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Existence of limits of analytic one-parameter semigroups of copulas

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Abstract. A 2-copula F is idempotent if $F * F = F$. Here $*$ denotes the product defined in [1]. An idempotent copula F is said to be a unit for a 2-copula A if $F * A = A * F = A$. An idempotent copula is said to annihilate a 2-copula A if $F * A = A * F = F$.

If F is a unit for A and s is a non-negative real number, define

$$\exp_F(sA) = F + sA + \frac{s^2}{2!} A * A + \frac{s^3}{3!} A * A * A + \dots .$$

For any copula A and any idempotent copula F which is a unit for A, the set

$$C_s = e^{-s} \exp_F(sA), \quad s \in [0, \infty)$$

is a semigroup of copulas under the $*$ operation, which is homomorphic to the semigroup $[0, \infty)$ under addition. We call this set an analytic one-parameter semigroup of copulas. C_s can be defined also for $s < 0$, and $C_{-s} * C_s = C_s * C_{-s} = F$, but in general C_s is not a copula for $s < 0$.

We show that for any such analytic one-parameter semigroup, the limit $\lim_{s\to\infty} C_s = E$ exists. We show also that the limit E has the following properties:

(i) E is idempotent.
(ii) E annihilates A, F and C_s.
(iii) E is the greatest annihilator of A and of C_s, $s \in (0, \infty)$.

It is also true that F is the least unit for C_s, $s \in [0, \infty)$. We give a geometrical interpretation of this result, and we comment on the use of analytic semigroups to construct Markov processes with continuous parameter.

Keywords: copula, idempotent, star product

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Notation and background

The $*$ product of two 2-copulas A and B is defined as follows:

$$A * B(x, y) = \int_0^1 A_{,2}(x, t) B_{,1}(t, y)\, dt.$$

Here, and throughout this paper, $C_{,1}$ and $C_{,2}$ denote the partial derivatives of a copula C with respect to its first and second arguments, respectively. For any copulas A and B, $A * B$ is a copula. The $*$ product is associative: $A * (B * C) = (A * B) * C$ for any copulas A, B and C. Furthermore, the $*$ product is continuous in each place. If $A_n \to A$ uniformly, then $A_n * B \to A * B$ and $B * A_n \to B * A$ uniformly. But the $*$ product is not jointly continuous: if $A_n \to A$ uniformly and $B_n \to B$ uniformly, it need not be the case that $A_n * B_n \to A * B$. We shall, however, establish and use a limited joint continuity result in Lemma 3 below. For proofs of these and other properties of the $*$ product, see [1].

We will normally write AB for $A * B$, omitting the $*$. We will sometimes put the $*$ back in for emphasis or clarity.

The following result is well known; it is proved exactly like the corresponding result for stochastic matrices. We give a proof here, since the proof of our main result is based directly on this classic proof.

Theorem 1. *Let A be a 2-copula. The sequence of $*$ powers A^k possesses a Cesaro limit E. E is a copula, and E satisfies $E^2 = E$ and $EA = AE = E$. Furthermore, if F is any copula satisfying $F^2 = F$ and $FA = AF = F$, then $FE = EF = F$.*

Proof. Given a copula A, set

$$S_n = \frac{1}{n}\sum_{k=1}^{n} A^k.$$

Observe that S_n is a convex combinations of copulas, hence a copula; we will call S_n a "Cesaro sum."

Since the copulas are a closed and equicontinuous subset of L^∞, they are a compact set in the topology of uniform convergence. See, e.g., [2] for a discussion of compactness of the set of copulas. It follows that the sequence S_n always possesses a convergent subsequence. Let S_{n_k} be a convergent subsequence, and call its limit E. Now

$$AS_n = S_nA = \frac{1}{n}(A^{n+1} - A + \sum_{k=1}^{n} A^k) = \frac{1}{n}(A^{n+1} - A) + S_n$$

for all n. Inserting S_{n_k} into this expression and taking the limit, using the one-sided continuity of the $*$ product, we obtain $AE = EA = E$, since A^{n+1}/n and A/n both converge to 0 uniformly as $n \to \infty$. Observe that $EA = AE = E$ implies that $EA^k = A^kE = E$ for all k, whence, by taking an appropriate convex combination of powers, that

$$ES_n = S_nE = E \tag{1}$$

for all n. Insert the convergent sequence S_{n_k} into (1) and take the limit to obtain $E^2 = E$.

Now let S_{n_ℓ} be another convergent subsequence of S_n, with limit, say H. Inserting this subsequence into (1) and taking the limit yields $EH = HE = E$. Reversing the roles of E and H yields $HE = EH = H$. We conclude that $E = H$, necessarily, and by a standard argument, S_n itself converges and has limit E.

It remains to show that if F is any other copula satisfying $F^2 = F$ and $AF = FA = F$, then $EF = FE = F$. This follows from logic similar to that which led to (1): $FA = AF = F$ implies that $FA^k = A^kF = F$ for all k, whence, by taking an appropriate convex combination of powers, that

$$FS_n = S_nF = F.$$

Take the limit to obtain $FE = EF = F$. $\boxed{QED}$

The argument used in the proof of Theorem 1 extends to many kinds of rule-based convex combinations of powers of a copula. It is by no means limited to Cesaro sums. For example, one can show that the conclusions of Theorem 1 hold for the convex combinations

$$\frac{2}{n(n+1)}\sum_{k=1}^{n} kA^k \quad \text{and} \quad \frac{1}{n}\sum_{k=n+1}^{2n} A^k,$$

both of which exhibit features similar to the convex combinations addressed here in Lemma 2.

