Note di Matematica **23,** n. 1, 2004, 61–70.

Irrational sequences of rational numbers

Jaroslav Hančl[i]
Department of Mathematics, University of Ostrava,
Dvořákova 7, 701 03 Ostrava 1, Czech Republic
hancl@osu.cz

Received: 4/4/2003; accepted: 19/9/2003.

Abstract. The main result of this paper is a criterion for irrational sequences which consist of rational numbers for which the corresponding convergent series does not converge very fast.

Keywords: Irrationality, infinite series, infinite sequences.

MSC 2000 classification: 11J72.

1 Introduction

One approach to proving the irrationality of sums of infinite series is due to Mahler. A nice survey of this kind of result can be found in the book of Nishioka [7]. Other methods are described in Badea [1], [2], Sándor [8] and Duverney [3].

In 1975 in [4] Erdős proved the following theorem.

1 Theorem. *(Erdős) Let $a_1 < a_2 < a_3 < \cdots$ be an infinite sequence of integers satisfying*

$$\limsup_{n\to\infty} a_n^{1/2^n} = \infty$$

and

$$a_n > n^{1+\epsilon}$$

for some fixed $\epsilon > 0$ and $n > n_0(\epsilon)$. Then

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n}$$

is irrational.

In this paper he also introduced the notion of irrational sequences of positive integers and proved that the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ is irrational. Later in [5] the author extended this definition of irrational sequences to sequences of positive real numbers.

[i]Supported by the grant 201/04/0393 of the Czech Grant Agency

2 Definition. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the sum $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ is an irrational number, then the sequence is called irrational. If the sequence is not irrational then it is called rational.

In this paper he also proved the following theorem.

3 Theorem. *Let $\{r_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of positive real numbers such that $\lim_{n\to\infty} r_n = \infty$. Let B be a positive integer, and let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers such that*

$$b_{n+1} \leq r_n^B$$

and

$$a_n \geq r_n^{2^n}$$

hold for every large n. Then

$$A = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$$

is irrational. Furthermore the sequence $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ is irrational.

2 Main results

Theorem 4 deals with a criterion for irrationality of sums of infinite series and sequences consisting of rational numbers which depends on the speed and character of the convergence. In particular it does not depend on arithmetical properties like divisibility.

For notational convenience we define the sequence of iterated logarithm functions by $L_0(x) = x$ and $L_{j+1}(x) = \log_2 L_j(x)$ $j = 0, 1, \ldots$.

4 Theorem. *Let α and ϵ be positive real numbers with $0 < \alpha < \epsilon$. Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that $\{a_n\}_{n=1}^{\infty}$ is nondecreasing. Assume also that for a positive integer s*

$$\limsup_{n\to\infty} L_s^{\frac{1}{2^n}}(a_n) > 1, \tag{1}$$

$$a_n > \left(\prod_{j=0}^{s} L_j(n)\right) L_s^{\epsilon}(n), \tag{2}$$

and

$$b_n < L_s^{\alpha}(a_n) \tag{3}$$

for all sufficiently large positive integers n. Then the sequence $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ are irrational.

5 Remark. Theorem 4 does not hold if $s = 0$. To see this let a_0 be a positive integer greater than 1 and for every positive integer n, $a_n = a_{n-1}^2 - a_{n-1} + 1$. Then $\limsup_{n\to\infty} a_n^{\frac{1}{2^n}} > 1$ and the series $\alpha = \sum_{n=0}^{\infty} \frac{1}{a_n} = \frac{1}{a_0 - 1}$ is a rational number.

6 Example. Let $\{d_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $d_1 = 2^{2^2}$ and for every positive integer n, $d_{n+1} = 2^{2^{2^{d_n}}}$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_1 = d_1$ and for every positive integer n and $k = d_n + 1, \ldots, d_{n+1}$, $a_k = 2^{2.2^{d_n}} d_{n+1} + k$. Then the sequences

$$\{\frac{a_n}{[\sqrt{\log_2 n}]}\}_{n=1}^{\infty}, \quad \{\frac{a_n}{[(\frac{n}{p_n})^{\frac{3}{4}}]}\}_{n=1}^{\infty}, \quad \text{and} \quad \{\frac{a_n}{[(\log_2 q_n)^{\frac{2}{3}}]}\}_{n=1}^{\infty}$$

are irrational, where $[x]$ is the greatest integer less than or equal to x, p_n is the number of primes less than or equal to n, and q_n is the number of divisors of the number n. It follows that for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the series

$$\sum_{n=1}^{\infty} \frac{[\sqrt{\log_2 n}]}{a_n c_n}, \quad \sum_{n=1}^{\infty} \frac{[(\frac{n}{p_n})^{\frac{3}{4}}]}{a_n c_n}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{[(\log_2 q_n)^{\frac{2}{3}}]}{a_n c_n}$$

are irrational numbers.

