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Locally affine geometries of order 2 where shrinkings are affine expansions

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Abstract. Given a locally affine geometry Γ of order 2 and a flag-transitive subgroup $G \leq \mathrm{Aut}(\Gamma)$, suppose that the shrinkings of Γ are isomorphic to the affine expansion of the upper residue of a line of Γ by a homogeneous representation in a 2-group. We shall prove that, under certain hypotheses on the stabilizers G_p and G_l of a point p and a line l, we have $G = RG_p$ for a representation group R of $\mathrm{Res}(p)$. We also show how to apply this result in the classification of flag-transitive c-extended P- and T-geometries.

Keywords: shrinkings, affine expansions, representation groups, sporadic groups

MSC 2000 classification: primary 51E24, secondary 20D08, 20C34

1 Introduction

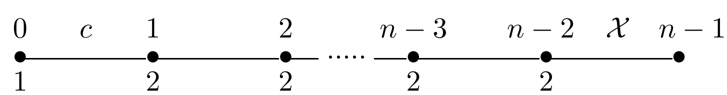
This paper is a continuation of a previous paper [14], by C. Wiedorn and myself. In [14], developing an idea of Stroth and Wiedorn [17] (but exploited also in [4], [9] and [8]) we built up a general theory of local parallelisms, geometries at infinity and shrinkings for geometries with string diagrams (called poset-geometries in [14]). We applied that theory to a number of examples taken from the literature, with special emphasis on the investigation of flag-transitive c-extensions of P- and T-geometries (Fukshansky and Wiedorn [3] and Stroth and Wiedorn [17]; see also Stroth and Wiedorn [18] for examples not considered in [3] and [17]). In particular, in Proposition 7.8 of [14] we put in full evidence the role that a combined analysis of shrinkings and structures at infinity had in [17]. However, by that method, we can only get control over c-extended P-geometries of rank $n \geq 4$ where, by repeating the shrinking procedure $n-3$ times, we end up with the $c.P$-geometry for $3^{\cdot}S_6$, which has the tilde geometry as its structure at infinity. In all but one of these geometries the structures at infinity are T-geometries, whence known objects (see Ivanov and Shpectorov [6]). So, we can compare feasible geometries at infinity with feasible shrinkings. The latter have rank $n-1$ and, if we work inductively, have already been classified at a previous step. In this way, one can classify the rank n case, too.

In the remaining cases allowed by the hypotheses of [17] things go differently.

Apart from two c-extended P-geometries of rank 4 related to $U_6(2)$ and $2^{\cdot}U_6(2)$, in all remaining cases the $(n-3)$th repeated

shrinking is either the $c.P$-geometry for $2^6:S_5$ or $2^5:S_5$ or the $c.T$-geometry for $2^6:3^{\cdot}S_6$. In these cases the geometries at infinity, albeit locally projective, belong to diagrams that have never been considered in the literature. So, they are not so useful to get informations on the c-extended geometry Γ we want to describe. However, it turns out that Γ now arises from a representation of a point-residue of Γ. (Note that the above mentioned geometries for $2^6:S_5$, $2^5:S_5$ and $2^6:3^{\cdot}S_6$ are indeed affine expansions of abelian representations of the dual Petersen graph and the tilde geometry, respectively.) This result is obtained in [17] by a detailed group-theoretical analysis, but one might ask for a more geometric approach. Let $\mathcal{C}$ be the class of c-extended P- or T-geometries of rank > 3 satisfying the hypotheses of [17] and such that, by repeatedly applying the shrinking procedure to them, we eventually get the $c.P$-geometry for $2^6:S_5$ or $2^5:S_5$ or the $c.T$-geometry for $2^6:3^{\cdot}S_6$. By definition, $\mathcal{C}$ contains the shrinkings of all of its members of rank > 4. Let Γ be a member of $\mathcal{C}$ of rank $n > 4$ and suppose that we have already proved that every $\Sigma \in \mathcal{C}$ of rank $n-1$ is the affine expansion of a representation of a point-residue of Σ. Then the shrinking of Γ, being a member of $\mathcal{C}$ of rank $n-1$, is the affine expansion of a representation of the upper residue of a line of Γ. If this is sufficient to claim that Γ itself arises from a representation of a point-residue of Γ, then Γ itself is determined, provided that we know all representations of the point-residues of Γ (as it happens for the point-residues allowed by the hypotheses of [17]).

In this paper, inspired by a lemma of Stroth and Wiedorn [17, Lemma 6], we shall prove two theorems that can do the above job. Referring to Section 5 for their precise statements, here we only give a rough exposition of their content. Let Γ be a flag-transitive geometry of rank $n \geq 4$, with diagram and order as follows, where $\mathcal{X}$ denotes a class of partial linear spaces, no matter which.



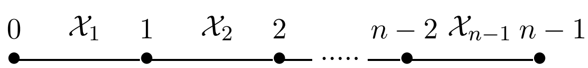
Let Σ be the shrinking of Γ and $\{p, l, P\}$ be a $\{0, 1, 2\}$- flag of Γ. Suppose that Γ satisfies the Intersection Property and Σ is the affine expansion of a representation of $\mathrm{Res}^+(l)$ in a 2-group. Then, under certain hypotheses on the stabilizers of l, p and P in $\mathrm{Aut}(\Gamma)$, the group $\mathrm{Aut}(\Gamma)$ is essentially a semi-direct product of a representation group for $\mathrm{Res}(p)$ by the stabilizer of p in $\mathrm{Aut}(\Gamma)$.

The paper is organized as follows. In Section 2 we recall some basics on geometries with string diagram and the definition of locally affine and locally projective geometries. In Section 3 we recall the essentials on shrinkings, but we only consider locally affine geometries, in order to avoid complications unnecessary in this paper. In this way, however, we do not go very far beyond [17]. In Section 4 we discuss representations of locally projective geometries of order 2 and their affine expansions. Most of what we say in Section 3 is taken from [12], [14, Section 2.8] and [18, Section 2], but a couple of results are also proved that do not appear in any of the above references. The main theorems of this paper are stated in Section 5 and proved in Section 6. In Section 7 (Theorem 60) we show how those theorems can be applied to the geometries considered in [17]. We gain a remarkable simplification of the arguments of [17]. We also make a little progress with respect to [17], giving a characterization of the c-extended P-geometry for $J_4 \wr 2$ (Theorem 61), which is not included in the classification of [17] since, contrary to the hypotheses of [17], it involves the P-geometry for $3^{\cdot}M_{22}$ as a residue.

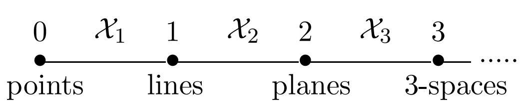
2 Basics on geometries with string diagram

2.1 Terminology and notation

We follow [11] for basic notions of diagram geometry. In particular, all geometries are residually connected and firm, by definition. Let Γ be a geometry of rank n, with string diagram and types $0, 1, \ldots, n-1$ given in increasing order from left to right, as in the following picture:



where the labels $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{n-1}$ denote classes of rank 2 geometries, no matter which. In the sequel, we take the writing $x \in \Gamma$ as a shortening of the phrase "x is an element of Γ" and we denote by $t(x)$ the type of an element $x \in \Gamma$. Given two elements $x, y \in \Gamma$, we write $x < y$ (respectively $x \leq y$) when x and y are incident and $t(x) < t(y)$ (resp. $t(x) \leq t(y)$). When $x < y$ we will freely use expressions as "x is *below* y", "y is *above* x", "y *contains* x", "x *belongs* to y", and others in this style. The elements of type 0, 1 and 2 are called *points*, *lines* and *planes*, respectively. Two points (lines) are said to be *collinear* (*coplanar*) if they belong to a common line (plane). The elements of type 3 will also be called 3-*spaces*.



The elements of type i are called i-*elements* and we denote by Γ^i the set of i-elements of Γ. For a subset $J \subseteq I := \{0, 1, \ldots, n-1\}$, we put $\Gamma^J := \cup_{j \in J} \Gamma^j$,

$\Gamma^{>0} := \Gamma^{\{1,2,\ldots,n-1\}}$ and $\Gamma^{<n-1} := \Gamma^{\{0,1,\ldots,n-2\}}$. Given $x \in \Gamma$, we denote by $\Gamma^i(x)$ the set of elements of type i incident to x. When $i = 0, 1, 2$ we also use the following notation: $P(x) := \Gamma^0(x)$, $\mathcal{L}(x) := \Gamma^1(x)$ and $\mathcal{P}(x) := \Gamma^2(x)$.

If $J \subset I$, the *J-truncation* $\mathrm{Tr}_J(\Gamma)$ of Γ is the geometry induced by Γ on $\Gamma^{I \setminus J}$. The residue of an element $x \in \Gamma$ will be denoted by $\mathrm{Res}_\Gamma(x)$ (also $\mathrm{Res}(x)$ when no ambiguity arises). If $0 < t(x) < n-1$, the *lower residue* $\mathrm{Res}^-_\Gamma(x)$ of x (the *upper residue* $\mathrm{Res}^+_\Gamma(x)$) is the poset-geometry induced by Γ on the set of elements below (above) x.

2.2 The intersection property

The Intersection Property ((IP) for short) can be formulated in various equivalent ways (see [11, Chapter 6]). We choose the following formulation: we say that a geometry Γ with string diagram satisfies (IP) if the both the following hold for any two elements $X, Y \in \Gamma$:

(IP1) if $P(X) \cap P(Y) \neq \emptyset$ then $P(X) \cap P(Y) = P(Z)$ for some $Z \in \Gamma$.

(IP2) if $P(X) \subseteq P(Y)$ then $X \leq Y$.

In particular, if (IP2) holds then no two distinct elements of Γ have the same set of points. In this case, if $P(X) = \{x_1, x_2, \ldots, x_m\}$ we may write $X = \{x_1, x_2, \ldots, x_m\}$, but we will use this shortened notation only for lines.

By [11, Lemma 7.25], when Γ is locally affine or locally projective (see the next subsection), (IP) can be formulated in the following way, easier to check in practice:

(LL) no two distinct lines of Γ are incident with the same pair of points and the same property holds in $\mathrm{Res}^+(X)$, for every $X \in \Gamma$ of type $t(X) < n-2$ (where $n = \mathrm{rank}(\Gamma)$).

2.3 Pre-parallelisms, structures at infinity and parallelisms

Let Γ be a geometry with string diagram and $\mathrm{rank}(\Gamma) = n \geq 2$. A *pre-parallelism* of Γ is an equivalence relation π on $\Gamma^{>0}$ such that no two elements of $\Gamma^{>0}$ of different type correspond in π. (Pre-parallelisms are called type-compatible equivalence relations in [14].) Given a pre-parallelism π of Γ, when two elements $X, Y \in \Gamma^{>0}$ correspond in π, we write $X\pi Y$ and we say that X and Y are *parallel* in π (also *π-parallel*). The class of π containing X will be called the *parallel class* of X in π and will be denoted by X^π.

The *structure at infinity* Γ/π of (Γ, π) is the incidence structure of rank $n-1$ over the set of types $\{0, 1, \ldots, n-2\}$, defined as follows: For $i = 0, 1, \ldots, n-2$, the elements of Γ/π of type i are the parallel classes of the $(i+1)$-elements of Γ and two parallel classes X^π and Y^π are declared to be incident in Γ/π when some member of X^π is incident in Γ to some member of Y^π. The function p_π sending $X \in \Gamma^{>0}$ to X^π is a surjective morphism from $\mathrm{Tr}_0(\Gamma)$ to Γ/π. We call it the *projection* of Γ onto Γ/π. We warn that Γ/π is not a geometry in general, but it is a geometry in many interesting cases.

An *automorphism* of (Γ, π) is an automorphisms of Γ that permute the classes of π. The automorphisms of (Γ, π) form a subgroup of $\mathrm{Aut}(\Gamma)$, denoted by $\mathrm{Aut}(\Gamma, \pi)$. Clearly, $\mathrm{Aut}(\Gamma, \pi)$ induces in Γ/π a subgroup of $\mathrm{Aut}(\Gamma/\pi)$.

Following Buekenhout, Huybrechts and Pasini [1], we say that a pre-parallelism π of Γ is a *partial parallelism* of Γ if it satisfies the following: For any choice of $X, Y, X', Y' \in \Gamma^{>0}$ with $X\pi X'$, $Y\pi Y'$ and $X \leq Y$, if $P(X') \cap P(Y') \neq \emptyset$ then $X' \leq Y'$.

Let π be a partial parallelism. Then no two distinct π-parallel elements of $\Gamma^{>0}$ have any point in common (see [14]). Therefore, given a point p and an element X of type $t(X) = i > 0$, at most one element of $\Gamma^i(p)$ is π-parallel to X. That element, if it exists, will be denoted by $\pi(p, X)$. A partial parallelism π is said to be a *parallelism* if $\pi(p, X)$ exists for any $X \in \Gamma^{>0}$ and any point p.

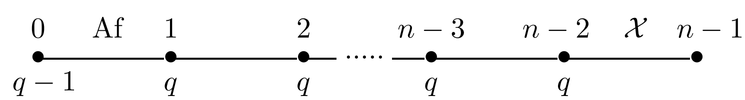
1 Proposition (Pasini and Wiedorn [14, section 2.5]). *Let π be a pre-parallelism of Γ. Then:*

(1) π is a partial parallelism if and only if, for every point p, the projection p_π of Γ onto Γ/π induces an isomorphism from $\mathrm{Res}_\Gamma(p)$ to the structure induced by Γ/π on the set $p_\pi(\mathrm{Res}_\Gamma(p))$.

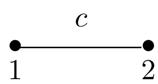
(2) π is a parallelism if and only if, for every point p, the projection p_π induces an isomorphism from $\mathrm{Res}_\Gamma(p)$ to Γ/π.

2.4 Locally affine and locally projective geometries

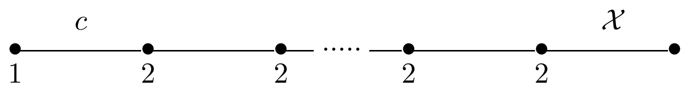
Given integers $q > 1$ and $n > 2$, a *locally affine geometry* of *order* q and rank n is a geometry Γ with diagram and orders as follows, where the label Af denotes the class of affine planes and $\mathcal{X}$ is a given class of rank 2 geometries, no matter which:



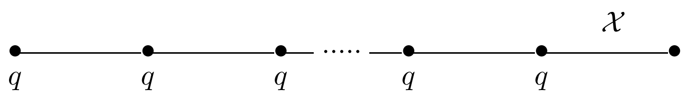
(We do not assume that Γ admits order at the type $n-1$.) It follows from the diagram that $\mathrm{Res}(x) \cong AG(n-1, q)$, for $x \in \Gamma^{n-1}$. The class of affine planes of order 2 is also denoted by the following symbol:



Accordingly, the following diagram describes locally affine geometries of order 2:



A *locally projective geometry* of *order* q is a geometry with diagram and orders as follows:



3 Shrinkings

3.1 The local parallelism of a locally affine geometry

Let Γ be a locally affine geometry of rank $n \geq 3$. As $\mathrm{Res}(A)$ is an affine geometry of rank $n-1$ for $A \in \Gamma^{n-1}$, a unique parallelism π_A is defined in $\mathrm{Res}(A)$. These parallelisms form a coherent system, namely:

(LP) For any element $X \in \Gamma^{<n-1}$ of type $t(X) > 1$ and any $(n-1)$-elements $A, B > X$, π_A and π_B induce the same parallelism on $\mathrm{Res}^{-}(X)$.

We call the family $\gamma := \{\pi_A\}_{A \in \Gamma^{n-1}}$ the *local parallelism* of Γ. The members of γ are equivalence relations on certain subsets of $\Gamma^{>0} \cap \Gamma^{<n-1}$. As relations are sets of pairs, we can form the union $\cup\gamma := \cup(\pi_A \mid A \in \Gamma^{n-1})$ of the members of γ. The relation $\cup\gamma$ is reflexive and symmetric, but it is not transitive, in general. We call its transitive closure the *closure* of γ and we denote it by $\lfloor\gamma\rfloor$. Clearly, $\lfloor\gamma\rfloor$ is a pre-parallelism of $\mathrm{Tr}_{n-1}(\Gamma)$, but possibly not a partial parallelism. Also, for every $A \in \Gamma^{n-1}$, the natural parallelism π_A is a (possibly proper) refinement of the relation $\lfloor\gamma\rfloor_A$ induced by $\lfloor\gamma\rfloor$ on $\mathrm{Res}(A)$.

