Note di Matematica **24,** n. 2, 2005, 1–7.

Canonical coordinate systems and exponential maps of n-loop

Henrietta Tomán
Faculty of Informatics, University of Debrecen
Egyetem tér 1. H-4032 Debrecen, Hungary
toman@inf.unideb.hu

Received: 02/09/2004; accepted: 27/10/2004.

Abstract. This paper is devoted to the study of canonical coordinate systems and the corresponding exponential maps of n-ary differentiable loops and to the discussion of their differentiability properties. Canonical coordinate systems can be determined by the canonical normal form of the power series expansion of the n-th power map $x \to x \circ x \circ \cdots \circ x \circ x$.

Keywords: loops, n-ary systems, local Lie groups

MSC 2000 classification: 20N05, 20N15, 22E05

1 Introduction

The canonical coordinate systems of Lie groups are important tools for the investigation of local properties of group manifolds. They can be generalized for non-associative differentiable loops. The first study of the expansion of analytical loop multiplication in a canonical coordinate system using formal power series was given in the paper [1] by M. A. Akivis in 1969, (cf. [6, Chapter 2]). The convergence conditions of power series expansions of loop multiplications were investigated later in [2] (1986). E. N. Kuzmin in [9] (1971) treated the local Lie theory of analytic Moufang loops using power series expansion in canonical coordinate systems and gave a generalization of the classical Campbell-Hausdorff formula. V. V. Goldberg introduced canonical coordinates using power series expansions in local analytic n-ary loops, (cf. [6, Chapter 3]).

As it is well-known differentiable groups are automatically (analytic) Lie groups. But in the case of non-associative loop theory the class of $\mathcal{C}^k$-differentiable loops contains the class of $\mathcal{C}^l$-differentiable loops for any $k < l; k, l = 0, 1, \dots, \infty$, as a proper subclass (cf. P. T. Nagy – K. Strambach [10] (2002)).

The theory of normal forms of $\mathcal{C}^\infty$-differentiable n-ary loop multiplications has been investigated in the paper of J-P. Dufour and P. Jean [4], (1985) by the application of S. Sternberg's linearization theorem to the coordinate representation of $n+1$-webs, which are the differential geometric structures determined by the level manifolds of n-ary loop multiplications and its inverse operations. J. Kozma in [8] (1987) defined the canonical coordinates of binary $\mathcal{C}^\infty$-loops by the linearizing coordinate systems of the square map $x \to x \circ x$. For Lie groups these canonical coordinate systems coincide with the classical systems defined with help of one-parameter subgroups.

Now, we consider a natural generalization of Kozma's construction to n-ary $\mathcal{C}^k$-differentiable loops. According to Sternberg's linearization theorem the linearizing coordinate system of the n-th power map $x \to x \circ x \circ \cdots \circ x \circ x$ has the same differentiability property as the n-ary loop multiplication map if $k \geq 2$. Hence in the following we will assume that the differentiability class $\mathcal{C}^k$ of the investigated n-ary loops satisfies $k \geq 2$. Similar construction for canonical coordinate systems was introduced by V. V. Goldberg in [6, Chapter 3], in the case of analytic n-loop multiplications using formal power series expansions.

The author expresses her sincere thanks to Professor Péter T. Nagy for his valuable suggestion and help.

2 Canonical coordinate systems of n-loops

1 Definition. Let H be a differentiable manifold of class $\mathcal{C}^k$, let $e \in H$ be a given element and let $m\colon H^n \to H$, $\delta_i\colon H^n \to H$ be differentiable maps of class $\mathcal{C}^k$, where $i = 1, \dots, n$. Then $\mathcal{H} = (H, e, m, \delta_1, \dots, \delta_n)$ is called a $\mathcal{C}^k$-*differentiable n-ary loop* (or shortly *n-loop*) with unit element e if the multiplication m and the i-th divisions δ_i, $i = 1, \dots n$, satisfy the following identities:

(1) $m(\overset{(1)}{e}, \dots, \overset{(i-1)}{e}, \overset{(i)}{a}, \overset{(i+1)}{e}, \dots, \overset{(n)}{e}) = a$, for all $a \in H$, $(1 \leq i \leq n)$, where $\overset{(i)}{x}$ means that the i-th argument has the value x,

(2) $m(a_1, a_2, \dots, a_{i-1}, \delta_i(b, a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n), a_{i+1}, \dots, a_n) = b$ for all $a_i \in H$, $(1 \leq i \leq n), b \in H$,

(3) $\delta_i(m(a_1, a_2, \dots, a_n), a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n) = a_i$ for all $a_i \in H$, $(1 \leq i \leq n), b \in H$.

