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On Norden-Walker 4-manifolds

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Abstract. A Walker 4-manifold is a semi-Riemannian manifold (M_4, g) of neutral signature, which admits a field of parallel null 2-plane. The main purpose of the present paper is to study almost Norden structures on 4-dimensional Walker manifolds with respect to a proper and opposite almost complex structures. We discuss sequently the problem of integrability, Kähler (holomorphic), isotropic Kähler and quasi-Kähler conditions for these structures. The curvature properties for Norden-Walker metrics is also investigated. Also, we give counterexamples to Goldberg's conjecture in the case of neutral signature.

Keywords: Walker 4-manifolds, Proper almost complex structure, Opposite almost complex structure, Norden metrics, Holomorphic metrics, Goldberg conjecture

MSC 2000 classification: primary 53C50, secondary 53B30

1 Introduction

Let M_{2n} be a Riemannian manifold with neutral metric, i.e., with pseudo-Riemannian metric g of signature (n, n). We denote by $\Im_q^p(M_{2n})$ the set of all tensor fields of type (p, q) on M_{2n}. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

Let (M_{2n}, φ) be an almost complex manifold with almost complex structure φ. Such a structure is said to be integrable if the matrix $\varphi = (\varphi_j^i)$ is reduced to constant form in a certain holonomic natural frame in a neighborhood U_x of every point $x \in M_{2n}$. In order that an almost complex structure φ be integrable, it is necessary and sufficient that there exists a torsion-free affine connection ∇ with respect to which the structure tensor φ is covariantly constant, i.e.,$\nabla\varphi = 0$. It is also know that the integrability of φ is equivalent to the vanishing

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of the Nijenhuis tensor $N_\varphi \in \Im_2^1(M_{2n})$. If φ is integrable, then φ is a complex structure and, moreover, M_{2n} is a C-holomorphic manifold $X_n(C)$ whose transition functions are holomorphic mappings.

1.1 Norden metrics

A metric g is a Norden metric [18] if

$$g(\varphi X, \varphi Y) = -g(X, Y)$$

or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_{2n})$. Metrics of this type have also been studied under the other names: pure metrics, anti-Hermitian metrics and B-metrics (see [5], [6], [10], [17], [19], [23], [25]). If (M_{2n}, φ) is an almost complex manifold with Norden metric g, we say that (M_{2n}, φ, g) is an almost Norden manifold. If φ is integrable, we say that (M_{2n}, φ, g) is a Norden manifold.

1.2 Holomorphic (almost holomorphic) tensor fields

Let $\overset{*}{t}$ be a complex tensor field on a C-holomorphic manifold $X_n(C)$. The real model of such a tensor field is a tensor field on M_{2n} of the same order irrespective of whether its vector or covector arguments is subject to the action of the affinor structure φ. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e.g., [10], [20], [21], [23], [24], [25], [27]). In particular, for a $(0, q)$-tensor field ω, the purity means that for any $X_1, ..., X_q \in \Im_0^1(M_{2n})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, ..., X_q) = \omega(X_1, \varphi X_2, ..., X_q) = ... = \omega(X_1, X_2, ..., \varphi X_q).$$

We define an operator

$$\Phi_\varphi : \Im_q^0(M_{2n}) \to \Im_{q+1}^0(M_{2n})$$

applied to a pure tensor field ω by (see [27])

$$\begin{aligned}(\Phi_\varphi \omega)(X, Y_1, Y_2, ..., Y_q) &= (\varphi X)(\omega(Y_1, Y_2, ..., Y_q)) - X(\omega(\varphi Y_1, Y_2, ..., Y_q)) \\ &\quad + \omega((L_{Y_1}\varphi)X, Y_2, ..., Y_q) + ... + \omega(Y_1, Y_2, ..., (L_{Y_q}\varphi)X),\end{aligned}$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a complex structure on M_{2n} and the tensor field $\Phi_\varphi \omega$ vanishes, the complex tensor field $\overset{*}{\omega}$ on $X_n(C)$ is said to be holomorphic (see [10], [23], [27]). Thus, a holomorphic tensor field $\overset{*}{\omega}$ on $X_n(C)$ is realized on M_{2n} in the form of a pure tensor field ω, such that

