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L_{10}-free $\{p, q\}$-groups

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Abstract. If L is a lattice, a group is called L-free if its subgroup lattice has no sublattice isomorphic to L. It is easy to see that L_{10}, the subgroup lattice of the dihedral group of order 8, is the largest lattice L such that every finite L-free p-group is modular. In this paper we continue the study of L_{10}-free groups. We determine all finite L_{10}-free $\{p, q\}$-groups for primes p and q, except those of order $2^\alpha 3^\beta$ with normal Sylow 3-subgroup.

Keywords: subgroup lattice, sublattice, finite group, modular Sylow subgroup

MSC 2000 classification: 20D30

1 Introduction

This paper contains the results presented in the second part of our talk on "L_{10}-free groups" given at the conference "Advances in Group Theory and Applications 2009" in Porto Cesareo. The first part of the talk mainly contained results out of [6]. In that paper we introduced the class of L_{10}-free groups; here L_{10} is the subgroup lattice of the dihedral group D_8 of order 8 and for an arbitrary lattice L, a group G is called *L-free* if its subgroup lattice $L(G)$ has no sublattice isomorphic to L. It is easy to see that L_{10} is the unique largest lattice L such that every L-free p-group has modular subgroup lattice. So the finite L_{10}-free groups form an interesting, lattice defined class of groups lying between the modular groups and the finite groups with modular Sylow subgroups. Therefore in [6] we studied these groups and showed that every finite L_{10}-free group G is soluble and the factor group $G/F(G)$ of G over its Fitting subgroup is metacyclic or a direct product of a metacyclic $\{2, 3\}'$-group with the (non-metacyclic) group $Q_8 \times C_2$ of order 16. However, we were not able to determine the exact structure of these groups as had been done in the cases of L-free groups for certain sublattices L of L_{10} (and therefore subclasses of the class of L_{10}-free groups) in [2], [5] and [1].

In the present paper we want to determine the structure of L_{10}-free $\{p, q\}$-groups where p and q are different primes. As mentioned above, the Sylow subgroups of an L_{10}-free group have modular subgroup lattice. Hence a nilpotent group is L_{10}-free if and only if it is modular and the structure of these groups is well-known [4, Theorems 2.3.1 and 2.4.4]. So we only have to study non-nilpotent L_{10}-free $\{p, q\}$-groups G. The results of [6] show that one of the Sylow subgroups of G is normal – we shall choose our notation so that this is the Sylow p-subgroup P of G – and the other is cyclic or a quaternion group of order 8 or we are in the exceptional situation $p = 3$, $q = 2$. So there are only few cases to be considered (see Proposition 1 for details) and we handle all of them except the case $p = 3$, $q = 2$. Unfortunately, however, in the main case that $P = C_P(Q) \times [P, Q]$ where $[P, Q]$ is elementary abelian and Q is cyclic, the structure of G depends on the relation of q and $|Q/C_Q(P)|$ to $p - 1$ (see Theorems 1–3). For example, if $q \nmid p - 1$, then $C_P(Q)$ may be an arbitrary (modular) p-group, whereas $C_P(Q)$ usually has to be small if $q \mid p - 1$. The reason for this and for similar structural peculiarities are the technical lemmas proved in §2, the most interesting being that a direct product of an elementary abelian group of order p^m and a nonabelian P-group of order $p^{n-1}q$ is L_{10}-free if and only if one of the ranks m or n is at most 2 (Lemma 3 and Theorem 2).

All groups considered are finite. Our notation is standard (see [3] or [4]) except that we write $H \cup K$ for the group generated by the subgroups H and K of the group G. Furthermore, p and q always are different primes, G is a finite $\{p, q\}$-group, $P \in \operatorname{Syl} p(G)$ and $Q \in \operatorname{Syl} q(G)$. For $n \in \mathbb{N}$,

C_n is the cyclic group of order n,
D_n is the dihedral group of order n (if n is even),
Q_8 is the quaternion group of order 8.

2 Preliminaries

By [6, Lemma 2.1 and Proposition 2.7], the Sylow subgroups of an L_{10}-free $\{p, q\}$-group are modular and one of them is normal. So we only have to consider groups satisfying the assumptions of the following proposition.

Proposition 1. *Let $G = PQ$ where P is a normal modular Sylow p-subgroup and Q is a modular Sylow q-subgroup of G operating nontrivially on P. If G is L_{10}-free, then one of the following holds.*

I. $P = C_P(Q) \times [P, Q]$ *where* $[P, Q]$ *is elementary abelian and* Q *is cyclic.*

II. $[P, Q]$ *is a hamiltonian* 2*-group and* Q *is cyclic.*

III. $p > 3$, $Q \simeq Q_8$ *and* $C_Q(P) = 1$.

IV. $p = 3$, $q = 2$ *and* Q *is not cyclic.*

Proof. Since Q is not normal in G, by [6, Proposition 2.6], Q is cyclic or $Q \simeq Q_8$ or $p = 3$, $q = 2$. By [6, Lemma 2.2], $[P,Q]$ is a hamiltonian 2-group or $P = C_P(Q) \times [P,Q]$ with $[P,Q]$ elementary abelian. In the first case, $q \neq 2$ and hence II. holds. In the other case, I. holds if Q is cyclic. And if $Q \simeq Q_8$, then clearly III. or IV. is satisfied or $C_Q(P) \neq 1$. In the latter case, $\phi(Q) \trianglelefteq G$ and $G/\phi(Q)$ is L_{10}-free with nonnormal Sylow 2-subgroup $Q/\phi(Q)$; again [6, Proposition 2.6] implies that $p = 3$ and hence IV. holds.

Definition 1. We shall say that an L_{10}-free $\{p,q\}$-group $G = PQ$ is of type I, II, III, or IV if it has the corresponding property of Proposition 1.

We want to determine the structure of L_{10}-free $\{p,q\}$-groups of types I–III. So we have to study the operation of Q on $[P,Q]$ and for this we need the following technical results. The first one is Lemma 2.8 in [6].

Lemma 1. *Suppose that $G = (N_1 \times N_2)Q$ with normal p-subgroups N_i and a cyclic q-group Q which operates irreducibly on N_i for $i = 1,2$ and satisfies $C_Q(N_1) = C_Q(N_2)$. If G is L_{10}-free, then $|N_1| = p = |N_2|$ and Q induces a power automorphism in $N_1 \times N_2$.*

An immediate consequence is the following.

Lemma 2. *Suppose that $G = NQ$ with normal p-subgroup N and a cyclic q-group Q operating irreducibly on N. If G is L_{10}-free, then every subgroup of Q either operates irreducibly on N or induces a (possibly trivial) power automorphism in N; in particular, G is L_7-free.*

Proof. Suppose that $Q_1 \leq Q$ is not irreducible on N and let N_1 be a minimal normal subgroup of NQ_1 contained in N. Then $N = \langle N_1^x \mid x \in Q \rangle$ and so $N = N_1 \times \cdots \times N_r$ with $r > 1$ and $N_i = N_1^{x_i}$ for certain $x_i \in Q$. For $i > 1$, $C_{Q_1}(N_i) = C_{Q_1}(N_1)^{x_i} = C_{Q_1}(N_1)$ and hence Lemma 1 implies that a generator x of Q_1 induces a power automorphism in $N_1 \times N_i$. This power is the same for every i and thus x induces a power automorphism in N. This proves the first assertion of the lemma; that G then is L_7-free follows from [5, Lemma 3.1].

