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Groups and set theoretic solutions of the Yang-Baxter equation

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Abstract. We discuss some recent results on the link between group theory and set theoretic involutive non-degenerate solutions of the Yang-Baxter equation. Some problems are included.

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1 Introduction

In a paper on statistical mechanics by Yang [21], the quantum Yang-Baxter equation appeared. It turned out to be one of the basic equations in mathematical physics and it lies at the foundation of the theory of quantum groups. One of the important unsolved problems is to discover all the solutions R of the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where V is a vector space, $R : V \otimes V \to V \otimes V$ is a linear map and R_{ij} denotes the map $V \otimes V \otimes V \to V \otimes V \otimes V$ that acts as R on the (i, j) tensor factor (in this order) and as the identity on the remaining factor. In recent years, many solutions have been found and the related algebraic structures have been intensively studied (see for example [17]). Drinfeld, in [6], posed the question of finding the simplest solutions, that is, the solutions R that are induced by a linear extension of a mapping $\mathcal{R} : X \times X \to X \times X$, where X is a basis for V. In this case, one says that $\mathcal{R}$ is a set theoretic solution of the quantum Yang-Baxter equation.

Let $\tau : X^2 \to X^2$ be the map defined by $\tau(x, y) = (y, x)$. Observe that $\mathcal{R}$ is a set theoretical solution of the quantum Yang-Baxter equation if and only if the mapping $r = \tau \circ \mathcal{R}$ is a solution of the braided equation (or a solution of the Yang-Baxter equation, in the terminology used for example in [10, 12])

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}.$$

Set theoretic solutions $\mathcal{R} : X^2 \rightarrow X^2$ of the quantum Yang-Baxter equation (with X a finite set) that are (left) non-degenerate and such that $r = \tau \circ \mathcal{R}$ is involutive (i.e., r^2 is the identity map on X^2) have received recently a lot of attention by Etingof, Schedler and Soloviev [9], Gateva-Ivanova and Van den Bergh [10, 13], Lu, Yan and Zhu [18], Rump [19, 20], Jespers and Okniński [15, 16] and others. (The set theoretical solutions $\mathcal{R}$ such that r is involutive are called unitary in [19], and in [9] one then says that (X, r) is a symmetric set.)

Recall that a bijective map

$$\begin{array}{llcl} r\colon & X \times X & \longrightarrow & X \times X \\ & (x,y) & \mapsto & (f_x(y), g_y(x)) \end{array}$$

is said to be left (respectively, right) non-degenerate if each map f_x (respectively, g_x) is bijective.

Gateva-Ivanova and Van den Bergh in [13], and Etingof, Schedler and Soloviev in [9], gave a beautiful group theoretical interpretation of involutive non-degenerate solutions of the braided equation. In order to state this, we need to introduce some notation. Let FaM_n be the free abelian monoid of rank n with basis $u_1, \ldots, u_n$. A monoid S generated by a set $X = \{x_1, \ldots, x_n\}$ is said to be of left I-type if there exists a bijection (called a left I-structure)

$$v\colon \mathrm{FaM}_n \longrightarrow S$$

such that

$$v(1) = 1 \quad \text{and} \quad \{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\},$$

for all $a \in \mathrm{FaM}_n$. In [13] it is shown that these monoids S have a presentation

$$S = \langle x_1, \ldots, x_n \mid x_i x_j = x_k x_l \rangle,$$

with $\binom{n}{2}$ defining relations so that every word $x_i x_j$, with $1 \leq i, j \leq n$, appears at most once in one of the relations. Such a presentation induces a bijective map $r\colon X \times X \longrightarrow X \times X$ defined by

$$r(x_i, x_j) = \begin{cases} (x_k, x_l), & \text{if } x_i x_j = x_k x_l \text{ is a defining relation for } S; \\ (x_i, x_j), & \text{otherwise.} \end{cases}$$

Furthermore, r is an involutive right non-degenerate solution of the braided equation. Conversely, for every involutive right non-degenerate solution of the braided equation $r\colon X \times X \longrightarrow X \times X$ and every bijection $v : \{u_1, \ldots, u_n\} \to X$ there is a unique left I-structure $v : \mathrm{FaM}_n \to S$ extending v, where S is the semigroup given by the following presentation

$$S = \langle X \mid ab = cd, \text{ if } r(a,b) = (c,d)\rangle$$

([16, Theorem 8.1.4.]).