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, ..., Y_q) = 0$$

for any $X, Y_1, ..., Y_q \in \Im_0^1(M_{2n})$. Such a tensor field ω on M_{2n} is also called holomorphic tensor field. When φ is an almost complex structure on M_{2n}, a tensor field ω satisfying $\Phi_\varphi \omega = 0$ is said to be almost holomorphic.

1.3 Holomorphic Norden (Kähler-Norden or anti-Kähler) metrics

On a Norden manifold, a Norden metric g is called a *holomorphic* if

$$(\Phi_\varphi g)(X, Y, Z) = -g((\nabla_X \varphi)Y, Z) + g((\nabla_Y \varphi)Z, X) + g((\nabla_Z \varphi)X, Y) = 0 \tag{1}$$

for any $X, Y, Z \in \Im_0^1(M_{2n})$.

By setting $X = \partial_k,\ Y = \partial_i,\ Z = \partial_j$ in equation (1), we see that the components $(\Phi_\varphi g)_{kij}$ of $\Phi_\varphi g$ with respect to a local coordinate system $x^1, ..., x^n$ can be expressed as follows:

$$(\Phi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im}\partial_j \varphi_k^m.$$

If (M_{2n}, φ, g) is a Norden manifold with holomorphic Norden metric, we say that (M_{2n}, φ, g) is a *holomorphic Norden manifold*.

In some aspects, holomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is an analogue to the next known result: an almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

Theorem 1. *[8] (For a paracomplex version see [22]) For an almost complex manifold with Norden metric g , the condition $\Phi_\phi g = 0$ is equivalent to $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g.*

A *Kähler-Norden* manifold can be defined as a triple (M_{2n}, φ, g) which consists of a manifold M_{2n} endowed with an almost complex structure φ and a pseudo-Riemannian metric g such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g and the metric g is assumed to be a Norden one. Therefore, there exists a one-to-one correspondence between *Kähler-Norden* manifolds and Norden manifolds with *holomorphic metric.* Recall that the Riemannian curvature tensor of such a manifold is pure and holomorphic, and the scalar curvature is locally holomorphic function (see [8], [19]).

Remark 1. We know that the integrability of an almost complex structure φ is equivalent to the existence of a torsion-free affine connection with respect to which the equation $\nabla\varphi = 0$ holds. Since the Levi-Civita connection ∇ of g is a torsion-free affine connection, we have: if $\Phi_\varphi g = 0$, then φ is integrable. Thus, almost Norden manifold with conditions $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$, i.e., *almost holomorphic Norden manifolds (analogues of almost Kähler manifolds with closed Kähler form) do not exist.*

1.4 Quasi-Kähler manifolds

The basis class of non-integrable almost complex manifolds with Norden metric is the class of the quasi-Kähler manifolds. An almost Norden manifold (M_{2n}, φ, g) is called a quasi-Kähler [17], if

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0,$$

where σ is the cyclic sum by three arguments.

From (1) and the last equation we have

$$(\Phi_\varphi g)(X, Y, Z) + 2g((\nabla_X \varphi)Y, Z) = \underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0,$$

which is satisfied by the Norden metric in the quasi-Kähler manifold.

1.5 Twin Norden metrics

Let (M_{2n}, φ, g) be an almost Norden manifold. The associated Norden metric of almost Norden manifold is defined by

$$G(X, Y) = (g \circ \varphi)(X, Y)$$

for all vector fields X and Y on M_{2n}. One can easily prove that G is a new Norden metric, which is also called the twin(or dual) Norden metric of g.

