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Large quartic groups on translation planes, III—groups with common centers

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Abstract. The 'desirable' planes odd order of Biliotti, Jha, Johnson, Menichetti are shown to be exactly the class of planes that admit two trivially intersecting quartic groups of order q with common centers.

Keywords: quartic group, translation plane, Ott-Schaeffer plane

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1 Introduction

In two previous articles, the authors classify translation planes with spreads in $PG(3, q)$ that admit 'large quartic groups'.

1 Definition. Let π be a translation plane of order q^2, $q = p^r$, p a prime, with spread in $PG(3, q)$. A 'quartic group' T is an elementary Abelian p-group all of whose non-identity elements are quartic (i. e. have minimal polynomials $(x-1)^4$) and which fix the same 1-dimensional $GF(q)$-subspace pointwise. The fixed-point space is called the 'quartic center' of the group and the unique component of π containing the center is called the 'quartic axis'.

If T and S are quartic groups, we shall say that T and S are 'skew' if and only if their quartic centers are distinct (i. e. if and only if $t \in T - \{1\}$ and $s \in S - \{1\}$).

The authors have classified translation planes admitting two large skew quartic groups as follows:

2 Theorem (Biliotti, Jha and Johnson [1]). *Let π be a translation plane with spread in $PG(3,q)$, where $q = p^r$, p a prime. If π admits at least two skew quartic p-groups of orders $> \sqrt{q}$, then*

(1) the group generated by the quartic p-elements is isomorphic to $SL(2,q)$, and

(2) the plane is one of the following planes:

(a) a Hering plane or

(b) $q = 5$ and the plane is one of the three exceptional Walker planes.

The previous definition applies to odd order planes. For even order planes, there is a similar definition.

3 Definition. By a 'quartic 2-group', acting on a translation plane π of even order 2^{2r} with spread in $PG(3,2^r)$, it is meant that the 2-group contains no elations so all involutions are Baer and any two distinct non-identity involutions fix Baer subplanes pointwise that share exactly one component and share on that component a common 1-dimensional K-subspace, where K is isomorphic to $GF(2^r)$. The component containing the common 1-dimensional fixed point space is called the 'quartic axis' of the quartic 2-group and the fixed point space on the quartic axis is called the 'center' of the group.

Let T and S be quartic 2-groups. We shall say that T and S are 'skew' if and only if the quartic center of T and the quartic center of S are disjoint subspaces.

The main result concerning planes of even order admitting two large trivially intersection quartic groups is as follows:

4 Theorem (Biliotti, Jha and Johnson [2]). *Let π be a translation plane of even order with spread in $PG(3,q)$ that admits at least two skew quartic 2-groups of orders at least $2\sqrt{q}$ then π is an Ott-Schaeffer plane.*

Let T and S be distinct quartic groups of order q of a translation plane of order q^2 with spread in $PG(3,q)$. If T and S are skew then it is thus known that $SL(2,q)$ is generated and the plane is either Hering or Walker of order 25, if q is odd and the plane is Ott-Schaeffer if q is even.

In this article, we completely determine the translation planes of order q^2 that admit at least two trivially intersecting quartic groups of order q with common axis. It is shown that q must be odd for this to occur and the planes are determined relative to a set of functions and are known as the 'desirable' translation planes.

Furthermore, when q is even, it is then possible to remove the hypothesis that the quartic groups are skew and show that all that is required is that groups are trivially intersecting to show that the plane is an Ott-Schaeffer plane.

2 The main theorems

Suppose that T and S are disjoint quartic groups but are not trivially intersecting. That is, we assume that T and S are disjoint groups with common quartic center X and consequently common quartic axis L_X. Since the group $\langle T, S\rangle$ is in $GL(4,q)$, and $T \mid L_X = S \mid L_X$ then there is an elation group E of order q in $\langle T, S\rangle$. Assume that E^+ is the full collineation group of order $p^a q$, where $q = p^r$. Then T normalizes E^+ and E^+T has order $q^2 p^a$. Clearly, we then have a Baer group of order p^a.