In [15] Jespers and Okniński proved that a monoid S is of left I-type if and only if it is of right I-type; one calls them simply monoids of I-type. Hence, it follows that an involutive solution of the braided equation is right non-degenerate if and only if it is left non-degenerate (see [15, Corollary 2.3] and [16, Corollary 8.2.4]).

Jespers and Okniński in [15] also obtained an alternative description of monoids of I-type. Namely, it is shown that a monoid is of I-type if and only if it is isomorphic to a submonoid S of the semi-direct product $\mathrm{FaM}_n \rtimes Sym_n$, with the natural action of Sym_n on FaM_n (that is, $\sigma(u_i) = u_{\sigma(i)}$ for $\sigma \in Sym_n$), so that the projection onto the first component is a bijective map, that is

$$S = \{(a, \phi(a)) \mid a \in \mathrm{FaM}_n\},$$

for some map $\phi\colon \mathrm{FaM}_n \to Sym_n$. It then follows that S has a (two-sided) group of quotients (one needs to invert the central element $((u_1 \cdots u_n)^k, 1)$; where k is the order of the permutation $\phi(u_1 \cdots u_n)$). Of course the group of quotients $\mathcal{G} = S^{-1}S$ of the monoid of I-type S is defined by the same generators and relations as S. These groups have been investigated by Etingof, Guralnick, Schedler and Soloviev in [8, 9], where they are called structural groups. They are simply called groups of I-type.

The group $\mathcal{G}$ can also be described as follows. The map ϕ extends uniquely to a map $\phi\colon \mathrm{Fa}_n \to Sym_n$, where Fa_n is the free abelian group of rank n, and the group $\mathcal{G}$ is isomorphic to a subgroup of the semi-direct product $\mathrm{Fa}_n \rtimes Sym_n$ so that the projection onto the first component is a bijective map, that is

$$\mathcal{G} = \{(a, \phi(a)) \mid a \in \mathrm{Fa}_n\}. \tag{1}$$

Note that if we put $f_{u_i} = \phi(u_i)$ then $S = \langle (u_i, f_{u_i}) \mid 1 \leq i \leq n\rangle$ and one can easily obtain the associated involutive non-degenerate solution $r : X^2 \to X^2$ defining the monoid of I-type. Indeed, if we set $X = \{u_1, \ldots, u_n\}$, then $r(u_i, u_j) = (f_{u_i}(u_j), f^{-1}_{f_{u_i}(u_j)}(u_j))$. Obviously, $\phi(\mathrm{Fa}_n) = \langle \phi(a) \mid a \in \mathrm{FaM}_n\rangle = \langle f_{u_i} \mid 1 \leq i \leq n\rangle$ (we will denote this group also as G_r). Note that, because of Proposition 2.2 in [9], if $(x, g) \mapsto (f_x(y), g_y(x))$ is an involutive non-degenerate solution of the braided equation then $T^{-1}{g_x}^{-1}T = f_x$, where $T : X \to X$ is the bijective map defined by $T(y) = {g_y}^{-1}(y)$. Hence $\langle f_x : x \in X\rangle$ is isomorphic with $\langle g_x : x \in X\rangle$.

2 Groups of I-type

In order to describe all involutive non-degenerate solutions of the braided equation (equivalently the non-degenerate unitary set theoretic solutions of the quantum Yang-Baxter equation) one needs to solve the following problem.

Problem 1: Characterize the groups of I-type.