We say a copula E is idempotent if $E^2 = E$. If E and F are commuting idempotents which satisfy $EF = FE = E$, we say $E \leq F$. An idempotent E annihilates a copula C if $EC = CE = E$. Call the limit of the Cesaro sums in Theorem 1 E_A. Then Theorem 1 says that for each copula A there is a greatest annihilator E_A, and it is the Cesaro limit of the powers of A.

An idempotent F is a unit for a copula A if $FA = AF = A$. For each copula A there is a least unit F_A, that is, an idempotent copula F_A which is a unit for A and satisfies $FF_A = F_AF = F_A$, that is, $F_A \leq F$, whenever F is another idempotent unit for A. A proof is outlined in [3].

A copula A and a unit F for A generate an analytic one-parameter semigroup of copulas via

$$C_s = \exp_F(s(A-F)) = F + s(A-F) + s^2(A-F)^2/2! + \ldots. \tag{2}$$

The last expression defines the others. The series is uniformly convergent for all s. A proof is given in [2]. (The proof depends on the choice of a suitable norm on the span of copulas for which the span of copulas is a Banach algebra. There is such a norm; it is used below in the proof of Lemma 3.) Observe that $\exp_F(s(A-F))$ is the usual operator series for $\exp(s(A-F))$, with the identity operator in the zeroth order term replaced by F, which may, but need not, be the identity copula $M(x,y) = \min(x,y)$. We use the subscript F in the notation $\exp_F$ as a reminder that F is is used in the zeroth term, though we will sometimes drop the subscript when there is no cause for confusion. Since F is idempotent and is a unit for A and F, $\exp_F(s(A-F))$ has the usual properties of the exponential operator series, in particular, $\exp_F(s(A-F))\exp_F(t(A-F)) = \exp_F((s+t)(A-F))$. Also, since A and F commute, we can write

$$C_s = \exp_F(-sF)\exp_F(sA) = e^{-s}\exp_F(sA), \tag{3}$$

using the fact that $\exp_F(-sF) = e^{-s}F$ (which follows from $F^k = F$ for all k), and that F is a unit for each term in the expansion of $\exp_F(sA)$. Equation (3) is useful for some purposes, for example, see the proof of both Theorem 2 and Theorem 4 below. Since C_s is analytic in s, we can take the derivative of C_s with respect to s, and we can otherwise do calculus on C_s in the usual way. We say that $A - F$ is the generator of the semigroup C_s.

The use of the term one-parameter semigroup of copulas for $C_s, s \geq 0$ is justified by the following theorem:

Theorem 2. *Let A be a copula and let F be an idempotent copula which is a unit for A. Let C_s be as defined in (2). Then C_s is a copula for all $s \geq 0$ and $C_sC_t = C_{s+t}$, hence, $\{C_s\}_{s\geq 0}$ is a semigroup under the $*$ product which is homomorphic to $[0,\infty)$ under addition.*

Proof. The proof uses equation (3). For $s > 0$ define $q_n = \sum_{k=0}^{n}\frac{s^k}{k!}$ and define $S_n = F + \sum_{k=1}^{n}\frac{A^k}{k!}$. Then S_n/q_n is a convex combination of copulas, hence a copula, for all n. Thus, its pointwise limit as $n \to \infty$, which exists and equals C_s by inspection, is necessarily a copula. That the map $s \to C_s$ is a homomorphism is a known property of exponential operator series, as remarked above. QED

Remark: We conjecture that if there exists $s < 0$ for which C_s is a copula, then necessarily $A = F$, and we have the trivial case $C_s = F$ for all $s \in (-\infty, \infty)$. This assertion is true when $F = M$, for if C_{-s} is a copula for some $s > 0$, then $C_{-s}C_s = M$ implies C_s has a left inverse with respect to M among the set of copulas, hence must have a first partial derivative $C_{s,1}$ which is 0 or 1 almost everywhere, [1], Theorem 7.1. But we have

$$C_{s,1} = e^{-s}M_{,1} + e^{-s}sA_{,1} + \cdots + e^{-s}\frac{s^k}{k!}(A^k)_{,1} + \ldots.$$

The sum of the coefficients on the right hand side is 1, and the first partial derivative of any copula exists almost everywhere and lies in the interval $[0, 1]$ whereever it exists. Consider the subset of $[0, 1]^2$ where $M_{,1} = 1$. Since the coefficient of $M_{,1}$ is $e^{-s} > 0$ in the expansion above, necessarily $C_{s,1} > 0$ a.e. on this set, hence it must be true that $C_{s,1} = 1$ a.e. in the set. By analogous argument, on the set where $M_{,1} = 0$, necessarily $C_{s,1} < 1$, hence $C_{s,1} = 0$ a.e. on the set. We conclude that $C_s = M$. A similar argument shows $A = M$ whence for all $t \in (-\infty, \infty)$, $C_t = M$. This argument extends to show that the assertion holds for all nonatomic idempotents F, but the argument is somewhat involved, and we omit it. For terminology, see [3]. It is an open question whether the assertion holds for atomic idempotents F; we conjecture that it does. If the conjecture holds, one cannot extend the range of the parameter s for the semigroup C_s without simultaneously going outside the set of copulas.