2.1 Odd order

If q is odd, if there is a non-trivial Baer p-group then, since there are elations, this is a contradiction by Foulser [4]. In this case, when q is odd, E thus has order q. Thus, we have a collineation group of order q^2 in $GL(4,q)$ and we may apply Johnson-Wilke [6].

5 Lemma. *The plane π cannot be a semifield plane.*

Proof. If π is a semifield plane then there is an elation group of order q^2 and a quartic group of order q, which are necessarily trivially intersecting. Hence, there is a p-group of order q^3, which is in $GL(4,q)$. Hence, there is a Baer group of order q, a contradiction. $\boxed{QED}$

So, if q is odd, using Johnson-Wilke [6], ET may be represented as follows:

$$\left\langle \tau_{u,a} = \begin{bmatrix} 1 & a & u & ua - \frac{a^3}{3} + J(a) + m(u) \\ 0 & 1 & a & u \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}; u, a \in GF(q) \right\rangle, \tag{1}$$

where $J(a+b) + m(ab) = J(a) + J(b)$ and m is an additive function.

6 Lemma.

$$\tau_{u,a}\tau_{v,b} = \tau_{u+ab+v,a+b}\,.$$

Proof. Direct calculation. $\boxed{QED}$

7 Lemma. $\tau_{u,a}^k = \tau_{ku+\frac{(k-1)k}{2}a^2,ka}$.

Proof. Clearly valid for $k = 1$, assume true for k. Now note that

$$\tau_{u,a}^k \tau_{u,a} = \tau_{ku+\frac{(k-1)k}{2}a^2,ka}\tau_{u,a} = \tau_{(k+1)a+\frac{(k-1)k}{2}a^2+ka^2,(k+1)a}\,,$$

by Lemma 6. Since $\frac{(k-1)k}{2} + k = \frac{k(k+1)}{2}$, we have the proof to the lemma. $\boxed{QED}$

8 Lemma. *For every u, a not both zero, $\tau_{u,a}$ has order p.*

Proof. We note that $\tau_{u,a}^p$ must be in the associated elation group. Hence, $\tau_{u,a}$ has order p or p^2. Letting $k = p$ in Lemma 7, we have

$$\tau_{u,a}^p = \tau_{pu+\frac{(p-1)p}{2}a^2,pa}$$

and since p is odd $\frac{(p-1)p}{2}$ is a factor of p. Hence, $\tau_{u,a}^p = \tau_{0,0}$. $\boxed{QED}$

9 Lemma. *$E = \langle \tau_{u,0};\ u \in GF(q) \rangle$ and $T = \left\langle \tau_{f(a),a};\ a \in GF(q) \right\rangle$, for some function f on $GF(q)$.*

Proof. In the give representation, T is a subgroup of ET that induces a faithful group of order q on $x = 0$. The group induced on $x = 0$ is represented by

$$\left\langle \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix};\ a \in GF(q) \right\rangle.$$

Hence, for each element $a \in GF(q)$, there is a unique element $\tau_{f(a),a} \in T$. Hence, f is a function from $GF(q)$ to $GF(q)$. $\boxed{QED}$

10 Lemma.

$$f(a+b) = f(a) + ab + f(b).$$

Proof. We note that

$$\tau_{u,a}\tau_{v,b} = \tau_{u+ab+v,a+b}$$

by direct calculation. Hence, we must have

$$\tau_{f(a),a}\tau_{f(b),b} = \tau_{f(a)+ab+f(b),a+b} = \tau_{f(a+b),a+b}\ ,$$

since T is a group. Hence, we obtain:

$$* : f(a+b) = f(a) + ab + f(b), \forall a, b \in GF(q).$$

$$\begin{aligned} **: (f(a+b))(a+b) - \frac{(a+b)^3}{3} + J(a+b) + m(f(a+b)) \\ = \left(f(a)a - \frac{a^3}{3} + J(a) + m(f(a)) + af(b) \right) \\ + \left(f(b)b - \frac{b^3}{3} + J(b) + m(f(b)) + bf(a) \right), \end{aligned}$$

which actually is an identity. $\boxed{QED}$

11 Lemma. *$\langle S, T \rangle = ET$.*

Given an element $\tau_{f(t),t}$ of T, there exists an elation $\tau_{r(t),0}$ such that

$$\tau_{f(t),t}\tau_{r(t),0}$$

is in S. Hence, $\langle S,T\rangle = ET$. Therefore, the following lemma is clear:

12 Lemma.

$$S = \left\langle \tau_{g(a),a};\ a \in GF(q) \right\rangle,$$

for some function g on $GF(q)$ such that

$$g(a+b) = g(a) + ab + g(b).$$

We now determine the functions f and g.

Let

$$f(a) = \sum_{i=0}^{q-1} f_i a^i.$$

Since $f(0)$ must be 0 as this is the identity arises from the identity element, it follows that

$$f(a) = \sum_{i=1}^{q-1} f_i a^i.$$

Letting $b = -a$, we obtain:

$$f(0) = f(a) - a^2 + f(-a).$$

Hence,

$$\sum_{i=1}^{q-1} f_i a^i + \sum_{i=1}^{q-1} f_i\,(-a)^i = a^2,\ \forall a \in GF(q).$$

This implies $2f_2 = 1$ and $f_{2j} = 0$ for $j \neq 1$. Hence,

$$\frac{a^2}{2} + \sum_{k=1}^{(q-1)/2} f_{2k-1}a^{2k-1} = f(a).$$

Now letting $a = b$, we have

$$f(2a) = 2f(a) + a^2.$$

This implies that

$$\frac{(2a)^2}{2} + \sum_{k=1}^{(q-1)/2} f_{2k-1}\,(2a)^{2k-1} = f\,(2a) = 2\left(\frac{a^2}{2} + \sum_{k=1}^{(q-1)/2} f_{2k-1}a^{2k-1}\right) + a^2.$$

But, this then says that

$$f_{2k-1}2^{2k-1} = 2f_{2k-1}, \forall k.$$

Thus,

$$f_{2k-1} = 0,\ k \neq 1.$$

Therefore:

13 Lemma.

$$f(a) = f_1 a + \frac{a^2}{2}.$$

Similarly,

$$g(a) = g_1 a + \frac{a^2}{2}.$$

14 Lemma. *Let π be any desirable translation plane with group G represented as in* (1)*. Then there are at least two trivially intersecting quartic groups S and T of orders q with common center such that $G = ST$.*

Proof. Let $f(a) = f_1 a + \frac{a^2}{2}$ and $g(a) = g_1 a + \frac{a^2}{2}$, where $f_1 \neq g_1$. Let $h_1 = f_1 - g_1$. Then

$$GF(q) = \{\, h_1 a;\ a \in GF(q) \,\}.$$

$$\tau_{f(a),a}\tau_{g(-a),-a} = \tau_{h_1 a,0}.$$

Hence, we obtain that

$$T = \left\langle\, \tau_{f(a),a};\ a \in GF(q) \,\right\rangle,$$

$$S = \left\langle\, \tau_{g(a),a};\ a \in GF(q) \,\right\rangle,$$

and

$$\tau_{f(a),a} = \tau_{(f_1-g_1)a,0}\tau_{g(a),a}\,.$$

This shows that we obtain trivially intersecting quartic groups S and T of orders q with common centers such that $ST = G$. $\boxed{QED}$

Hence, for q odd, we obtain the following theorem.

15 Theorem. *Let π be a translation plane of odd order q^2 and spread in $PG(3, q)$ that admits two trivially intersecting quartic groups with orders q that share their quartic centers.*

(1) Then π is a ‘desirable’ plane.

(2) If π is a desirable plane then there group G of order q^2 contains two trivially intersecting quartic groups T and S with the same quartic axis such that $\langle T, S\rangle = G$.