A pre-parallelism π of Γ is called an *extension* of γ if π induces π_A in $\mathrm{Res}(A)$ for every $A \in \Gamma^{n-1}$. If moreover π is a partial-parallelism, then we say that it is a *strong extension* of γ. We say that γ is *extensible* if it admits an extension. If γ admits a strong extension, then we say it is *strongly extensible.* Note that, if π is an extension of γ, then $\lfloor\gamma\rfloor$ is a refinement of the pre-parallelism induced by π on $\mathrm{Tr}_{n-1}(\Gamma)$. We say that a pre-parallelism π of Γ is a *completion* of γ if it induces $\lfloor\gamma\rfloor$ on $\mathrm{Tr}_{n-1}(\Gamma)$. Completions always exist, even if γ is non-extensible. All of them are joins $\lfloor\gamma\rfloor \cup \gamma_{n-1}$, where γ_{n-1} is an equivalence relations on Γ^{n-1}. The completion obtained by choosing the identity relation as γ_{n-1} is the *minimal completion* of γ. We denote it by $[\gamma]$. The *canonical completion* of γ, denoted by $\langle\gamma\rangle$, is obtained by choosing γ_{n-1} as follows: Two elements $A, B \in \Gamma^{n-1}$ correspond in γ_{n-1} if and only if, for every $X \in \mathrm{Res}(A) \cap \Gamma^{>0}$, we have $X\lfloor\gamma\rfloor Y$ for at least one $Y < B$, and the same holds if we permute the roles of A and B. We call $\Gamma/[\gamma]$ and $\Gamma/\langle\gamma\rangle$ the *finest* and the *canonical structure at infinity* of (Γ, γ). By Pasini and Wiedorn [14, Theorem 3.11], if γ is extensible then both $\Gamma/[\gamma]$ and $\Gamma/\langle\gamma\rangle$ are geometries, locally projective when $n > 3$.

Since $[\gamma]$ and $\langle\gamma\rangle$ are uniquely determined by γ, which in its turn is uniquely determined by Γ, we have $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma, [\gamma]) = \mathrm{Aut}(\Gamma, \langle\gamma\rangle)$. Clearly, the identity automorphism ι of Γ induces a surjective morphism from $\Gamma/[\gamma]$ to $\Gamma/\langle\gamma\rangle$. Similarly, if π is a refinement of $\langle\gamma\rangle$ (or a completion or an extension of γ) then ι induces a surjective morphism from Γ/π to $\Gamma/\langle\gamma\rangle$ (respectively, from $\Gamma/[\gamma]$ to Γ/π). In particular, $\Gamma/\langle\gamma\rangle$ is a homomorphic image of $\Gamma/[\gamma]$.

3.2 Shrinkings

Given a locally affine geometry Γ of rank $n \geq 3$, let $\gamma = \{\pi_A\}_{A\in\Gamma^{n-1}}$ be its local parallelism. Given an element X of type $1 < t(X) < n-1$, we denote by π_X the parallelism induced by π_A on $\mathrm{Res}^-(X)$, for $A \in \Gamma^{n-1}(X)$. By (LP), π_X does not depend on the choice of $A \in \Gamma^{n-1}(X)$ and, for any $Y > X$, π_Y induces π_X on $\mathrm{Res}^-(X)$. Let $\overline{\Sigma}$ be the incidence structure of rank $n-1$ defined as follows:

Elements. The 0-elements of $\overline{\Sigma}$ (also called 'points' of $\overline{\Sigma}$) are the lines of Γ. For $i = 1, 2, \ldots, n-2$, the i-elements of $\overline{\Sigma}$ are the pairs (X, L) with $X \in \Gamma^{i+1}$ and L a class of the equivalence relation induced by π_X on the set $\mathcal{L}(X) = \Gamma^1(X)$.

Incidence. A point l of $\overline{\Sigma}$ and an element (X, L) are declared to be incident when $l < X$ and $l \in L$. Two elements (X, L) and (Y, M) with $t(X) \leq t(Y)$ are incident when $X \leq Y$ in Γ and $L \subseteq M$.

The structure $\overline{\Sigma}$ is not connected, in general. However,

2 Proposition (Stroth and Wiedorn [17], Pasini and Wiedorn [14]). *Let Σ be a connected component of $\overline{\Sigma}$. Then Σ is a geometry. Furthermore:*

(1) If $n > 3$ then Σ is locally affine, with the same order as Γ.

(2) $\mathrm{Res}_\Sigma(l) \cong \mathrm{Res}^+_\Gamma(l)$ for every line l of Γ belonging to Σ. More explicitly, the mapping sending $X \in \mathrm{Res}^+_\Gamma(l)$ to the pair (X, L), where L is the unique class of π_X containing l, is an isomorphism from $\mathrm{Res}^+_\Gamma(l)$ to $\mathrm{Res}_\Sigma(l)$.

(3) The lines of Γ *belonging to* Σ *form a class of the equivalence relation* $\lfloor\gamma\rfloor_1$ *induced by* $\lfloor\gamma\rfloor$ *on* Γ^1.

(4) If Γ *satisfies the Intersection Property (IP), then* Σ *also satisfies (IP).*

The connected components of $\overline{\Sigma}$ are called *shrinkings* of Γ. Given a class Λ of $\lfloor\gamma\rfloor_1$, let Σ be the shrinking of Γ having Λ as the point-set. Given a subgroup $G \leq \mathrm{Aut}(\Gamma)$, let G_Σ be the set-wise stabilizer of Λ. Clearly, G_Σ stabilizes Σ as a whole, acting on it as group of automorphisms. Let K_Σ be kernel of that action. (Note that K_Σ is contained in the element-wise stabilizer K_Λ of Λ but, if Σ does not satisfies (IP), then K_Σ might be smaller than K_Λ.) The claims gathered in the next proposition easily follow from Proposition 2 (2):

3 Proposition. *Given a line* $l \in \Lambda$*, let* G_l *be the stabilizer of* l *in* G *and* K_l^+ *be the element-wise stabilizer of* $\mathrm{Res}_\Gamma^+(l)$*. Then* $G_l \leq G_\Sigma$*,* $K_\Sigma \trianglelefteq K_l^+$ *and* K_l^+/K_Σ *is the element-wise stabilizer of* $\mathrm{Res}_\Sigma(l)$ *in* G_Σ/K_Σ*, namely* G_l *acts on* $\mathrm{Res}_\Sigma(l)$ *in the same way as on* $\mathrm{Res}_\Gamma^+(l)$.

4 Corollary. *If* G *is flag-transitive in* Γ*, then* G_Σ/K_Σ *is flag-transitive in* Σ.

When $n > 3$, Σ is locally affine of rank $n-1 \geq 3$. So, we can consider a shrinking of Σ, too. Continuing in this way, we obtain a series of repeated shrinkings $\Sigma_1 = \Sigma, \Sigma_2, \ldots, \Sigma_{n-2}$, of rank $n-1, n-2, \ldots, 2$. When Γ is flag-transitive, every member of this series is uniquely determined up to isomorphism. In this case we call Σ_i the ith-*shrinking* of Γ. In particular, $\Sigma = \Sigma_1$ is the *first shrinking* of Γ. The $(n-2)$th-shrinking is actually the last one but, in general, it saves almost no track of the structure of Γ. So, people generally stop the shrinking process at step $n-3$. Accordingly, we call Σ_{n-3} the *ultimate shrinking* of Γ.

4 Representations and affine expansions of locally projective geometries of order 2

4.1 Representations

Throughout this section R is a given group and Δ is either a locally projective geometry of order 2 and rank $n > 2$ or a geometry of rank $n = 2$ where every line has exactly 3 points. We assume that Δ satisfies the 'weak intersection property' (IP2). Following Ivanov and Shpectorov [6], we say that a mapping $\rho : \Delta^0 \to R$ is a *representation* of Δ in R if it satisfies the following:

(R1) $\rho(x)^2 = 1$ for every point $x \in \Delta^0$;

(R2) if $l = \{x, y, z\}$ is a line of Δ, then $\rho(z) = \rho(x)\rho(y)$;

(R3) $R = \langle \rho(x) \rangle_{x \in \Delta^0}$.

We extend ρ to Δ by putting $\rho(X) := \langle \rho(x) \rangle_{x \in P(X)}$ for every $X \in \Delta$. (Note that, in this way, when $x \in \Delta^0$ the symbol $\rho(x)$ can be read in two ways, either as an element of R or as the group generated by that element, but this ambiguity will cause no confusion in the sequel.) By (R1) and (R2), $\rho(X)$ is an elementary abelian 2-group of order $\leq 2^{i+1}$, for every $X \in \Delta^i$. In particular, if l is a line then $\rho(l)$ is elementary abelian of order $1, 2$ or 4. (Note that $\rho(x) = 1$ is allowed in (R1).) The *image* $\rho(\Delta)$ of Δ by ρ is the poset $\{\rho(X)\}_{X \in \Delta}$ of the ρ-images of the elements of Δ, equipped with the inclusion relation. Clearly, ρ induces a homomorphism of posets from Δ to $\rho(\Delta)$. In the sequel we also use the letter ρ to denote this homomorphism; the context will make it clear if we refer to the representation or to the homomorphism induced by it.

We say that ρ is *locally faithful* if $\rho(x) \neq 1$ for every $x \in \Delta^0$. In this case, $\rho(l)$ is elementary abelian of order 4, for every line $l \in \Delta^1$. Hence $\rho(x) \neq \rho(y)$ for any two collinear points $x, y \in \Delta^0$ and $\rho(X)$ is elementary abelian of order 2^{i+1}, for every $X \in \Delta^i$. Nevertheless, ρ might be non-injective, as it might map distinct non-collinear points onto the same involution of R. If ρ is injective then we say that ρ is *faithful*. In view of (IP2), ρ is faithful if and only if the homomorphism $\rho : \Delta \to \rho(\Delta)$ is an isomorphism.

Let $\mathrm{Aut}(\rho)$ be the set-wise stabilizer of $\rho(\Delta)$ in $\mathrm{Aut}(R)$. We say that an automorphism g of Δ *lifts* to $\mathrm{Aut}(\rho)$ if $\rho g = \alpha_g \rho$ for a (unique) $\alpha_g \in \mathrm{Aut}(\rho)$. The automorphisms of Δ that lift to $\mathrm{Aut}(\rho)$ form a subgroup $\mathrm{Aut}_\rho(\Delta)$ of $\mathrm{Aut}(\Delta)$ and the mapping $\rho_{\mathrm{Aut}} : \mathrm{Aut}_\rho(\Delta) \to \mathrm{Aut}(\rho)$ that maps $g \in \mathrm{Aut}_\rho(\Delta)$ to its lifting α_g is a homomorphism from $\mathrm{Aut}_\rho(\Delta)$ into $\mathrm{Aut}(\rho)$. Following Ivanov and Shpectorov [6], we say that a subgroup $G \leq \mathrm{Aut}(\Delta)$ is *ρ-admissible* if $G \leq \mathrm{Aut}_\rho(\Delta)$. If $\mathrm{Aut}_\rho(\Delta) = \mathrm{Aut}(\Delta)$ then ρ is said to be *homogeneous*.

Following [6], we say that a representation $\rho : \Delta \to R$ is *universal* if the relations embodied by (R1) and (R2) give a presentation of R. Universal representations are unique modulo isomorphisms and every representation of Δ is a homomorphic image of the universal one. More explicitly, if $\rho_1 : \Delta \to R_1$ and $\rho_2 : \Delta \to R_2$ are representations of Δ and ρ_1 universal, then $\rho_2 = \varphi\rho_1$ for a unique homomorphism $\varphi : R_1 \to R_2$. If moreover ρ_2 is also universal, then φ is an isomorphism. As a consequence, universal representations are homogeneous.

A representation $\rho : \Delta \to R$ is *abelian* if R is abelian (whence it is an elementary abelian 2-group). An abelian representation is *universal* (as an abelian representation) if R is the abelian group presented by the set of relations (R1), (R2). Universal abelian representations are also homogeneous and

every abelian representation of a given geometry Δ is a homomorphic image of the universal abelian representation of Δ.

Remark. A representation is an embedding in the sense of [14] and [12] if and only if it is faithful. If R is elementary abelian, then ρ is a projective embedding in the sense of Ronan [15] if and only if it is locally faithful.

4.2 Affine expansions

Given Δ as in the previous subsection, let $\rho : \Delta \to R$ be a faithful representation of Δ. The *affine expansion* of Δ to R by ρ is the geometry $\mathrm{Ex}_\rho(\Delta)$ of rank $n+1$ defined as follows: The 0-elements of $\mathrm{Ex}_\rho(\Delta)$ (also called points of $\mathrm{Ex}_\rho(\Delta)$) are the elements of R and, for every $i = 1, 2, \ldots, n$, the i-elements of $\mathrm{Ex}_\rho(\Delta)$ are the right cosets $\rho(X)r$ for $r \in R$ and $X \in \Delta^{i-1}$. The incidence relation is the natural one, namely inclusion. (We warn that many authors, as Stroth and Wiedorn [18] for instance, call affine expansions *affine extensions*.)

Throughout the rest of this section we put $\Gamma := \mathrm{Ex}_\rho(\Delta)$, for short. If $X \in \Delta^i$, then ρ induces a faithful representation ρ_X of $\mathrm{Res}^-_\Delta(X)$ into $\rho(X)$. As $V_X := \rho(X)$ is elementary abelian of order 2^{i+1} and $\mathrm{Res}^-_\Delta(X) \cong PG(i,2)$, ρ_X realizes $\mathrm{Res}^-_\Delta(X)$ as $PG(V_X)$ in V_X, the latter being regarded as a $GF(2)$-vector space. So, $\mathrm{Ex}_{\rho_X}(\mathrm{Res}^-_\Delta(X)) \cong AG(i+1,2)$. On the other hand, $\mathrm{Res}^-_\Gamma(\rho(X)r) \cong \mathrm{Ex}_{\rho_X}(\mathrm{Res}^-_\Delta(X))$ for every $r \in R$. Hence $\mathrm{Res}^-_\Gamma(\rho(X)r) \cong AG(i+1,2)$. Thus, Γ is locally affine, of order 2 and rank $n+1$. Clearly, the residues of the points of Γ are isomorphic to Δ. Moreover, Γ inherits (IP2) from Δ.

The relation 'being cosets of the same subgroup' is a parallelism of Γ. We call it the *natural parallelism* of Γ. Throughout the sequel, we denote the natural parallelism of Γ by the symbol π_ρ. As the point-residue of Γ are isomorphic to Δ, Proposition 1 (2) implies the following:

5 Proposition. $\Gamma/\pi_\rho \cong \Delta$.

Denoted by γ the local parallelism of Γ, π_ρ is an extension of γ. However, γ is non-strongly extensible in general. So, in general, π_ρ is not a completion of γ and $\Delta \cong \Gamma/\pi_\rho$ is a proper homomorpic image of $\Gamma/\langle\gamma\rangle$. More explicitly, the following holds:

6 Proposition (Pasini and Wiedorn [14, Prop. 2.4]). *For $k < n-1$ and an element $X \in \Delta^k$, put $R[X] := \langle \rho(Y) \rangle_{Y \in \Delta^{k+1}(X)}$. Then the classes of $\lfloor \delta \rfloor$ contained in the π_ρ-parallel class $\{\rho(X)r\}_{r \in R}$ bijectively correspond to the right cosets of $R[X]$ in R.*

In particular, given a point $p \in \Delta^0$ and an element $r_0 \in R$, the induced sub-geometry of Γ formed by the cosets $\rho(X)rr_0$ for $X \geq p$ and $r \in R[p]$ is the shrinking of Γ containing the line $\rho(p)r_0 \in \Gamma^1$.

7 Corollary (Pasini and Wiedorn [14, Cor. 2.2]). *π_ρ is a completion of γ if and only if $R = R[X]$ for every $X \in \Delta^{n-2}$.*

We shall now describe $\mathrm{Aut}(\Gamma, \pi_\rho)$. Note first that, in general, $\mathrm{Aut}(\Gamma, \pi_\rho) < \mathrm{Aut}(\Gamma)$. The action of R on itself by right multiplication induces on Γ a subgroup T_R of $\mathrm{Aut}(\Gamma, \pi_\rho)$. We call T_R the *translation group* of (Γ, π_ρ) (also the *translation group of the affine expansion* Γ). Clearly, T_R acts regularly on Γ^0. For an element $r \in R$, we denote by t_r the element of T_R corresponding to r. By (R3), $T_R = \langle t_{\rho(x)} \rangle_{x \in \Delta^0}$. Also, for $r \in \Gamma^0$, we denote by L_r the stabilizer of r in $\mathrm{Aut}(\Gamma, \pi_\rho)$ and by K_r the element-wise stabilizer of $\mathrm{Res}_\Gamma(r)$ in $\mathrm{Aut}(\Gamma)$. So, $L_r \cap K_r$ is the kernel of the action of L_r in $\mathrm{Res}_\Gamma(r)$. In view of the isomorphism $\mathrm{Res}_\Gamma(r) \cong \Gamma/\pi_\rho$, $L_r \cap K_r$ is also the kernel of the action of L_r in $\Gamma/\pi_\rho \cong \Delta$.