Results

It is clear from the definition (2) that $\lim_{s\downarrow 0} C_s = F$. The principal issue addressed here is the existence of the limit $\lim_{s\uparrow\infty} C_s$. It this limit exists, its properties are easy to establish:

Theorem 3. *Let A be a copula, let F be a unit for A, and let C_s be the analytic semigroup generated by $A - F$, per equation (2). Then F is the least unit for C_s, $s \in [0, \infty)$. Furthermore, if $\lim_{s\uparrow\infty} C_s = E$ exists then:*
(i) $E^2 = E$;
(ii) E is the greatest annihilator of C_s for all $s > 0$;
(iii) $E \leq F$, that is, $FE = EF = E$; and
(iv) E is the greatest annihilator of A.

Proof. We show first that F is the least unit for C_s. Clearly, F is a unit for C_s for all s; this follows from the fact F is a unit for each term in the expansion (2) and an appropriate continuity argument. If H is any unit for C_s, then $HC_s = C_s$ and $C_sH = C_s$. Post- and pre-multiply these equations by C_{-s} to obtain $HF = F$ and $FH = F$, using $C_sC_{-s} = C_{-s}C_s = F$. It follows that $F \leq H$, so that F is the least unit for C_s, as claimed.

Now assume that $\lim_{s\to\infty} C_s = E$ exists. Then for $s > 0$ and k a positive integer, $C_s^k = C_{ks}$, so $\lim_{k\to\infty} C_s^k = E$ exists. When the limit of the sequence of powers exists, the Cesaro limit must exist and be equal to it; thus, E is the greatest annihilator of C_s, by Theorem 1 above. This is conclusion (ii). Take the limit of $EC_s = C_sE = E$ as $s \uparrow \infty$ to obtain conclusion (i). Take the limit of $EC_s = C_sE = E$ as $s \downarrow 0$ to obtain conclusion (iii).

It remains to show that E is the greatest annihilator of A. Differentiate $EC_s = C_sE = E$ with respect to s and set $s = 0$ to obtain

$$E(A - F) = (A - F)E = 0.$$

Then rearrange terms and use (iii) to obtain $EA = AE = E$. This says that E annihilates A. Write E_A for the greatest annihilator of A, per Theorem 1. We will show that $E_A = E$. Since

E annihilates A, we have necessarily $E \leq E_A$, so if we can show that $E_A \leq E$, we are done. Now $E_A \leq F$, since we can write

$$E_A F = (E_A A)F = E_A(AF) = E_A A = E_A,$$

and similarly $FE_A = E_A$. Thus, $E_A(A - F) = (A - F)E_A = 0$. It follows that

$$E_A(A - F)^k = (A - F)^k E_A = 0$$

for all positive integers k. Multiply the power series (2) by E_A, and use the fact that the series converges absolutely and that the $*$ product is continuous in each place to get a resultant series, every term in which except the first vanishes, and the first term is $E_A F = E_A$. Conclude that $E_A C_s = C_s E_A = E_A$ for all s. Take the limit as $s \uparrow \infty$ to obtain $E_A E = E E_A = E_A$, that is $E_A \leq E$. This completes the proof. QED

Our principal result here is that the limit $\lim_{s\to\infty} C_s$ does always in fact exist:

Theorem 4. *Let A be a copula, let F be a unit for A, and let C_s be the analytic one-parameter semigroup generated by $A - F$, per equation (2). Then $\lim_{s\uparrow\infty} C_s = E$ exists.*

The proof is similar to the proof of Theorem 1, but more involved. We proceed by way of 3 lemmas.

Lemma 1. *Let $N \geq 3$ denote a positive integer and set $L(N) = [\sqrt{N}\ln N]$, where $[\cdot]$ denotes the greatest integer function. Define $U(N)$, $V(N)$ and $W(N)$ as follows:*

$$U(N) = e^{-N} \sum_{k=0}^{N-L(N)-1} N^k/k!$$

$$V(N) = e^{-N} \sum_{k=-L(N)}^{L(N)} N^{(N+k)}/(N+k)!$$

$$W(N) = e^{-N} \sum_{k=N+L(N)+1}^{\infty} N^k/k!$$

Then:
(i) $\lim_{N\to\infty} U(N) = 0$;
(ii) $\lim_{N\to\infty} V(N) = 1$; *and*
(iii) $\lim_{N\to\infty} W(N) = 0$.

Proof. The proof shows that the term $V(N)$ behaves asymptotically like

$$\frac{1}{\sqrt{2\pi N}} \int_{-\sqrt{N}\ln N}^{\sqrt{N}\ln N} e^{-x^2/2N}\, dx.$$

This is shown by an argument based on Stirling's Theorem and some Taylor expansions. Thus $V(N)$ behaves asymptotically like the integral of a normal density over an interval extending $\ln N$ standard deviations on either side of its mean, and the desired conclusions readily follow from this fact. The details are as follows.