2 Definition. If H is a differentiable manifold of class $\mathcal{C}^k$, $e \in H$ is a given element and $m\colon H^n \to H$, $\delta_i\colon H^n \to H$ are differentiable maps of class $\mathcal{C}^k$, $i = 1, \dots n$, which are defined in a neighbourhood of $e \in H$, then $\mathcal{H} = (H, e, m, \delta_1, \dots, \delta_n)$ is called a $\mathcal{C}^k$-*differentiable local n-loop* with unit element e, provided that the multiplication m and the i-th divisions δ_i, $i = 1, \dots n$ satisfy the following identities:

(1) $m(\overset{(1)}{e}, \dots, \overset{(i-1)}{e}, \overset{(i)}{a}, \overset{(i+1)}{e}, \dots, \overset{(n)}{e}) = a$, for all $a \in H$, $(1 \leq i \leq n)$, where $\overset{(i)}{x}$ means that the i-th argument has the value x,

(2) $m(a_1,a_2,\dots,a_{i-1},\delta_i(b,a_1,a_2,\dots,a_{i-1},a_{i+1},\dots,a_n),a_{i+1},\dots,a_n)=b$ for all $a_i\in H$, $(1\le i\le n), b\in H$,

(3) $\delta_i(m(a_1,a_2,\dots,a_n),a_1,a_2,\dots,a_{i-1},a_{i+1},\dots,a_n)=a_i$ for all $a_i\in H$, $(1\le i\le n), b\in H$

in a neighbourhood of $e\in H$.

3 Definition. Let $\mathcal{H}=(H,e,m,\delta_1,\dots,\delta_n)$ be a $\mathcal{C}^k$-differentiable local n-loop. A coordinate map $\varphi\colon U\to\mathbb{R}^q$ of class $\mathcal{C}^k$ of the open neighbourhood $U\subset H$ of $e\in H$ into the coordinate space $\mathbb{R}^q$ is called a *canonical coordinate system* of $\mathcal{H}$ if $\varphi(e)=0$ and the coordinate function

$$M=\varphi\circ m\circ(\varphi^{-1}\times\cdots\times\varphi^{-1})\colon\varphi(U)\times\cdots\times\varphi(U)\to\mathbb{R}^q$$

of the multiplication map $m\colon H^n\to H$ satisfies

$$M(x,x,\dots,x)=n\,x$$

for all $x\in\varphi(U)$.

We will need the following assertions in the investigation of canonical coordinate systems:

4 Lemma. *Let be $k\ge 2$ and ϕ a local $\mathcal{C}^k$-diffeomorphism of $\mathbb{R}^q$ keeping $0\in\mathbb{R}^q$ fixed which is defined in some neighbourhood of $0\in\mathbb{R}^q$ and let $\phi_*|_{(0)}$ denote the tangent map of ϕ at $0\in\mathbb{R}^q$. We assume that ϕ satisfies $\phi_*|_{(0)}=\lambda\,\mathrm{id}_{\mathbb{R}^q}$ with $\lambda\ne 0,1,-1$. Then there exists a unique local $\mathcal{C}^k$-diffeomorphism ρ of $\mathbb{R}^q$ keeping $0\in\mathbb{R}^q$ fixed such that $\rho\cdot\phi\cdot\rho^{-1}=\phi_*|_{(0)}$ and $\rho_*|_{(0)}=\mathrm{id}_{\mathbb{R}^q}$.*

Proof. The existence of a local $\mathcal{C}^k$-diffeomorphism ρ of $\mathbb{R}^q$ satisfying the conditions of the assertion follows from Sternberg's Linearization Theorem for local contractions (cf. [11]) since either the map ϕ or its inverse ϕ^{-1} is a local contraction, the minimum and maximum of eigenvalues of its tangent map coincide, $k\ge 2$ and it satisfies the so called resonance condition $\lambda\ne\lambda^m$ for any $m>1$. The unicity of the map ρ follows from the ideas of the proof of Sternberg's Theorem, since the difference of two solutions must be a fixed point of a contractive operator on a linear space of differentiable maps. Hence the difference of these solution is 0. *QED*

5 Lemma. *Let κ be a differentiable map of a star shaped neighbourhood $W\subset\mathbb{R}^p$ into $\mathbb{R}^q$ with $\kappa(0)=0$. If there exists a real number $0<r<1$ such that $\kappa(r\,x)=r\,\kappa(x)$ holds for all $x\in W$ then κ is the restriction of a linear map.*