The following two lemmas yield further restrictions on the structure of L_{10}-free $\{p,q\}$-groups. In the proofs we have to construct sublattices isomorphic to L_{10} in certain subgroup lattices. For this and also when we assume, for a contradiction, that a given lattice contains such a sublattice, we use the standard notation displayed in Figure 1 and the following obvious fact.

Remark 1. Let L be a lattice.

(a) A 10-element subset $\{A,B,C,D,E,F,S,T,U,V\}$ of L is a sublattice isomorphic to L_{10} if the following conditions are satisfied :

(1.1) $D \cup S = D \cup T = S \cup T = A$ and $D \cap S = D \cap T = S \cap T = E$,

(1.2) $D \cup U = D \cup V = U \cup V = C$ and $D \cap U = D \cap V = U \cap V = E$, (1.3) $A \cup B = B \cup C = F$ and $A \cap B = A \cap C = B \cap C = D$,

(1.4) $S \cup U = S \cup V = T \cup U = T \cup V = F$.

(b) Conversely, every sublattice of L isomorphic to L_{10} contains 10 pairwise different elements $A, \ldots, V$ satisfying (1.1)–(1.4).

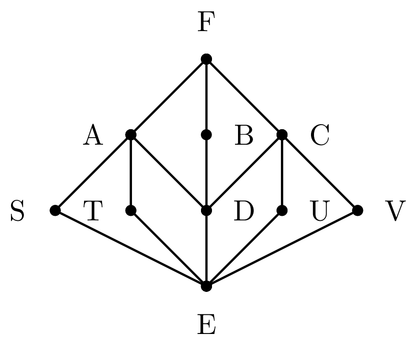

Figure 1

Lemma 3. *If $G = M \times H$ where M is a modular p-group with $|\Omega(M)| \geq p^3$ and H is a P-group of order $p^{n-1}q$ with $3 \leq n \in \mathbb{N}$, then G is not L_{10}-free.*

Proof. By [4, Lemma 2.3.5], $\Omega(M)$ is elementary abelian. So G contains a subgroup $F = F_1 \times F_2$ where $F_1 \leq M$ is elementary abelian of order p^3 and $F_2 \leq H$ is a P-group of order p^2q; let $F_1 = \langle a, b, c\rangle$ and $F_2 = \langle d, e\rangle\langle x\rangle$ where a, b, c, d, e all have order p, $o(x) = q$ and x induces a nontrivial power automorphism in $\langle d, e\rangle$. We let $E = 1$ and define every $X \in \{A, B, C, D, U, V\}$ as a direct product $X = X_1 \times X_2$ with $X_i \leq F_i$ in such a way that (1.2) and (1.3) hold for the X_i in F_i $(i = 1, 2)$ and then of course also for the direct products in F. For this we may take $A_1 = \langle a, b\rangle$, $B_1 = \langle a, bc\rangle$, $U_1 = \langle c\rangle$, $V_1 = \langle ac\rangle$, hence $D_1 = \langle a\rangle$ and $C_1 = \langle a, c\rangle$, and similarly $A_2 = \langle d, e\rangle$, $B_2 = \langle d, ex\rangle$, $U_2 = \langle x\rangle$, $V_2 = \langle dx\rangle$, and hence $D_2 = \langle d\rangle$ and $C_2 = \langle d, x\rangle$. Since $q \mid p-1$, we have $p > 2$ and so we finally may define $S = \langle ae, bd\rangle$ and $T = \langle ae^2, bd^2\rangle$.

Then $A = \langle a, b, d, e\rangle$ is elementary abelian of order p^4 and $D = \langle a, d\rangle$; therefore $D \cup S = D \cup T = S \cup T = A$. Since S, T, D all have order p^2, it follows that $D \cap S = D \cap T = S \cap T = 1$ and so also (1.1) holds. Now x and dx operate in the same way on A and do not normalize $\langle ae^i\rangle$ or $\langle bd^i\rangle$ (i=1,2); hence all the groups $S \cup U$, $S \cup V$, $T \cup U$, $T \cup V$ contain $A = S \cup S^x = T \cup T^x$. Since $A \cup U = A \cup V = F$, also (1.4) holds. Thus $\{A, \ldots, V\}$ is a sublattice of $L(G)$ isomorphic to L_{10}.

We remark that Theorem 2 will show that if $|\Omega(M)| \leq p^2$ or $n \leq 2$ in the group G of Lemma 3, then G is L_{10}-free.

Lemma 4. *Let $k,l,m \in \mathbb{N}$ such that $k \le l < m$ and $q^m \mid p-1$. Suppose that $G = PQ$ where $P = M_1 \times M_2 \times M$ is an elementary abelian normal p-subgroup of G with $|M_i| \ge p$ for i=1,2 and $|M| \ge p^2$ and where Q is cyclic and induces power automorphisms of order q^k in M_1, q^l in M_2, and of order q^m in M. Then G is not L_{10}-free.*

Proof. We show that $G/C_Q(P)$ is not L_{10}-free and for this we may assume that $C_Q(P) = 1$, that is, $|Q| = q^m$. Then G contains a subgroup $F = AQ$ where $A = \langle a,b,c,d\rangle$ is elementary abelian of order p^4 with $a \in M_1$, $b \in M_2$ and $c,d \in M$. We let $E = 1$, $D = \langle a,c\rangle$, $S = \langle acd, bcd^{-1}\rangle$, $T = \langle acd^2, bc^{-1}d^{-1}\rangle$, $U = Q$, $V = Q^{ac}$, $C = DQ$, $B = DQ^{bd}$ and claim that these groups satisfy (1.1)–(1.4).

This is rather obvious for (1.1) since $|D| = |S| = |T| = p^2$ and, clearly, $D \cup S = D \cup T = S \cup T = A$. By [4, Lemma 4.1.1], $Q \cup Q^{ac} = [ac,Q]Q$ and $Q \cap Q^{ac} = C_Q(ac)$; since Q induces different powers in $\langle a\rangle$ and $\langle c\rangle$, we have $[ac,Q] = \langle a,c\rangle$ and $C_Q(ac) = C_Q(c) = 1$. It follows that (1.2) is satisfied. Since $G/D \simeq \langle b,d\rangle Q$ and $Q \cap Q^{bd} = C_Q(bd) = 1$, we have $B \cap C = D$ and so (1.3) holds. Finally, since a generator of Q (or of Q^{ac}) induces different powers in M_i and M, $S \cup U$ and $S \cup V$ contain $\langle a, cd, b, cd^{-1}\rangle = A$; similarly $T \cup U$ and $T \cup V$ both contain $\langle a, cd^2, b, c^{-1}d^{-1}\rangle = A$. Thus also (1.4) holds and $\{A,\dots,V\}$ is a sublattice of $L(G)$ isomorphic to L_{10}.

To show that the groups in our characterizations indeed are L_{10}-free, we shall need the following simple properties of sublattices isomorphic to L_{10}.

Lemma 5. *Let M and N be lattices. If M and N are L_{10}-free, then so is $M \times N$.*

Proof. This follows from the fact that L_{10} is subdirectly irreducible; see [5, Lemma 2.2] the proof of which (for $k = 7$) can be copied literally.

Lemma 6. *Let G be a group and suppose that $A,\dots,V \in L(G)$ satisfy (1.1)–(1.4). If $W \le G$ such that $F \not\le W$, then either $S \not\le W$ and $T \not\le W$ or $U \not\le W$ and $V \not\le W$.*

Proof. Otherwise there would exist $X \in \{S,T\}$ and $Y \in \{U,V\}$ such that $X \le W$ and $Y \le W$. But then $F = X \cup Y \le W$, a contradiction.