Observe first that the terms in the sums defining $U(N)$, $V(N)$ and $W(N)$ are positive for all N and that $U(N) + V(N) + W(N) = e^{-N}e^{N} = 1$ for all N, so that conclusion (ii) implies conclusions (i) and (iii). Observe also that, by the same reasoning, $V(N) < 1$ for all N, so conclusion (ii) follows if we can show that, given $\epsilon > 0$, $V(N) > 1 - \epsilon$ for all sufficiently large N. We use Stirling's formula for the approximation of $n!$ The kth term of V_N can be written

$$\frac{e^{-N}N^{N+k}}{(N+k)!} = \frac{e^{-N}N^{N+k}}{e^{-(N+k)}(N+k)^{(N+k)}\sqrt{2\pi(N+k)}}R_{N+k}. \tag{4}$$

Here R_n denotes the ratio

$$R_n = \frac{e^{-n}n^n\sqrt{2\pi n}}{n!}. \tag{5}$$

By Stirling's theorem, $\lim_{n\to\infty} R_n = 1$. Given $\epsilon > 0$, let K be so large that $n \geq K$ implies $R_n > 1 - \epsilon/5$. Let N_0 be so large that $N \geq N_0$ implies $N - L(N) > K$. Then we have, from (4), for all $N \geq N_0$ and all k between $-L(N)$ and $L(N)$,

$$\frac{e^{-N}N^{N+k}}{(N+k)!} > \frac{e^{k}N^{N+k}}{(N+k)^{(N+k)}\sqrt{2\pi(N+k)}}(1-\epsilon/5). \tag{6}$$

Next, we write

$$\frac{N^{N+k}}{(N+k)^{(N+k)}} = \big(1 - \frac{k}{N+k}\big)^{N+k}.$$

Now

$$\begin{aligned}\ln(1-\frac{x}{n})^n &= n\ln(1-\frac{x}{n})\\ &= -n(\frac{x}{n} + \frac{1}{2!}(\frac{x}{n})^2 + \frac{1}{3!}(\frac{x}{n})^3 + \dots)\\ &= -x - \frac{1}{2}\frac{x^2}{n} - \frac{1}{6}\frac{x^3}{n^2} + \dots.\end{aligned} \tag{7}$$

Observe that, in our context,

$$x^2/n \simeq L(N)^2/N \simeq (\ln N)^2$$

so that the second term in (7) cannot be made small by taking N to be large. On the other hand, the term

$$x^3/n^2 \simeq L(N)^3/N^2 \simeq (\ln N)^3/N^{1/2}$$

and subsequent terms can be made arbitrarily small by taking N to be large enough. Using (7), we write

$$\begin{aligned}\ln\big(1-\frac{k}{N+k}\big)^{N+k} &= -k - \frac{1}{2}\frac{k^2}{N+k} - \frac{1}{6}\frac{k^3}{(N+k)^2} + \dots\\ &= -k - \frac{1}{2}\frac{k^2}{N} + \frac{1}{2}\frac{k^3}{N(N+k)} - \frac{1}{6}\frac{k^3}{(N+k)^2} + \dots.\end{aligned}$$

Let N_1 be so large that for all $N > N_1$ and all k between $-L(N)$ and $L(N)$,

$$\ln\big(\frac{N^{N+k}}{(N+k)^{(N+k)}}\big) = \ln\big(1-\frac{k}{N+k}\big)^{N+k} > -k - \frac{1}{2}\frac{k^2}{N} + \ln(1-\epsilon/5)$$

so that

$$\frac{N^{N+k}}{(N+k)^{(N+k)}} > e^{-k}e^{-k^2/2N}(1-\epsilon/5).$$

Then, using (6), for $N > \max(N_0, N_1)$ and all k between $-L(N)$ and $L(N)$, we have,

$$\frac{e^{-N}N^{N+k}}{(N+k)!} > \frac{1}{\sqrt{2\pi(N+k)}}e^{-k^2/2N}(1-\epsilon/5)^2. \tag{8}$$

Next, write

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \dots .$$

Using this expansion, we obtain

$$\frac{1}{\sqrt{2\pi(N+k)}} = \frac{1}{\sqrt{2\pi N}}(1 - \frac{1}{2}\frac{k}{N} + \frac{3}{8}\frac{k^2}{N^2} - \dots).$$

Let N_2 be so large that $N \geq N_2$ implies

$$\frac{1}{\sqrt{2\pi(N+k)}} \geq \frac{1}{\sqrt{2\pi N}}(1-\epsilon/5)$$

for all k between $-L(N)$ and $L(N)$. Then we have, for $N \geq \max(N_0, N_1, N_2)$ and k between $-L(N)$ and $L(N)$, we have

$$\frac{N^{N+k}}{(N+k)!} > \frac{1}{\sqrt{2\pi N}}e^{-k^2/2N}(1-\epsilon/5)^3. \tag{9}$$

Substitute (9) into the expression for $V(N)$ and conclude that

$$V(N) > \big(\frac{1}{\sqrt{2\pi N}}\sum_{k=-L(N)}^{L(N)} e^{-k^2/2N}\big)(1-\epsilon/5)^3 \tag{10}$$

for $N \geq \max(N_0, N_1, N_2)$.