Proof. Since the map $\kappa\colon W\to\mathbb{R}^p$ is differentiable one can define the continuous map $\omega\colon W\to\mathbb{R}^p$ satisfying

$$\kappa(x)=\kappa_*|_{(0)}(x)+\|x\|\omega(x),\quad \omega(0)=0.$$

Hence

$$\kappa(r\,x) = r\,(\kappa_*|_{(0)}(x) + \|x\|\omega(r\,x)) = r\,\kappa(x) = r\,(\kappa_*|_{(0)}(x) + \|x\|\omega(x)).$$

It follows $\omega(x) = \omega(r^m\,x)$ for any natural number $m \in \mathbb{N}$ and hence

$$\omega(x) = \lim_{m\to\infty} \omega(r^m) = \omega(0) = 0$$

for all $x \in W$. QED

6 Theorem. *For any $\mathcal{C}^k$-differentiable local n-loop $\mathcal{H} = (H, e, m, \delta_1, \ldots, \delta_n)$ with $k \geq 2$ there exists a canonical coordinate system.*

If (U, φ) is a canonical coordinate system of $\mathcal{H}$ then for any linear map $\lambda : \mathbb{R}^q \to \mathbb{R}^q$ the pair $(U, \lambda \circ \varphi)$ is a canonical coordinate system of $\mathcal{H}$, too.

If $\varphi \colon U \to \mathbb{R}^q$ and $\psi \colon U \to \mathbb{R}^q$ are the coordinate maps of canonical coordinate systems of $\mathcal{H}$ defined on the same neighbourhood U then $\varphi \circ \psi^{-1}$ is the restriction of a linear map $\mathbb{R}^q \to \mathbb{R}^q$.

Proof. Let $(\bar{U}, \bar{\varphi})$ be a coordinate system of $\mathcal{H}$, let $\bar{M}$ be the coordinate function of the local n-loop multiplication m with respect to $(\bar{U}, \bar{\varphi})$. Now, we introduce the map $\bar{G} \colon \bar{\varphi}(\bar{U}) \to \mathbb{R}^q$ defined by $\bar{G}(x) = \bar{M}(x, x, \ldots, x)$. Clearly one has $\bar{G}(0) = 0$. Since $\bar{M}(0, \ldots, 0, x, 0, \ldots, 0) = x$ the tangent map $\bar{G}_*|_0 \colon \mathbb{R}^q \to \mathbb{R}^q$ of $\bar{G}$ at the point 0 satisfies $\bar{G}_*|_0 = n\,\mathrm{id}_{\mathbb{R}^q}$. The map $\bar{G}$ is of class C^k in a neighborhood of 0 and hence it has an inverse map in a neighborhood of 0 of the same class $\mathcal{C}^k$. We can apply Lemma 4 for $\bar{G}^{-1}$. It follows that there exists a local $\mathcal{C}^k$-diffeomorphism ρ keeping $0 \in \mathbb{R}^q$ fixed such that $(\rho \circ \bar{G} \circ \rho^{-1})_*|_0 = \rho \circ \bar{G} \circ \rho^{-1}$. We consider the composed map $\varphi = \rho \circ \bar{\varphi}$ as the coordinate map of a new coordinate system (U, φ) with a suitable neighborhood U. The coordinate function of the multiplication map $m \colon H^n \to H$ satisfies $M = \rho \circ \bar{M} \circ \rho^{-1}$. Let Q be the following function

$$Q \colon x \mapsto Q(x) = (x, x, \ldots, x) \colon \mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}^q \times \cdots \times \mathbb{R}^q.$$

Then we have the equation

$$G = M \circ Q = (\rho \circ \bar{M} \circ \rho^{-1})(\rho \circ Q \circ \rho^{-1}) = \rho \circ \bar{G} \circ \rho^{-1} = (\rho \circ \bar{G} \circ \rho^{-1})_*|_0 = n\,\mathrm{id}_{\mathbb{R}^q}.$$

Hence (U, φ) is a canonical coordinate system of $\mathcal{H}$.