Lemma 7. *Let $\overline{P} \trianglelefteq G$ such that $|G : \overline{P}|$ is a power of the prime q and suppose that Q_0 is the unique subgroup of order q in G. If $\overline{P}$ and G/Q_0 are L_{10}-free, then so is G.*

Proof. Suppose, for a contradiction, that $\{A,\dots,V\}$ is a sublattice of $L(G)$ isomorphic to L_{10} and satisfying (1.1)–(1.4). Since $\overline{P}$ is L_{10}-free, $F \not\le \overline{P}$. By Lemma 6, either S and T or U and V are not contained in $\overline{P}$ and therefore have order divisible by q. Hence either $Q_0 \le S \cap T = E$ or $Q_0 \le U \cap V = E$; in both cases, G/Q_0 is not L_{10}-free, a contradiction.

In the inductive proofs that the given $\{p,q\}$-group $G = PQ$ is L_{10}-free, the above lemma will imply that $C_Q(P) = 1$. And the final result of this section handles a situation that shows up in nearly all of these proofs.

Lemma 8. *Let $G = PQ$ where P is a normal Sylow p-subgroup of G and Q is a nontrivial cyclic q-group or $Q \simeq Q_8$; let $Q_0 = \Omega(Q)$ be the minimal subgroup of Q.*

Assume that every proper subgroup of G is L_{10}-free and that there exists a minimal normal subgroup N of G such that $P = N \times C_P(Q_0)$; in addition, if $Q \simeq Q_8$, suppose that every subgroup of order 4 of Q is irreducible on N.

Then G is L_{10}-free.

Proof. Suppose, for a contradiction, that G is not L_{10}-free and let $\{A,\dots,V\}$ be a sublattice of $L(G)$ isomorphic to L_{10}; so assume that (1.1)–(1.4) hold. Since every proper subgroup of G is L_{10}-free, $F = G$.

By assumption, $G = NC_G(Q_0)$; hence $Q_0^G \le NQ_0$ and $[P,Q_0] \le N$. Since $P = [P,Q_0]C_P(Q_0)$ (see [4, Lemma 4.1.3]), it follows that

$$[P,Q_0] = N \quad \text{and} \quad Q_0^G = NQ_0. \tag{1}$$

Suppose first that E is a p-group. By Lemma 6, we have $S,T \not\le P\phi(Q)$ or $U,V \not\le P\phi(Q)$; say $U,V \not\le P\phi(Q)$. Then U and V both contain Sylow q-subgroups of G, or subgroups of order 4 of G if $Q \simeq Q_8$. Since $U \cap V = E$ is a p-group, $C = U \cup V$ contains two different subgroups of order q and hence by (1), $C \cap N \ne 1$. Since U is irreducible on N, it follows that $N \le C$. Therefore $Q_0^G = NQ_0 \le C$ and so C contains every subgroup of order q of G. Since $S \cap C = T \cap C = E$ is a p-group, it follows that S and T are p-groups. Hence $A = S \cup T \le P$; but then also $B \cap C = D \le A$ is a p-group and therefore $B \le P$. So, finally, $G = A \cup B \le P$, a contradiction.

Thus E is not a p-group and therefore contains a subgroup of order q. If we conjugate our L_{10} suitably, we may assume that

$$Q_0 \le E. \tag{2}$$

Every subgroup X of G containing Q_0 is of the form $X = (X \cap P)Q_1$ where $Q_0 \le Q_1 \in \mathrm{Syl}\, q(X)$; since $X \cap P = [X \cap P, Q_0]C_{X\cap P}(Q_0)$ and $[X \cap P, Q_0] \le X \cap N$, it follows that

$$X \le C_G(Q_0) \ \text{ if } \ Q_0 \le X \ \text{ and } \ X \cap N = 1. \tag{3}$$

Since $G = A \cup B = A \cup C = B \cup C$, at least two of the three groups A, B, C are not contained in $P\phi(Q)$ and hence contain Sylow q-subgroups of G, or subgroups of order 4 of G if $Q \simeq Q_8$. Similarly, two of the groups A, B, C are not contained in $C_G(Q_0)$ and hence, by (2) and (3), have nontrivial intersection with N. So there exists $X \in \{A, B, C\}$ having both properties. Since the Sylow q-subgroups of X are irreducible on N, it follows that $N \leq X$. Let $Y, Z \in \{A, B, C\}$ with $Y \neq X \neq Z$ such that $Y \cap N \neq 1$ and Z contains a Sylow q-subgroup of G, or a subgroup of order 4 of G if $Q \simeq Q_8$. Then $1 < Y \cap N \leq Y \cap X = D$ and hence also $Z \cap N \neq 1$. Thus $N \leq Z$ and so

$$N \leq X \cap Z = D. \tag{4}$$

Therefore $S \cap N = S \cap D \cap N = E \cap N$ and $U \cap N = E \cap N$; so if $E \cap N = 1$, then (2) and (3) would imply that $G = S \cup U \leq C_G(Q_0)$, a contradiction. Thus $E \cap N \neq 1$. Again by Lemma 6, $U, V \nleq P\phi(Q)$, say. So $U \cap N \neq 1 \neq V \cap N$ and U and V are irreducible on N; it follows that $N \leq U \cap V = E$. But by assumption, $G = NC_G(Q_0)$ and $N \cap C_G(Q_0) = 1$ so that $G/N \simeq C_G(Q_0)$ is L_{10}-free, a final contradiction.

3 Groups of type I

Unfortunately, as already mentioned, this case splits into three rather different subcases according to the relation of q and $|Q/C_Q(P)|$ to $p-1$. We start with the easiest case that q does not divide $p-1$. In the whole section we shall assume the following.

Hypothesis I. Let $G = PQ$ where P is a normal p-subgroup of G with modular subgroup lattice, Q is a cyclic q-group and $P = C_P(Q) \times [P,Q]$ with $[P,Q]$ elementary abelian and $[P,Q] \neq 1$.

Theorem 1. *Suppose that G satisfies Hypothesis I and that $q \nmid p-1$.*

Then G is L_{10}-free if and only if $P = C_P(Q) \times N_1 \times \cdots \times N_r$ $(r \geq 1)$ and for all $i, j \in \{1, \ldots, r\}$ the following holds.

(1) Every subgroup of Q operates trivially or irreducibly on N_i.

(2) $C_Q(N_i) \neq C_Q(N_j)$ for $i \neq j$.

Proof. Suppose first that G is L_{10}-free. By Maschke's theorem, Q is completely reducible on $[P,Q]$ and hence $[P,Q] = N_1 \times \cdots \times N_r$ with $r \geq 1$ and Q irreducible on N_i for all $i \in \{1, \ldots, r\}$. By Lemma 2, every subgroup of Q either is irreducible on N_i or induces a power automorphism in N_i. But since $q \nmid p-1$, there is no power automorphism of order q of an elementary abelian p-group and hence all these induced power automorphisms have to be trivial. Thus (1) holds and (2) follows from Lemma 1.

To prove the converse, we consider a minimal counterexample G. Then G satisfies (1) and (2) but is not L_{10}-free. Every subgroup of G also satisfies (1) and (2) or is nilpotent with modular subgroup lattice; the minimality of G implies that every proper subgroup of G is L_{10}-free.