Now, as noted above, the right hand side of equation (10) is like the integral of the density of a normal distribution, integrated over an interval extending $\ln N$ standard deviations to either side of the mean, since $L(N) = [N^{1/2}\ln N]$. The integral can be made as close as we please to 1, by choosing N large enough, and it turns out that the approximation error between the integral and the corresponding sum can also be made small for sufficiently large N. We outline the argument. Since

$$\begin{aligned}\frac{1}{\sqrt{2\pi N}}\int_{L(N)}^{\infty} e^{-x^2/2N}\,dx &\leq \frac{1}{\sqrt{2\pi N}}\int_{L(N)}^{\infty}\frac{x}{L(N)}e^{-x^2/2N}\,dx\\ &\simeq \frac{1}{\sqrt{2\pi}\ln N}e^{-(\ln N)^2/2},\end{aligned}$$

which can clearly be made as small as we please for large N, it follows that there exists N_3 such that for all $N > N_3$

$$\frac{1}{\sqrt{2\pi N}}\int_{-L(N)}^{L(N)} e^{-x^2/2N}\,dx > 1-\epsilon/5. \tag{11}$$

As to the approximation error, if we integrate $f(x) = e^{-x^2/2N}/\sqrt{2\pi N}$ from $-L(N)$ to $L(N)$ numerically, using the composite trapezoid rule with cell size 1, we obtain

$$\frac{1}{\sqrt{2\pi N}}\int_{-L(N)}^{L(N)} e^{-x^2/2N}\,dx = \frac{1}{\sqrt{2\pi N}}\sum_{k=-L(N)}^{L(N)-1} e^{-k^2/2N} + E_N$$

where E_N denotes the approximation error. (The endpoints of the interval of integration each receive weight 1/2 instead of 1 and $f(-L(N)) = f(L(N))$ here; this accounts for the upper summation limit being $L(N)-1$ instead of $L(N)$.) It follows that

$$\frac{1}{\sqrt{2\pi N}}\sum_{k=-L(N)}^{L(N)} e^{-k^2/2N} = \frac{1}{\sqrt{2\pi N}}\int_{-L(N)}^{L(N)} e^{-x^2/2N}\,dx + \frac{1}{\sqrt{2\pi N}}e^{-L(N)^2/2N} - E_N. \tag{12}$$

It is well known that in the approximation of the integral of a C^2 function f over an interval $[a,b]$ by the composite trapezoid rule using cell size h, the following estimate holds for the error E:

$$|E_N| \le C(b-a)h^2 \sup_{a\le x\le b}|f''(x)|,$$

where C denotes a constant of order 1 independent of f, of the interval $[a,b]$ and of the the cell size h. E.g., [4], p. 446. In our case, $f(x) = e^{-x^2/2N}/\sqrt{2\pi N}$, and it is easy to verify that maximum value of $|f''(x)|$ occurs at $x=0$ and is equal to $1/(N\sqrt{2\pi N})$. The cell size here is $h=1$, and the interval length is $2L \le 2N^{1/2}\ln N$. Thus, $|E_N| \le 2C\ln N/N$ which can be made as small as we please for large N. The extra $e^{-L(N)^2/2N}/\sqrt{2\pi N}$ term in (12) can also clearly be made arbitrarily small for large N. Thus, we may choose N_4 so large that $N \ge N_4$ implies

$$\frac{1}{\sqrt{2\pi N}}\sum_{k-L(N)}^{L(N)} e^{-k^2/2N} > (1-\epsilon/5)\frac{1}{\sqrt{2\pi N}}\int_{-L(N)}^{L(N)} e^{-x^2/2N}\,dx. \tag{13}$$

To complete the proof, put (10), (11) and (13) together, and conclude that for $N > \max(N_0, N_1, N_2, N_3, N_4)$,

$$V(N) > (1-\epsilon/5)^5 > 1-\epsilon.$$

Since ϵ is arbitrary and necessarily $V(N) < 1$, this shows that $\lim_{N\to\infty} V(N) = 1$. This is conclusion (ii), and since (ii) implies (i) and (iii), we have the the desired conclusions. QED

Lemma 2. *Let N, $L(N)$ and $V(N)$ be as in Lemma 1. Let A be a copula. Define*

$$S(N) = \frac{e^{-N}}{V(N)}\sum_{k=-L(N)}^{L(N)} \frac{N^{N+k}}{(N+k)!}A^{N+k}.$$

Then $\lim_{N\to\infty} S(N) = E$ exists, the limit E is idempotent, and E is in fact the greatest annihilator of A.

Proof. The proof rests on estimates similar to those used in the proof of Lemma 1. The idea here is to take the terms in the expansion of $C_N = \exp_F(N(A-F)) = e^{-N}\exp_F(NA)$ corresponding to the terms included in $V(N)$ in Lemma 1, then divide by $V(N)$, so as to obtain a convex combination of copulas, hence a copula. The resulting sequence of convex combinations behaves like the sequence of Cesaro sums addressed in Theorem 1. The details are as follows.

Observe first that for any N,

$$\begin{aligned}AS(N) &= S(N)A\\ &= S(N) + \frac{e^{-N}}{V(N)}\Big(\sum_{k=-L+1}^{L+1}\frac{N^{N+k-1}}{(N+k-1)!}A^{N+k} - \sum_{k=-L}^{L}\frac{N^{N+k}}{(N+k)!}A^{N+k}\Big)\\ &= S(N) + \frac{1}{V(N)}\Big(e^{-N}\frac{N^{N+L}}{(N+L)!}A^{N+L+1} - e^{-N}\frac{N^{N-L}}{(N-L)!}A^{N-L}\\ &\qquad + e^{-N}\sum_{k=-L+1}^{L}\frac{k}{N}\frac{N^{N+k}}{(N+k)!}A^{N+k}\Big),\end{aligned} \tag{14}$$

where we have temporarily suppressed the N-dependence of L in the interest of readability. Now $V(N) \to 1$ as $N \to \infty$, by Lemma 1 above. We will show that each of the three terms on the right hand side of (14) which are multiplied by $1/V(N)$ approach 0 as $N \to \infty$. This will imply that $AS(N) - S(N) \to 0$ and $S(N)A - S(N) \to 0$ as $N \to \infty$.