8 Proposition.

(1) $L_r \cap K_r = 1$ *for any* $r \in \Gamma^0$.

(2) T_R *is the kernel of the action of* $\mathrm{Aut}(\Gamma, \pi_\rho)$ *on* Γ/π_ρ.

(3) $N_{\mathrm{Aut}(\Gamma)}(T_R) = \mathrm{Aut}(\Gamma, \pi_\rho)$.

PROOF. Let $k \in H_r := L_r \cap K_r$. Then $k \in L_s$ for every $s \in \Gamma^0$ collinear with r, since the lines of Γ have size 2. As k acts trivially on Γ/π_ρ, k also belongs to H_s. The connectedness of Γ now implies that $k \in H_r$ for every $r \in \Gamma^0$. Claim (1) is proved. Turning to (2), let K be the kernel of the action of $\mathrm{Aut}(\Gamma, \pi_\rho)$ on Γ/π_ρ. By (1), K acts semi-regularly on Γ^0. On the other hand, K contains T_R, which is transitive on Γ^0. Hence $K = T_R$.

We shall now prove (3). As $T_R \trianglelefteq \mathrm{Aut}(\Gamma, \pi_\rho)$, we only must prove that $N_{\mathrm{Aut}(\Gamma)}(T_R) \leq \mathrm{Aut}(\Gamma, \pi_\rho)$. By way of contradiction, suppose that g normalizes T_R but does not preserve π_ρ. Then, for some $i = 1, 2, \ldots, n$ there are i-elements X_1, X_2, Y_1, Y_2 of Γ such that $g(X_1) = X_2$, $g(Y_1) = Y_2$, $X_1 \pi_\rho Y_1$ but X_2 and Y_2 are not π_ρ-parallel. Let $t \in T_R$ map Y_1 onto X_1. Then $tgt^{-1}g^{-1}$ maps X_2 onto an element $Z := t(Y_2)$ which, being π_ρ-parallel to Y_2, cannot be π_ρ-parallel to X_2. On the other hand $gt^{-1}g^{-1} \in T_R$, as g normalizes T_R. Hence $tgt^{-1}g^{-1} = t_1 \in T_R$. Therefore $Z = t_1(X_2)$ must be π_ρ-parallel to X_2. We have reached a contradiction. $\boxed{QED}$

9 Corollary. *Suppose that* $K_r = 1$ *and that* L_r *induces on* Δ *its full automorphism group. Then* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma, \pi_\rho)$.

By Proposition 8, $\mathrm{Aut}(\Gamma, \pi_\rho)$ is the semi-direct product of T_R and L_r, for $r \in \Gamma^0$. The group $\mathrm{Aut}(\rho)$ acts naturally on Γ as a subgroup of L_u, where u stands for the point of Γ corresponding to the unit element $1 \in R$. In general, L_u is larger than $\mathrm{Aut}(\rho)$. However:

10 Corollary. *If* ρ *is homogeneous then* $L_u = \mathrm{Aut}(\rho)$. *If moreover* $K_u = 1$, *then* $T_R \,\mathrm{Aut}(\rho) = \mathrm{Aut}(\Gamma, \pi_\rho) = \mathrm{Aut}(\Gamma)$.

Let $\rho_1 : \Delta \to R_1$ be another faithful representation of Δ and φ be a *morphism* from ρ to ρ_1, namely a homomorphism $\varphi : R \to R_1$ such that $\varphi\rho = \rho_1$. Then the mapping $\mathrm{Ex}(\varphi) : \Gamma \to \Gamma_1 := \mathrm{Ex}_{\rho_1}(\Delta)$ sending $\rho(X)r$ to $\rho_1(X)\varphi(r)$ is a covering. The universal cover of $\mathrm{Ex}_\rho(\Delta)$ is the affine expansion of the universal representation of Δ (Pasini [12, (3.3)]).

4.3 A characterization of affine expansions

Let $\Gamma = \mathrm{Ex}_\rho(\Delta)$. For every $X \in \Delta^{<n-1}$, the subgroup $\rho(X) < R$, regarded as a subgroup of the stabilizer in $\mathrm{Aut}(\Gamma)$ of the point $1 \in \Gamma^0$, is contained in the element-wise stabilizer of $\mathrm{Res}^+_\Gamma(\rho(X))$. This remark entails a characterization of affine expansions:

11 Proposition. *Let Γ be a locally affine geometry of order 2, satisfying (IP2). For a point $p_0 \in \Gamma^0$, put $\Delta := \mathrm{Res}_\Gamma(p_0)$ and suppose that $\mathrm{Aut}(\Gamma)$ admits a subgroup R with the following properties:*

(1) R acts regularly on Γ^0;

(2) there exists a faithful representation $\rho : \Delta \to R$ such that, for every $l \in \mathcal{L}(p_0)$, $\rho(l)$ belongs to the element-wise stabilizer of $\mathrm{Res}^+_\Gamma(l)$ in $\mathrm{Aut}(\Gamma)$.

Then $\Gamma \cong \mathrm{Ex}_\rho(\Delta)$ and R is the translation group of the expansion Γ.

Proof. By (1), for every point $x \in \Gamma^0$ there exists a unique element $r_x \in R$ such that $r_x(p_0) = x$. We shall first prove the following:

$$r_x^{-1}(X) = r_y^{-1}(X) \text{ for every } X \in \Gamma^{>0} \text{ any two points } x, y \in P(X). \qquad (^\circ)$$

Assume first that x and y are collinear. Put $m = \{x, y\}$ and $l = r_x^{-1}(m)$. By the regularity of R on Γ^0, $r_y = r_x\rho(l)$. Hence $r_y^{-1}(X) = \rho(l)r_x^{-1}(X)$. However, either $r_x^{-1}(X) = l$ or $r_x^{-1}(X) \in \mathrm{Res}^+_\Gamma(l)$. By (2), $\rho(l)$ stabilizes $r_x^{-1}(X)$. So, $r_y^{-1}(X) = r_x^{-1}(X)$. Claim ($^\circ$) follows from this and the connectedness of $\mathrm{Res}^-_\Gamma(X)$ when $t(X) > 1$. Clearly, $r_x^{-1}(X) \in \mathrm{Res}_\Gamma(p_0)$. Put $\alpha(x) = r_x$ for $x \in \Gamma^0$ and $\alpha(X) = \rho(r_x^{-1}(X))r_x$ for $X \in \Gamma^{>0}$. By(2), this definition is consistent. It is straightforward to check that α is an isomorphism from Γ to $\mathrm{Ex}_\rho(\Delta)$. $\boxed{QED}$

5 Main results

In this section Γ is a given finite locally affine geometry of rank $n \geq 4$ and order 2. We assume that Γ is flag-transitive and satisfies the Intersection Property (IP).

Henceforth, $G \leq \mathrm{Aut}(\Gamma)$ is a given flag-transitive automorphism group of Γ. For an element $X \in \Gamma$, we denote by G_X the stabilizer of X in G. The element-wise stabilizer of $\mathrm{Res}(X)$ (respectively, $\mathrm{Res}^+(X)$, $\mathrm{Res}^-(X)$) in G_X will be denoted by K_X (resp. K_X^+, K_X^-). For $X, Y \in \Gamma$ we put $G_{X,Y} = G_X \cap G_Y$. Note that, for a point-line flag $\{p, l\}$, $K_p \leq K_l = K_l^+ \cap G_p$ and $G_{p,l}/K_l$ is the group induced by $G_{p,l}$ on $\mathrm{Res}(p,l) = \mathrm{Res}^+(l)$. Clearly, $|G_l : G_{p,l}| = 2 \geq |K_l^+ : K_l|$. So, $G_{p,l}/K_l$ has index 2 in G_l/K_l. Also, $G_{p,l}K_l^+/K_l^+ \cong G_{p,l}/K_l$. If $|K_l^+ : K_l| = 2$, then $G_{p,l}K_l^+/K_l^+ = G_l/K_l^+$ whereas, if $K_l^+ = K_l$ then $G_l/K_l^+ = G_l/K_l$ contains $G_{p,l}/K_l$ as a subgroup of index 2. Note also that, for a plane $P > l$, G_P/K_P^- is isomorphic to either A_4 or S_4, the latter being always the case when $n > 4$.

We put $C_l := \cap_{P \in \mathcal{P}(l)} K_P^-$ where, according to the conventions stated in Subsection 2.1, $\mathcal{P}(l)$ stands for the set of planes on l. In other words, C_l is the stabilizer in $G_{p,l}$ of all lines on p coplanar with l. (Indeed, by (IP), if a subgroup of $G_{p,l}$ stabilizes all lines on p coplanar with l, then it also stabilizes all planes $P > l$.) Also, C_l is the stabilizer of all lines coplanar with l, no matter if they contain p or not. Clearly, $K_p \trianglelefteq C_l$ and, by (IP), $C_l \trianglelefteq K_l$. The following conditions, stated for a given point-line flag $\{p, l\}$, will be assumed in our two main theorems.

(A1) $|K_l^+ : K_l| = 2$, namely $G_{p,l}/K_l \cong G_l/K_l^+$.

(A2) $C_l < K_l$.

(A3) $|C_l : K_p| \leq 2$.

We will also assume the following, where $\Sigma(l)$ denotes the shrinking of Γ containing the line l as a point:

(B1) $\Sigma(l) \cong \mathrm{Ex}_\varepsilon(\mathrm{Res}^+(l))$ for a suitable faithful representation $\varepsilon : \mathrm{Res}^+(l) \to E$, where E is a 2-group.

In view of the conditions we are going to consider next, we need to state a few preliminary conventions. Let $\Lambda(l)$ be the point-set of $\Sigma(l)$, namely the parallel class of l in the closure of the local parallelism of Γ. The set-wise and the element-wise stabilizers of $\Lambda(l)$ in G will be denoted by $G_{\Sigma(l)}$ and $K_{\Sigma(l)}$, respectively. Clearly, $G_{\Sigma(l)}/K_{\Sigma(l)}$ is a subgroup of $\mathrm{Aut}(\Sigma(l))$. Chosen an isomorphism $\alpha : \mathrm{Ex}_\varepsilon(\mathrm{Res}^+(l)) \to \Sigma(l)$, let π_ε^α be the α-image of the natural parallelism π_ε of $\mathrm{Ex}_\varepsilon(\mathrm{Res}^+(l))$, namely: two elements X and Y of $\Sigma(l)$ correspond in π_ε^α precisely when their pre-images $\alpha^{-1}(X)$ and $\alpha^{-1}(Y)$ correspond in π_ε. Then π_ε^α is a parallelism of $\Sigma(l)$ and extends the local parallelism σ of $\Sigma(l)$. However, in general, π_ε^α is not a completion of σ (see Corollary 7), hence $\mathrm{Aut}(\Sigma(l), \pi_\varepsilon^\alpha)$ might be a proper subgroup of $\mathrm{Aut}(\Sigma(l))$. We assume that, nevertheless, (B2) $G_{\Sigma(l)}/K_{\Sigma(l)} \leq \mathrm{Aut}(\Sigma(l), \pi_\varepsilon^\alpha)$.

By (B2), we can define the action $G_{\Sigma(l)}^\infty$ of $G_{\Sigma(l)}$ on $\mathrm{Res}^+(l)$, the latter being regarded as the geometry at infinity $\Sigma(l)/\pi_\varepsilon^\alpha$ of $(\Sigma(l), \pi_\varepsilon^\alpha)$. As $G_l < G_{\Sigma(l)}$, an action $G_l^\infty \leq G_{\Sigma(l)}^\infty$ of G_l on $\Sigma(l)/\pi_\varepsilon^\alpha$ is also defined. The following is our final assumption on $\Sigma(l)$:

(B3) $G_l^\infty = G_{\Sigma(l)}^\infty$.

The next lemma, to be proved in Subsection 6.1, is useful for a better understanding of the hypotheses of the following two theorems, which are the main results of this paper.

12 Lemma. *Assume that conditions (A2) and (A3) hold. Then $|K_p| \leq 2$ and one of the following occurs:*

(I) $K_p = C_l = 1$.

(II) $K_p = 1$ *and* $|C_l| = 2$.

(III) $|K_p| = 2$ *and* $C_l = K_p \times K_q$, *where q is the point of l different from p.*

13 Theorem (Outer Representation). *Assume that the pair (Γ, G) satisfies (A1)-(A3), (B1)-(B3) and the following condition, where p and l are as above and $P \in \mathcal{P}(l)$:*

(C) $K_l \cap Z(G_P) \leq C_l$.

Moreover, when $C_l \neq 1$ we also assume the followings:

(D1) $C_{G_p}(C_{K_l}(G_l)) = G_{p,l}$.

(D2) Let $X < G_{p,l}$ be such that $|G_{p,l} : X| = |K_l : K_l \cap X| = 2$ and either $G_{p,l} = X \times K_p$ (when $|K_p| = 2$) or $C_l \leq X$ (when $K_p = 1$). Then $\langle X^{G_p} \rangle < G_p$.

Then there exist a subgroup $R_o \leq G$ and a faithful representation $\rho_o : \mathrm{Res}(p) \to R_o$ such that $G = R_o G_p$ (namely, R_o is transitive on the point-set of Γ). Moreover all the following hold, where $\overline{G}_p$ is the normalizer of ρ_o in G_p, namely

$$\overline{G}_p := \{\, g \in N_{G_p}(R_o) \mid \rho_o(l)^g = \rho_o(g(l)) \text{ for all lines } l > p \,\}.$$

(1) $\overline{G}_p$ has index ≤ 2 in G_p and it acts flag-transitively on $\mathrm{Res}(p)$.

(2) If $C_l = 1$ then $\overline{G}_p = G_p$.

(3) If $|K_p| = 2$, then $G_p = K_p \times \overline{G}_p$.

(4) *If* $\overline{G}_p < G_p$, *then* $\mathrm{Res}(p)$ *admits another representation* $\rho_o^* : \mathrm{Res}(p) \to R_o^* \leq G$, *isomorphic to* ρ_o, $\overline{G}_p$ *also normalizes* ρ_o^* *and* $G_p \setminus \overline{G}_p$ *permutes* ρ_o *with* ρ_o^*.

(5) *If* $R_o \cap G_p = 1$ *then* $\Gamma \cong \mathrm{Ex}_{\rho_o}(\mathrm{Res}(p))$ *and* R_o *is the translation group of the affine expansion* Γ.

14 Theorem (Inner Representation). *Assume that the pair* (Γ, G) *satisfies (A2), (A3), (B1)-(B3) and that* $C_l \neq 1$. *When* $K_p = 1$, *we also assume that* (Γ, G) *satisfies hypothesis (C) of Theorem 13. Then there exist a normal subgroup* $R_i \trianglelefteq G_p$ *and a locally faithful representation* $\rho_i : \mathrm{Res}(p) \to R_i$ *such that the action of* G_p *on* R_i *by conjugation coincides with the action induced by* ρ. *Moreover:*

(1) *If* $K_p \neq 1$ *then* ρ_i *is faithful.*

(2) *Let* $K_p = 1$, *but assume that* (Γ, G) *satisfies (D1) of Theorem 13. Then* ρ_i *is faithful.*

We will prove theorems 13 and 14 in Section 5. The next corollary will be proved at the end of Section 5:

15 Corollary. *Suppose that* (Γ, G) *satisfies the hypotheses of Theorem 13 and let* ρ_o, R_o *and* $\overline{G}_p$ *be as in that theorem. Suppose moreover that* G_p/K_p *is simple. Then the followings hold:*

(1) *Either* $R_o \cap G_p = 1$ *and* $\Gamma \cong \mathrm{Ex}_{\rho_o}(\mathrm{Res}(p))$, *or* $\overline{G}_p \leq R_o$ *and* $G = R_o K_p$.

(2) *Assume that* $C_l \neq 1$ *and let* R_i *be as in Theorem 14. Then* $R_i = \overline{G}_p$ $(= G_p$ *if* $K_p = 1)$ *and* $G = R_o R_i K_p$.