If $C_Q(P) \neq 1$, then $Q_0 := \Omega(Q)$ would be the unique subgroup of order q in G and again the minimality of G would imply that G/Q_0 would be L_{10}-free. Since also P is L_{10}-free, Lemma 7 would yield that G is L_{10}-free, a contradiction. Thus $C_Q(P) = 1$ and hence there is at least one of the N_i, say N_1, on which Q_0 acts nontrivially and hence irreducibly. By (2), Q_0 centralizes the other N_j so that $P = N_1 \times C_P(Q_0)$. By Lemma 8, G is L_{10}-free, a final contradiction.

We come to the case that G satisfies Hypothesis I and $q \mid p-1$. Then again by Maschke's theorem, $[P,Q] = N_1 \times \cdots \times N_r$ $(r \geq 1)$ with irreducible $GF(p)Q$-modules N_i; but this time some of the N_i might be of dimension 1. In fact, if the order of the operating group $Q/C_Q(P)$ divides $p-1$, then $|N_i| = p$ for all i (see [3, II, Satz 3.10]). Therefore a generator x of Q induces power automorphisms in all the N_i and $[P,Q]$ is the direct product of nontrivial eigenspaces of x. We get the following result in this case.

Theorem 2. *Suppose that G satisfies Hypothesis I and that $|Q/C_Q(P)|$ divides $p-1$; let $Q = \langle x \rangle$.*

Then G is L_{10}-free if and only if $P = C_P(Q) \times M_1 \times \cdots \times M_s$ $(s \geq 1)$ with eigenspaces M_i of x satisfying (1) and (2).

(1) $C_Q(M_s) < C_Q(M_{s-1}) < \cdots < C_Q(M_1) < Q$

(2) One of the following holds:

(2a) $|M_i| = p$ for all $i \in \{1, \dots, s\}$,

(2b) $|M_1| \geq p^2$, $|M_i| = p$ for all $i \neq 1$ and $|\Omega(C_P(Q))| \leq p^2$,

(2c) $|M_2| \geq p^2$, $|M_i| = p$ for all $i \neq 2$ and $C_P(Q)$ is cyclic.

Proof. Suppose first that G is L_{10}-free. As mentioned above, since $|Q/C_Q(P)|$ divides $p-1$, $[P,Q]$ is a direct product of eigenspaces $M_1, \dots, M_s$ of x. By Lemma 1, $C_Q(M_i) \neq C_Q(M_j)$ for $i \neq j$ and we can choose the numbering of the eigenspaces in such a way that (1) holds.

If $|M_i| = p$ for all i, then (2a) is satisfied. So suppose that $|M_k| \geq p^2$ for some $k \in \{1, \dots, s\}$. Then by (1), $K := C_Q(M_k) < C_Q(M_i)$ for all $i < k$. Therefore if $k \geq 3$, then x would induce power automorphisms of pairwise different orders $|Q/C_Q(M_i)|$ in M_i for $i \in \{1, 2, k\}$, contradicting Lemma 4. So $k \leq 2$, that is, $|M_i| = p$ for all $i > 2$; and if $k = 2$, again Lemma 4 implies that also $|M_1| = p$.

Let $K < Q_1 \leq Q$ such that $|Q_1 : K| = q$. Then $K \leq Z(H)$ if we put $H = (C_P(Q) \times M_1 \times \cdots \times M_k)Q_1$ and M_kQ_1/K is a P-group of order $p^{n-1}q$

with $n \geq 3$. So if $k = 2$, then by (1), $Q_1 \leq C_Q(M_1)$ and hence $H/K = (C_P(Q) \times M_1)K/K \times M_2Q_1/K$; by Lemma 3, $|\Omega(C_P(Q) \times M_1)| \leq p^2$. Thus $C_P(Q)$ is cyclic and (2c) holds. Finally, if $|M_2| = p$, then $k = 1$ and Lemma 3 applied to H/K yields that $|\Omega(C_P(Q))| \leq p^2$. So (2b) is satisfied and G has the desired structure.

To prove the converse, we again consider a minimal counterexample G. Then G satisfies (1) and (2) and $L(G)$ contains 10 pairwise different elements $A, \ldots, V$ satisfying (1.1)–(1.4).

Every subgroup of G is conjugate to a group $H = (H \cap P)\langle y \rangle$ with $y \in Q$. By (1) there exists $k \in \{0, \ldots, s\}$ such that y has $M_{k+1}, \ldots, M_s$ as nontrivial eigenspaces; and (2) implies that if $|H \cap M_i| \geq p^2$ for some $i \in \{k+1, \ldots, s\}$, then either $k = 0$ or $k = 1$ and $i = 2$. In the first case, H trivially satisfies (1) and (2); in the other case, G satisfies (2c) and (2b) holds for H. The minimality of G implies :

$$\text{Every proper subgroup of } G \text{ is } L_{10}\text{-free and } F = G. \tag{3}$$

Again let $Q_0 := \Omega(Q)$. If $C_Q(P) \neq 1$, then G/Q_0 and, by Lemma 7, also G would be L_{10}-free, a contradiction. Thus

$$C_Q(P) = 1. \tag{4}$$

By (1), $C_Q(M_s) = C_Q(P) = 1$ and Q_0 centralizes $M_1, \ldots, M_{s-1}$; furthermore Q_0 induces a power automorphism of order q in M_s. Thus

$$P = M_s \times C_P(Q_0) \text{ and } Q_0^G = M_sQ_0 \text{ is a } P\text{-group.} \tag{5}$$

If $|M_s| = p$, then by Lemma 8, G would be L_{10}-free, a contradiction. Thus $|M_s| > p$ and hence $s \leq 2$, by (2); in fact, (2) implies that there are only two possibilities for the M_i.

$$\text{Let } M_0 := C_P(Q). \text{ Then one of the following holds :} \tag{6}$$

(6a) $P = M_0 \times M_1$ where $|\Omega(M_0)| \leq p^2$ and $|M_1| \geq p^2$,

(6b) $P = M_0 \times M_1 \times M_2$ where M_0 is cyclic, $|M_1| = p$ and $|M_2| \geq p^2$.

By Lemma 6, either $S, T \not\leq P\phi(Q)$ or $U, V \not\leq P\phi(Q)$; say $U, V \not\leq P\phi(Q)$. Then

$$U \text{ and } V \text{ contain Sylow } q\text{-subgroups of } G. \tag{7}$$

We want to show next that $E = 1$. For this note that by (5), $G = M_sC_G(Q_0)$ and $M_s \cap C_G(Q_0) = 1$. Since every subgroup of M_s is normal in G, the map

$$\phi : L(M_s) \times [C_G(Q_0)/Q_0] \longrightarrow [G/Q_0]; (H, K) \longmapsto HK$$

is well-defined. Every $L \in [G/Q_0]$ is of the form $L = (L \cap P)Q_1$ where $Q_0 \leq Q_1 \in \mathrm{Syl}\, q(L)$; since $M_s = [P, Q_0]$, we have $L \cap P = (L \cap M_s)C_{L \cap P}(Q_0)$. Hence $L = (L \cap M_s)C_L(Q_0)$ and the map

$$\psi : [G/Q_0] \longrightarrow L(M_s) \times [C_G(Q_0)/Q_0]; L \longmapsto (L \cap M_s, C_L(Q_0))$$

is well-defined and inverse to ϕ. Thus $[G/Q_0] \simeq L(M_s) \times [C_G(Q_0)/Q_0]$. By (3), $C_G(Q_0)$ is L_{10}-free and then Lemma 5 implies that also $[G/Q_0]$ is L_{10}-free. So $[G/Q_0^g]$ is L_{10}-free for every $g \in G$ and this implies that E is a p-group.