An important first step in this direction is to classify the finite groups that are of the type $\phi(\mathrm{Fa}_n)$ for some group of I-type $\mathcal{G}$, as in (1) (equivalently the groups of the form $\langle f_x : x \in X\rangle$, for $(x, y) \in X^2 \mapsto (f_x(y), g_y(x))$ a non-degenerate involutive solution of the braided equation). As in [4], a finite group with this property is called an *involutive Yang-Baxter* (IYB, for short) group. So to tackle the above problem we will need to solve the following two problems.

Problem 1a: Classify involutive Yang-Baxter groups.

Problem 1b: Describe all groups of I-type that have a fixed associated IYB group G.

In [4] these problems are being investigated and in this section we report on the main results of that paper.

Recall that Etingof, Schedler and Soloviev in [9, Theorem 2.15] proved that any group of I-type is solvable. As a consequence, every IYB group is solvable.

Verifying that a finite group is IYB seems to be a non-trivial task. Hence it is useful the give several equivalent properties that guarantee this property. For this we first recall some terminology of [4]. For a finite set X we denote by Sym_X the symmetric group on X. An *involutive Yang-Baxter map* (IYB map, for short) on a finite set X is a map $\lambda : X \to Sym_X$ satisfying

$$\lambda(x)\lambda(\lambda(x)^{-1}(y)) = \lambda(y)\lambda(\lambda(y)^{-1}(x)) \quad (x, y \in X). \tag{2}$$

The justification for this terminology is based on the fact that each IYB map yields an involutive non-degenerate set theoretical solution of the Yang-Baxter equation and conversely. Indeed, let $r : X^2 \to X^2$ be a bijective map. As before, denote $r(x, y) = (f_x(y), g_y(x))$. From the proof of [3, Theorem 4.1], it follows that $r : X^2 \to X^2$ is an involutive non-degenerate set theoretical solution of the Yang-Baxter equation if and only if $f_x \in Sym_X$ for all $x \in X$ and the map $\lambda\colon X \to Sym_X$ defined by $\lambda(x) = f_x$, for all $x \in X$, is an IYB map.

Theorem 1. *[4] The following conditions are equivalent for a finite group G.*

(1) *G is an IYB group, that is, there is a map $\phi : \mathrm{Fa}_n \to Sym_n$ such that $\{(a, \phi(a)) : a \in \mathrm{Fa}_n\}$ is a subgroup of $\mathrm{Fa}_n \rtimes Sym_n$ and G is isomorphic to $\phi(\mathrm{Fa}_n)$.*

(2) *There is an abelian group A, an action of G on A and a group homomorphism $\rho : G \to A \rtimes G$ such that $\pi_G \rho = \mathrm{id}_G$ and $\pi_A \rho : G \to A$ is bijective, where π_G and π_A are the natural projections on G and A respectively.*

(3) *There is an abelian group A, an action of G on A and a bijective 1-cocycle $G \to A$.*

(4) *There exists an IYB map $\lambda \colon A \cup X \to Sym_{A\cup X}$ satisfying the following conditions:*

a. *$\lambda(A)$ is a subgroup of $Sym_{A\cup X}$ isomorphic to G,*

b. *$A \cap X = \emptyset$,*

c. *$\lambda(x) = \mathrm{id}_{A\cup X}$ for all $x \in X$,*

d. *$\lambda(a)(b) \in A$ for all $a, b \in A$ and*

e. *$\lambda|_A$ is injective.*

(5) *$G \cong \lambda(X)$ for some IYB map $\lambda : X \to Sym_X$ whose image is a subgroup of Sym_X.*

(6) *$G \cong \langle \lambda(X)\rangle$ for some IYB map $\lambda : X \to Sym_X$.*

(7) *There exist a group homomorphism $\mu : G \to Sym_G$ satisfying*

$$x\mu(x)^{-1}(y) = y\mu(y)^{-1}(x), \tag{3}$$

for all $x, y \in G$.