We denote by ∇_g the covariant differentiation of the Levi-Civita connection of Norden metric g. Then, we have

$$\nabla_g G = (\nabla_g g) \circ \varphi + g \circ (\nabla_g \varphi) = g \circ (\nabla_g \varphi),$$

which implies $\nabla_g G = 0$ by virtue of Theorem 1. Therefore we have: *the Levi-Civita connection of Kähler-Norden metric g coincides with the Levi-Civita connection of twin metric G (i.e. nonuniqueness of the metric for the Levi-Civita connection in Kähler-Norden manifolds).*

2 Norden-Walker metrics

In the present paper, we shall focus our attention to Norden manifolds of dimension four. Using a Walker metric we construct new Norden-Walker metrics together with a proper and opposite almost complex structures.

2.1 Walker metric g

A neutral metric g on a 4-manifold M_4 is said to be a Walker metric if there exists a 2-dimensional null distribution D on M_4, which is parallel with respect to g. From Walker's theorem [26], there is a system of coordinates (x, y, z, t) with respect to which g takes the following local canonical form

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{2}$$

where a, b, c are smooth functions of the coordinates (x, y, z, t). The paralel null 2-plane D is spanned locally by $\{\partial_x, \partial_y\}$, where $\partial_x = \frac{\partial}{\partial x},\ \partial_y = \frac{\partial}{\partial y}$.

2.2 Almost Norden-Walker manifolds

Let F be an almost complex structure on a Walker manifold M_4, which satisfies

i) $F^2 = -I$,

ii) $g(FX, Y) = g(X, FY)$ (Nordenian property),

iii) $F\partial_x = \partial_y,\ F\partial_y = -\partial_x$ (F induces a positive $\frac{\pi}{2}$−rotation on D).

We easily see that these three properties define F non-uniquely, i.e.,

$$\begin{cases} F\partial_x = \partial_y, \\ F\partial_y = -\partial_x, \\ F\partial_z = \alpha\partial_x + \frac{1}{2}(a+b)\partial_y - \partial_t, \\ F\partial_t = -\frac{1}{2}(a+b)\partial_x + \alpha\partial_y + \partial_z \end{cases}$$

and F has the local components

$$F = (F_j^i) = \begin{pmatrix} 0 & -1 & \alpha & -\frac{1}{2}(a+b) \\ 1 & 0 & \frac{1}{2}(a+b) & \alpha \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

with respect to the natural frame $\{\partial_x, \partial_y, \partial_z, \partial_t\}$, where $\alpha = \alpha(x, y, z, t)$ is an arbitrary function.

Therefore, we now put $\alpha = c$. Then g defines a unique almost complex structure

$$\varphi = (\varphi_j^i) = \begin{pmatrix} 0 & -1 & c & -\frac{1}{2}(a+b) \\ 1 & 0 & \frac{1}{2}(a+b) & c \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3}$$

The triple (M_4, φ, g) is called almost Norden-Walker manifold. In conformity with the terminology of [3], [4], [14], [15] we call φ the proper almost complex structure.

We note that the typical examples of Norden-Walker metrics with proper almost complex structure

$$J = (J_j^i) = \begin{pmatrix} 0 & -1 & -c & \frac{1}{2}(a-b) \\ 1 & 0 & \frac{1}{2}(a-b) & c \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

are studied in [2].

2.3 Isotropic Kähler-Norden-Walker structures

A proper almost complex structure φ on Norden-Walker manifold (M_4, φ, g) is said to be *isotropic Kähler* if $\|\nabla\varphi\|^2 = 0$, but $\nabla\varphi \neq 0$. Examples of isotropic Kähler structures were given first in [7] in dimension 4, subsequently in [1] in dimension 6 and in [3] in dimension 4. Our purpose in this section is to show that a proper almost complex structure on almost Norden-Walker manifold (M_4, φ, g) is isotropic Kähler as we will see Theorem 2.