6 Proof of theorems 13 and 14

We shall prove Lemma 12 first (Subsection 6.1). In the proof of Theorems 13 and 14, we shall discuss each of the cases (I), (II) and (III) of Lemma 12 separately. However, some preliminary work can be done before to split our discussion according to those cases. We shall do that in Subsections 6.1 and 6.2. The proof of Theorems 13 and 14 will take Subsections 6.3, 6.4, 6.5, 6.6 and 6.7. We will consider case (I) of Lemma 12 first (subsection 6.3). After that, we will turn to case (III), constructing the inner representation in Subsection 6.4 and the outer representation in Subsection 6.5. Case (II) will be examined for last, in subsections 6.6 and 6.7. Corollary 15 will be proved in Subsection 6.8.

6.1 Proof of lemma 12 and more on C_l, K_l and K_l^+

Throughout this subsection we only assume that (Γ, G) satisfies (A2) and (A3) for a given point-line flag $\{p, l\}$. We shall state a few preliminary results before to tackle the proof of Lemma 12.

16 Lemma. *Given a plane $P \in \Gamma^2$, G_P induces S_4 on $\mathrm{Res}^-(P)$.*

PROOF. By (A2) and the flag-transitivity of G, for every line m of P, K_m contains elements that fix m point-wise but permute the two points of P exterior to m. Hence the group G_P/K_P^- induced by G_P on the point-set X of P contains all transpositions of X. Namely, $G_P/K_P^- \cong S_4$. $\boxed{QED}$

17 Corollary. *G_p is transitive on the set of ordered pairs of coplanar lines through p.*

PROOF. This immediately follows from Lemma 16. $\boxed{QED}$

18 Lemma. *If m is a line through p coplanar with l but distinct from l, then $C_m \cap C_l = K_p$.*

PROOF. Suppose the contrary, namely $C_l \cap C_m > K_p$. Then $C_l = C_m$ by (A3). Corollary 17 and the connectedness of $\mathrm{Res}(p)$ now imply that $C_m = C_l$ for every line m on p. Hence C_l fixes all lines on p. By (IP), $C_l = K_p$, contrary to the initial assumption $C_l \cap C_m > K_p$. $\boxed{QED}$

PROOF OF LEMMA 12. We are now ready to prove Lemma 12. By (A3), either $C_l = K_p$ or $|C_l : K_p| = 2$. Suppose first that $C_l = K_p$. No mention of the particular point p of l is made in the definition of C_l. So, if $l = \{p, q\}$ we also have $C_l = K_q$. It follows that $K_p = K_q$ for any two collinear points p, q. The connectedness of Γ forces $K_p = 1$, and we have case (I). Suppose that $|C_l : K_p| = 2$ and $K_p \neq 1$ and let $l = \{p, q\}$. Then $K_p \neq K_q$. Hence $K_q < K_pK_q = \langle K_p, K_q \rangle \leq C_l$. Assumption (A3) now forces $K_pK_q = C_l$ and $|K_pK_q : K_q| = |K_p : (K_p \cap K_q)| = 2$. Let r be a third point, coplanar with l, and $m = \{p, r\}$, $n = \{q, r\}$ be the lines joining r with p and q. Then $C_m = K_pK_r$ and $C_n = K_qK_r$. Also, $C_m \cap C_n = K_r$, by Lemma 18. Therefore $K_pK_r \cap K_qK_r = K_r$. Hence $K_p \cap K_q \leq K_r$. Accordingly, $K_p \cap K_q \leq K_p \cap K_r$. By symmetry, $K_p \cap K_q = K_p \cap K_r$. Let Φ be the graph with the lines of Γ as vertices, where two lines are adjacent when they are coplanar and meet in a point. Then Φ is connected, by the residual connectedness of Γ. Hence $K_p \cap K_q = K_{p_1} \cap K_{q_1}$ for any line $l_1 = \{p_1, q_1\}$. Therefore, $K_p \cap K_q = 1$. Hence $|K_p| = |K_pK_q : K_q| = |C_l : K_q| = 2$ and $C_l = K_p \times K_q$. $\boxed{QED}$

19 Lemma. *Let X be a subgroup of G_l, transitive on $\mathcal{P}(l)$.*

(1) If either $X \leq G_{p,l}$ or $K_p = 1$, then $C_{K_l}(X) = C_l$.

(2) If $X \not\leq G_{p,l}$ and $|K_p| = 2$, then $C_{K_l}(X) = \{1, ij\} < C_l$, where i and j are the involutions of K_p and K_q respectively (and $l = \{p, q\}$).

In particular, $C_{K_l}(G_{p,l}) = C_l$ in any case. If $K_p = 1$ then $C_{K_l}(G_l) = C_l$. When $|K_p| = 2$, then $C_{K_l}(G_l) = \{1, ij\}$.

Proof. Let $g \in K_l \setminus C_l$. Then g acts non-trivially on $\mathrm{Res}^-(P)$, for some plane $P \in \mathcal{P}(l)$. Explicitly, g fixes both points of l and permutes the remaining two points of P. Let S be a 3-space on P. As $g \in K_l$, g stabilizes S and the three planes of S through l. It is now easy to see that g acts trivially on exactly one of those three planes, say P', and $P' \neq P$. On the other hand, as X is assumed to be transitive on $\mathcal{P}(l)$, X contains an element f mapping P onto P'. Clearly, g cannot centralize f. Hence $g \notin C_{K_l}(X)$. Therefore, $C_{K_l}(X) \leq C_l$. On the other hand, $C_l \trianglelefteq G_l$. Hence (1) holds when $|C_l| \leq 2$. Assume that $|K_p| = 2$. Both K_p and K_q are central in $G_{p,l}$. Hence $G_{p,l}$ also centralizes $C_l = K_p \times K_q$. On the other hand, if $g \in G_l \setminus G_{p,l}$, then g permutes p and q, and we have (2). $\boxed{QED}$

By Lemma 19, an action of K_l/C_l on $G_{p,l}$ is also defined. The following can be proved by the same argument used for (1) of Lemma 19.

20 Lemma. *$C_{K_l/C_l}(G_{p,l}) = 1$.*

As C_l is also normal in K_l^+, we can consider the quotient K_l^+/C_l. Clearly, $k^2 \in C_l$ for every $k \in K_l^+$. By this remark we immediately obtain the following:

21 Lemma. *K_l^+/C_l is an elementary abelian 2-group.*

For $k \in K_l/C_l$, put $\pi(k) = \{\, P \in \mathcal{P}(l) \mid k \in K_P^-/C_l \,\}$. (Note that $C_l \trianglelefteq K_P^-$ for every $P \in \mathcal{P}(l)$).

22 Lemma.

(1) If $k \neq 1$, then $\pi(k)$ is a geometric hyperplane of the point-line system $(\mathcal{P}(l), \Gamma^3(l))$ of $\mathrm{Res}^+(l)$.

(2) $\pi(k_1k_2) = (\pi(k_1) \cap \pi(k_2)) \cup [\mathcal{P}(l) \setminus (\pi(k_1) \cup \pi(k_2))]$.

(3) If $k_1 \neq k_2$ then $\pi(k_1) \neq \pi(k_2)$.

Proof. Let $k \in K_l/C_l$, $k \neq 1$. Then $P(k) \neq \mathcal{P}(l)$. Let S be a 3-space on l. As k fixes both points of l, it acts trivially on either exactly one or each of the three planes of S on l. So, claim (1) holds. If K_P^-/C_l contains either both k_1 and k_2 or none of them, then it also contains $k_3 = k_1k_3$. On the other hand, if K_P^-/C_l contains only one of k_1 or k_2, then it does not contain k_3. Equality (2) follows from these remarks. Finally, let $\pi(k_1) = \pi(k_2)$. Then $k_2^{-1}k_1 \in K_P^-/C_l$ for every $P \in \mathcal{P}(l)$, namely $k_2^{-1}k_1 = 1$. $\boxed{QED}$

Conversely, for $P \in \mathcal{P}(l)$, put $\pi^*(P) := (K_l \cap K_P^-)/C_l = \{ k \in K_l/C_l \mid P \in \pi(k) \}$. The mapping π^* sending $P \in \mathcal{P}(l)$ to $\pi^*(P)$ is a locally faithful representation of $\mathrm{Res}^+(l)$ in the dual of K_l/C_l, the latter being regarded as a $GF(2)$-vector space, as we may in view of Lemma 21. More explicitly:

23 Lemma.

(1) $|K_l/C_l : \pi^*(P)| = 2$ *for every* $P \in \mathcal{P}(l)$.

(2) If P_1, P_2, P_3 *are the three planes through* l *in a given* 3*-space* $S > l$*, then* $\pi^*(P_3) \supset \pi^*(P_1) \cap \pi^*(P_2)$ *and* $\pi^*(P_1) \neq \pi^*(P_2)$.

PROOF. If $k_1, k_2 \in K_l/C_l \setminus K_P^-/C_l$, then $k_1 k_2 \in K_P^-/C_l$. Hence $|K_l/C_l : \pi^*(P)| \leq 2$. If $\pi^*(P) = K_l/C_l$, namely $K_P^- = K_l$, then $K_l = \cap_{X \in \mathcal{P}(l)} K_X^- = C_l$ by the transitivity of $G_{p,l}$ on $\mathcal{P}(l)$. So, $K_l = C_l$, contrary to (A1). Claim (1) is proved. Given P_1, P_2, P_3 in the same 3-space, the inclusion $\pi^*(P_1) \cap \pi^*(P_2) \subseteq \pi^*(P_3)$ is obvious. It remains to prove that $\pi^*(P_1) \neq \pi^*(P_2)$. Suppose to the contrary that $\pi^*(P_1) = \pi^*(P_2) = H$, say. Then $P_3 \in \pi(h)$ for every $h \in H$, as $\{P_1, P_2, P_3\}$ is a line of $\mathrm{Res}^+(l)$ and $P_1, P_2 \in \pi(h)$, which is a subspace of the point-line system of $\mathrm{Res}^+(l)$ (Lemma 22 (1)). On the other hand, let $k \in K_l/C_l \setminus H$. Then $\pi(k)$, being a hyperplane of $\mathrm{Res}^+(l)$, meets the 'line' $\{P_1, P_2, P_3\}$ non-trivially. However, $\pi(k)$ contains neither P_1 nor P_2. Hence $P_3 \in \pi(k)$. It follows that $P_3 \in \pi(k)$ for all $k \in K_l/C_l$, namely $\pi^*(P_3) = K_l/C_l$. This contradicts (1). $\boxed{QED}$

Let $\Lambda_1(l)$ be the set of lines parallel to l in the local parallelism of Γ. That is, $\Lambda_1(l)$ is the neighborhood of l in the collinearity graph of $\Sigma(l)$, with $l \in \Lambda_1(l)$ by convention. The following lemma, albeit quite trivial, is crucial for the sequel.

24 Lemma. $\cap_{m \in \Lambda_1(l)} K_m \leq C_l$.

We finish this subsection by showing that assuming condition (C) of Theorem 13 is equivalent to assume that $K_l \cap Z(G_P) = 1$.

25 Lemma. $C_{C_l}(G_P) = 1$*, for every plane* $P > l$.

PROOF. Let $z \in C_{C_l}(G_P)$. By the flag-transitivity of G_P on $\mathrm{Res}^-(P)$, $z \in C_{C_m}(G_P)$ for every line m of P. By Lemma 18, $z \in K_q$ for every point q of P. Hence $z = 1$, by Lemma 12. $\boxed{QED}$

6.2 Lemmas on $G_{\Sigma(l)}$ and $K_{\Sigma(l)}$

In this subsection we assume that (Γ, G) satisfies (B1)-(B3). As in the paragraph before (B2), α is a given isomorphism from $\mathrm{Ex}_\varepsilon(\mathrm{Res}^+(l))$ to $\Sigma(l)$. Let $T := T_E^{\alpha^{-1}}$ be the α^{-1}-image of the translation group T_E of $\mathrm{Ex}_\varepsilon(\mathrm{Res}^+(l))$. For every plane $P \in \mathcal{P}(l)$, put $t_P := \alpha t_e \alpha^{-1}$, where e is the involution of $\varepsilon(P)$. Then $T = \langle t_P \rangle_{P \in \mathcal{P}(l)}$, as $E = \langle \varepsilon(P) \rangle_{P \in \mathcal{P}(l)}$. Moreover, T is regular on the point-set

$\Lambda(l)$ of Σ and normal in $\mathrm{Aut}(\Sigma(l), \pi_\varepsilon^\alpha)$. In fact, by Proposition 8, T is the kernel of the action of $\mathrm{Aut}(\Sigma(l), \pi_\varepsilon^\alpha)$ on $\Sigma(l)/\pi_\varepsilon^\alpha \cong \mathrm{Res}^+(l)$.

26 Lemma. *T is a normal subgroup of $G_{\Sigma(l)}/K_{\Sigma(l)}$. Hence $G_{\Sigma(l)}/K_{\Sigma(l)}$ is a semi-direct product of T by $G_\Sigma^\infty = G_l^\infty$.*

Proof. By (B2), $G_{\Sigma(l)}/K_{\Sigma(l)}$ normalizes T. It remains to prove that $T \leq G_{\Sigma(l)}/K_{\Sigma(l)}$. Given $P \in \mathcal{P}(l)$, let $g \in G$ map l onto the line l_1 of P parallel to l. By (B3), there is an element $f \in G_l$ that acts on $\Sigma(l)/\pi_\varepsilon^\alpha$ in the same way as g. So, gf^{-1} maps l onto l_1 and acts trivially on $\Sigma(l)/\pi_\varepsilon^\alpha$. By Proposition 8, gf^{-1} induces t_P on $\Sigma(l)$. Hence $t_P \in G_{\Sigma(l)}/K_{\Sigma(l)}$. Therefore $T \leq G_{\Sigma(l)}/K_{\Sigma(l)}$. The second claim of the lemma follows from the first one and the regularity of T on $\Lambda(l)$. $\boxed{QED}$

27 Corollary. $K_l^+ = K_{\Sigma(l)}$.

Proof. Clearly, $K_{\Sigma(l)} \leq K_l^+$. Conversely, as K_l^+ acts trivially on $\mathrm{Res}_l^+ \cong \Sigma(l)/\pi_\varepsilon^\alpha$ and, by (B2), $K_l^+/K_{\Sigma(l)}$ preserves π_ε^α by (B2), K_l^+ also acts trivially in $\mathrm{Res}^+(m)$ for every line $m \in \Lambda(l)$ coplanar with l in Γ. Namely, $K_m^+ = K_l^+$ for every such line m. By the connectedness of $\Sigma(l)$, $K_m^+ = K_l^+$ for every $m \in \Lambda(l)$. Hence $K_l^+ = K_{\Sigma(l)}$. $\boxed{QED}$

Let $\widetilde{T}$ be the pre-image of T in the projection of $G_{\Sigma(l)}$ onto $G_{\Sigma(l)}/K_{\Sigma(l)}$ and put $Z = C_{K_{\Sigma(l)}}(\widetilde{T})$. Clearly, $Z(G_{\Sigma(l)}) \leq Z \leq Z(K_{\Sigma(l)})$ and, since both $K_{\Sigma(l)}$ and $\widetilde{T}$ are normal in $G_{\Sigma(l)}$, the latter normalizes Z. In particular, if $|Z| \leq 2$, then $Z = Z(G_{\Sigma(l)})$.

28 Lemma. *Suppose that (Γ, G) satisfies (A3). Then $\widetilde{T}$ is a 2-group and $Z \cap K_l \leq C_l$. If moreover $K_l^+ \neq 1$ (as when (Γ, G) satisfies (A1) or (A2)), then $1 \neq Z$.*

Proof. T is a 2-group by (B1) and K_l^+ is a 2-group by lemmas 12 and 21. Hence $\widetilde{T}$ is a 2-group by Corollary 27. Clearly, $Z \cap K_l \leq K_m$ for every line $m \in \Lambda_1(l)$. Hence $Z \cap K_l \leq C_l$ by Lemma 24. Finally, if $K_l^+ \neq 1$ then $Z \neq 1$, as both $K_{\Sigma(l)} = K_l^+$ and $\widetilde{T}$ are 2-groups. $\boxed{QED}$

6.3 Outer representation when $C_l = 1$

In this subsection we assume that (Γ, G) satisfies (A1)-(A3), (B1)-(B3) and (C) of Theorem 13, and that $K_{p_0} = C_{l_0} = 1$ for a given point-line flag $\{p_0, l_0\}$. We shall use the following shortened notation: $G_0 := G_{p_0}$, $G_1 := G_{l_0}$, $G_{01} := G_{p_0, l_0}$, $K_0 := K_{p_0}$, $K_1 := K_{l_0}$, $K_1^+ := K_{l_0}^+$, $C := C_{l_0}$, $\Sigma := \Sigma(l_0)$ and $\Lambda := \Lambda(l_0)$.