(8) *There exist a generating subset Z of G and a group homomorphism $\mu : G \to Sym_Z$ satisfying (3) for all $x, y \in Z$.*

One obtains some constructions of IYB-groups from a give IYB-groups.

Corollary 1. *[4]*

(1) *If G is an IYB group then its Hall subgroups are also IYB.*

(2) *The class of IYB groups is closed under direct products.*

Theorem 2. *[4] Let G be a finite group such that $G = AH$, where A is an abelian normal subgroup of G and H is an IYB subgroup of G. Suppose that there is a bijective 1-cocycle $\pi : H \to B$, with respect to an action of H on the abelian group B such that $H \cap A$ acts trivially on B. Then G is an IYB group.*

In particular, every semi-direct product $A \rtimes H$ of a finite abelian group A by an IYB group H is IYB.

Theorem 3. *[4] Let N and H be IYB groups and let $\pi_N : N \to A$ be a bijective 1-cocycle with respect to an action of N on an abelian group A. If $\gamma : H \to \mathrm{Aut}\,(N)$ and $\delta : H \to \mathrm{Aut}\,(A)$ are actions of H on N and A respectively such that $\delta(h)\pi_N = \pi_N\gamma(h)$ for every $h \in H$, then the semi-direct product $N \rtimes H$, with respect to the action γ, is an IYB group.*

Corollary 2. *[4]*

(1) Let G be an IYB group and H an IYB subgroup of Sym_n. Then the wreath product $G \wr H$ of G and H is an IYB group.

(2) Any finite solvable group is isomorphic to a subgroup of an IYB group.

(3) Let n be a positive integer. Then the Sylow subgroups of Sym_n are IYB groups.

(4) Any finite nilpotent group is isomorphic to a subgroup of an IYB nilpotent group.

The next result yields many examples of IYB groups.

Theorem 4. *[4] Let G be a finite group having a normal sequence*

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$$

satisfying the following conditions:

(i) for every $1 \leq i \leq n$, $G_i = G_{i-1}A_i$ for some abelian subgroup A_i;

(ii) $(G_{i-1} \cap (A_i \cdots A_n), G_{i-1}) = 1$;

(iii) A_i is normalized by A_j for every $i \leq j$.

Then G is an IYB group.

Corollary 3. *[4]*

(1) Let G be a finite group. If $G = NA$, where N and A are two abelian subgroups of G and N is normal in G, then G is an IYB group. In particular, every abelian-by-cyclic finite group is IYB.

(2) Every finite nilpotent group of class 2 *is IYB.*

It is unclear whether the class of IYB groups is closed for taking subgroups. As a consequence, it is unknown whether the class of IYB groups contains all finite solvable groups. Hence, our next problem.

Problem 2: Are all finite solvable groups involutive Yang-Baxter groups, i.e. does the class of IYB groups coincide with that of all solvable finite groups.

The results in [4] also indicate that there is no obvious inductive process to prove that solvable finite groups are IYB. Indeed for such a process to exist one would like to be able to lift the IYB structure from subgroups H or quotient groups $\overline{G}$ of a given group G to G. However, in [4], examples are given that show that not every IYB homomorphism of a quotient of G can be lifted to an IYB homomorphism of G.

Concerning Problem 1b: If $r(x_1, x_2) = (f_{x_1}(x_2), g_{x_2}(x_1))$ is an involutive non-degenerate solution on a finite set X of the braided equation then it is easy to produce, in an obvious manner, infinitely many solutions with the same associated IYB group, namely for every set Y let $r_Y : (X \cup Y)^2 \to (X \cup Y)^2$ be given by $r_Y((x_1, y_1), (x_2, y_2)) = ((f_{x_1}(x_2), y_1), (g_{x_2}(x_1), y_2))$. In [4] an alternative way of obtaining another involutive non-degenerate solution on $X \times X$ of the braided equation with the same associated IYB group is given. It follows that, in a non-obvious fashion, infinitely many set theoretic solutions of the Yang-Baxter equation are obtained for the same IYB group.

