla(e_i, e_j)], \varphi\rangle = 0.\end{aligned}$$

since $\langle \psi, [\eta, \xi]\rangle = -\langle [\eta, \psi], \xi\rangle$ for all $\eta,\ \psi,\ \xi \in F = \mathrm{End}(E, h)$.

Now ∇ is a bi-Yang-Mills field, $(\delta^\nabla d^\nabla + \mathcal{R}^\nabla)(\delta^\nabla R^\nabla) = 0$, so that we have

$$\begin{aligned} -\mathcal{R}^\nabla(\delta^\nabla R^\nabla) &= \delta^\nabla d^\nabla(\delta^\nabla R^\nabla) \\ &= \Delta^\nabla(\delta^\nabla R^\nabla) \quad \text{(by (10.7))} \\ &= \nabla^*\nabla(\delta^\nabla R^\nabla) + \delta^\nabla R^\nabla \circ \mathrm{Ric} + \mathcal{R}^\nabla(\delta^\nabla R^\nabla), \end{aligned}$$

by (10.6). Thus, we have

$$-\nabla^*\nabla(\delta^\nabla R^\nabla) = 2\mathcal{R}^\nabla(\delta^\nabla R^\nabla) + \delta^\nabla R^\nabla \circ \mathrm{Ric}. \tag{144}$$

Substituting (10.8) into the first term of the RHS of (10.5), we have (10.3). QED

Proof of Theorem 10.1. By Integrating (10.3) over M, and by Green's theorem, we have

$$\begin{aligned} 2\int_M \langle \mathcal{R}^\nabla(\delta^\nabla R^\nabla), \delta^\nabla R^\nabla\rangle v_g + \int_M \langle \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla\rangle v_g \\ + \int_M \sum_{i=1}^m \langle \nabla_{e_i}(\delta^\nabla R^\nabla), \nabla_{e_i}(\delta^\nabla R^\nabla)\rangle v_g = 0. \end{aligned} \tag{145}$$

Notice here that

$$|\langle \mathcal{R}^\nabla(\alpha), \alpha\rangle| \leq \|R^\nabla\|\,\|\alpha\|^2, \quad \alpha \in \Omega^1(F). \tag{146}$$

Indeed, by Lemma 9.1, and Schwarz inequality, we have

$$\begin{aligned} |\langle \mathcal{R}^\nabla(\alpha), \alpha\rangle| &= |\langle R^\nabla, [\alpha \wedge \alpha]\rangle| \\ &\leq \|R^\nabla\|\,\|[\alpha\wedge\alpha]\| \\ &\leq \|R^\nabla\|\,\|\alpha\|^2, \end{aligned} \tag{147}$$

of which the last ienquality follows from

$$\begin{aligned} \|[\alpha\wedge\alpha]\|^2 &= \sum_{i<j} \|[\alpha\wedge\alpha](e_i, e_j)\|^2 \\ &= \frac{1}{2}\sum_{i,j=1}^m \|[\alpha\wedge\alpha](e_i, e_j)\|^2 \\ &\leq \frac{1}{2}\sum_{i,j=1}^m 2\|\alpha(e_i)\|^2\,\|\alpha(e_j)\|^2 \quad \text{(Lemma (2.30) in [3], p.197)} \\ &= \left(\sum_{i=1}^m \|\alpha(e_i)\|^2\right)^2 \end{aligned}$$

which is equal to $\|\alpha\|^4$. We have (10.11).

Furthermore, by the assumption of the Ricci curvature of (M, g), we have

$$\langle \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla\rangle \geq k\|\delta^\nabla R^\nabla\|^2. \tag{148}$$

Indeed, since, at each point $x \in M$, we may choose an orthonormal basis $\{e_i\}_{i=1}^m$ of (T_xM, g_x) in such a way that

$$\mathrm{Ric}(e_i) = \mu_i e_i \quad (i = 1\ldots, m)$$

where μ_i $(i = 1, \dots, m)$ are bigger than or equal to $k > 0$. Then,

$$\begin{aligned}\langle \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla \rangle &= \sum_{i=1}^{m} \langle \delta^\nabla R^\nabla(\mathrm{Ric}(e_i)), \delta^\nabla R^\nabla(e_i)\rangle \\ &= \sum_{i=1}^{m} \mu_i \|\delta^\nabla R^\nabla(e_i)\|^2 \\ &\geq k\|\delta^\nabla R^\nabla\|^2.\end{aligned}$$