Open problem. *Let the sequence $\{d_n\}_{n=1}^{\infty}$ be defined as in Example 6. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_1 = d_1$ and for every positive integer n and $k = d_n + 1, \ldots, d_{n+1}$, $a_k = 2^{2d_n} 2^{2^{d_n}} d_{n+1} + k$. It is an open problem to determine if the sequence $\{a_n\}_{n=1}^{\infty}$ is irrational or not.*

3 Proof

PROOF. (of Theorem 4) Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the sequences $\{a_n c_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ also satisfy conditions (1)-(3) and there exists a bijection $\varphi \colon \mathbb{N} \to \mathbb{N}$ such that the sequence $\{a_{\varphi(n)} c_{\varphi(n)}\}_{n=1}^{\infty}$ is nondecreasing. From the definition of the bijection φ and the fact that the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{a_{\varphi(n)} c_{\varphi(n)}\}_{n=1}^{\infty}$ are nondecreasing we obtain that $a_{\varphi(n)} c_{\varphi(n)} \geq a_n$ holds for all $n \in \mathbb{N}$. This implies that the sequences $\{a_{\varphi(n)} c_{\varphi(n)}\}_{n=1}^{\infty}$ and $\{b_{\varphi(n)}\}_{n=1}^{\infty}$ will satisfy (1)-(3) also. Thus it suffices to prove that the series $\beta = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is an irrational number, where the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy all the conditions of Theorem 4. Arguing as in Theorem 3 of [6], for every $\delta > 0$ we find a positive integer n such that

$$(\prod_{j=1}^{n} a_j) \sum_{j=1}^{\infty} \frac{b_{n+j}}{a_{n+j}} < \delta. \tag{4}$$

Inequality (1) and the fact that $s > 0$ (for $s = 0$ Theorem 4 does not hold) imply that

$$\limsup_{n\to\infty} a_n^{\frac{1}{3^n}} = \infty.$$

From this we obtain that for infinitely many n

$$a_{n+1}^{\frac{1}{3^{n+1}}} > \max_{1\le j\le n} a_j^{\frac{1}{3^j}}.$$

Thus

$$a_{n+1}^{\frac{1}{2}} > \big(\max_{1\le j\le n} a_j^{\frac{1}{3^j}}\big)^{\frac{3^{n+1}}{2}} > \big(\max_{1\le j\le n} a_j^{\frac{1}{3^j}}\big)^{\frac{3^{n+1}-1}{2}} =$$

$$\big(\max_{1\le j\le n} a_j^{\frac{1}{3^j}}\big)^{3^n+3^{n-1}+\cdots+1} > \prod_{j=1}^{n} a_j. \tag{5}$$

From (1) we also obtain that there is a real number $c > 1$ such that for infinitely many n

$$L_s^{\frac{1}{2^n}}(a_n) > c > 1. \tag{6}$$

Now the proof falls into two parts.

1. First assume that

$$a_n > 2^n \tag{7}$$

for every sufficiently large n. From (3) we obtain that for every large positive integer n

$$\sum_{j=n+1}^{\infty} \frac{b_j}{a_j} \le \sum_{j=n+1}^{\infty} \frac{L_s^\alpha(a_j)}{a_j} = \sum_{n<j\le \log_2 a_{n+1}} \frac{L_s^\alpha(a_j)}{a_j} + \sum_{\log_2 a_{n+1}<j} \frac{L_s^\alpha(a_j)}{a_j}. \tag{8}$$

Now we will estimate these summands on the right hand side of inequality (8).