As in the proof of Lemma 1, we use Stirling's formula for the approximation of $n!$. Let R_n be as in (5) above. By Stirling's theorem, $\lim_{n\to\infty} R_n = 1$. We can write

$$\|\frac{e^{-N}N^{N+L(N)}}{(N+L(N))!}A^{N+L(N)+1}\| = \frac{e^{-N}N^{N+L(N)}}{(N+L(N))!},$$

where $\|\cdot\|$ denotes the L^∞ norm, since A^k is a copula for all k and hence has norm 1. Now

$$\frac{e^{-N}N^{N+L(N)}}{(N+L(N))!} = \frac{e^{L(N)}}{\sqrt{2\pi(N+L(N))}}\big(1-\frac{L(N)}{N+L(N)}\big)^{N+L(N)}R_{N+L(N)} \le O(\frac{1}{\sqrt{N}}),$$

since, by reasoning like that in Lemma 1, $(1 - L(N)/(N+L(N)))^{N+L(N)}e^{L(N)}$ is bounded uniformly in N. Thus, this term can be made as small as we please for sufficiently large N. Similarly,

$$\|\frac{e^{-N}N^{N-L(N)}}{(N-L(N))!}A^{N-L(N)}\| \to 0$$

as $N \to \infty$. It remains to address the sum on the right hand side of (14). We have

$$\|e^{-N}\sum_{k=-L(N)+1}^{L(N)}\frac{k}{N}\frac{N^{N+k}}{(N+k)!}A^{N+k}\| \le e^{-N}\sum_{k=-L(N)+1}^{L(N)}\frac{|k|}{N}\frac{N^{N+k}}{(N+k)!},$$

again using the fact that $\|A^{N+k}\| = 1$ for all N and k. Now

$$e^{-N}\sum_{k=-L+1}^{L}\frac{|k|}{N}\frac{N^{N+k}}{(N+k)!} = \sum_{k=-L+1}^{L}\frac{|k|}{N}\frac{e^k}{\sqrt{2\pi(N+k)}}\big(1-\frac{k}{N+k}\big)^{N+k}R_{N+k},$$

where we have once again temporarily supressed the N dependence of L. Given $\epsilon > 0$, we get, by arguments directly analogous to those which led to equation (10) above, a number N_1 such that $N \ge N_1$ implies

$$\|e^{-N}\sum_{k=-L(N)+1}^{L(N)}\frac{k}{N}\frac{N^{N+k}}{(N+k)!}A^{N+k}\| \le \frac{1}{N\sqrt{2\pi N}}\sum_{k=-L(N)+1}^{L(N)}|k|e^{-k^2/2N}(1+\epsilon). \tag{15}$$

We complete the argument in a manner analogous to what was done in the proof of Lemma 1. One computes that

$$\frac{1}{N\sqrt{2\pi N}}\int_0^{L(N)} xe^{-x^2/2N}\,dx \leq \frac{1}{\sqrt{2\pi N}}.$$

One likewise computes that the second derivative of $xe^{-x^2/2N}$ is bounded above by some multiple of $1/\sqrt{N}$, hence that the numerical quadrature error in approximating

$$\frac{1}{N\sqrt{2\pi N}}\int_0^{L(N)} xe^{-x^2/2N}\,dx$$

by

$$\frac{1}{N\sqrt{2\pi N}}\Big(\sum_{k=1}^{L(N)-1} ke^{-k^2/2N} + \frac{1}{2}e^{-L(N)^2/2N}\Big)$$

is $O(N^{-3/2}\ln N)$. It follows that the sum on the right hand side of (15) can be made as small as we please for sufficiently large N.

What we have shown so far is that $\|AS(N) - S(N)\| = \|S(N)A - S(N)\|$ converges to 0 as $N \to \infty$. Now $S(N)$ is a convex combination of copulas, hence a copula, for all N, and since the copulas are a compact subset of L^∞, $S(N)$ possesses a convergent subsequence, call it $S(N_k)$, and call its limit E. By the result obtained above, $AS(N_k) = S(N_k)A$ converges to the limit of $S(N_k)$, which is E, and we conclude that $AE = EA = E$, hence that E annihilates A. It follows readily that E annihilates A^k for all k, hence, since $S(N)$ is a convex combination of powers of A, that

$$S(N)E = ES(N) = E \tag{16}$$

for all N. Insert $S(N_k)$ in (16) and take the limit as $k \to \infty$; conclude that $E^2 = E$, that is, that E is idempotent. Now let $S(N_\ell)$ be any other convergent subsequence of $S(N)$, and call its limit F. Insert $S(N_\ell)$ in (16) and take the limit as $\ell \to \infty$. Conclude that $FE = EF = E$. Reverse the roles of E and F in this argument and conclude that $F = E$. It follows that every subsequence of $S(N)$ possesses a sub-subsequence converging to E, hence that $\lim_{N\to\infty} S(N)$ exists and equals E.