Under the assumption that $\|R^\nabla\| < \frac{k}{2}$ at each point of M, we have

$$\langle 2\mathcal{R}^\nabla(\delta^\nabla R^\nabla) + \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla\rangle \geq 0, \tag{149}$$

equality holds if and only if $\delta^\nabla R^\nabla = 0$.
Because, by (10.10) and (10.12), we have

$$\begin{aligned}&\langle 2\mathcal{R}^\nabla(\delta^\nabla R^\nabla) + \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla\rangle \\ &\qquad \geq (-2\|R^\nabla\| + k)\, \|\delta^\nabla R^\nabla\|^2 \\ &\qquad \geq 0,\end{aligned}$$

and equality holds if and only if $\|\delta^\nabla R^\nabla\| = 0$ by the assumption $\|\delta^\nabla R^\nabla\| < \frac{k}{2}$.

Now due to (10.13), both the sum of the first and second terms of th LHS of (10.9), and the third term in the same one are bigger than or equal to 0. Thus, (10.9) implies that the sum of the first and second term of (10.9) is 0, and by (10.13), we have $\delta^\nabla R^\nabla = 0$ everywhere on M.

Remark 1. (1) In the case $\|R^\nabla\| = \frac{k}{2}$, we can also conclude $\nabla_X(\delta^\nabla R^\nabla) = 0$ for all $X \in \mathfrak{X}(M)$. (2) In tha case of the unit sphere $(M, g) = (S^m, \mathrm{can})$, $k = m - 1$.

Proof of Theorem 10.2. For a bi-Yang-Mills field $\nabla \in \mathcal{C}(E, h)$, we have (10.9) which we can estimated by (10.10) and (10.12) as follows.

$$\begin{aligned}0 &= 2\int_M \langle \mathcal{R}^\nabla(\delta^\nabla R^\nabla), \delta^\nabla R^\nabla\rangle v_g + \int_M \langle \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla\rangle v_g \\ &\quad + \int_M \|\nabla(\delta^\nabla R^\nabla)\|^2 v_g \\ &\geq \int_M \|\nabla(\delta^\nabla R^\nabla)\|^2 v_g + k\int_M \|\delta^\nabla R^\nabla\|^2 v_g - 2\int_M \|R^\nabla\|\, \|\delta^\nabla R^\nabla\|^2 v_g \\ &\geq \|\nabla(\delta^\nabla R^\nabla)\|_{L^2}{}^2 + k\|\delta^\nabla R^\nabla\|_{L^2}{}^2 - 2\|R^\nabla\|_{L^2}\, \|\delta^\nabla R^\nabla\|_{L^4}{}^2\end{aligned} \tag{150}$$

by Schwarz inequality.

Now let us recall (cf. [19], p. 160) the Sobolev inequality for a four dimensional Riemannian manifold (M, g):

$$\|\nabla f\|_{L^2}{}^2 \geq \frac{\sqrt{c_1}}{18}\|f\|_{L^4}{}^2 - \frac{1}{9}\left(\frac{c_1}{\mathrm{Vol}(M, g)}\right)^{1/2} \|f\|_{L^2}{}^2, \quad f \in H_1^2(M), \tag{151}$$

where $H_1^2(M)$ is the Sobolev space of (M, g). By applying (10.15) to the first term of (10.14), we have

$$\begin{aligned}\text{the RHS of (10.14)} &\geq \frac{\sqrt{c_1}}{18}\|\delta^\nabla R^\nabla\|_{L^4}{}^2 - \frac{1}{9}\left(\frac{c_1}{\mathrm{Vol}(M,g)}\right)^{1/2}\|\delta^\nabla R^\nabla\|_{L^2}{}^2 \\ &\quad + k\|\delta^\nabla R^\nabla\|_{L^2}{}^2 - 2\|R^\nabla\|_{L^2}\|\delta^\nabla R^\nabla\|_{L^4}{}^2 \\ &= \left(\frac{\sqrt{c_1}}{18} - 2\|R^\nabla\|_{L^2}\right)\|\delta^\nabla R^\nabla\|_{L^4}{}^2 \\ &\quad + \left(k - \frac{1}{9}\left(\frac{c_1}{\mathrm{Vol}(M,g)}\right)^{1/2}\right)\|\delta^\nabla R^\nabla\|_{L^2}{}^2. \end{aligned} \tag{152}$$