For the first summand. The facts that the function $g(x) = L_s^\alpha(x)x^{-1}$ is decreasing for sufficiently large x and the sequence $\{a_n\}_{n=1}^\infty$ is nondecreasing imply that for every sufficiently large positive integer n

$$\sum_{n<j\le \log_2 a_{n+1}} \frac{L_s^\alpha(a_j)}{a_j} \le \frac{L_s^\alpha(a_{n+1}) \log_2 a_{n+1}}{a_{n+1}}. \tag{9}$$

For the second summand. The fact that the function $g(x) = L_s^\alpha(x)x^{-1}$ is decreasing for sufficiently large x and (7) imply that for every sufficiently large positive integer n

$$\sum_{\log_2 a_{n+1}\le j} \frac{L_s^\alpha(a_j)}{a_j} \le \sum_{\log_2 a_{n+1}\le j} \frac{L_s^\alpha(2^j)}{2^j} \le \sum_{\log_2 a_{n+1}\le j} \frac{j^\alpha}{2^j} \le \frac{\log_2^{\alpha+1} a_{n+1}}{a_{n+1}}. \tag{10}$$

From (8), (9) and (10) we obtain that for every sufficiently large positive integer n

$$\sum_{j=n+1}^{\infty} \frac{b_j}{a_j} \le \sum_{n<j\le \log_2 a_{n+1}} \frac{L_s^{\alpha}(a_j)}{a_j} + \sum_{\log_2 a_{n+1}<j} \frac{L_s^{\alpha}(a_j)}{a_j} \le$$

$$\frac{L_s^{\alpha}(a_{n+1})\log_2 a_{n+1}}{a_{n+1}} + \frac{\log_2^{\alpha+1} a_{n+1}}{a_{n+1}} \le \frac{\log_2^{\alpha+2} a_{n+1}}{a_{n+1}}. \tag{11}$$

The inequalities (5) and (11) yield

$$\left(\prod_{j=1}^{n} a_j\right) \sum_{j=1}^{\infty} \frac{b_{n+j}}{a_{n+j}} < a_{n+1}^{\frac{1}{2}} \frac{\log_2^{\alpha+2} a_{n+1}}{a_{n+1}} < a_{n+1}^{-\frac{1}{4}}$$

for infinitely many n and (4) follows.

2. Now assume

$$a_n < 2^n \tag{12}$$

for infinitely many n. A brief sketch of the proof is following. First we prove that for some real number K with $1 < K < 2$ there exist infinitely many positive integers t such that $a_t > 2^{K^t} \prod_{j=1}^{t-1} a_j$. Then we will estimate $\sum_{j=1}^{\infty} \frac{b_{t+j}}{a_{t+j}}$ and finally we will prove (4) for every positive real number δ.

Let k be a sufficiently large positive integer satisfying (6). It follows that

$$L_s(a_k) > c^{2^k} = 2^{(\log_2 c)2^k}. \tag{13}$$

Let k_0 be the largest positive integer not greater than k and satisfying (12). Let t be the least positive integer greater than k_0 such that

$$a_{t+1}^{\frac{1}{2^{t+1}}} > \left(1 + \frac{1}{t^2}\right) \max_{k_0<j\le t} a_j^{\frac{1}{2^j}}. \tag{14}$$

(By the way the factor $(1 + \frac{1}{t^2})$ can be substitute by $(1 + \frac{1}{t^{\gamma}})$ where $1 < \gamma$.) Such a number t must exist with $t < k$, otherwise from (12), (6) and (13) we obtain that

$$1 < c < L_s^{\frac{1}{2^k}}(a_k) < a_k^{\frac{1}{2^k}} \le$$

$$\left(1 + \frac{1}{(k-1)^2}\right) \max_{k_0<j\le k-1} a_j^{\frac{1}{2^j}} \le \cdots \le \prod_{j=k_0}^{k} \left(1 + \frac{1}{j^2}\right) a_{k_0}^{\frac{1}{2^{k_0}}} = D(k). \tag{15}$$

From (12) and the definition of the number k_0 we obtain that $a_{k_0}^{\frac{1}{2^{k_0}}} \to 1$ as $k_0 \to \infty$ with k. This and the fact that the product $\prod_{j=1}^{\infty}(1 + \frac{1}{j^2})$ is convergent imply $D(k) \to 1$ as $k_0 \to \infty$ with k, a contradiction with (15).