It remains to show that the limit E of $S(N)$ is the greatest annihilator of A. We have shown that E annihilates A. If F is any other annihilator of A, then F annihilates $S(N)$ for all N, so we have $FS(N) = S(N)F = F$ for all N. Take the limit and obtain $FE = EF = F$, that is, $F \leq E$. QED

Let $\mathcal{C}$ denote the collection of all 2-copulas and span$\mathcal{C}$ its linear span, that is, the collection of all linear combinations of elements of $\mathcal{C}$. An element $A \in \text{span}\mathcal{C}$ can always be written in the form $A = sB - tC$, where $s \geq 0$, $t \geq 0$ and $B, C \in \mathcal{C}$. Furthermore, the quantity $\|A\|_M$ defined by

$$\|A\|_M = \inf\{s + t \,|\, s \geq 0, t \geq 0, B, C \in \mathcal{C}, A = sB - tC\}$$

is a norm on span$\mathcal{C}$, and span$\mathcal{C}$ is a Banach algebra under this norm. The subscript M is for Minkowski; the norm is a Minkowski norm on span$\mathcal{C}$. The Minkowski norm $\|\cdot\|_M$ dominates the L^∞ norm on span$\mathcal{C}$. These results are proved in [2].

Lemma 3. *Let $s \to C_s = \exp(s(A - F))$ be an analytic one-parameter semigroup of copulas. Then for any copula $B \in \mathcal{C}$, and any $s, t \in [0, \infty)$,*

$$\|C_sB - C_tB\|_\infty \leq |e^{2s} - e^{2t}|, \text{ and} \tag{17}$$

$$\|BC_s - BC_t\|_\infty \le |e^{2s} - e^{2t}|. \tag{18}$$

Proof. Using (2) and the one-sided continuity of the $*$-product, we can write

$$C_sB - C_tB = \sum_{k=0}^{\infty} \frac{s^k - t^k}{k!}(A-F)^k B.$$

Using the properties of the Minkowski norm $\|\cdot\|_M$, we obtain

$$\|(A-F)^k B\|_M \le \|A-F\|_M^k \|B\|_M \le 2^k,$$

since the Minkowski norm of a copula is necessarily 1 and the Minkowski norm of a difference of copulas is necessarily less than or equal to 2. Accordingly, if $s > t$,

$$\begin{aligned}\|C_sB - C_tB\|_\infty &\le \|C_sB - C_tB\|_M \\ &\le \sum_{k=0}^{\infty} \frac{s^k - t^k}{k!}\|A-F\|_M^k \|B\|_M \\ &\le \sum_{k=0}^{\infty} \frac{s^k - t^k}{k!}2^k \\ &\le e^{2s} - e^{2t}.\end{aligned}$$

Similarly, if $s < t$, $\|C_sB - C_tB\|_\infty \le e^{2t} - e^{2s}$, and if $s = t$, the norm of the difference is 0. This completes the proof of (17), and the proof of (18) is analogous. QED

Remark: The interesting part of Lemma 3 is that the estimates in (17) and (18) are independent of B and, for that matter, also of the generator $A - F$ of the analytic one-parameter semigroup. While the $*$-product is not jointly continuous in the uniform norm, Lemma 3 leads to a limited joint continuity result: If $s_n \to s$ and B_n is a copula for all n and $B_n \to B$, then $\|C_{s_n}B_n - C_sB\|_\infty \to 0$. This is an immediate consequence of Lemma 3. The proof is left to the reader.

We return now to the proof of Theorem 4.

Proof of Theorem 4. Same notation as in Lemmas 1 and 2. Lemma 2 states that $\lim_{N\to\infty} S(N) = E$, where E is the greatest annihilator of A. We claim first that $\lim_{N\to\infty} \|C_N - E\|_\infty = 0$, where C_N is C_s evaluated at $s = N$, and E is the greatest annihilator of A. To see this, observe that we can write

$$\begin{aligned}\|C_N - E\| \le &\sum_{k=0}^{N-L(N)-1} e^{-N}\frac{N^k}{k!}\|A^k\| + \|V(N)S(N) - E\| \\ &+ \sum_{k=N+L(N)+1}^{\infty} e^{-N}\frac{N^k}{k!}\|A^k\| \\ \le\; & U(N) + \|V(N)S(N) - S(N)\| + \|S(N) - E\| + W(N) \\ \le\; & U(N) + |V(N) - 1|\|S(N)\| + \|S(N) - E\| + W(N),\end{aligned} \tag{19}$$

using once again the fact that A^k is a copula for all k, hence has norm 1. By Lemma 2, $\|S(N) - E\| \to 0$. Since $S(N)$ is convergent, $\|S(N)\|$ is bounded uniformly in N. By Lemma

1, $U(N) \to 0$, $V(N) \to 1$ and $W(N) \to 0$. Thus, all terms on the right in (19) vanish in the limit. This completes the proof of the claim.

To complete the proof of Theorem 4, we have to show that for any $\epsilon > 0$, there exists a real number s_0 such that $s > s_0$ implies $\|C_s - E\| < \epsilon$. For this we will use Lemma 3. First, let K be a positive integer for which $|e^{2\ell/K} - e^{2(\ell+1)/K}| < \epsilon/2$ for all $\ell = 0, 1, \ldots, K-1$. We can find such a K because the exponential function is uniformly continuous on the compact set $[0, 2]$. Next, observe that it follows from the claim proved just above and the one-sided continuity of the $*$-product that for all $s \in [0, \infty)$,

$$C_{N+s} = C_N C_s \to EC_s = E.$$

This uses the fact the E is an annihilator of A, hence, by reasoning used toward the end of the proof of Theorem 3 above, E annihilates C_s for all s. Hence, we can choose N_0 so large that $N \geq N_0$ implies