For a canonical coordinate system (U, φ) of the local n-loop $\mathcal{H}$ the coordinate function

$$M = \varphi \circ m \circ (\varphi^{-1} \times \cdots \times \varphi^{-1}) \colon \varphi(U) \times \cdots \times \varphi(U) \to \mathbb{R}^q$$

of the multiplication map $m \colon H^n \to H$ satisfies $M(x, x, \ldots, x) = n\,x$ for all $x \in \varphi(U)$. Hence for arbitrary linear map $\lambda \colon \mathbb{R}^n \to \mathbb{R}^n$ one has

$$\lambda \circ M(\lambda^{-1}y, \ldots, \lambda^{-1}y) = \lambda(n\,\lambda^{-1}y) = n\,y, \quad y \in \lambda \circ \varphi(U).$$

It follows that $(U, \psi = \lambda \circ \varphi)$ is also a canonical coordinate system of $\mathcal{H}$.

Let (U, φ) and (U, ψ) be canonical coordinate systems of $\mathcal{H}$ given on the same neighbourhood U and let M_φ and M_ψ be the coordinate functions of the multiplication map $m\colon H^n \to H$. We denote $\kappa = \varphi \circ \psi^{-1}\colon \psi(U) \to \varphi(U)$. For all $x \in \varphi(U)$ and $y \in \psi(U)$ we have

$$M_\varphi(x, x, \dots, x) = n\,x \quad \text{and} \quad M_\psi(y, y, \dots, y) = n\,y.$$

Since

$$M_\varphi(\kappa(y), \kappa(y), \dots, \kappa(y)) = \kappa(M_\psi(y, y, \dots, y))$$

we obtain $n\,\kappa(y) = \kappa(n\,y)$. Putting $z = n\,y$ we get $\kappa(r\,z) = r\,\kappa(z)$ for all $z \in \psi(U)$, where $r = \frac{1}{n}$. It follows by Lemma 5 that the map $\kappa = \psi \circ \varphi^{-1}$ is the restriction of a linear map. QED

7 Example. The local non-associative loop-multiplication $f(x, y) = x + y + x^2 y(x - y)$ is defined in a canonical coordinate system.

3 Exponential map

There are different natural possibilities for the definition of the exponential map $W \to H$ with $0 \in W \subset T_e H$ of $\mathcal{C}^k$-differentiable local n-loops. One of them is analogous to the usual construction in Lie group theory, namely the map exp could be determined by the integral curves of vector fields defined by the i-th translations of tangent vectors at the unit element of the n-loop. In binary Lie groups these curves are 1-parameter subgroups, but for smooth loops it is not always the case (cf. J. Kozma [8]). An other disadvantage of such construction is that one can expect only $\mathcal{C}^{k-1}$-differentiability of the the map $W \to H$ with $0 \in W \subset T_e H$ which is determined by integral curves of $\mathcal{C}^{k-1}$-differentiable vector fields defined by the i-th translations of tangent vectors.

An alternative natural possibility for the definition of the exponential map is given by using the construction of canonical coordinate systems studied in the previous section.

8 Theorem. *Let $\mathcal{H} = (H, e, m, \delta_1, \dots, \delta_n)$ be a $\mathcal{C}^k$-differentiable local n-loop with $k \geq 2$. There exists a unique local $\mathcal{C}^k$-diffeomorphism* $\exp\colon W \to H$*, where W is a neighbourhood of $0 \in T_e H$, such that the following conditions hold:*

(i) $\exp(0) = e$ *and* $\exp(n\,x) = m(\exp(x), \dots, \exp(x))$,

(ii) $\exp_*|_0 = \mathrm{id}_{T_e H}$.

Proof. Let $\varphi\colon U \to \mathbb{R}^q$ be the coordinate map of a canonical coordinate system (U, φ) of the local n-loop $\mathcal{H}$. According to Theorem 6 $(U, \varphi_*|_0^{-1} \circ \varphi)$ is also a canonical coordinate system of $\mathcal{H}$ where the vector space T_eH is the coordinate space and $\varphi_*|_0^{-1} \circ \varphi \colon U \to T_eH$ is the coordinate map. Let $W \subset \varphi_*|_0^{-1} \circ \varphi(U)$ be a neighbourhood of $0 \in T_eH$. Then the coordinate function

$$M = \varphi_*|_0^{-1} \circ \varphi \circ m \circ ((\varphi_*|_0^{-1} \circ \varphi)^{-1} \times \cdots \times (\varphi_*|_0^{-1} \circ \varphi)^{-1}) \colon W \times \cdots \times W \to \ T_eH$$

of the multiplication map $m\colon H^n \to H$ satisfies $M(x,\ldots,x) = n\,x$, or equivalently

$$m(\varphi^{-1} \circ \varphi_*|_0(x), \ldots, \varphi^{-1} \circ \varphi_*|_0(x)) = \varphi^{-1} \circ \varphi_*|_0(nx)$$

for any $x \in W$. Moreover one has $(\varphi^{-1} \circ \varphi_*|_0)_*|_0 = \mathrm{id}_{T_eH}$. Hence we can define $\exp = \varphi^{-1} \circ \varphi_*|_0$ and this map satisfies the conditions given in the assertion.