It follows that for every $r = k_0, k_0+1, \ldots, t-1$

$$a_{r+1}^{\frac{1}{2^{r+1}}} \le (1+\frac{1}{r^2}) \max_{k_0<j\le r} a_j^{\frac{1}{2^j}} \le$$

$$\prod_{j=k_0}^{r} (1+\frac{1}{j^2}) a_{k_0}^{\frac{1}{2^{k_0}}} \le \prod_{j=k_0}^{k} (1+\frac{1}{j^2}) a_{k_0}^{\frac{1}{2^{k_0}}} = D(k).$$

Thus

$$a_{r+1} < (D(k))^{2^{r+1}} = 2^{\log_2(D(k))2^{r+1}}. \tag{16}$$

The fact that the sequence $\{a_n\}_{n=1}^{\infty}$ is nondecreasing and the definition of the number k_0 imply that for every $j = 1, 2, \ldots, k_0$

$$a_j \le a_{k_0} < 2^{k_0}.$$

Hence

$$\prod_{j=1}^{k_0} a_j < 2^{k_0^2}. \tag{17}$$

From this and (16) we obtain that

$$\prod_{j=1}^{t} a_j = \prod_{j=1}^{k_0} a_j \prod_{j=k_0+1}^{t} a_j \le 2^{k_0^2} \prod_{j=k_0+1}^{t} (D(k))^{2^j} =$$

$$2^{k_0^2} (D(k))^{\sum_{j=k_0+1}^{t} 2^j} \le 2^{k_0^2} 2^{\log_2(D(k))2^{t+1}}. \tag{18}$$

On the other hand (14) implies

$$a_{t+1} > (1+\frac{1}{t^2})^{2^{t+1}} (\max_{k_0<j\le t} a_j^{\frac{1}{2^j}})^{2^{t+1}} >$$

$$(1+\frac{1}{t^2})^{2^{t+1}} (\max_{k_0<j\le t} a_j^{\frac{1}{2^j}})^{2^t+2^{t-1}+\cdots+1} > (1+\frac{1}{t^2})^{2^{t+1}} \prod_{j=k_0+1}^{t} (\max_{k_0<j\le t} a_j^{\frac{1}{2^j}})^{2^j}.$$

From this and (17) we obtain that

$$a_{t+1} > (1+\frac{1}{t^2})^{2^{t+1}} \prod_{j=k_0+1}^{t} (\max_{k_0<j\le t} a_j^{\frac{1}{2^j}})^{2^j} > (1+\frac{1}{t^2})^{2^{t+1}} (\prod_{j=1}^{t} a_j)(\prod_{j=1}^{k_0} a_j)^{-1} >$$

$$(1+\frac{1}{t^2})^{2^{t+1}} (\prod_{j=1}^{t} a_j) 2^{-k_0^2}. \tag{19}$$

We have

$$\sum_{j=1}^{\infty} \frac{b_{t+j}}{a_{t+j}} = \sum_{j=t+1}^{k-1} \frac{b_j}{a_j} + \sum_{k \le j \le a_k^{\frac{1}{2}}} \frac{b_j}{a_j} + \sum_{a_k^{\frac{1}{2}} < j} \frac{b_j}{a_j}. \tag{20}$$

We now estimate the summands on the right hand side of equation (20).

For the first summand. The facts that $a_k > 2^k$ and k_0 is the largest positive integer less than to k and satisfying (12) imply that for every $j = k_0 + 1, \dots, k$

$$a_j \ge 2^j.$$

From this and (11) we obtain

$$\sum_{j=t+1}^{k-1} \frac{b_j}{a_j} \le \frac{\log_2^{\alpha+2} a_{t+1}}{a_{t+1}}. \tag{21}$$

For the second summand. Inequality (3) and the fact that the function $g(x) = L_s^{\alpha}(x)x^{-1}$ is decreasing for sufficiently large x imply that for every k large enough

$$\sum_{k \le j \le a_k^{\frac{1}{2}}} \frac{b_j}{a_j} \le a_k^{\frac{1}{2}} \frac{L_s^{\alpha}(a_k)}{a_k} < a_k^{-\frac{1}{4}}. \tag{22}$$