$$\|C_{N+\ell/K} - E\| < \epsilon/2$$

for all $\ell = 0, 1, \ldots, K-1$. Given $s > N_0$, set $N = [s]$, where $[\cdot]$ denotes the greatest integer function, and set $\xi = N - s$. Since $\xi \in [0, 1)$, there exists an integer $\ell \in \{0, 1, \ldots, K-1\}$ such that $\ell/K \leq \xi < (\ell+1)/K$, and since the exponential function is increasing, we have

$$0 \leq e^{2\xi} - e^{2\ell/K} < e^{2(\ell+1)/K} - e^{2\ell/K} < \epsilon/2,$$

by the choice of the integer K. It follows from Lemma 3 (with C_N in the role of B) that

$$\begin{aligned} \|C_s - E\| &= \|C_N C_\xi - E\| \\ &\leq \|C_N C_\xi - C_N C_{\ell/K}\| + \|C_N C_{\ell/K} - E\| \\ &< |e^{2\xi} - e^{2\ell/K}| + \epsilon/2 < \epsilon. \end{aligned}$$

This completes the proof of Theorem 4. QED

Remarks

1. Suppose $t \to X_t$ is a process for which X_s and X_t have continuous cumulative distribution functions F_s and F_t and have the joint distribution $A_{st}(F_s(x), F_t(y))$ when $s \leq t$. Then A_{st} is a copula, and the family of copulas A_{st} satisfies the condition $A_{su} * A_{ut} = A_{st}, s \leq u \leq t$ if and only if the process satisfies the Chapman-Kolmogorov equations. This is Theorem 3.2 of [1]. For this reason, we say the copulas in such a family are the copulas of a Markov process. Given an analytic one-parameter semigroup C_s, define $A_{st} = C_{t-s}$ whenever $s \leq t$. Then A_{st} are the copulas of a Markov process, as is readily verified.

2. It is a curious fact that defining $B_{st} = C_{\frac{1}{s} - \frac{1}{t}}$ also gives the copulas of a Markov process. For the process B_{st}, we have

$$\begin{aligned} &\lim_{t\uparrow\infty} B_{st} = C_{1/s} \\ &\lim_{s\downarrow 0} B_{st} = P. \end{aligned}$$

The last equation above says that for $t > 0$ the random variable X_t in the process B_{st} is independent of X_0.

3. We remark that there are many nontrivial copulas whose greatest annihilator is P and whose least unit is M, including the hat copula Λ given by

$$\Lambda(x,y) = \begin{cases} x, & 0 \leq x \leq 1/2, 2x \leq y \leq 1 \\ y/2, & 0 \leq x \leq 1/2, 0 \leq y \leq 2x \\ y/2, & 1/2 \leq x \leq 1, 0 \leq y \leq 2(1-x) \\ x+y-1, & 1/2 \leq x \leq 1, 2(1-x) \leq y \leq 1. \end{cases}$$

That Λ has least unit M follows from the fact that it has a left inverse with respect to M. That it has greatest annihilator P follows from the facts that, on the vertical line $x = 1/2$, $\Lambda(1/2, y) = P(1/2, y)$ for all y, and that, when one forms powers of Λ, the verticle lines $x = a$ on which $\Lambda(a, y) = P(a, y)$ for all y proliferate, eventually becoming as close together as we please. It then follows that, given $\epsilon > 0$, there exists N such that $\|\Lambda^n - P\|_\infty < \epsilon$ for all $n \geq N$. Details are messy and are left to the reader.

4. One can obtain an interesting geometric picture of the set of copulas and the place of analytic one-parameter semigroups in the set of copulas as follows: Every copula A has a greatest annihilator E_A and a least unit F_A, by remarks above. Both E_A and F_A are idempotent copulas, and $E_A \leq F_A$, as is readily verified. Define a relation among copulas by setting $A \sim B$ if $E_A = E_B$ and $F_A = F_B$, that is, A is similar to B if their greatest annihilators are the same and their least units are the same. It is easy to verify that this is an equivalence relation, hence splits up the set of copulas into disjoint equivalence classes, which we label $\{E \leq F\}$ since each such equivalence class is associated with a pair E and F of idempotents with $E \leq F$. Each such class is nonempty: if $E = F$, E is the sole member of the class, since any copula A which has E as both a greatest annihilator and a least unit must be equal to E. If $E < F$, then for example the copula $A = (E + F)/2 \in \{E \leq F\}$, as is readily verified. Neither E nor F is in $\{E \leq F\}$ in this case, since as was just shown, each has its own equivalence class, of which it is the sole member. Now let $A \in \{E \leq F\}$ and consider the analytic one-parameter semigroup C_s with generator $A - F$. By Theorem 3, $C_s \in \{E \leq F\}$ for all $s \in (0, \infty)$. If we set $C_\infty = E$, as Theorem 4 suggests doing, we have a smooth closed arc of copulas connecting E and F, all points of which, except the two endpoints, lie in $\{E \leq F\}$. Furthermore, if we take any other member $B \sim A$, we get another such closed arc, possibly not geometrically distinct, since possibly it is just a reparameterization of the first one. There is nothing in the definition of C_s which requires us to use the least unit of A as the zeroth term in the expansion; any unit G for S can be used. If $A \in \{E \leq F\}$, and $F < G$, so that G is a unit for A but not the least such, and $C_s = \exp_G(s(A - G))$, then by Theorem 3, $C_s \in \{E \leq G\}$ for $s \in (0, \infty)$. This picture suggests a number of questions, some of which it might be interesting to explore further.

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