The inverse of the metric tensor (2), $g^{-1} = (g^{ij})$, given by

$$g^{-1} = \begin{pmatrix} -a & -c & 1 & 0 \\ -c & -b & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4}$$

For the covariant derivative $\nabla\varphi$ of the almost complex structure put $(\nabla\varphi)_{ij}^k = \nabla_i\varphi_j^k$. Then, after some calculations we obtain

$$\begin{aligned}
\nabla_x\varphi_z^x &= \nabla_x\varphi_t^y = c_x, \nabla_y\varphi_z^x = \nabla_y\varphi_t^y = c_y, \\
\nabla_z\varphi_x^x &= -\nabla_z\varphi_y^y = \nabla_z\varphi_z^z = -\nabla_z\varphi_t^t = \frac{1}{2}a_y + \frac{1}{2}c_x, \\
\nabla_z\varphi_x^y &= \nabla_z\varphi_y^x = \nabla_z\varphi_z^t = \nabla_z\varphi_t^z = -\frac{1}{2}a_x + \frac{1}{2}c_y, \\
\nabla_z\varphi_z^x &= 2c_z + ca_x - a_t - \frac{1}{2}cc_y - \frac{1}{2}ac_x + \frac{1}{2}ba_y, \\
\nabla_z\varphi_z^y &= a_z + \frac{1}{4}ac_y - \frac{1}{4}bc_y + ca_y + \frac{3}{4}aa_x + \frac{1}{4}ba_x, \\
\nabla_z\varphi_t^x &= \frac{1}{4}aa_x - \frac{1}{4}ba_x + ca_y + \frac{3}{4}bc_y + cc_x + \frac{1}{4}ac_y, \\
\nabla_z\varphi_z^y &= a_z + \frac{1}{4}ac_y - \frac{1}{4}bc_y + ca_y + \frac{3}{4}aa_x + \frac{1}{4}ba_x, \\
\nabla_z\varphi_t^x &= \frac{1}{4}aa_x - \frac{1}{4}ba_x + ca_y + \frac{3}{4}bc_y + cc_x + \frac{1}{4}ac_y,
\end{aligned} \tag{5}$$

$$\begin{aligned}
\nabla_z\varphi_t^y &= 2c_z+\frac{1}{2}cc_y-a_t+\frac{1}{2}ba_y+\frac{1}{2}ca_x-\frac{1}{2}ac_x,\\
\nabla_t\varphi_x^x &= -\nabla_t\varphi_y^y=\nabla_t\varphi_z^z=-\nabla_t\varphi_t^t=\frac{1}{2}c_y+\frac{1}{2}b_x,\\
\nabla_t\varphi_x^y &= \nabla_t\varphi_y^x=\nabla_t\varphi_z^t=\nabla_t\varphi_t^z=-\frac{1}{2}c_x+\frac{1}{2}b_y,\\
\nabla_t\varphi_z^x &= \frac{3}{2}cc_x+b_z-\frac{1}{2}cb_y-\frac{1}{2}ab_x+\frac{1}{2}bc_y,\\
\nabla_t\varphi_z^y &= \frac{1}{4}ab_y-\frac{1}{4}bb_y-\frac{1}{4}ac_x+\frac{1}{4}bc_x,\\
\nabla_t\varphi_t^x &= \frac{1}{4}ac_x-\frac{1}{4}bc_x+cc_y+\frac{1}{4}bb_y+cb_x-\frac{1}{4}ab_y\\
\nabla_t\varphi_t^y &= \frac{1}{2}cb_y+b_z+\frac{1}{2}bc_y+\frac{1}{2}cc_x-\frac{1}{2}ab_x.
\end{aligned}$$

Now a long but straightforward calculation shows that

$$\|\nabla\varphi\|^2=g^{ij}g^{kl}g_{ms}(\nabla\varphi)_{ik}^m(\nabla\varphi)_{jl}^s=0.$$

Theorem 2. *A proper almost complex structure on almost Norden-Walker manifold* (M_4, φ, g) *is isotropic Kähler.*

2.4 Integrability of φ

We consider the general case.