For the third summand. From (2) we obtain that for k large enough

$$\sum_{a_k^{\frac{1}{2}} < n} \frac{b_n}{a_n} \le \sum_{a_k^{\frac{1}{2}} < n} \frac{L_s^{\alpha}(a_n)}{a_n} \le \sum_{a_k^{\frac{1}{2}} < j} \frac{L_s^{\alpha}(\prod_{j=0}^{s} L_j(n))L_s^{\epsilon}(n))}{(\prod_{j=0}^{s} L_j(n))L_s^{\epsilon}(n)} <$$

$$2\int_{a_k^{\frac{1}{2}}-1}^{\infty} \frac{dx}{(\prod_{j=0}^{s} L_j(x))L_s^{\epsilon-\alpha}(x)} = \frac{2}{(\epsilon-\alpha)L_s^{\epsilon-\alpha}(a_k^{\frac{1}{2}}-1)} < \frac{1}{L_s^{\frac{\epsilon-\alpha}{3}}(a_k)} \tag{23}$$

and so (13), (20), (21), (22) and (23) imply that

$$\sum_{j=1}^{\infty} \frac{b_{t+j}}{a_{t+j}} \le \frac{\log_2^2 a_{t+1}}{a_{t+1}} + a_k^{-\frac{1}{4}} + L_s^{-\frac{\epsilon-\alpha}{3}}(a_k) \le \frac{\log_2^2 a_{t+1}}{a_{t+1}} + \frac{1}{L_s^{\frac{\epsilon-\alpha}{4}}(a_k)} \le$$

$$\frac{\log_2^2 a_{t+1}}{a_{t+1}} + \frac{1}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}}. \tag{24}$$

Now we consider two cases.

2a. Let us assume

$$a_{t+1} < 2^{2^{t+1}}. \tag{25}$$

From (24) we obtain

$$(\prod_{j=1}^{t} a_j)\sum_{j=1}^{\infty}\frac{b_j}{a_j} \le \frac{(\prod_{j=1}^{t} a_j)(\log_2^2 a_{t+1})}{a_{t+1}} + \frac{\prod_{j=1}^{t} a_j}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}}. \tag{26}$$

We will estimate these summands on the right hand side of inequality (26).

For the first summand. Inequality (19) implies

$$\frac{(\prod_{j=1}^{t} a_j)\log_2^2 a_{t+1}}{a_{t+1}} \le \frac{\log_2^2 a_{t+1}}{(1+\frac{1}{t^2})^{2^{t+1}}2^{-k_0^2}}.$$

From this and (25) we obtain

$$\frac{(\prod_{j=1}^{t} a_j)\log_2^2 a_{t+1}}{a_{t+1}} \le \frac{\log_2^2 a_{t+1}}{(1+\frac{1}{t^2})^{2^{t+1}}2^{-k_0^2}} < \frac{\log_2^2 2^{2^{t+1}}}{(1+\frac{1}{t^2})^{2^{t+1}}2^{-k_0^2}} =$$

$$= \frac{2^{2(t+1)}}{(1+\frac{1}{t^2})^{2^{t+1}}2^{-k_0^2}} = 2^{2(t+1)+k_0^2-\log_2(1+\frac{1}{t^2})2^{t+1}} \le 2^{2(t+1)+t^2-\frac{1}{t^3}2^{t+1}}.$$

This and the fact that

$$\lim_{x\to\infty}(2(x+1)+x^2-\frac{1}{x^3}2^{x+1}) = -\infty$$

imply that

$$\frac{(\prod_{j=1}^{t} a_j)\log_2^2 a_{t+1}}{a_{t+1}} < \frac{\delta}{2}. \tag{27}$$

For the second summand. From (18) and the fact that $t \le k$ we obtain that

$$\frac{\prod_{j=1}^{t} a_j}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}} \le \frac{2^{k_0^2}2^{\log_2(D(k))2^{t+1}}}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}} \le \frac{2^{k_0^2+\log_2(D(k))2^{k+1}}}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}} =$$

$$2^{k_0^2-(\frac{\epsilon-\alpha}{4}\log_2 c-2\log_2(D(k)))2^k} \le 2^{k^2-(\frac{\epsilon-\alpha}{4}\log_2 c-2\log_2(D(k)))2^k}. \tag{28}$$

If k tends to infinity then $D(k)$ tends to 1. Hence

$$\lim_{x\to\infty}(x^2-(\frac{\epsilon-\alpha}{4}\log_2 c-\log_2(D(x)))2^x) = -\infty.$$

From this and (28) we obtain that

$$\frac{\prod_{j=1}^{t} a_j}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}} < \frac{\delta}{2}. \tag{29}$$

Inequalities (26), (27) and (29) imply (4).