As $C = 1$ by assumption, $K_1^+ = K_\Sigma$ is an elementary abelian 2-group (Lemma 21). Hence $Z = C_{K_\Sigma}(T)$.

29 Lemma. *$|Z| = 2$, $Z \cap K_1 = 1$ and $Z = Z(G_\Sigma)$. Moreover, $\cap_{t \in T} K_1^t = 1$.*

Proof. We have $|Z| = 2$ and $Z \cap K_1 = 1$ by Lemma 28 and $Z = Z(G_\Sigma)$ because $|Z| = 2$. The last claim of the lemma follows from Lemma 24 and the transitivity of T on the point-set Λ of Σ. $\boxed{QED}$

We denote by z_Σ the unique involution of Z.

30 Lemma. *The involution z_Σ is the only element of K_Σ that permutes the two points of l, for every line $l \in \Lambda$.*

Proof. Clearly, z_Σ has the above property, as it switches the two points of l_0 and is centralized by T, which is transitive on Λ. Conversely, suppose that $z \in K_\Sigma$ satisfies the above property. Then $zz_\Sigma \in K_l$ for every line $l \in \Lambda$. Hence $zz_\Sigma = 1$ by the third claim of Lemma 29. $\boxed{QED}$

As z_Σ is uniquely determined by Σ and the latter is uniquely determined by any of the lines $l \in \Lambda$, we can also write z_l instead of z_Σ, for $l \in \Lambda$. With this notation, we can state the following:

31 Lemma. $[z_l, z_m] = 1$ *for any two coplanar lines l, m on p_0.*

Proof. Let P be the plane on l and m. By applying Lemma 30 to $\Sigma(m)$ we see that z_m permutes l with the line l_1 of P parallel to l. Hence, it permutes z_l and z_{l_1}. However, $\Sigma(l) = \Sigma(l_1)$, whence $z_l = z_{l_1}$. Therefore z_m commutes with z_l. $\boxed{QED}$

Given a plane P on p_0, let l, m, n be the three lines of P through p_0 and put $h_P = z_l z_m z_n$. In view of Lemma 31, this definition is consistent, namely it does not depend on which order is put on the triple $\{l, m, n\}$. Note also that, if we replace p_0 with any other point p_1 of P, and l_1, m_1, n_1 are the lines of P through p_1 parallel to l, m and n, then $z_{l_1} = z_l$, $z_{m_1} = z_m$ and $z_{n_1} = z_n$, whence $h_P = z_{l_1} z_{m_1} z_{n_1}$.

32 Lemma. $h_P \in K_l$.

Proof. By Lemma 23, $\pi^*(P)$ is a hyperplane of K_l. Pick $k \in K_l \setminus \pi^*(P)$. Then k permutes m and n. Accordingly, $z_m^k = z_n$, namely

$$k^{z_m} = z_m z_n k. \tag{1}$$

(Recall that all elements involved here are involutions.) On the other hand, z_m permutes l with l_1, whence it stabilizes $\Sigma(l)$. Hence z_m normalizes $K_{\Sigma(l)} = K_l\langle z_l \rangle$. Therefore,

$$k^{z_m} = z_l^e k_1 \tag{2}$$

for a suitable $k_1 \in K_l$ and $e \in \{0, 1\}$. On the other hand, as $k \notin K_P^-$ and z_m permutes l with l_1, we also have

$$k^{z_m} \in K_{l_1} \setminus K_P^-. \tag{3}$$

If $e = 0$, (2) and (3) imply $k_1 \in (K_l \cap K_{l_1}) \setminus K_P^-$, which is a contradiction, since $K_l \cap K_{l_1} \subset K_P^-$. Therefore $e = 1$, that is:

$$k^{z_m} = z_1 k_1. \tag{4}$$

By comparing (1) with (4) we obtain that $z_l k_1 = z_m z_n k$, namely $z_l z_m z_n = k_1 k^{-1}$. Hence $h_P \in K_l$, as $k_1 k^{-1} \in K_l$. $\boxed{QED}$

33 Corollary. $h_P = 1$.

Proof. As G_P permutes l, m, n, it centralizes h_P. Hence $h_P = 1$ by Lemma 32, assumption (C) and Corollary 25. $\boxed{QED}$

34 Lemma. *For any two lines l, m on p_0, if $z_l = z_m$ then $l = m$.*

Proof. Put $p_1 = z_l(p_0)$. Then $l = \{p_0, p_1\}$. So, if $z_l = z_m$ then l and m have the same points, whence $l = m$ by (IP). $\boxed{QED}$

We can now construct a representation $\rho_o : \mathrm{Res}(p_0) \to R_o := \langle z_l \rangle_{l \in \mathcal{L}(p_0)}$, where $\mathcal{L}(p_0)$ is the set of lines on p_0. We put $\rho_o(l) = z_l$ for every $l \in \mathcal{L}(p_0)$. By Corollary 33, ρ_o is indeed a representation, faithful by Lemma 34.

35 Lemma. *R_o is transitive on the point-set of Γ and G_0 normalizes R_o.*

Proof. The second claim is obvious. We shall prove the first claim by induction on the distance $d(p, p_0)$ of a point p from p_0 in the collinearity graph of Γ. Suppose first that $d(p, p_0) = 1$. Then $p = z_l(p_0)$ where $l = \{p, p_0\}$. If $d(p, p_0) = d > 1$, let q be a point at distance $d - 1$ from p_0 and collinear with p. Then $q = r(p_0)$ for a suitable $r \in R_o$, by induction. The point $r^{-1}(p)$ is collinear with p_0. Hence $r^{-1}(p) = z_l(p_0)$ for $l = \{p_0, r^{-1}(p)\}$. Finally, $p = r z_l(p_0)$. $\boxed{QED}$

By Lemma 35, $R_o \trianglelefteq G = R_o G_0$. In this case, all claims of Theorem 13 are proved. (Claim (5) follows from Proposition 11.)

6.4 Inner representation when $|K_p| = 2$

In this subsection we assume that (Γ, G) satisfies (A2), (A3) and (B1)-(B3), and that $|K_{p_0}| = |C_{l_0} : K_{p_0}| = 2$ for a given point-line flag $\{p_0, l_0\}$. We use the same shortened notation as in Subsection 6.3, thus writing C for C_{l_0}, G_1 for G_{l_0}, and so on. Moreover, for a point p we denote by i_p the involution of K_p. So, if $l_0 = \{p_0, q_0\}$, the element $u_{l_0} := i_{p_0} i_{q_0}$ is the unique involution of C centralized by G_1 (see Lemma 19 (2)). When $K_1^+ > K_1$, u_{l_0} is also the unique involution of C centralized by K_1^+ and neither i_{p_0} nor i_{q_0} centralizes the elements of $K_1^+ \setminus K_1$. By comparing these remarks with Lemma 28, we obtain the following:

36 Lemma. *$C \geq Z \geq \langle u_{l_0} \rangle = Z(G_\Sigma)$. If $K_1^+ > K_1$, then $Z = \langle u_{l_0} \rangle$.*

The previous lemma shows that u_{l_0} is uniquely determined by Σ. The latter is uniquely determined by any of the lines $l \in \Lambda$, $u_l = u_{l_0}$ for any $l \in \Lambda$.

37 Lemma. *Given a plane P on p_0, let l, m, n be the three lines of P through p_0. Then*$[u_l, u_m] = 1$ *and* $u_l u_m u_n = 1$.

Proof. Let $\{p_0, p, q, r\}$ be the point-set of P, where $l = \{p_0, p\}$, $m = \{p_0, q\}$ and $n = \{p_0, r\}$. Then $u_l = i_{p_0} i_p$ and $u_m = i_{p_0} i_q$. In order to show that $[u_l, u_m] = 1$ we only need to prove that $[i_p, i_q] = 1$. This can be done as follows: p and q are collinear, as the lines l and m are coplanar. Therefore, if $n_1 = \{p, q\}$, we have $C_{n_1} = K_p \times K_q$, hence $[i_p, i_q] = 1$. We shall now prove that $u_l u_m u_n = 1$. By definition, $u_l u_m u_n = i_{p_0} i_p i_{p_0} i_q i_{p_0} i_r$. As shown above, the involutions i_{p_0}, i_p, i_q, i_r pairwise commute, as the points p_0, p, q, r are mutually collinear. Hence $u_l u_m u_n = i_{p_0} i_p i_q i_r = u_l u_{l_1}$, where $l_1 = \{q, r\}$. However, $l_1 \in \Lambda(l)$. Therefore $u_{l_1} = u_l$. So, $u_l u_m u_n = 1$. $\boxed{QED}$

38 Lemma. *For any two lines l, m on p_0, if $u_l = u_m$ then $l = m$.*

Proof. Let $l = \{p_0, p\}$ and $m = \{p_0, q\}$. Then $u_l = i_{p_0} i_p$ and $u_m = i_{p_0} i_q$. If $u_l = u_m$ then $i_p = i_q$, whence $p = q$. Consequently, $l = m$. $\boxed{QED}$

We define the representation $\rho_i : \mathrm{Res}(p_0) \to R_i := \langle z_l \rangle_{l \in \mathcal{L}(p_0)}$ by putting $\rho_i(l) = u_l$ for every $l \in \mathcal{L}(p_0)$. Lemmas 37 and 38 imply that ρ_i is indeed a faithful representation. As $u_l \in G_0$ for every $l \in \mathcal{L}(p_0)$, the group R_i is contained in G_0. Clearly, it is normal in G_0, as claimed in Theorem 14.

6.5 Outer representation when $|K_p| = 2$

We keep the hypotheses and the notation of Subsection 6.4, but now we assume that (Γ, G) also satisfies (A1), (C), (D1) and (D2). By Lemma 36, $Z = \langle u_{l_0} \rangle$. Put $\widetilde{Z} = C_{K_\Sigma/Z}(\widetilde{T})$.

39 Lemma. $|\widetilde{Z}| = 2$ *and* $\widetilde{Z} \cap (K_1/Z) = 1$.

Proof. We have $\widetilde{Z} \neq 1$ as both K_Σ/Z and $\widetilde{T}$ are 2-groups. Moreover, $\widetilde{Z} \cap (K_1/Z) \leq C/Z$ by Lemma 24. So, either $\widetilde{Z} \cap (C/Z) = 1$ and $|\widetilde{Z}| = 2$, or $C/Z \leq \widetilde{Z}$. Assume the latter. Given $P \in \mathcal{P}(l_0)$, let $l = \{p, q\}$ be the line of P parallel to $l_0 = \{p_0, q_0\}$ and t be a representative of t_P in $\widetilde{T}$. So, $C^t = C_l$. On the other hand, $C^t = C$, as we have assumed that $C/Z \leq \widetilde{Z}$. Hence $C = C_l$. With no loss, we may assume that t maps p_0 onto p and q_0 onto q. Hence $K_0^t = K_p$ and, since $C/Z \leq \widetilde{Z}$, $K_0^t Z = K_0 Z$. Therefore i_{p_0} is equal to either i_p or $i_p u_l = i_p(i_p i_q) = i_q$. However, this is impossible, since p_0 is collinear with either of p and q, and $K_x \neq K_y$ if x, y are collinear points. Hence $\widetilde{Z} \cap (K_1/Z) = 1$. $\boxed{QED}$

Let $\overline{Z}$ be the pre-image of $\widetilde{Z}$ in the projection of K_Σ onto K_Σ/Z. By the above, $|\overline{Z}| = 4$, with $\overline{Z} \cap K_1 = Z$. Let i_Σ, j_Σ be the two elements of $\overline{Z} \setminus Z$. The group $\overline{Z}$ might be either elementary abelian of order 2^2 or cyclic of order 4. However, in any case, $i_\Sigma j_\Sigma = u_{l_0} = i_{p_0} i_{q_0}$. The group $\overline{Z}$ is characteristic in $\widetilde{T} \trianglelefteq G_\Sigma$. Hence it is normalized by G_Σ and, since $|\overline{Z} : Z| = 2$, the group

$\overline{G}_\Sigma := C_{G_\Sigma}(\overline{Z})$ has index 2 in G_Σ. Put $\overline{G}_{01} = G_{01} \cap \overline{G}_\Sigma$, $\overline{G}_1 = G_1 \cap \overline{G}_\Sigma$, $\overline{K}_1 = K_1 \cap \overline{G}_\Sigma$, $\overline{K}_\Sigma = K_\Sigma \cap \overline{G}_\Sigma$ and $\overline{T} = \overline{G}_\Sigma \cap \widetilde{T}$.

40 Lemma. *We have $C\overline{Z} \cong D_8$, with $Z(C\overline{Z}) = Z$.*

Proof. The group $C\overline{Z}$ has order 8 and contains at least three involutions, namely i_{p_0}, i_{q_0} and $u_{l_0} = i_{p_0} i_{q_0}$, the latter being in the center of $C\overline{Z}$. On the other hand, i_Σ and j_Σ belong to $K_1^+ \setminus K_1$, hence they permute i_{p_0} with i_{q_0}. It follows that $C\overline{Z} \cong D_8$. Clearly, $Z(C\overline{Z}) = Z$. $\boxed{QED}$

41 Corollary. *We have $K_1 = \overline{K}_1 \times K_0$ and $G_{01} = \overline{G}_{01} \times K_0$. Moreover, $\overline{K}_1^+ = \overline{K}_1\overline{Z}$, $\overline{G}_1 = \overline{G}_{01}\overline{Z}$, $\overline{T}/\overline{K}_\Sigma \cong \widetilde{T}/K_\Sigma \cong T$, $\overline{G}_\Sigma/\overline{K}_\Sigma \cong G_\Sigma/K_\Sigma$.*

Proof. By Lemma 40, $K_0 \cap \overline{G}_{01} = 1$. All claims of the corollary follow from this remark. $\boxed{QED}$

42 Lemma. $C_{\overline{K}_\Sigma}(\overline{T}) = \overline{Z}$

Proof. Suppose that $\overline{T}$ centralizes $k \in \overline{K}_\Sigma$. Modulo replacing k with ki_Σ, we may assume that $k \in \overline{K}_1$. However, $\overline{T}/\overline{K}_\Sigma \cong T$ by Corollary 40. Hence, for every plane $P \in \mathcal{P}(l_0)$, t_P has a representative $t \in \overline{T}$. So, $k^t = k$, hence $k \in K_1^t = K_l$, where l is the line of P parallel to l_0. As P is an arbitrary plane on l_0, $k \in \cap_{l \in \Lambda_1(l_0)} K_l$ and Lemma 24 forces $k \in C$. Hence $k \in Z$, as $\overline{K}_1 \cap C = Z$. $\boxed{QED}$

43 Lemma. *The elements i_Σ and j_Σ are involutions.*

Proof. Suppose they are not. Then $i := i_\Sigma$ and $j := j_\Sigma$ have order 4, $j = i^3$ and $ij = i^2 = j^2 = u := u_{l_0}$. The elements of $\overline{K}_\Sigma$ also have order 2 or 4, since $\overline{K}_\Sigma = \overline{K}_1\overline{Z}$ by Corollary 41, $|\overline{Z}| = 4$ and $\overline{K}_1/Z$ is an elementary abelian 2 group, by Lemma 21. Let U_2 be the set of elements of $\overline{K}_1$ of order 2 different from u and U_4 be the set of elements of $\overline{K}_1$ of order 4. Similarly, let V_2 be the set of elements of $\overline{K}_1^+ \setminus \overline{K}_1$ of order 2 and V_4 the set of elements of $\overline{K}_1^+ \setminus \overline{K}_1$ of order 4, but different from i and j. If $x \in U_2$ then $xi \in V_4$. Let $x \in U_4$. Then $x^2 \in Z = \overline{K}_1 \cap C$, namely $x^2 = u$. As $i^2 = u$ we obtain that $(xi)^2 = u^2 = 1$, whence $xi \in V_2$. Therefore, $|U_2| = |V_4|$ and $|U_4| = |V_2|$. Let W_2 be the set of elements of $\overline{K}_1^+$ of order 2 different from u and W_4 the set of elements of $\overline{K}_1^+$ of order 4 different from i and j. Then $W_2 = U_2 \cup V_2$ and $W_4 = U_4 \cup V_4$. By the above, $|W_2| = |W_4|$. So, if $n := |W_2| = |W_4|$ and $2^d := |\overline{K}_1^+|$, we have

$$n = (2^d - 4)/2 = 2^{d-1} - 2. \qquad (^\diamond)$$

In particular, n is divisible by 4 only if $d = 2$. Suppose $d > 2$ and consider the orbit of $\overline{T}$ on $\overline{K}_1^+$. As $\overline{T}$ fixes each of the elements u, i and j, it stabilizes W_2 and W_4. Moreover, by Lemma 41, no element of $W_2 \cup W_4$ is fixed by $\overline{T}$. However, by $(^\diamond)$, $n/2$ is odd whereas $\overline{T}$ is a 2-group. Hence at least one orbit of $\overline{T}$ on W_2 has size 2. The same holds for the orbits on W_4. Let $O_1 = \{k, h\}$ be one of those orbits. Suppose that $h = ku$. For every plane $P \in \mathcal{P}(l_0)$, t_P has a representative $t \in \overline{T}$. As $O_1 = \{k, ku\}$, either $k^t = k$ or $k^t = ku$. Modulo replacing k with ki (as we may, since $\overline{T}$ centralizes i) we may assume that $k \in \overline{K}_1$. So, the relation $k^t = ku$ implies that $k \in K_1^t$. Lemma 24 now forces $k \in C$. (Compare the proof of Lemma 42.) Hence $k = u$, as $\overline{K}_1 \cap C = Z$. We have reached a contradiction.