3 Groups of I-type and poly-$\mathbb{Z}$ groups

There are two other approaches, both originating from the work of Etingof, Schedler and Soloviev, [9], that could lead to successfully classify all possible set theoretic solutions of the Yang-Baxter equation. The idea is to show that every solution can be built in a recursive way from certain solutions of smaller cardinality.

Let (X, r) be a set theoretical solution on the finite set X. The first alternative approach is based on the retract relation $\sim$ on the set X, introduced in [9], and defined by

$$x_i \sim x_j \quad \text{if } \sigma_i = \sigma_j$$

(here we denote by σ_i the permutation f_{x_i}; by γ_j or γ_{x_j} we denote the map g_{x_j}). There is a natural induced solution

$$Ret(X, r) = (X/\sim, \tilde{r}),$$

and it is called the retraction of X. A solution (X, r) is called a *multipermutation solution of level* m if m is the smallest nonnegative integer such that the solution $Ret^m(X, r)$ has cardinality 1. Here one defines

$$Ret^k(X, r) = Ret(Ret^{k-1}(X, r))$$

for $k > 1$. If such an m exists then one also says that the solution is *retractable*. In this case, the group $G(X, r)$ is a poly-$\mathbb{Z}$ group (see [16, Proposition 8.2.12]). Recall that a group is called poly-$\mathbb{Z}$ (or, poly-infinite cyclic) if it has a finite subnormal series with factors that are infinite cyclic groups.

The second alternative approach is based on the notion of generalized twisted union. In order to state the definition, first notice that there is a natural action of the associated involutive Yang-Baxter group G_r on X defined by $\sigma(x_i) = x_{\sigma(i)}$. A set theoretic involutive non-degenerate solution (X, r) is called a *generalized twisted union* of solutions (Y, r_Y) and (Z, r_Z) if X is a disjoint union of two G_r-invariant non-empty subsets Y, Z such that for all $z, z' \in Z, y, y' \in Y$ we have

$$\sigma_{\gamma_y(z)|Y} = \sigma_{\gamma_{y'}(z)|Y} \tag{4}$$

$$\gamma_{\sigma_z(y)|Z} = \gamma_{\sigma_{z'}(y)|Z}. \tag{5}$$

Here, to simplify notation, we write σ_x for σ_i if $x = x_i$, and similarly for all γ_i. If, moreover, (X, r) is a square free solution (that is, every defining relation is without words of the form x_i^2), then conditions (4) and (5) are equivalent to

$$\sigma_{\sigma_y(z)|Y} = \sigma_{z|Y} \tag{6}$$

$$\sigma_{\sigma_z(y)|Z} = \sigma_{y|Z}, \tag{7}$$

(see [10, Proposition 8.3]). Let $G_{r,Y}$ be the subgroup of G_r generated by the set $\{\sigma_y \mid y \in Y\}$ and let $G_{r,Z}$ be the subgroup of G_r defined in a similar way. Then (6) and (7) amount to saying that the elements of the same $G_{r,Y}$-orbit on Z determine the same permutation of Y and the elements of the same $G_{r,Z}$-orbit on Y determine the same permutation of Z. The simplest example (called a twisted union in [9]) motivating this definition is obtained by choosing any permutations $\sigma_1, \sigma_2 \in Sym_n$, $n = |X|$, such that $\sigma_i(Y) = Y$ for $i = 1, 2$, and $\sigma_y = \sigma_1$ for every $y \in Y$ and $\sigma_z = \sigma_2$ for every $z \in Z$. An important step supporting this approach was made by Rump [19], who showed that the number of G_r-orbits on X always exceeds 1 if (X, r) is a non-degenerate involutive square free solution with $|X| > 1$.

The following conjectures were formulated by Gateva-Ivanova in [10].