2.2 Even order

Now we consider the even order case. In this setting, T is a quartic group, meaning that all involutions in T are Baer and T fixes exactly one 1-dimensional $GF(q)$-space, the 'quartic center' X and the unique component L_X containing X is called the 'quartic axis'. Suppose S is a quartic group having center X and hence axis L_X, such that S and T are trivially intersecting. Let E^+ denote the elation group of order $2^a q$ with axis L_X in $\langle T, S\rangle = G$. Then E^+T has order $2^a q^2$.

16 Lemma. *Assume that E^+T has order $> q^2$. Then π is a semifield plane.*

Proof. If $a > 1$, then we may apply Biliotti, Jha, Johnson, Menichetti [3], section 5, to show that the plane if a semifield plane, as we can find a subgroup of order $2q^2$. $\boxed{QED}$

17 Lemma. *π is not a semifield plane.*

Proof. Semifield planes has elation groups of order q^2, implying that we have a Baer group of order q, since we now have a group of order q^3. But, is implies that $q = 2$ by Jha-Johnson [5]. $\boxed{QED}$

Hence, the order of E is q and $G^* = ET$ has order q^2.

18 Lemma. *The group $G^* = ET$ of order q^2 cannot act regularly on the components other than the quartic axis.*

Proof. We have Baer involutions in T. Hence, ET cannot be regular. $\boxed{QED}$

Hence, ET must have two component orbits of length $q^2/2$ by Theorem (3.1) of [3]. Thus, applying the main result (3.14) of [3], we may represent the group G^* in the following form:

$$G^* = \langle\, \tau_{u,a},\, \rho_{u,a};\, u \in GF(q),\, a \in \Sigma\,\rangle\,,$$

where

$$\tau_{u,a} = \begin{bmatrix} 1 & T(a) & u + a + m(a) + aT(a) & m(u) + uT(a) + R(a) \\ 0 & 1 & a & u \\ 0 & 0 & 1 & T(a) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\rho_{u,a} = \begin{bmatrix} 1 & T(a)+1 & u + m(a) + aT(a) & m(u) + uT(a) + u + R(a) \\ 0 & 1 & a & u \\ 0 & 0 & 1 & T(a) \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where Σ is a subgroup of $(GF(q),+)$ of order $q/2$, T is a function from Σ to $GF(q)$, m is an additive function on $GF(q)$ and R is a function on $GF(q)$, where the functions T, m and R have certain restrictions as listed in the theorem (3.14) of [3].

19 Lemma. *$\tau_{u,a}^2 = \tau_{aT(a),0}$ and $\rho_{u,a}^2 = \tau_{a(T(a)+1)}$.*

Proof. Simply use the forms and calculate the square of each element. $\boxed{QED}$

Now we have a quartic group T of order q, an elementary Abelian 2-group consisting of Baer involutions. Thus, each non-identity element of T must be either an element $\tau_{u,a}$ for $T(a) \neq 0$ or an element $\rho_{u,a}$ for $T(a)+1 \neq 0$. But, in order to obtain a quartic group T of order q, we must have

$$T(a) = 0 \text{ or } 1, \ \forall a \in GF(q).$$

Furthermore, the elements of T must induce on $x = 0$ i. e. L_X, elements of the following form:

$$\begin{bmatrix} 1 & W(a) \\ 0 & 1 \end{bmatrix}$$

where

$$W(a) = T(a) \text{ or } T(a)+1, \ \forall a \in GF(q).$$

Since $T(a) = 0$ or 1, it follows that $W(a) = 0$ or 1. However, W must be a $1-1$ function since T is a quartic group of order q. Hence, $q = 2$. However, planes of order 4 are Desarguesian and cannot admit quartic groups.

Hence, we have the following result—noting that we now do not require that the two groups are 'skew'.

20 Theorem. *Let π be a translation plane of even order q^2 with spread in $PG(3,q)$ admitting two trivially intersecting quartic groups of order q. Then the group generated by the quartic groups is isomorphic to $SL(2,q)$ and the plane is an Ott-Schaeffer plane of order q^2.*

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