Therefore $h \neq ku$. This shows that there is another orbit of size 2, obtained from O_1 by multiplication by u, say $O_2 = \{ku, hu\}$. So, the orbits of $\overline{T}$ on W_2 of size 2 are partitioned in pairs, two orbits in the same pair being permuted by multiplication by u. The same holds for the orbits on W_4, but now we may forget about them. Let s be the number of orbits of $\overline{T}$ on W_2 of size 2 and X be the union of those orbits. By the above, s is even. Hence $|X| = 2s$ is multiple of 4. The set $W_2 \setminus X$ is partitioned in orbits of size 2^r for suitable exponents $r > 1$. Thus, 4 divides n. Therefore $d = 2$, contrary to our assumption. $\boxed{QED}$

44 Lemma. *The group* $\overline{G}_0 := \langle \overline{G}_{01}^{G_0} \rangle$ *acts flag-transitively on* $\mathrm{Res}(p_0)$.

Proof. $\overline{G}_{01}$ acts as G_{01} in $\mathrm{Res}^+(l_0)$. In order to obtain the conclusion, we only must show that, if $\overline{G}_{0,P} := \overline{G}_0 \cap G_P$ for a given plane $P \in \mathcal{P}(l_0)$, then $\overline{G}_{0,P}$ induces S_3 on the triple $X := \{l_0, l_1, l_2\}$ of lines of P through p_0. By Lemma 23, K_1 contains an element k that fixes l_0 and permutes l_1 with l_2. As $K_1 = K_0 \times \overline{K}_1$, we can assume that $k \in \overline{K}_1$. Let $g \in G_0 \cap G_P$ be such that $g(l_0) = l$. Then $k^g \in \overline{G}_{0,P}$ permutes l_0 with l_2. It is now clear that $\overline{G}_{0,P}$ acts as S_3 on X. $\boxed{QED}$

45 Corollary. $\overline{G}_0 \cap G_{01} = \overline{G}_{01}$ *and* $|G_0 : \overline{G}_0| = 2$.

Proof. Clearly, $\overline{G}_{01} \leq \overline{G}_0 \cap G_1$ and, as $|G_{01} : \overline{G}_{01}| = 2$, either $\overline{G} \cap G_{01} = \overline{G}_{01}$ or $G_{01} \leq \overline{G_0}$. Moreover, $\overline{G}_0$ is transitive on $\mathcal{L}(p_0)$. Hence $|\overline{G}_0| = |\mathcal{L}(p_0)| \cdot |\overline{G}_0 \cap G_{01}| \leq |\mathcal{L}(p_0)| \cdot |G_{01}| = |G_0|$. On the other hand, $\overline{G}_0 < G_0$ by (D2). Hence $\overline{G}_0 \cap G_{01} \neq G_{01}$. Therefore $\overline{G}_0 \cap G_{01} = \overline{G}_{01}$. $\boxed{QED}$

We shall now consider the action of $\overline{G}_0 = \langle \overline{G}_{01}^{G_0} \rangle$ by conjugation on the set $\mathcal{I}$ of ordered pairs $(i_{\Sigma(l)}, j_{\Sigma(l)})$ for $l > p_0$. By Lemma 44, $\overline{G}_0$ is transitive on the set of unordered pairs $\{i_{\Sigma(l)}, j_{\Sigma(l)}\}$ with $l > p_0$. Hence $\overline{G}_0$ has at most two orbits on $\mathcal{I}$.

46 Lemma. $\overline{G}_0$ *has exactly two orbits on* $\mathcal{I}$.

Proof. Suppose to the contrary that $(i_\Sigma^g, j_\Sigma^g) = (j_\Sigma, i_\Sigma)$ for some $g \in \overline{G}_0$. Then $u_{l_0}^g = u_{l_0}$, since $i_\Sigma j_\Sigma = u_{l_0}$. However, $\langle u_{l_0} \rangle = C_{K_1}(G_1)$ by Lemma 19. Hence $g \in G_{01}$ by assumption (D1). So, $g \in \overline{G}_{01} = G_{01} \cap \overline{G}_0$. However, i_Σ and j_Σ are central in $\overline{G}_{01}$. We have reached a contradiction. $\boxed{QED}$

Let I and J be the two orbits of $\overline{G}_0$ on $\mathcal{I}$. Pick one of them, say I. For every line $l > p_0$ we put $z_l := i_{\Sigma(l)}$ and $z_l^* := j_{\Sigma(l)}$, where $(i_{\Sigma(l)}, j_{\Sigma(l)}) \in I$. By Lemma 43, z_l an z_l^* are involutions. They can be characterized as follows:

(z_l, z_l^*) is the unique pair of I formed by involutions of $K_{\Sigma(l)}$ that permute the points of every line of $\Lambda(l)$. In particular, if $l_1 \in \Lambda(l)$ then $z_{l_1} = z_l$ and $z_{l_1}^* = z_l^*$.

47 Lemma. *$[z_l, z_m] = [z_l^*, z_m^*] = 1$ for any two distinct coplanar lines $l, m > p_0$.*

Proof. We only need to prove that $[z_l, z_m] = 1$. If $[z_l, z_m] \neq 1$, then $z_m z_l z_m = z_l^* = z_l u_l$. Similarly, $z_l z_m z_l = u_m z_m$. Therefore $z_l z_l u_l = z_l z_m z_l z_m = u_m z_m z_m$. Hence $u_l = u_m$, which is a contradiction. $\boxed{QED}$

48 Lemma. *Give a plane $P > p_0$, let l, m, n be the three lines of P through p_0. Put $h_P := z_n z_m z_l$ and $h_P^* := z_n^* z_m^* z_l^*$ and $\overline{K}_l := K_l \cap \overline{G}_0$. Then h_P and h_P^* belong to $\overline{K}_l$ and they are centralized by $\overline{G}_0 \cap G_P$.*

Proof. The proof that $h_P \in \overline{K}_l$ is similar to that of Lemma 32. We first choose $k \in \overline{K}_l \setminus K_P^-$. As $k \in \overline{G}_0$, it preserves I. Hence $z_m^k = z_n$. Now we can continue as in the proof of Lemma 32, but recalling that all elements involved here, namely k, z_n, z_m and z_l, belong to $\overline{K}_{\Sigma(l)}$. We leave details to the reader. The second claim of lemma follows from the fact that $\overline{G}_0 \cap G_P$ permutes the lines l, m, n and stabilizes each of I and J. $\boxed{QED}$

49 Corollary. *Given P, l, m, n as in Lemma 48, we have $h_P = h_P^* = 1$.*

Proof. We have already proved that $u_l u_m u_n = 1$ (see Lemma 37). On the other hand, $h_P = z_l z_m z_n = z_l^* u_l z_m^* u_m z_n^* u_n = h_P^* u_l u_m u_n$. Hence $h_P = h_P^*$. It remains to show that $h_P = 1$. In view of (C), Corollary 25 and Lemma 48, we only need to prove that G_P centralizes h_P. By Lemma 47 and the second claim of Lemma 48, h_P is centralized by $\overline{G}_P := \langle \overline{G}_0 \cap G_P, z_l \rangle$. As $|G_0 : \overline{G}_0| = 2$, we have $|G_P : \overline{G}_P| \leq 2$. In fact $G_P = \overline{G}_P K_0$. On the other hand, if $k := i_{p_0}$ then $h_P^k = h_P^*$. However, $h_P^* = h_P$. So, G_P centralizes h_P. $\boxed{QED}$

By Corollary 49, the mapping $\rho_o : \mathrm{Res}(p_0) \to R_o := \langle z_l \rangle_{l \in \mathcal{L}(p_0)}$ is a representation of $\mathrm{Res}(p_0)$. As in Lemma 34, $z_l \neq z_m$ if $l \neq m$. Hence ρ_o is faithful. As in Lemma 35, one can prove that R_o is transitive on Γ^0. Hence $G = R_o G_0$. Clearly, $\overline{G}_0$ normalizes ρ_o and K_0 switches ρ_o with $\rho_o^* : l \to z_l^*$.

6.6 Inner representation when $K_p = 1$ and $|C_l| = 2$

Now (Γ, G) satisfies (A2)-(A3), (B1)-(B3) and (C), and we assume that $K_0 = 1$ and $|C| = 2$ (notation as in the previous three subsections, relatively to a given point-line flag $\{p_0, l_0\}$). By Lemma 28, Z has order 2 or 4.

50 Lemma. *$C \leq Z$.*

Proof. Suppose to the contrary that $Z \cap C = 1$. Let $\widetilde{Z} = C_{K_\Sigma/Z}(\widetilde{T})$. Then $\widetilde{Z} \neq 1$, as both K_Σ/Z and $\widetilde{T}$ are 2-groups. Moreover, $\widetilde{Z} \leq CZ/Z$ by Lemma 24. Hence $\widetilde{Z} = CZ/Z$. It follows that CZ contains exactly two subgroups $C^+ := C$

and C^- such that, for every $l \in \Lambda$, either $C_l = C^+$ or $C_l = C^-$. For $\varepsilon \in \{+,-\}$, let $\Lambda^\varepsilon = \{\, l \in \Lambda \mid C_l = C^\varepsilon \,\}$. Then $\{\Lambda^+, \Lambda^-\}$ is a partition of Λ, $l_0 \in \Lambda^+$ and $\widetilde{T}$ permutes Λ^+ with Λ^-, whereas G_1 stabilizes both Λ^+ and Λ^-, as it centralizes $C = C^+$. Suppose that Λ^+ contains a line $l \neq l_0$ coplanar with l. As G_1 is transitive on $\mathcal{P}(l_0)$, Λ^+ contains all lines of Λ coplanar with l. Hence, by the transitivity of T on Λ, any two coplanar lines of Λ belong to the same class Λ^ε. This forces $\Lambda^+ = \Lambda$, which is a contradiction. Therefore, $l \in \Lambda^-$ for every line $l \neq l_0$ coplanar with l_0 and, by the transitivity of T on Λ, any two distinct coplanar lines of Λ belong to opposite classes. Consider now a 3-space S on l_0 and let l_1, l_2, l_3 be the lines of S parallel to l_0 but distinct from l_0. By the above, $l_1, l_2, l_3 \in \Lambda^-$. On the other hand, l_1 and l_2 are coplanar, hence they belong to opposite classes. We have reached a final contradiction. $\boxed{QED}$

Let u_l be the involution of C_l. By Lemma 50, $u_l = u_{l_0}$ for every line $l \in \Lambda$. Hence $[u_l, u_m] = 1$ for any two coplanar lines $l, m > p_0$.

Given a plane $P > p_0$, put $h_P = u_l u_m u_n$ where l, m, n are the three lines of P through p_0.

51 Lemma. *$h_P \in K_l$.*

Proof. The proof is similar to that of Lemma 32, but easier. Given $k \in K_l \setminus \pi^*(P)$, we get $u_m^k = u_n$, hence:

$$k^{u_m} = u_m u_n k. \tag{5}$$

On the other hand,

$$k^{u_m} = k_1 \tag{6}$$

for a suitable $k_1 \in K_l$, as u_m stabilizes p_0 and l. By comparing (5) and (6) we see that $u_m u_n = k_1 k^{-1} \in K_l$. Hence $h_P = u_l k_1 k^{-1} \in K_l$. $\boxed{QED}$

52 Corollary. *$h_P = 1$.*

Proof. This follows from Lemma 51 by applying (C), as in the proof of Corollary 33. $\boxed{QED}$

53 Lemma. *Suppose that (D1) holds. Then, for any two lines $l, m \in \mathcal{L}(p_0)$, we have $u_l = u_m$ only if $l = m$.*

Proof. Suppose that $u_l = u_m$ and let $g \in G_0$ map l onto m. Then g centralizes $C_l = C_m$. By (D1), $g \in G_l$. Hence $l = m$. $\boxed{QED}$

We put $R_i := \langle u_l \rangle_{l \in \mathcal{L}(p_0)}$ and $\rho_i(l) = u_l$ for every line $l \in \mathcal{L}(p_0)$. By Corollary 52, ρ_i is a locally faithful representation of $\mathrm{Res}(p_0)$. By Lemma 53, if (D1) holds then ρ_i is faithful.

6.7 Outer representation when $K_p = 1$ and $|C_l| = 2$

Again, $K_0 = 1$ and $|C| = 2$, but now (Γ, G) is assumed to satisfy (A1)-(A3), (B1)-(B3) and (C), (D1), (D2). By Lemmas 28 and 50, either $Z = C$ or $C < Z$ with $|Z : C| = 2$. The arguments of Subsection 6.5 work well for either of these cases. In the latter case we only need to rephrase the second part of Subsection 6.5, with G_0 now playing the role of $\overline{G}_0$.

When $Z = C$, the arguments of Subsection 6.5 apply but for two exceptions, when we must prove that $|K_1 : \overline{K}_1| = 2$ and that $\overline{G}_0$ is flag-transitive. Explicitly, we consider $\widetilde{Z} := C_{K_\Sigma/Z}(\widetilde{T})$ and its pre-image $\overline{Z}$ in K_Σ. As in Subsection 6.5, $\overline{Z}$ has order 4 and one can prove that the elements of $\overline{Z} \setminus Z$, say i_Σ and j_Σ, are involutions. Put $\overline{G}_{01} = G_{01} \cap \overline{G}_\Sigma$ and $\overline{K}_1 = \overline{G}_{01} \cap K_1$. Clearly, $C \leq \overline{K}_1$ and $|K_1 : \overline{K}_1| \leq 2$.

54 Lemma. *$|K_1 : \overline{K}_1| = 2$.*

Proof. Suppose to the contrary that $\overline{K}_1 = K_1$. Then $\overline{Z} \leq Z(K_1^+)$. Therefore the action of $t_P \in T$ on $\overline{Z}$ does not depend on the choice of its representative in $\widetilde{T}$. On the other hand, the centralizer $\overline{T}$ of $\overline{Z}$ in $\widetilde{T}$ has index 2 in $\widetilde{T}$, as $C_{K_\Sigma}(\widetilde{T}) = Z < \overline{Z}$. Hence $\widehat{T} := C_T(\overline{Z})$ has index 2 in T. If $t_P \in \widehat{T}$ for a plane $P \in \mathcal{P}(l_0)$, then $t_P^g \in \widehat{T}$ for every $g \in G_{01}$. Indeed, t_P fixes each of i_Σ and j_Σ whereas g stabilizes $\{i_\Sigma, j_\Sigma\}$ as a set. By the same argument, if $t_P \in T \setminus \widehat{T}$ then $t_P^g \in T \setminus \widehat{T}$ for every $g \in G_{01}$. As G_{01} is transitive on $\mathcal{P}(l_0)$, either $t_P \in \widehat{T}$ for every $P \in \mathcal{P}(l_0)$ or $t_P \in T \setminus \widehat{T}$ for every such P. In the first case we have $T = \widehat{T}$, contrary to what we have established above. Hence $t_P \in T \setminus \widehat{T}$ for every $P \in \mathcal{P}(l_0)$. Consider now a 3-space S on l_0 and let P_1, P_2, P_3 the three planes of S through l_0. Then $t_{P_1} t_{P_2} t_{P_3} = 1$. On the other hand, as $|T : \widehat{T}| = 2$, a product of three elements of $T \setminus \widehat{T}$ is always $\neq 1$. We have reached a final contradiction. $\boxed{QED}$

55 Lemma. *The group $\overline{G}_0 := \langle \overline{G}_{01}^{G_0} \rangle$ acts flag-transitively on* $\mathrm{Res}(p_0)$.