(GI 1) Every set theoretic involutive non-degenerate square free solution (X, r) of cardinality $n \geq 2$ is a multipermutation solution of level $m < n$.

(GI 2) Every multipermutation square free solution of level m and of cardinality $n \geq 2$ is a generalized twisted union of multipermutation solutions of level less than m.

Notice that the square free assumption in (GI 1) is essential. Indeed in [16, Example 8.2.13] an example is given of a set theoretic involutive non-degenerate solution of cardinality 4 that is not a multipermutation solution.

In a recent paper [5] Cedó, Jespers and Okniński investigated these conjectures and obtained the following results. Recall that a set theoretic solution (X, r) is said to be trivial if $r(x_i, x_j) = (x_j, x_i)$ for every i, j. This is equivalent to saying that σ_i is the identity map for every i.

Theorem 5. *[5] Assume that (X, r) is a set theoretic involutive non-degenerate square free solution with abelian associated IYB-group G_r. If r is not trivial then there exist $i, j \in \{1, \ldots, n\}$ such that $\sigma_i = \sigma_j$, $i \neq j$ and x_i, x_j in one G_r-orbit.*

An application is an affirmative answer for (GI 1) in case the IYB-group is abelian. Actually a stronger statement is proved. For this the notion of strong retractability of (X, r) was introduced. Let ρ denote the refining of the relation $\sim$ on X by requesting additionally that the elements are in the same G_r-orbit on X. Then, let $Ret_\rho(X, r) = (X/\rho, \bar{r})$ denote the induced solution. One says in [5] that (X, r) is strongly retractable if there exists $m \geq 1$ such that applying m times the operator Ret_ρ we get a trivial solution.

Note that the IYB group corresponding to the solution $(X/\rho, \bar{r})$ also is abelian if G_r is abelian.

Corollary 4. *[5] Assume that (X, r) is a set theoretic involutive non-degenerate square free solution with abelian associated IYB-group G_r. Then (X, r) is strongly retractable.*

The conjecture can also be confirmed in some cases that are not covered by the previous result.

Theorem 6. *[5] Let (X, r) be a set theoretic involutive non-degenerate square free solution with associated IYB group G_r, such that its generators σ_i, $i = 1, \ldots, n$, are cyclic permutations. Then, (X, r) is strongly retractable. Moreover, if $|X| > 1$ then (X, r) is a generalized twisted union.*

The following result shows that (GI 2) is not true in general.

Theorem 7. *[5] There exists a multipermutation square free solution of level 3 (on 24 generators and with 3 orbits) that it is not a generalized twisted union. Furthermore, the associated IYB group is abelian.*

In a recent paper [2] Cameron and Gateva-Ivanova introduced the notion of strong twisted union. This notion is weaker than that of a generalized twisted union and is defined as follows. A set theoretic involutive non-degenerate solution (X, r) with associated IYB-group G_r is called a *strong twisted union* of solutions if X is a disjoint union $X = X_1 \cup \cdots \cup X_m$ of G_r-invariant non-empty subsets X_i so that $(X_i \cup X_j, r_{X_i \cup X_j})$ is a generalized twisted union. A new conjecture is then stated.

(GI 2a) Every involutive non-degenerate square-free multipermutation solution of level m is a strong twisted union of multipermutation solutions of levels less than m.

This conjecture is confirmed by Cameron and Gateva-Ivanova in the following cases.

Theorem 8. *[2] The statement (GI 2a) holds in the following cases:*

(1) the associated IYB-group is abelian,

(2) the solution is retractable of multipermutation level not exceeding 3.

We finish this section with posing two problems.

Problem 3: Prove (GI 1) for arbitrary IYB-groups.

Problem 4: Classify the multipermutation square free solutions of level m and of cardinality $n \geq 2$ that are a generalized twisted union of multipermutation solutions of level less than m.