Now suppose, for a contradiction, that $E \neq 1$. By (6), the M_i are eigenspaces (and centralizer) of every Sylow q-subgroup of G. Therefore by (7), $U \cap P$ and $V \cap P$ are direct products of their intersections with the M_i and hence this also holds for $(U \cap P) \cap (V \cap P) = E \cap P = E$. The minimality of G implies that $E_G = 1$. Hence $E \cap M_1 = E \cap M_2 = 1$ and so $E \leq M_0$ and $|\Omega(M_0)| = p^2$. If two of the groups S, T, U, V would contain $\Omega(M_0)$, then $\Omega(M_0) \leq E$, contradicting $E_G = 1$. Hence there are $X \in \{S, T\}$ and $Y \in \{U, V\}$ such that $X \cap M_0$ and $Y \cap M_0$ are cyclic. Since $E \leq M_0$, it follows that $E \trianglelefteq X \cup Y = G$, a contradiction. We have shown that

$$E = 1 \tag{8}$$

and come to the crucial property of G.

(9) Let $X, Y \leq G$ such that Y contains a Sylow q-subgroup of G; let $|X| = p^j q^k$ where $j, k \in \mathbb{N}_0$. Then $|X \cup Y| \leq p^{j+2}|Y|$.

Proof. Conjugating the given situation suitably, we may assume that $Q \leq Y$. Suppose first that X is a p-group and let $H = M_0$ and $K = M_1$ if (6a) holds, whereas $H = M_0 \times M_1$ and $K = M_2$ if (6b) holds. Then $X \leq P = H \times K$ where H is modular of rank at most 2 and K is elementary abelian. Let $X_1 = XK \cap H$, $X_2 = XH \cap K$ and $X_0 = (X \cap H) \times (X \cap K)$. Then by [4, 1.6.1 and 1.6.3], $X_1/X \cap H \simeq X_2/X \cap K$ and X/X_0 is a diagonal in the direct product $(X_1 \times X_2)/X_0 = X_1X_0/X_0 \times X_2X_0/X_0$. Since $X_2/X \cap K$ is elementary abelian and $X_1/X \cap H$ has rank at most 2, we have $|(X_1 \times X_2) : X| = |X_1/X \cap H| \leq p^2$.

Now $X \cup Y \leq (X_1 \times X_2) \cup Y$. Since $L(P)$ is modular, any two subgroups of P permute [4, Lemma 2.3.2]; furthermore, Q normalizes X_2. So if Q also normalizes X_1, then $X_1 \times X_2$ permutes with Y and $|X \cup Y| \leq |X_1 \times X_2| \cdot |Y| \leq |X| \cdot p^2 \cdot |Y|$, as desired. If Q does not normalize X_1, then (6b) holds and X_1 is cyclic since every subgroup of $H = M_0 \times M_1$ containing M_1 is normal in G. Then $X_1/X \cap H$ is cyclic and elementary abelian and hence $|(X_1 \times X_2) : X| = |X_1/X \cap H| \leq p$. It follows that $|X \cup Y| \leq |(X_1M_1 \times X_2)Y| \leq |X| \cdot p^2 \cdot |Y|$. Thus (9) holds if X is a p-group.

Now suppose that X is not a p-group; so $X = (X \cap P)Q_1^a$ where $1 \neq Q_1 \leq Q$ and $a \in [P,Q]$. If (6a) holds, then by (4), $M_0 = C_P(Q_1)$ and M_1 is a nontrivial eigenspace of Q_1; hence $X \cap P = (X \cap M_0) \times (X \cap M_1)$. Since every subgroup of M_0 is permutable and every subgroup of M_1 is normal in G, we have that $\langle a \rangle \trianglelefteq G$ and $X \cup Y = (X \cap P)(Y \cap P)(Q \cup Q_1^a) \leq (X \cap P)Y\langle a \rangle$; thus $|X \cup Y| \leq p^j \cdot |Y| \cdot p$. Finally, if (6b) holds, then $C_P(Q_1) = M_0$ or $C_P(Q_1) = M_0 \times M_1 = H$; in any case, $X \cap P = (X \cap H) \times (X \cap M_2)$. Since P is abelian, $(X \cap H)M_1$, $X \cap M_2$ and $Y \cap P$ are normal in G and $a = a_1a_2$ with $a_i \in M_i$. Hence $X \cup Y \leq ((X \cap H)M_1 \times (X \cap M_2))(Y \cap P)Q\langle a_2 \rangle$ and so $|X \cup Y| \leq p^{j+1} \cdot |Y| \cdot p$, as claimed.

Since U and V contain Sylow q-subgroups of G, we may apply (9) with $X \in \{S,T\}$ and $Y \in \{U,V\}$. Then since $X \cap C = E = 1$, we obtain, if $|X| = p^jq^k$, that $p^jq^k|C| = |XC| \leq |G| = |X \cup Y| \leq p^{j+2}|Y|$ and hence

$$|C : Y| \leq \frac{p^2}{q^k} \quad \text{for} \quad Y \in \{U,V\}. \tag{10}$$

Similarly, $A \cap Y = 1$ and therefore $|A||Y| = |AY| \leq |G| = |X \cup Y| \leq p^{j+2}|Y|$; hence $|A| \leq p^{j+2}$, that is

$$|A : X| \leq \frac{p^2}{q^k} \quad \text{for} \ \ X \in \{S,T\}. \tag{11}$$

Since $S \cap T = 1 = D \cap T$, we have $|S|, |D| \leq |A : T|$ and $|T| \leq |A : S|$; similarly $|U| \leq |C : V|$ and $|V| \leq |C : U|$. Thus (10) and (11) yield that

$$S, T, D, U, V \quad \text{all have order at most } p^2. \tag{12}$$

In particular, $|S| \leq p^2$ and $|U| \leq pq^m$ where $q^m = |Q|$ and hence by (9), $|G| = |S \cup U| \leq p^5q^m$. If $|P| = p^2$, then since $|M_s| \geq p^2$, we would have that $G = M_1Q$; by [5, Lemma 3.1], G then even would be L_7-free, a contradiction. Thus

$$p^3 \leq |P| \leq p^5. \tag{13}$$

Now suppose, for a contradiction, that $A \not\leq P$. Since $A = S \cup T$, one of these subgroups, say S, has to contain a Sylow q-subgroup of A; so if we take $X = S$ above, then $k \geq 1$ in (10) and (11). By (10), $|C : V| < p^2$ and since $|C : V|$ is a power of p, it follows that $|C : V| = p$. Hence $|U| \leq p$ and since $q^m \mid |U|$, we have $|U| = q^m$. By (11), $|A : S| < p^2$ and since $|A : S|$ is a power of p, also $|A : S| = p$ and hence $|T| \leq p$. If T would be a q-group, then by (9), $|G| = |T \cup U| \leq p^2q^m$, contradicting (13). Thus $|T| = p$ and $|G| = p^3q^m$. But then $P = H \times M_s$ where $H \trianglelefteq G$ and $|H| = p$; it follows that $HT \trianglelefteq G$ and then $|G| = |HTU| \leq p^2q^m$, again contradicting (13). Thus A is a p-group. Hence $L(A)$ is modular and so by (8), $|A| = |S||T| = |S||D| = |T||D|$. Therefore $|S| = |T| = |D|$ and by (13),