Proof. By Lemma 54, $\overline{G}_{01}$ acts as G_{01} in $\mathrm{Res}^+(l_0)$. As in the proof of Lemma 44, we must show that, given a plane $P \in \mathcal{P}(l_0)$, the group $\overline{G}_{0,P} := \overline{G}_0 \cap G_P$ induces S_3 on the triple $X := \{l_0, l_1, l_2\}$ of lines of P through p_0. In view of this, we must prove that $\overline{K}_1$ contains an element $k \notin K_P^-$. Suppose it doesn't. Then $\overline{K}_1 \subset K_P^-$, and this happens for every $P \in \mathcal{L}(l_0)$, since $\overline{G}_{01}$ is transitive on $\mathcal{P}(l_0)$. Therefore $\overline{K}_1 = C$. Consequently, $\pi^*(P)$ is trivial for every $P \in \mathcal{P}(l_0)$, by Lemma 23 (1). Hence $\mathcal{P}(l_0)$ only contains one plane, by Lemma 23 (2). But this conclusion is absurd. Therefore $\overline{K}_1$ contains an element $k \notin K_P^-$. Now we can go on as in the proof of Lemma 44. We omit the details. $\boxed{QED}$

Having proved the previous two lemmas, we can continue just as in Subsection 6.5. Eventually, we get a faithful representation $\rho_o : \mathrm{Res}(p_0) \to R_o :=$

$\langle z_l \rangle_{l \in \mathcal{L}(p_0)}$. We leave details for the reader.

6.8 Proof of corollary 15

Assume that (Γ, G) satisfies the hypotheses of Theorem 13 and that $C_l \neq 1$ and G_p/K_p is simple. All claims of Corollary 15 easily follow from theorems 13 and 14, except the following:

56 Lemma. *Let* $|K_p| = 2$ *and* $R_o \cap G_p \neq 1$. *Then* $R_o \geq \overline{G}_p$.

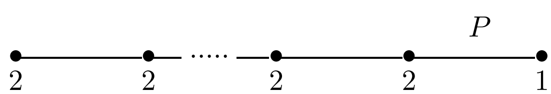
PROOF. Suppose that $R_o \not\geq \overline{G}_p$. Then, since $\overline{G}_p \cong G_p/K_p$ is simple and normalizes R_o, and $R_o \cap G_p \neq 1$ by assumption, $R_o \cap G_p = K_p$. Hence $R_o = R_o^*$, as K_p switches ρ_o with ρ_o^*. Hence $u_l = z_l z_l^* \in R_o$, for every line $l > p$. Therefore, for every such line $l = \{p, q\}$, R_o contains $i_q = u_l i_p$. It follows that $C_l \leq R_o$ for every $l \in \mathcal{L}(p)$. However, $\langle C_l \rangle_{l \in \mathcal{L}(p)} = G_p$, since G_p/K_p is simple by assumption. Hence $R_o \geq G_p$, contrary to the assumption that $R_o \not\geq \overline{G}_p$. $\boxed{QED}$

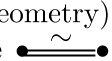
7 Two applications of theorems 13 and 14

7.1 A survey of a class of c-extended P- and T-geometries

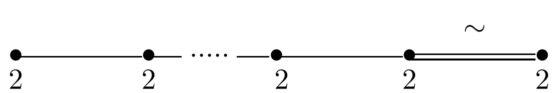
We recall that a *P-geometry* of rank $n \geq 2$ is a geometry for the following diagram, where P is the dual of the Petersen graph (as in [5] and [6]):

(P_n)



We also recall that the generalized quadrangle $W(2)$ admits a triple cover T, called the *tilde geometry*, with $\mathrm{Aut}(T) \cong 3^{\cdot}S_6$ (Ronan and Stroth [16, page 67]; see also Pasini and Van Maldeghem [13] for more information on this geometry). A *T-geometry* is a geometry belonging to the following diagram, where  stands for the tilde geometry:

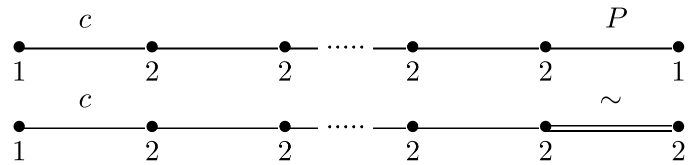
(T_n)



A *c-extended P-geometry* (a *c-extended T-geometry*) of rank $n \geq 3$ is a locally affine geometry of order 2 where point-residues are P-geometries (T-geometries) of rank $n-1$. The following diagrams describe c-extended P- and T-geometries:

$(c.P_{n-1})$

$(c.T_{n-1})$



In the sequel, c-extended P- and T-geometries of rank n will also be called $c.P_{n-1}$- and $c.T_{n-1}$-geometries, for short.

Flag-transitive P- and T-geometries are classified (Ivanov and Shpectorov [7]; see also Ivanov [5] and Ivanov and Shpectorov [6]). We summarize that classification in the following table. We put the type of the geometry in the first column of the table, with the convention that P_n (respectively, T_n) means 'P-geometry (T-geometry) of rank n'. The full automorphism group of the geometry is recorded in the second column. In the third column we give the considered geometry a name, for further reference. Isomorphism types of point-residues are recorded in the fourth column. In the last column we note if the geometry is a 2-quotient of another geometry of the list. If nothing is written in that column, then the considered geometry is 2-simply connected. In the last row, $e(n) := (2^n - 1)(2^{n-1} - 1)/3$ and, when $n = 3$, $T_{n-1}(3S_6) = T$ (the tilde geometry).

Table 1. Flag-transitive P- and T-geometries of rank ≥ 3

type	group	name	residue	
P_3	$3^{\cdot}\mathrm{Aut}(M_{22})$	$P_3(3M_{22})$	P	
P_3	$\mathrm{Aut}(M_{22})$	$P_3(M_{22})$	P	quot. of $P_3(3M_{22})$
P_4	M_{23}	$P_4(M_{23})$	$P_3(M_{22})$	
P_4	$3^{23\cdot}Co_2$	$P_4(3^{23}Co_2)$	$P_3(3M_{22})$	
P_4	Co_2	$P_4(Co_2)$	$P_3(M_{22})$	quot. of $P_4(3^{23}Co_2)$
P_4	J_4	$P_4(J_4)$	$P_3(3M_{22})$	
P_5	$3^{4371\cdot}BM$	$P_5(3^{4371}BM)$	$P_4(3^{23}Co_2)$	
P_5	BM	$P_5(BM)$	$P_4(Co_2)$	quot. of $P_5(3^{4371}BM)$
T_3	M_{24}	$T_3(M_{24})$	T	
T_3	He	$T_3(He)$	T	
T_4	Co_1	$T_4(Co_1)$	$T_3(M_{24})$	
T_5	M	$T_5(M)$	$T_4(Co_1)$	
T_n	$3^{e(n)\cdot}S_{2n}(2)$	$T_n(3S_6)$	$T_{n-1}(3S_6)$	

Flag-transitive $c.P_{n-1}$- and $c.T_{n-1}$-geometries exist where the upper residues of the elements of Γ^{n-4} are isomorphic to $P_3(3M_{22})$, $T_3(3S_6(2))$ or $T_3(He)$. We refer to Stroth and Wiedorn [18] for a survey of examples of this kind. One of them, of type $c.P_4$ and with point-residues isomorphic to $P_4(J_4)$, will be discussed at the end of this section. However, for the moment, we only consider flag-transitive $c.P_{n-1}$- and $c.T_{n-1}$-geometries of rank $n > 3$ satisfying the following:

($*$) the upper residues of the elements of Γ^{n-4} are isomorphic to $P_3(M_{22})$ or $T_3(M_{24})$, according to whether Γ is of type $c.P_{n-1}$ or $c.T_{n-1}$.

These geometries have been classified by Fukshansky and Wiedorn [3], who did the $c.P_3$-case, and Stroth and Wiedorn [17], who did the rest. The next table summarizes that classification. The table is organized in the same way as Table 1, except that now, when nothing is written in the last column, the geometry is $(n-1)$-simply connected, but it might not be 2-simply connected.

Table 2. Flag-transitive $c.P_{n-1}$- and $c.T_{n-1}$-geometries of rank $n > 3$, satisfying ($*$)

type	group	name	residue	
$c.P_3$	M_{24}	$EP_3(M_{24})$	$P_3(M_{22})$	
$c.P_3$	$2^{11}:\mathrm{Aut}(M_{22})$	$EP_3(2^{11}M_{22})$	$P_3(M_{22})$	
$c.P_3$	$2^{10}:\mathrm{Aut}(M_{22})$	$EP_3(2^{10}M_{22})$	$P_3(M_{22})$	quot. of $EP_3(2^{11}M_{22})$
$c.P_3$	$2^{\cdot}U_6(2):2$	$EP_3(2U_6(2))$	$P_3(M_{22})$	
$c.P_3$	$U_6(2):2$	$EP_3(U_6(2))$	$P_3(M_{22})$	quot. of $EP_3(2U_6(2))$
$c.P_4$	M_{24}	$EP_4(M_{24})$	$P_4(M_{23})$	
$c.P_4$	Co_1	$EP_4(Co_1)$	$P_4(Co_2)$	
$c.P_4$	$2^{23}:Co_2$	$EP_4(2^{23}Co_2)$	$P_4(Co_2)$	
$c.P_4$	$2^{22}:Co_2$	$EP_4(2^{22}Co_2)$	$P_4(Co_2)$	quot. of $EP_4(2^{23}Co_2)$
$c.P_5$	M	$EP_5(M)$	$P_5(BM)$	
$c.P_5$	$2^{\cdot}(BM \wr Z_2)$	$EP_5(2BM^2)$	$P_5(BM)$	
$c.P_5$	$BM \wr 2$	$EP_5(BM^2)$	$P_5(BM)$	quot. of $EP_5(2BM^2)$
$c.T_3$	$2^{11}:M_{24}$	$ET_3(2^{11}M_{24})$	$T_3(M_{24})$	
$c.T_4$	$2^{24}:Co_1$	$ET_4(2^{24}Co_1)$	$T_4(Co_1)$	
$c.T_5$	$M \wr 2$	$ET_5(M^2)$	$T_5(M)$	

As pointed out by Stroth and Wiedorn [18], if Γ is as in lines 2, 3, 8, 9 or 11-15 of Table 2, then Γ is the affine expansion of a homogeneous representation $\mathrm{Res}(p)$ for $p \in \Gamma^0$. In lines 2, 8, 11 and 13-15 that representation is universal. (Note that the universal representation groups of $P_5(BM)$ and $T_5(M)$ are the non-split central extension $2^{\cdot}BM$ of BM and the group M itself, respectively; see [6]).

The structures at infinity of $EP_3(M_{24})$, $EP_4(Co_1)$ and $EP_5(M)$ are isomorphic to $T_3(M_{24})$, $T_4(Co_1)$ and $T_5(M)$, respectively. The structure at infinity of $EP_4(M_{24})$ is the direct sum of a single point and a geometry dually isomorphic to the well known $C_2.L$-geometry for M_{24} (see Pasini and Wiedorn [14, Section 7.4]).

It is known (Meixner [10]) that only two simply connected flag-transitive $c.P_2$-geometries exist. They are infinite, with automorphism groups of the form $X:S_5$ and $Y:S_6$ for suitable infinite groups X and Y (in fact, X is the universal representation group of the dual Petersen graph P). They admit several (perhaps, infinitely many) flag-transitive finite quotients, but only three of them will be considered in the sequel. We list them below, together with the unique $c.T_2$-geometry we will consider in this section.

type	$c.P_2$	$c.P_2$	$c.P_2$	$c.T_2$
group	$3^{\cdot}S_6$	$2^6:S_5$	$2^5:S_5$	$2^6:3^{\cdot}S_6$
name	$EP_2(3S_6)$	$EP_2(2^6S_5)$	$EP_2(2^5S_5)$	$ET_2(2^63S_6)$

The geometry at infinity of $EP_2(3S_6)$ is isomorphic to the tilde geometry T (Pasini and Wiedorn [14, section 7.3]). $EP_2(2^6S_5)$ is the affine expansion of the universal abelian representation of P, whereas $EP_2(2^5S_5)$ and $ET_2(2^63S_6)$ arise from homogeneous but non-universal abelian representations of P and T, respectively. The $c.P_{n-1}$- and $c.T_{n-1}$-geometries of Table 2 form series of shrinkings, as shown in the following table, where the symbol $\prec$ stands for the words 'is a shrinking of':

Table 3.

$EP_2(3S_6)$	$\prec$	$EP_3(M_{24})$	$\prec$	$EP_4(M_{24})$		
$EP_2(3S_6)$	$\prec$	$EP_3(M_{24})$	$\prec$	$EP_4(Co_1)$	$\prec$	$EP_5(M)$
$EP_2(2^6S_5)$	$\prec$	$EP_3(2^{11}M_{22})$	$\prec$	$EP_4(2^{23}Co_2)$	$\prec$	$EP_5(2BM^2)$
$EP_2(2^5S_5)$	$\prec$	$EP_3(2^{10}M_{22})$	$\prec$	$EP_4(2^{22}Co_2)$	$\prec$	$EP_5(BM^2)$
?	$\prec$	$EP_3(2U_6(2))$				
?	$\prec$	$EP_3(U_6(2))$				
$ET_2(2^63S_6)$	$\prec$	$ET_3(2^{11}M_{24})$	$\prec$	$ET_4(2^{24}Co_1)$	$\prec$	$ET_5(M^2)$

The question marks in rows 5 and 6 are due to lack of information on the $c.P_2$-geometries that arise as shrinkings of $EP_3(2U_6(2))$ and $EP_3(U_6(2))$. Anyhow, $EP_3(U_6(2))$ and $EP_3(2U_6(2))$ will play almost no role in the sequel.

7.2 Characterizations by ultimate shrinkings

The following is proved in [14, Proposition 7.8] (see also Stroth and Wiedorn [17], where the same conclusions are obtained, but starting from $n \geq 5$ and exploiting the classification obtained by Fukshansky and Wiedorn [3] for the case of $n = 4$).

57 Proposition. *Let* Γ *be a flag-transitive* $c.P_{n-1}$*-geometry satisfying* $(*)$*, with* $n \geq 4$*. Suppose that the ultimate shrinkings of* Γ *are isomorphic to* $EP_2(3S_6)$*. Then* Γ *is one of* $EP_3(M_{24})$*,* $EP_4(M_{24})$*,* $EP_4(Co_1)$ *or* $EP_5(M)$*.*

In the sequel (Theorem 60) we will show how to exploit theorems 13 and 14 to classify flag-transitive $c.P_{n-1}$- and $c.T_{n-1}$-geometries satisfying $(*)$, with $n > 4$ and, in the $c.P_{n-1}$-case, with ultimate shrinkings not isomorphic to $EP_2(3S_6)$. We do not claim to prove anything new here. Indeed, the statement we will prove is a piece of the classification of [17]. We only offer a new proof.

We may assume $n \leq 6$, as no flag-transitive P_{n-1}- or T_{n-1}-geometry of rank $n-1 \geq 6$ exists where $(*)$ holds (see Ivanov and Shpectorov [6]). On the other hand, we will not consider the case of $n = 4$ since hypothesis (C) of Theorem 13 cannot be proved in that case. So, we take that case as settled and we will freely use the following:

58 Proposition. *All flag-transitive* $c.P_3$*- and* $c.T_3$*-geometries satisfying* $(*)$ *are mentioned in Table 2.*

(See Fukshansky and Wiedorn [3] for the $c.P_3$-case and Stroth and Wiedorn [17, Lemma 10] for the $c.T_2$-case.) The next lemma will also play a crucial role in the proof of Theorem 60.

59 Lemma. *Let* Γ *be a flag-transitive* $c.P_3$*-geometry and* $p \in \Gamma^0$*. Then the element-wise stabilizer of* $\mathrm{Res}(p)$ *in* $\mathrm{Aut}(\Gamma)$ *is trivial. The same holds for* $\Gamma = ET_3(2^{11}M_{24})$*.*

The first claim of this lemma, on $c.P_3$-geometries, is Lemma 1 of Fukshansky and Wiedorn [3]. (Actually, $(*)$ is assumed throughout [3], but that hypothesis plays no role in the proof of this lemma.) The second claim is contained in [17, Lemma 8].