Since $\|\delta^\nabla R^\nabla\|_{L^2}{}^2 \geq 0$ in (10.14), we also have

$$\text{the RHS of (10.14)} \geq k\|\delta^\nabla R^\nabla\|_{L^2}{}^2 - 2\|R^\nabla\|_{L^2}\|\delta^\nabla R^\nabla\|_{L^4}{}^2. \tag{153}$$

<u>Case 1</u>: $\|\delta^\nabla R^\nabla\|_{L^2}{}^2 \geq \frac{\mathrm{Vol}(M,g)^{1/2}}{2}\|\delta^\nabla R^\nabla\|_{L^4}{}^2$. In this case, if $\|\delta^\nabla R^\nabla\|_{L^4} > 0$, then

$$\begin{aligned}\text{the RHS of (10.17)} &> k\|\delta^\nabla R^\nabla\|_{L^2}{}^2 - \frac{k}{2}\mathrm{Vol}(M,g)^{1/2}\|\delta^\nabla R^\nabla\|_{L^4}{}^2 \\ &\qquad \left(\text{by } 2\|R^\nabla\|_{L^2} < \frac{k}{2}\mathrm{Vol}(M,g)^{1/2}\right) \\ &= k\left(\|\delta^\nabla R^\nabla\|_{L^2}{}^2 - \frac{\mathrm{Vol}(M,g)^{1/2}}{2}\|\delta^\nabla R^\nabla\|_{L^4}{}^2\right) \\ &\geq 0 \end{aligned}$$

which is a contradication. We have $\|\delta^\nabla R^\nabla\|_{L^4} = 0$, i.e., $\delta^\nabla R^\nabla = 0$.

<u>Case 2.</u>: $\|\delta^\nabla R^\nabla\|_{L^2}{}^2 \leq \frac{\mathrm{Vol}(M,g)^{1/2}}{2}\|\delta^\nabla R^\nabla\|_{L^4}{}^2$. In this case, if $\|\delta^\nabla R^\nabla\|_{L^2} > 0$, then

$$\begin{aligned}\text{the RHS of (10.16)} &= \left(\frac{\sqrt{c_1}}{18} - 2\|R^\nabla\|_{L^2}\right)\|\delta^\nabla R^\nabla\|_{L^4}{}^2 \\ &\quad + \left(k - \frac{1}{9}\left(\frac{c_1}{\mathrm{Vol}(M,g)}\right)^{1/2}\right)\|\delta^\nabla R^\nabla\|_{L^2}{}^2 \\ &\geq \left(\frac{\sqrt{c_1}}{18} - 2\|R^\nabla\|_{L^2}\right) 2\,\mathrm{Vol}(M,g)^{-1/2}\|\delta^\nabla R^\nabla\|_{L^2}{}^2 \\ &\quad + \left(k - \frac{1}{9}\left(\frac{c_1}{\mathrm{Vol}(M,g)}\right)^{1/2}\right)\|\delta^\nabla R^\nabla\|_{L^2}{}^2 \\ &\qquad \left(\text{by } \frac{\sqrt{c_1}}{18} - 2\|R^\nabla\|_{L^2} \geq 0\right) \\ &= \left\{\frac{\sqrt{c_1}}{9}\mathrm{Vol}(M,g)^{-1/2} - 2\|R^\nabla\|_{L^2}\cdot 2\mathrm{Vol}(M,g)^{-1/2} \right. \\ &\quad \left. + k - \frac{1}{9}\left(\frac{c_1}{\mathrm{Vol}(M,g)}\right)^{1/2}\right\}\|\delta^\nabla R^\nabla\|_{L^2}{}^2 \\ &= \left(k - 2\|R^\nabla\|_{L^2}\cdot 2\,\mathrm{Vol}(M,g)^{-1/2}\right)\|\delta^\nabla R^\nabla\|_{L^2}{}^2 \\ &> 0, \end{aligned}$$

which is also a contradiction. Thus, we have $\|\delta^\nabla R^\nabla\|_{L^2} = 0$, i.e., $\delta^\nabla R^\nabla = 0$.

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