2b. Let us assume

$$a_{t+1} \geq 2^{2^{t+1}}. \tag{30}$$

From (24) we obtain

$$(\prod_{j=1}^{t} a_j) \sum_{j=1}^{\infty} \frac{b_j}{a_j} \leq \frac{(\prod_{j=1}^{t} a_j) \log_2^2 a_{t+1}}{a_{t+1}} + \frac{\prod_{j=1}^{t} a_j}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}}. \tag{31}$$

Now we will estimate these summands on the right hand side of inequality (31).

For the first summand. Inequality (18) and the fact that $\log_2^2 x < x^{\frac{1}{2}}$ for every sufficiently large x imply

$$\frac{(\prod_{j=1}^{t} a_j) \log_2^2 a_{t+1}}{a_{t+1}} \leq \frac{\prod_{j=1}^{t} a_j}{a_{t+1}^{\frac{1}{2}}} \leq \frac{2^{k_0^2} 2^{\log_2(D(k))2^{t+1}}}{a_{t+1}^{\frac{1}{2}}}.$$

From this and (30) we obtain that

$$\frac{(\prod_{j=1}^{t} a_j) \log_2^2 a_{t+1}}{a_{t+1}} \leq \frac{2^{k_0^2} 2^{\log_2(D(k))2^{t+1}}}{a_{t+1}^{\frac{1}{2}}} \leq \frac{2^{k_0^2} 2^{(\log_2(D(k)))2^{t+1}}}{2^{2^t}} =$$

$$2^{k_0^2 + (2\log_2(D(k))-1)2^t} \leq 2^{t^2 - (1-2\log_2(D(k)))2^t} \tag{32}$$

If k tends to infinity then t tends to infinity and $D(k)$ tends to 1. Hence

$$\lim_{k\to\infty} (t^2 - (1 - 2(\log_2(D(k)))2^t) = -\infty.$$

From this and (32) we obtain that

$$\frac{(\prod_{j=1}^{t} a_j) \log_2^2 a_{t+1}}{a_{t+1}} < \frac{\delta}{2}. \tag{33}$$

For the second summand. Inequality (18) and the fact that $k_0 \leq t \leq k$ imply

$$\frac{\prod_{j=1}^{t} a_j}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}} \leq \frac{2^{k_0^2} 2^{(\log_2(D(k)))2^{t+1}}}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}} \leq \frac{2^{k^2} 2^{(\log_2(D(k)))2^{k+1}}}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}} =$$

$$2^{k^2 + (2\log_2(D(k)) - \frac{\epsilon-\alpha}{4}\log_2 c)2^k}. \tag{34}$$

If k tends to infinity then $D(k)$ tends to 1. Hence

$$\lim_{x\to\infty} (x^2 + (2\log_2(D(x)) - \frac{\epsilon-\alpha}{4}(\log_2 c))2^x) = -\infty.$$

From this and (34) we obtain that

$$\frac{\prod_{j=1}^{t} a_j}{2^{\frac{\epsilon-\alpha}{4}(\log_2 c)2^k}} < \frac{\delta}{2}. \tag{35}$$

Inequalities (31), (33) and (35) imply (4). This completes the proof of Theorem 4. QED

Acknowledgements. The author thanks Radhakrishnan Nair from the Department of Mathematical Sciences, University of Liverpool, James E. Carter of the College of Charleston and referee for their help with this article.

References

[1] C. Badea: *The irrationality of certain infinite series*, Glasgow Math. J. **29** (1987), 221–228.

[2] C. Badea: *A theorem on irrationality of infinite series and applications*, Acta Arith. LXIII, **4** (1993), 313–323.

[3] D. Duverney: *Sur les series de nombres rationnels a convergence rapide*, C. R. Acad. Sci. Paris Sér. I Math. **328**, n. 7 (1999), 553–556.

[4] P. Erdős: *Some problems and results on the irrationality of the sum of infinite series*, J. Math. Sci. **10** (1975), 1–7.

[5] J. Hančl: *Criterion for irrational sequences*, J. Number Theory **43**, no.1 (1993), 88–92.

[6] J. Hančl: *Irrational rapidly convergent series*, Rend. Sem. Mat. Univ. Padova **107** (2002), 225–231.

[7] K. Nishioka: *Mahler Functions and Transcendence*, Lecture Notes in Math 1631, Springer, (1996).

[8] J. Sándor: *Some classes of irrational numbers*, Studia Univ. Babeş-Bolyai, Math. XXIX (1984), 3–12.