60 Theorem. *Let* Γ *be a flag-transitive geometry of type* $c.P_4$*,* $c.P_5$*,* $c.T_4$ *or* $c.T_5$*, satisfying* $(*)$*. Suppose moreover that, when* Γ *is of type* $c.P_4$ *or* $c.P_5$*, its ultimate shrinkings are not isomorphic to* $EP_2(3S_6)$*. Then* Γ *is one of the followings:* $EP_4(2^{23}Co_2)$*,* $EP_4(2^{22}Co_2)$*,* $EP_5(2BM^2)$*,* $EP_5(BM^2)$*,* $ET_4(2^{24}Co_1)$ *or* $ET_5(M^2)$*.*

Proof. Throughout this proof G is a given flag-transitive subgroup of $\mathrm{Aut}(\Gamma)$. Given a point-line flag $\{p, l\}$ of Γ, we put $\Gamma_0 := \mathrm{Res}_\Gamma(p)$ and we denote by Σ the shrinking of Γ containing l. In order to apply theorems 13 and 14, we must know that Γ satisfies (IP). However, this is easy to see. It is well known that all flag-transitive P- and T-geometries satisfy (IP). Hence (IP) holds in $\mathrm{Res}(p)$. Moreover, in all cases to be considered in the sequel, G_p/K_p acts primitively on the set $\mathcal{L}(p)$ of lines through p. Hence no two lines of Γ can have the same points. Thus, Γ satisfies property (LL) of Subsection 2.2, which in this context is equivalent to (IP).

Suppose first that Γ is of type $c.P_4$. By $(*)$, Γ_0 is either $P_4(M_{23})$ or $P_4(Co_2)$ (see [6]). However, an easy counting argument shows that, if $\Gamma_0 \cong P_4(M_{23})$, then $|\Gamma^0| = 24$ (Stroth and Wiedorm [17, Lemma 9]). By Proposition 58, $\Sigma \cong EP_3(M_{24})$ is the only possibility that fits with this situation. However, $EP_3(M_{24})$ has shrinkings isomorphic to $EP_2(3S_6)$, which are excluded by the hypotheses of the theorem. Hence $\Gamma_0 \cong P_4(Co_2)$. Accordingly, $G_p/K_p = Co_2$ and $G_{p,l}/K_p = 2^{10}\colon M_{22}2$. As $2^{\cdot}U_6(2)2$ does not involve $2^{10}\colon M_{22}2$ (see [2]), Σ must be isomorphic to either $EP_3(2^{11}M_{22})$ or $EP_3(2^{10}M_{22})$.

Now we shall check if Γ and G satisfy the hypotheses of Section 4. Hypothesis (A1) holds, because $G_p/K_p \cong Co_2$ is simple. By the informations given on Co_2 in [2], $C_l = K_p < K_l$. Hence we are in case (I) of Lemma 12. Suppose $\Sigma \cong EP_3(2^{11}M_{22})$, to fix ideas. As $EP_3(2^{11}M_{22})$ is the affine expansion of the universal representation of $P_3(M_{22})$, hypotheses (B1) and (B2) of Section 4 hold. In particular, (B2) holds by Corollary 10 and Lemma 59. Condition (B3) holds because $G_{p,l}/K_l \cong M_{22}2$ is the full automorphism group of $\mathrm{Res}^+_\Gamma(l)$. In order to apply Theorem 13, we only must check hypothesis (C) of that theorem. In view of that, we need to determine the structure of G_P for a plane $P > l$. Considering that $|G_p : G_{p,P}| = |G_p : G_{p,l}| \cdot |G_{p,l} : G_{p,l,P}|/3 = 3586275$ and that $G_{p,P}$ is an extension of some 2-group by $S_3 \times S_5$, we recognize that $G_{p,P} = 2^{4+10}(S_3 \times S_5)$, which is a maximal subgroup of $G_p \cong Co_2$. However, $2^{4+10}(S_3 \times S_5)$ is not the centralizer of any involution of Co_2. Hence it cannot centralize any non-trivial element of K_l, which is elementary abelian (see also Lemma 21). Therefore, $G = R_oG_p$, where R_o is a representation group for $\Gamma_0 \cong P_4(Co_2)$. According to [6, Section 5.2], R_o is either the Co_2-submodule $\overline{\Lambda}^{23}$ of the Leech lattice, or its quotient $\overline{\Lambda}^{22}$. As $G_\Sigma = K_\Sigma T \cdot G_{p,l}/K_l = (K_l \times \langle z_l \rangle)T \cdot G_{p,l}/K_l = (2^{10} \times \langle z_l \rangle)2^{11}M_{22}2$ centralizes the element $z_l \in R_o$, $R_o = \overline{\Lambda}^{23}$ is the only possibility. Hence $\Gamma = EP_4(2^{23}Co_2)$. Similarly, if $\Sigma \cong EP_3(2^{10}M_{22})$ then $\Gamma = EP_4(2^{22}Co_2)$.

Let now Γ be of type $c.P_5$. Then $\Gamma_0 \cong P_5(BM)$ and $G_p/K_p \cong BM$, hence condition (A1) holds. $G_{p,l}/K_p = 2^{1+22}_+Co_2$, $K_l/K_p = 2^{1+22}_+$ and $C_l/K_p = 2$. So, both (A2) and (A3) hold. However, we are now in case (II) or (III) of Lemma 12. In view of the previous step, Σ is isomorphic to either $EP_4(2^{23}Co_2)$ or $EP_4(2^{22}Co_2)$. In any case, (B1), (B2), (B3) are satisfied. In particular, (B2) follows from Corollary 10 and the fact that, as shown in the previous paragraph, $EP_4(2^{23}Co_2)$ and $EP_4(2^{22}Co_2)$ are as in Case (I) of Lemma 12. As C_l/K_p is the center of K_l/K_p and the latter is a subgroup of $G_{p,P}/K_p$ for every plane $P > l$, condition (C) of Theorem 13 trivially holds. Condition (D1) follows from the fact that, according to the previous description of $G_{p,l}/K_p$, the latter is the centralizer of C_l/K_p in G_p/K_p. Finally, we check if (D2) holds. Note first that $G_{p,l}/K_p$ does not admit any subgroup of index 2. Hence in case (II) of Lemma 12 (D2) holds simply because its hypotheses are empty. Suppose we are in case (III) and $G_{p,l} = K_p \times X$ for a suitable subgroup $X < G_{p,l}$. Considering the orders of $G_{p,l}/K_p = 2_+^{1+22}Co_2$ and $G_p/K_p = BM$ we see that $G_{p,l}$ contains a Sylow 2-subgroup of G_p. As $G_{p,l}$ splits as $K_p \times X$ and $|K_p| = 2$, G_p also splits as $K_p \times Y$ for a suitable subgroup Y. As $|G_p : Y| = 2$, $|X : X \cap Y| \leq 2$. However, $X \cong G_{p,l}/K_p$ and the latter has no subgroups of index 2. Hence $X \leq Y$. Therefore $\langle X^{G_p} \rangle \leq Y$ (in fact, $\langle X^{G_p} \rangle = Y$). We can now apply (2) of Corollary 15, obtaining that G is either a product R_oR_i of two representation groups of $\Gamma_0 \cong P_5(BM)$, or it contains such a product as a subgroup of index 2. Moreover, as R_i is normal in G_p and G_p/K_p is simple, either $R_i = G_p$ or $|K_p| = 2$ and $G_p = R_i \times K_p$.

The representation groups of $P_5(BM)$ are known (see [6]): they are BM itself and its central non-split extension $2^{\cdot}BM$ (which the universal representation group). So, R_i and R_o are isomorphic to either BM or $2^{\cdot}BM$. In case (II) of Lemma 12 G_p is isomorphic to BM and normalizes R_o. In this case, $R_i = G_p \cong BM$. In case (III), $G_p = K_p \times N_{G_p}(R_o) \cong 2 \times BM$ by claim (4) of Theorem 13. In this case, as $2 \times BM$ is not a representation group of $P_5(BM)$, $R_i \neq G_p$, whence $R_i = N_{G_p}(R_o)$. So, in any case, R_i is isomorphic to BM, it normalizes R_o and acts on R_o by conjugation in the same ways as $\mathrm{Aut}(P_5(BM))$. Therefore, G can only have one of the following structures:

$$(2^{\cdot}BM) : BM, \quad ((2^{\cdot}BM) : BM)2, \quad BM : BM, \quad (BM : BM)2.$$

In the last two cases, the semi-direct product $R_o : R_i$ in fact entails a direct product. Indeed, as $\mathrm{Out}(BM) = 1$, for every $f \in R_i$ there is exactly one $g \in R_o$ such that $x^g = x^f$ for every $x \in R_o$. So, $g^{-1}f \in C_G(R_o)$. In the first two cases, the centralizer R_c of R_i in R_oR_i contains the center of R_o and R_oR_i is a central product $R_o * R_c$. The factor 2 on top in case (1.2) is contributed by K_p, it centralizes R_i but, according to claim (4) of Theorem 13, it replaces R_i with its twin R_i^*. In fact, $R_i^* = R_c$. By Corollary 15, if G is $(2^{\cdot}BM) : BM$ or $((2^{\cdot}BM) : BM)2$ then $\Gamma \cong EP_5(2BM^2)$ and G is either a central product of two copies of $2^{\cdot}BM$ or the extension of such a product by an involution that interchange the two factors. If G is described as $BM : BM$ or $(BM : BM)2$ then $\Gamma \cong EP_5(BM^2)$ and G is either $BM \times BM$ or $BM \wr 2$.

Let Γ be of type $c.T_4$. Then $\Sigma \cong ET_3(2^{11}M_{24})$ and $\Gamma_0 \cong T_4(Co_1)$. So, $C_p/K_p \cong Co_1$ and $G_{p,l}/K_p \cong 2^{11}M_{24}$, with $K_l/K_p \cong 2^{11}$. Hence $C_l/K_p = 1$.

Therefore (A1), (A2), (A3) hold, and $K_p = C_l = 1$ by Lemma 12. As Σ arises from the universal representation of $T_3(M_{24})$, condition (B1) holds. (B2) follows from Corollary 10 and Lemma 59. Condition (B3) holds because $G_{p,l}/K_l$ is the full automorphism group of $\mathrm{Res}^+(l)$. Hypothesis (C) of Theorem 13 remains to be proved. Likewise in the $c.P_4$-case, we recognize that $G_{p,P} = 2^{4+12}(S_3 \times 3S_6)$, which is maximal in Co_1 but does not centralize any involution. As K_l now is elementary abelian, (C) follows. (As the index of $2^{11}M_{24}$ in Co_1 is involved in the computation of $|G_p : G_{p,P}|$, we warn that a misprint occurs at page 183 of [2], where that index is recorded as 8282375 instead of 8292375.) By Theorem 13, $G = R_oG_p$ for a representation group R_o of $T_4(Co_1)$. The Leech lattice is the unique representation group for this geometry. Hence $G = 2^{24}Co_1$ and $\Gamma = ET_4(2^{24}Co_1)$.

Finally, let Γ be of type $c.T_5$. Now $\Gamma_0 \cong T_5(M)$, $G_p/K_p \cong M$, $G_{p,l}/K_p \cong 2_+^{1+24}Co_1$ with $K_l/K_p = 2_+^{1+24}$ and C_l/K_p is the center of K_l/K_p. So, (A1), (A2), (A3) hold and we are in case (II) or (III) of lemma 12. Moreover, $\Sigma \cong ET_4(2^{24}Co_1)$ by the previous step, whence (B1) holds. (B2) follows from the second claim of Proposition 58 and the fact that, as shown in the previous paragraph, $ET_4(2^{24}Co_1)$ is as in case (I) of Lemma 12. Condition (B3) holds because $G_{p,l}/K_l$ is the full automorphism group of $Res^+(l)$. Condition (C) holds because C_l/K_p is the center of K_l/K_p and (D1) holds because $G_{p,l}/K_p$ is the centralizer of C_l/K_p in G_p/K_p. No subgroup of index 2 exists in $G_{p,l}/K_p$. So, if we are in case (II) of Lemma 12 the hypotheses of (D2) are empty, whence (D2) holds. Suppose we are in case (III) and $G_{p,l} = K_p \times X$ for a suitable subgroup $X < G_{p,l}$. As M has trivial Schur multiplier, $G_p = K_p \times Y$ for a copy Y of M and $\langle X^{G_p} \rangle \leq Y$, by the same argument used in the $c.P_5$-case.

Corollary 15 (2) now implies that G is either a product R_oR_i of two representation groups of $\Gamma_0 \cong T_5(M)$, or it contains such a product as a subgroup of index 2. Moreover, as R_i is normal in G_p and G_p/K_p is simple, either $R_i = G_p$ or $|K_p| = 2$ and $G_p = R_i \times K_p$. It is known [6] that M is the unique representation group of $T_5(M)$. Hence $R_o \cong R_i \cong M$ and either $G = M{:}M$ or $G = (M{:}M)2$. As in the $c.P_5$-case one can see that $R_oR_i = R_o \times R_c$ where $R_c \cong M$ is the centralizer of R_o in R_oR_i. Hence either $G = M \times M$ or $G = M \wr 2$. In any case, $\Gamma = ET_5(M^2)$. $\boxed{QED}$

7.3 A characterization of $EP_4(J_4^2)$

We shall now consider a $c.P_4$-geometry that does not satisfy $(*)$ of Subsection 7.1. The universal representation group of $P_4(J_4)$ is J_4 itself (see Ivanov and Shpectorov [6]). The affine expansion of this representation is a flag-transitive $c.P_4$-geometry, denoted by $EP_4(J_4^2)$ in the sequel. Put $\Gamma := EP_4(J_4^2)$ and $\Delta := P_4(J_4)$, for short. Let ρ be the representation of Δ in $R := J_4$. So,

$\Gamma = \mathrm{Ex}_\rho(\Delta)$ and the translation group T_R of Γ is isomorphic to J_4. The group $G := \mathrm{Aut}(\Gamma, \pi_\rho)$ is a semidirect product $G = T_R G_p$, where $G_p \cong J_4$ is the stabilizer in G of a point p of Γ. However, by replacing G_p with $T := C_G(T_R)$, we see that $G = T_R \times T \cong J_4 \times J_4$ (compare the discussion of the $c.P_5$- and $c.T_5$-case in the proof of Theorem 60). $\mathrm{Aut}(\Gamma)$ contains an involution $i \notin G$ that permutes T_R with T and we have $\mathrm{Aut}(\Gamma) = G\langle i\rangle = J_4 \wr 2$ (see also Stroth and Wiedorn [18]). Let l be a line of Γ on p. The group $R[l] = \langle \rho(P)\rangle_{P\in\mathcal{P}(l)}$ (see Proposition 6) has the following structure: $R[l] = 2_+^{1+12}$ and $\rho(l) = Z(R[l])$ (compare [2, page 190]). The mapping ρ^l sending $P \in \mathcal{P}(l)$ to $\rho(P)/\rho(l)$ is a representation of $\mathrm{Res}_\Gamma^+(l) \cong P_3(3M_{22})$ in the abelian group $R[l]/\rho(l) \cong 2^{12}$. In fact, ρ^l is nothing but the (homogeneous) representation ρ^c of $P_3(3M_{22})$ in the representation module V^c of the enriched point-line system of $P_3(3M_{22})$ (Ivanov and Shpectorov [6, 4.4.2]). By Proposition 6, $\mathrm{Ex}_{\rho^l}(\mathrm{Res}_\Gamma^+(l))$ is just the shrinking $\Sigma(l)$ of Γ containing l. We shall prove that these features indeed characterize $EP_4(J_4^2)$.

61 Theorem. *Let Γ be a flag-transitive $c.P_4$-geometry with point-residues isomorphic to $P_4(J_4)$ and shrinkings isomorphic to the affine expansion of $P_3(3M_{22})$ by a homogeneous representation. Then $\Gamma \cong EP_4(J_4^2)$.*

Proof. The proof is basically the same as for the $c.T_5$-case of Theorem 60. Note first that Γ satisfies (IP) (this can be seen by the same argument as in the proof of Theorem 60). Given a point-line flag $\{p, l\}$ of Γ and a flag-transitive subgroup $G \le \mathrm{Aut}(\Gamma)$, we have $G_{p,l}/K_p = 2_+^{1+22}3M_{22}2$ with $K_l/K_p = 2_+^{1+22}$. Hence $C_l/K_p = Z(K_l/K_p)$, conditions (A1), (A2), (A3) hold and we are in case (II) or (III) of Lemma 12. Condition (B1) holds by assumption, (B2) follows from Corollary 10 and Lemma 59 and (B3) holds because $G_{p,l}/K_l \cong 3M_{22}2$ is the full automorphism group of $\mathrm{Res}^+(l)$. Conditions (C) and (D1) of Theorem 13 holds because C_l/K_p is the center of K_l/K_p and $G_{p,l}/K_p$ is the centralizer of C_l/K_p. As G_p/K_p is simple, the hypotheses of (D2) are empty in case (II). In case (II), we get $G_p = K_p \times \overline{G}_p$ for a copy $\overline{G}_p$ of J_4, because the Schur multiplier of J_4 is trivial. So, both theorems 13 and 14 can be applied, and we get the conclusion. $\boxed{QED}$

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