$$|A| = p^2 \quad \text{or} \quad |A| = p^4. \tag{14}$$

Suppose first that $|A| = p^2$. Then $|S| = |D| = p$ and by (12), $|U| \leq pq^m$. It follows from (9) that $|G| = |S \cup U| \leq p^4q^m$. So $|C_P(Q)| \leq p^2$ and hence P is abelian. Since $A \leq P$ and $G = A \cup B$, also B contains a Sylow q-subgroup of G; hence $B \cap P \trianglelefteq G$ and $C \cap P \trianglelefteq G$ and so $D = (B \cap P) \cap (C \cap P) \trianglelefteq G$. Therefore $C = DU$ and so $|C : U| = |D| = p$. It follows that $|V| = q^m$ and $|G| = |S \cup V| = p^3q^m$, by (9) and (13). Then again $P = H \times M_s$ with $H \trianglelefteq G$ and $|H| = p$ so that $|G| = |HSV| \leq p^2q^m$, a contradiction. Thus

$$|A| = p^4 \quad \text{and} \quad |S| = |T| = |D| = p^2. \tag{15}$$

Suppose first that $|U| = q^m$ or $|V| = q^m$, say $|U| = q^m$. Then by (9), $|G| = |S \cup U| \leq p^4q^m$ and since $|A| = p^4$, we have $A = P \trianglelefteq G$. Therefore $D = A \cap B \trianglelefteq B$ and $D \trianglelefteq C$ so that again $D \trianglelefteq G$. Furthermore $|V| = |G : A| = q^m$ and so $C = U \cup V \leq Q^G$. Since $|B : D| = |G : A| = q^m$, also $B \leq Q^G$; hence $G = B \cup C \leq Q^G$ so that $M_0 = 1$, by (6). By [5, Lemma 3.1], M_1Q is L_{10}-free; hence (6b) holds and $|M_2| = p^3$. It follows that Q induces a power automorphism either in D or in A/D; but in both groups $C = DU$ and $G/D = (A/D)(C/D)$ there exist two Sylow q-subgroups generating the whole group, a contradiction. So $|U| \neq q^m \neq |V|$ and by (12), $|U| = |V| = pq^m$. Since $A \cap U = E = 1$, it follows that $A < P$; so (13) and (15) yield that

$$|G| = p^5q^m \quad \text{and} \quad |U| = |V| = pq^m. \tag{16}$$

Since $L(P)$ is modular, $L(S) \simeq [A/D] \simeq L(T)$. So if S would be cyclic, then A would be of type (p^2, p^2) and hence by (6), $A \cap M_s = 1$ and $|P| \geq p^6$, a contradiction. Thus S and T are elementary abelian and so P is generated by elements of order p; by [4, Lemma 2.3.5], P is elementary abelian.

Now if (6a) holds, then $M_0S \trianglelefteq G$ and hence $G = M_0SU$. Since $|M_0| \leq p^2$, it follows from (16) that $|M_0| = p^2$ and $U \cap M_0 = 1$. Since $U \cap P \trianglelefteq G$, we have $U \cap P \leq M_1$ and so $U \leq Q^G = M_1Q$. Similarly, $V \leq Q^G$ and hence $C = U \cup V \leq Q^G$. Since $|C| \geq |D||U| = p^3q^m$ and $|M_1| = p^3$, it follows that $C = Q^G \trianglelefteq G$. But then $|B : D| = |G : C| = p^2$, so $|B| = p^4$ and $G = A \cup B \leq P$, a contradiction.

So, finally, (6b) holds and $P = M_0 \times M_1 \times M_2$ where $|M_0 \times M_1| \leq p^2$. This time $(M_0 \times M_1)S \trianglelefteq G$ and it follows from (16) that $|M_0 \times M_1| = p^2$ and $U \cap P \leq M_2$ and $V \cap P \leq M_2$. So $|M_2| = p^3$ and since $U \cap V = 1$, we have either $M_2 \leq C$ or $C \cap M_2 = (U \cap P) \times (V \cap P)$. In the first case, by (5), C

would contain every subgroup of order q of G; since $B \cap C = D$ is a p-group, it would follow that $B \leq P$ and hence $G = A \cup B \leq P$, a contradiction. So $|C \cap M_2| = p^2$ and if C_0, U_0, V_0 are the subgroups generated by the elements of order q of C, U, V, respectively, then by (5), C_0 is a P-group of order p^2q and U_0, V_0 are subgroups of order pq in C_0. So $U_0 \cap V_0 \neq 1$, but by (8), $U \cap V = 1$, the final contradiction.

We come to the third possibility for a group satisfying Hypothesis I.

Theorem 3. *Suppose that G satisfies Hypothesis I and that $q \mid p-1$ but $|Q/C_Q(P)|$ does not divide $p-1$; let $k \in \mathbb{N}$ such that q^k is the largest power of q dividing $p-1$.*

Then G is L_{10}-free if and only if there exists a minimal normal subgroup N of order p^q of G such that one of the following holds.

(1) $P = C_P(Q) \times N$ where $|\Omega(C_P(Q))| \leq p^2$.

(2) $P = C_P(Q) \times N_1 \times N$ where $N_1 \trianglelefteq G$, $|N_1| = p$ and $C_P(Q)$ is cyclic.

(3) $q = 2$, $k = 1$ and $P = M \times N$ where $|M| = p^2$, Q is irreducible on M and $C_Q(N) < C_Q(M)$.

(4) $P = M \times N$ where M is elementary abelian of order p^2 and Q induces a power automorphism of order q in M.

(5) $P = N_1 \times N_2 \times N$ where $N_i \trianglelefteq G$, $|N_i| = p$ for $i = 1,2$ and where $C_Q(N_1) < C_Q(N_2) = \phi(Q)$.

Proof. Suppose first that G is L_{10}-free. Again by Maschke's theorem, $[P,Q] = N_1 \times \cdots \times N_r$ $(r \geq 1)$ with Q irreducible on N_i and we may assume that $C_Q(N_r) \leq C_Q(N_i)$ for all i. Then $K := C_Q(P) = C_Q(N_r)$ and since $|Q/K|$ does not divide $p-1$, we have that $|N_r| > p$. By Lemma 2 and [5, Lemma 3.1], $|N_r| = p^q$ and $|Q/K| = q^{k+1}$, or $|Q/K| \geq q^{k+1} = 4$ in case $q = 2$, $k = 1$. We let $N := N_r$ and have to show that G satisfies one of properties (1)–(5).

For this put $M := C_P(Q) \times N_1 \times \cdots \times N_{r-1}$, so that $P = M \times N$, and let $Q_1 \leq Q$ such that $K < Q_1$ and $|Q_1 : K| = q$. By Lemma 2, Q_1 induces a power automorphism of order q in N; by Lemma 1, $C_Q(N) < C_Q(N_i)$ for all $i \neq r$ and hence Q_1 centralizes M. So $PQ_1/K = MK/K \times NQ_1/K$ where NQ_1/K is a P-group of order $p^q q$. By Lemma 3, $|\Omega(M)| \leq p^2$; in particular, $r \leq 3$.

If $r = 1$, then $M = C_P(Q)$ and (1) holds. If $r = 2$, then either $|N_1| = p$ and $C_P(Q)$ is cyclic, that is (2) holds, or $|N_1| = p^2$ and $C_P(Q) = 1$. In this case, since Q is irreducible on N_1 and, by Lemma 1, induces automorphisms of different orders in N and N_1, again Lemma 2 and [5, Lemma 3.1] imply that $q = 2$ and $k = 1$; thus (3) holds.