Let us assume that the map $\widetilde{\exp}\colon W \to H$ fulfills the conditions (i) and (ii). Then $(\widetilde{\exp}(W), \widetilde{\exp}^{-1})$ is a canonical coordinate system of the n-loop $\mathcal{H}$ and according to the previous theorem the map $\widetilde{\exp}^{-1} \circ \exp\colon W \to T_eH$ is the restriction of a linear map $\alpha\colon T_eH \to T_eH$. Since both of the maps $\widetilde{\exp}$ and exp satisfy the condition (ii) the linear map $\alpha\colon T_eH \to T_eH$ must be the identity map. Hence $\widetilde{\exp} = \exp\colon W \to H$ which proves that the map $\exp\colon W \to H$ is determined uniquely. QED

9 Theorem. *Let $\mathcal{H} = (H, e, m, \delta_1, \ldots, \delta_n)$ and $\mathcal{H}' = (H', e', m', \delta_1', \ldots, \delta_n')$ be $\mathcal{C}^k$-differentiable local n-loops and let $\exp\colon W \to H$, $\exp'\colon W' \to H'$ be the corresponding exponential maps, where $W \subset T_eH$ and $W' \subset T_{e'}H'$.*

If $\alpha\colon \mathcal{H} \to \mathcal{H}'$ is a continuous local homomorphism then the composed map $\exp'^{-1} \circ \alpha \circ \exp\colon W \to T_{e'}H'$ is locally linear.

Proof. Let us consider the $\mathcal{C}^k$-differentiable binary local loops $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}'}$ which are determined by the multiplication and division maps of $\mathcal{H}$ and $\mathcal{H}'$ in such a way that in the multiplication and division functions the j-th variable $(j \geq 3)$ is replaced by the identity element $e \in H$ and $e' \in H'$ respectively. The map $\alpha\colon H \to H'$ is clearly a continuous local loop homomorphism. According to the result of R. Bödi and L. Kramer [3] the map $\alpha\colon H \to H'$ is $\mathcal{C}^k$-differentiable. Hence according to Lemma 5 the identity

$$\exp'^{-1} \circ \alpha \circ \exp(nx) = n \exp'^{-1} \circ \alpha \circ \exp(x),$$

or equivalently

$$\exp'^{-1} \circ \alpha \circ \exp(ry) = r \exp'^{-1} \circ \alpha \circ \exp(y)$$

with $y = nx$ and $r = \frac{1}{n}$, implies the assertion. QED

References

[1] M. A. Akivis: *The canonical expansions of the equations of a local analytic quasigroup*, (Russian), Dokl. Akad. Nauk SSSR 188 (1969), 967–970.

[2] M. A. Akivis – A. M. Shelekov: *On the canonical coordinates in a local analytic loop*, (Russian), Webs and Quasigroups, Gos. Univ. Kalinin 130 (1986), 120–124.

[3] R. Bödi – L. Kramer: *Differentiability of continuous homomorphisms between smooth loops*, Result. Math. 25 (1994), 13–19.

[4] J-P. Dufour – P. Jean: *Rigidity of webs and family of hypersurfaces*, Singularites Dynamical Systems, S.N. Pnevmatikos (editor), Elsevier Science Publisher B.V., North-Holland (1985), 271–283.

[5] J-P. Dufour: *Rigidity of webs*, Web Theory and Related Topics, edited by J.Grifone and E.Salem, World Scientific (2001), 106–113.

[6] V. V. Goldberg: *Theory of Multidimensional (n+1)-Webs*, Kluwer Academic Publishers, (1988).

[7] J. Kozma: *On the differentiability of loopmultiplication in canonical coordinate system*, Publ. Math. Debrecen 37 (1990), 313–325.

[8] J. Kozma: *Loops with and without subloops,* Acta Sci. Math. (Szeged) 55 (1991), 21–31.

[9] E. N. Kuzmin: *The connection between Malcev algebras and analytic Moufang loops*, (Russian), Algebra i Logika 10 (1971), 3–22.

[10] P. T. Nagy – K. Strambach: *Loops in Group Theory and Lie Theory*, Expositions in Mathematics, Vol. 35. De Gruyter, (2002).

[11] S. Sternberg: *Local contractions and a Theorem of Poincaré*, Am. J. Math. 79 (1957), 809–824.