4 Algebras of groups of I-type

If $\mathcal{G} = \{(a, \phi(a)) \mid a \in \mathrm{Fa}_n\}$ is a group of I-type then the IYB group $G = \phi(\mathrm{Fa}_n)$ naturally acts on the quotient group $A = \mathrm{Fa}_n/K$, where $K = \{a \in \mathrm{Fa}_n \mid \phi(a) = 1\}$ and we obtain an associated bijective 1-cocycle $G \to A$ with respect to this action. By a result of Etingof and Gelaki [7], this bijective 1-cocycle yields a non-degenerate 2-cocycle on the semi-direct product $H = A \rtimes G$. This has been generalized by Ben David and Ginosar [1] to more general extensions H of A by G with a bijective 1-cocycle from G to A. This construction of Etingof and Gelaki and of Ben David and Ginosar gives rise to a group of central type in the sense of [1], i.e. a finite group H with a 2-cocycle $c \in Z^2(H, \mathbb{C}^*)$ such that the twisted group algebra $\mathbb{C}^c H$ is isomorphic to a full matrix algebra over the complex numbers, or equivalently $H = K/Z(K)$ for a finite group K with an irreducible character of degree $\sqrt{[K : Z(K)]}$. This provides a nice connection between IYB groups and groups of central type that should be investigated.

The above links groups of I-type (hence solutions of the Yang-Baxter equation) and ring theory. Another important link with ring theory is that the semigroup algebra FS of a monoid of I-type S over an arbitrary field F shares many properties with the polynomial algebra in finitely many commuting variables. For example, in [13], it is shown that FS is a domain that satisfies a polynomial identity and that it is a maximal order in its classical ring of quotients. In particular, the group of I-type SS^{-1} is finitely generated abelian-by-finite and torsion free, i.e., it is a Bieberbach group ([13, Theorem 1.7], see also [15, Corollary 8.27]). The homological properties for FS were the main reasons for studying monoids of I-type in [13] and it was inspired by earlier work of Tate and Van den Bergh on Sklyanin algebras.

Clearly the group algebra $F[S^{-1}S]$ is a central localization of FS and hence shares many properties with FS: it is a domain that satisfies a polynomial identity and it is a maximal order in its classical ring of quotients. Group algebras are a fundamental topic of research, as for example, they are a natural link between group theory and ring theory. Clearly the elements of the form fg, with $0 \neq f \in F$ and $g \in G$ are invertible in FG (these are called trivial units). In case G is torsion-free group, then there is a famous conjecture due to Kaplansky: are all units in FG trivial?

Problem 5: Determine the group of invertible elements in the group algebra FG, for a group G of I-type; i.e. verify Kaplansky's conjecture for such group algebras.

Note that if the group of I-type is poly-infinite cyclic then one obtains immediately that the group algebra FG is a domain that has only trivial units, i.e. all units are trivial. So, in case conjecture (GI 1) has a positive answer then Problem 5 only should be investigated for groups of I-type that are not square free. As mentioned earlier, an example of this type is the following ([16, Example 8.2.14]) $G = \langle x_1, x_2, x_3, x_4 \mid x_1x_2 = x_3x_3,\ x_2x_1 = x_4x_4,\ x_1x_3 = x_2x_4,\ x_1x_4 = x_4x_2,\ x_2x_3 = x_3x_1,\ x_3x_2 = x_4x_1\rangle$. This group is not poly-infinite cyclic as it contains the subgroup $\langle a, b \mid a^{-1}b^2a = b^{-2},\ b^{-1}a^2b = a^{-2}\rangle$ of which it is well known that it is not poly-infinite cyclic.

The problem of Kaplansky has been open for many decades and it appears to be notoriously difficult. Now, groups G of I-type are such that their group algebra FG behaves in many ways as commutative polynomial algebras. Hence, in this spirit, one would hope that such groups are ideal candidates for which the Kaplansky problem can be solved. The nice combinatorial nature of G should be of great help.

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