Finally, suppose that $r = 3$. Since $|\Omega(M)| \leq p^2$, it follows that $M = N_1 \times N_2$, $|N_1| = |N_2| = p$ and $C_P(Q) = 1$. If $q = 2$ and $k = 1$, then $Q = \langle x \rangle$ induces automorphisms of order 2 in N_1 and N_2; thus $a^x = a^{-1}$ for all $a \in M$ and (4) holds. So suppose that $q > 2$ or $q = 2$ and $k > 1$. Then $|Q/K| = q^{k+1}$ as mentioned above and so $|\phi(Q) : K| = q^k$ divides $p - 1$. Thus $H := P\phi(Q)$ is one of the groups in Theorem 2 and by Lemma 2, $\phi(Q)$ induces a power automorphism of order q^k in N. Since $[P, \phi(Q)] \leq [P, Q] = N_1 \times N_2 \times N$ and $C_Q(N) < C_Q(N_i)$ for $i \in \{1, 2\}$, N is one of the eigenspaces of x^p in $[P, \phi(Q)]$. Hence H satisfies (2b) or (2c) of Theorem 2. In the first case, $N = M_1$ in the notation of that theorem and $N_1 \times N_2 \leq C_P(\phi(Q))$ since $C_{\phi(Q)}(M_1)$ is the largest centralizer of a nontrivial eigenspace of x^p. So $C_Q(N_1) = \phi(Q) = C_Q(N_2)$ and by Lemma 1, Q induces a power automorphism of order q in $N_1 \times N_2$; thus (4) holds. In the other case, $N = M_2$ and $|M_1| = p$, so that $M_1 = N_1$, say, and then $N_2 \leq C_P(\phi(Q))$. Thus (5) holds and G has the desired properties.

To prove the converse, we again consider a minimal counterexample G. Then G has a minimal normal subgroup N of order p^q and satisfies one of the properties (1)–(5) but is not L_{10}-free. As in the proof of Theorem 1, by Lemma 7, $C_Q(P) = 1$.

Let H be a proper subgroup of G. Then either H contains a Sylow q-subgroup of G or $H \leq P\phi(Q)$. In the first case, $N \leq H$ or $H \cap N = 1$. Hence H satisfies the assumptions of Theorem 3 or Theorem 2 or is nilpotent; the minimality of G implies that H is L_{10}-free. So suppose that $H = P\phi(Q)$. A simple computation shows (see [5, p. 523]) that if $q > 2$ or if $q = 2$ and $k > 1$, then q^{k+1} is the largest power of q dividing $p^q - 1$. Therefore in these cases, by [3, II, Satz 3.10], a generator x of Q operates on $N = (GF(p^q), +)$ as multiplication with an element of order q^{k+1} of the multiplicative group of $GF(p^q)$. The q-th power of this element lies in $GF(p)$ and therefore fixes every subgroup of N. Thus $\phi(Q)$ induces a power automorphism of order q^k in N. So if G satisfies (1) or (4), then H satisfies $s = 1$ and (2b) of Theorem 2; the same holds if G satisfies (2) and $\phi(Q)$ centralizes N_1. If G satisfies (2) and $[\phi(Q), N_1] \neq 1$ or G satisfies (5), then (2c) of Theorem 2 holds for H. Finally, if $q = 2$ and $k = 1$, then either $\phi(Q)$ is irreducible on N or $|Q| = 4$; hence H satisfies the assumptions of Theorem 3 or 2. In all cases, Theorem 2 and the minimality of G imply that H is L_{10}-free.

Finally, $Q_0 = \Omega(Q)$ induces a power automorphism of order q in N and centralizes the complements of N in P given in (1)–(5). So $P = N \times C_P(Q_0)$ and by Lemma 8, G is L_{10}-free, the desired contradiction.

Note that in Theorem 1 and in (2a) of Theorem 2, $C_P(Q)$ may be an arbitrary modular p-group since by Iwasawa's theorem [4, Theorem 2.3.1], a direct product of a modular p-group with an elementary abelian p-group has modular subgroup lattice. In all the other cases of Theorems 2 and 3, Lemma 3 implied that $|\Omega(C_P(Q))| \leq p^2$; in (2b) of Theorem 2 and (1) of Theorem 3, $C_P(Q)$ may be an arbitrary modular p-group with this property.

4 Groups of type II and III

We now determine the groups of type II. Theorem 4 shows that modulo centralizers the only such group is $SL(2,3) \simeq Q_8 \rtimes C_3$.

Theorem 4. *Let $G = PQ$ where P is a normal Sylow 2-subgroup of G, Q is a cyclic q-group, $2 < q \in \mathbb{P}$, and $[P, Q]$ is hamiltonian.*

Then G is L_{10}-free if and only if $G = M \times NQ$ where M is an elementary abelian 2-group, $N \simeq Q_8$ and Q induces an automorphism of order 3 in N.

Proof. Suppose first that G is L_{10}-free. Then $L(P)$ is modular and since $[P, Q]$ is hamiltonian, it follows from [4, Theorems 2.3.12 and 2.3.8] that $P = H \times K$ where H is elementary abelian and $K \simeq Q_8$. Hence $\phi(P) = \phi(K)$ and $\Omega(P) = H \times \phi(P)$. By Maschke's theorem there are Q-invariant complements M of $\phi(P)$ in $\Omega(P)$ and $N/\phi(P)$ of $\Omega(P)/\phi(P)$ in $P/\phi(P)$. Then $\Omega(N) = \Omega(P) \cap N = \phi(P)$ implies that $N \simeq Q_8$ and since $[P, Q] \not\leq \Omega(P)$, Q operates nontrivially on N. Therefore $q = 3$ and Q induces an automorphism of order 3 in N.

Since P is a 2-group, $G/\phi(P)$ is an L_{10}-free $\{p, q\}$-group of type I with $q \nmid p - 1$. By Theorem 1, $P/\phi(P) = C_{P/\phi(P)}(Q) \times N_1 \times \cdots \times N_r$ with nontrivial $GF(2)Q$-modules N_i satisfying (1) and (2) of that theorem. By (1), the subgroup of order 3 of $Q/C_Q(N_i)$ is irreducible on N_i; therefore $|N_i| = 4$ and hence $C_Q(N_i) = \phi(Q)$ for all i. But then (2) implies that $r = 1$. It follows that $N_1 = N/\phi(P)$ and $[M, Q] \leq M \cap N = 1$; thus $G = M \times NQ$ as desired.

To prove the converse, we again consider a minimal counterexample G; let $\{A, \ldots, V\}$ be a sublattice of L(G) isomorphic to L_{10} and satisfying (1.1)–(1.4). The minimality of G implies that $F = G$ and, together with Lemma 7, that $C_Q(P) = 1$; hence $|Q| = 3$.

If A or C, say C, contains two subgroups of order 3, then $NQ \leq C$ and hence $C \trianglelefteq G$. Then $D = A \cap C = B \cap C \trianglelefteq A \cup B = G$ and $A/D \simeq G/C \simeq B/D$ are 2-groups; therefore G/D is a 2-group. Similarly, $E = S \cap D = U \cap D \trianglelefteq S \cup U = G$ and $S/E \simeq G/C$ and $U/E \simeq C/D$ are 2-groups. Thus G/E is a modular 2-group and hence L_{10}-free, a contradiction.

So A and C both contain at most one subgroup of order 3 and therefore are nilpotent. By Lemma 6, we have $U, V \not\leq P$, say; so U and V contain the subgroup Q_1 of order 3 of C and it follows that $Q_1 \leq U \cap V = E \leq A$. Hence $G = A \cup C \leq C_G(Q_1)$, a final contradiction.

We finally come to groups of type III; more generally, we determine all L_{10}-free $\{p,2\}$-groups in which Q_8 operates faithfully on P.

Theorem 5. *Let $G = PQ$ where P is a normal Sylow p-subgroup with modular subgroup lattice, $Q \simeq Q_8$ and $C_Q(P) = 1$.*

Then G is L_{10}-free if and only if $P = M \times N$ where $|N| = p^2$, Q operates irreducibly on N and one of the following holds :

(1) $p \equiv 3$ (mod 4), $M = C_P(Q)$ and $|\Omega(M)| \leq p^2$,

(2) $M = C_P(Q) \times M_1$ where $C_P(Q)$ is cyclic, $M_1 \trianglelefteq G$ and $|M_1| = 3$,

(3) $C_P(Q) = 1$ and $M = C_P(\Omega(Q))$ is elementary abelian of order 9.

Proof. Suppose first that G is L_{10}-free. By [6, Lemma 2.2], $P = C_P(Q) \times [P,Q]$ and $[P,Q]$ is elementary abelian; by Maschke's theorem, $[P,Q] = N_1 \times \cdots \times N_r$ with irreducible $GF(p)Q$-modules N_i. As $C_Q(P) = 1$, there exists $i \in \{1,\ldots,r\}$ such that $C_Q(N_i) = 1$; we choose the notation so that $i = r$ and let $N = N_r$, $M = C_P(Q) \times N_1 \times \cdots \times N_{r-1}$ and $Q_0 = \Omega(Q)$.

Clearly, $|N| \geq p^2$ and since $C_N(Q_0)$ is Q-invariant, $C_N(Q_0) = 1$; hence N is inverted by Q_0. It follows that if X is a maximal subgroup of Q, then $C_X(W) = 1$ for every minimal normal subgroup W of NX. By Lemma 1, either X is irreducible on N or it induces a power automorphism in N. Since Q is irreducible on N, at most one maximal subgroup of Q can induce power automorphisms in N and hence there are at least two maximal subgroups of Q which are irreducible on N. It follows that $|N| = p^2$ and $p \equiv 3$ (*mod* 4).

If there would exist $i \in \{1,\ldots,r-1\}$ such that $C_Q(N_i) = 1$, then there would exist a maximal subgroup X of Q which is irreducible on both N_i and N; but then $(N_i \times N)X$ would be L_{10}-free, contradicting Lemma 1. Thus $N = N_r$ is the unique N_i on which Q is faithful; it follows that $M = C_P(Q_0)$.

Since NQ_0 is a P-group of order $2p^2$, Lemma 3 yields that $|\Omega(M)| \leq p^2$. So if $r = 1$, then (1) holds; therefore assume that $r \geq 2$. Then $C_G(Q_0)/Q_0 = MQ/Q_0$ is L_{10}-free and has non-normal elementary abelian Sylow 2-subgroups of order 4. By [6, Proposition 2.6], $p = 3$. It follows that (2) holds if $r = 2$ and (3) holds if $r = 3$.

To show that, conversely, all the groups with the given properties are L_{10}-free, we consider a minimal counterexample G to this statement and want to apply Lemma 8.

Again since Q is irreducible on N and $|N| = p^2$, it follows that N is inverted by $Q_0 = \Omega(Q)$. By assumption, M is centralized by Q_0 and therefore we have that $P = N \times C_P(Q_0)$. Furthermore every subgroup of order 4 of Q is faithful on N and hence irreducible on N since $4 \nmid p-1$. So it remains to be shown that every proper subgroup H of G is L_{10}-free.

If $8 \nmid |H|$, then $H \leq PQ_1$ for some maximal subgroup Q_1 of Q. Since Q_1 is irreducible and faithful on N, the group PQ_1 is L_{10}-free by Theorem 3; thus also H is L_{10}-free. So suppose that H contains a Sylow 2-subgroup of G, say $Q \leq H$. Then either $N \leq H$ or $H \cap N = 1$ and then $H \leq MQ$. In the first case, the minimality of G implies that H is L_{10}-free. In the second case, we may assume that $H = MQ$. This group even is modular if (1) holds and by [6, Lemma 4.5], it is L_{10}-free if (2) is satisfied. So suppose that (3) holds. Then H/Q_0 is a group of order 36 so that it is an easy exercise to show that it is L_{10}-free (see also Remark 2); by Lemma 7, then also H is L_{10}-free. Thus every proper subgroup of G is L_{10}-free and Lemma 8 implies that G is L_{10}-free, the desired contradiction.

Remark 2. (a) Part (1) of Theorem 5 characterizes the L_{10}-free $\{p, q\}$-groups of type III and shows that also for $p = 3$ the corresponding groups are L_{10}-free.

(b) In addition, parts (2) and (3) of Theorem 5 show that for $p = 3$ there are exactly three further types of L_{10}-free $\{2, 3\}$-groups in which Q_8 operates faithfully. In these, $MQ/\Omega(Q)$ is isomorphic to

(i) $C_{3^n} \times D_6 \times C_2$ $(n \geq 0)$, or

(ii) $H \times C_2$ where H is a P-group of order 18, or

(iii) $D_6 \times D_6$.

(c) The groups in (ii) and (iii) both are subgroups of the group G in Example 4.7 of [6] and therefore are L_{10}-free.

Proof of (b). Clearly, the four group $Q/\Omega(Q)$ can only invert M_1 in (2) of Theorem 5; so we get the groups in (i). If (3) holds, then $M = M_1 \times M_2$ where $M_i \trianglelefteq MQ$ and $|M_i| = 3$. So if $C_Q(M_1) = C_Q(M_2)$, we obtain (ii) and if $C_Q(M_1) \neq C_Q(M_2)$, then $M_1C_Q(M_2)$ and $M_2C_Q(M_1)$ centralize each other modulo $\Omega(Q)$ and hence (iii) holds.

We finally mention that by Lemma 7, to characterize also the L_{10}-free $\{2, 3\}$-groups with Sylow 2-subgroup Q_8 operating non-faithfully on a 3-group P, it remains to determine the L_{10}-free $\{2, 3\}$-groups having a four group as Sylow 2-subgroup. This, however, is the crucial case in the study of L_{10}-free $\{2, 3\}$-groups since by [6, Lemma 2.9], in every such group PQ we have $|\Omega(Q/C_Q(P))| \leq 4$.

References

[1] S. Andreeva, R. Schmidt, I. Toborg: *Lattice defined classes of finite groups with modular Sylow subgroups.* To appear.

[2] C. Baginski and A. Sakowicz: *Finite groups with globally permutable lattice of subgroups*, Colloq. Math. **82** (1999), 65–77.

[3] B. Huppert: Endliche Gruppen I , vol. 1, Springer-Verlag, 1967.

[4] R. Schmidt: Subgroup lattices of groups, Expositions in Mathematics 14, de Gruyter, 1994.

[5] R. Schmidt: *L-free groups*, Illinois J. Math. **47** (2003), 515–528.

[6] R. Schmidt: L_{10}*-free groups*, J. Group Theory **10** (2007), 613–631.