The almost complex structure φ of an almost Norden-Walker manifold is integrable if and only if

$$(N_\varphi)_{jk}^i=\varphi_j^m\partial_m\varphi_k^i-\varphi_k^m\partial_m\varphi_j^i-\varphi_m^i\partial_j\varphi_k^m+\varphi_m^i\partial_k\varphi_j^m=0. \tag{6}$$

From (3) and (6) find the following integrability condition.

Theorem 3. *The proper almost complex structure* φ *of an almost Norden-Walker manifold is integrable if and only if the following PDEs hold:*

$$\begin{cases} a_x+b_x+2c_y=0,\\ a_y+b_y-2c_x=0.\end{cases} \tag{7}$$

From this theorem, we see that, in the case $a=-b$ and $c=0$, φ is integrable.

Let (M_4,φ,g) be a Norden-Walker manifolds ($N_\varphi=0$) and $a=b$. Then the equation (7) reduces to

$$\begin{cases} a_x=-c_y,\\ a_y=c_x,\end{cases} \tag{8}$$

from which follows

$$\begin{aligned} a_{xx}+a_{yy}=0,\\ c_{xx}+c_{yy}=0,\end{aligned} \tag{9}$$

e.g., the functions a and c are harmonic with respect to the arguments x and y.

Thus we have

Theorem 4. *If the triple* (M_4,φ,g) *is Norden-Walker and* $a=b$*, then* a *and* c *are all harmonic with respect to the arguments* x, y.

2.5 Example of Norden-Walker metric

We now apply the Theorem 4 to establish the existence of special types of Norden-Walker metrics. In our arguments, the harmonic function plays an important part.

Let $a = b$ and $h(x, y)$ be a harmonic function of variables x and y, for example $h(x, y) = e^x \cos y$. We put

$$a = a(x, y, z, t) = h(x, y) + \alpha(z, t) = e^x \cos y + \alpha(z, t)$$

where α is an arbitrary smooth function of z and t. Then, a is also hormonic with respect to x and y. We have

$$\begin{aligned} a_x &= e^x \cos y, \\ a_y &= -e^x \sin y. \end{aligned}$$

From (8), we have PDE's for c to satisfy as

$$\begin{aligned} c_x &= a_y = -e^x \sin x, \\ c_y &= -a_x = -e^x \cos y. \end{aligned}$$

For these PDE's, we have solutions

$$c = -e^x \sin y + \beta(z, t),$$

where β is arbitrary smooth function of z and t. Thus the Norden-Walker metric has components of the form

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & e^x \cos y + \alpha(z,t) & -e^x \sin y + \beta(z,t) \\ 0 & 1 & -e^x \sin y + \beta(z,t) & e^x \cos y + \alpha(z,t) \end{pmatrix}.$$

3 Holomorphic Norden-Walker(Kähler-Norden-Walker) and quasi-Kähler-Norden-Walker metrics on (M_4, φ, g)

Let (M_4, φ, g) be an almost Norden-Walker manifold. If

$$(\Phi_\phi g)_{kij} = \phi_k^m \partial_m g_{ij} - \phi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \phi_k^m - \partial_k \phi_i^m) + g_{im} \partial_j \phi_k^m = 0, \tag{10}$$

then, by virtue of Theorem 1, φ is integrable and the triple (M_4, φ, g) is called a holomorphic Norden-Walker or a Kähler-Norden-Walker manifold. Taking into account Remark 1, we see that an almost Kähler-Norden-Walker manifold with conditions $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$ does not exist.

Substitute (2) and (3) into (10), we see that the non-vanishing components of $(\Phi_\varphi g)_{kij}$

are

$$
\begin{aligned}
(\Phi_\varphi g)_{xzz} &= a_y,\ (\Phi_\varphi g)_{xzt} = (\Phi_\varphi g)_{xtz} = \frac{1}{2}(b_x - a_x) + c_y, \\
(\Phi_\varphi g)_{xtt} &= b_y - 2c_x,\ (\Phi_\varphi g)_{yzz} = -a_x, \\
(\Phi_\varphi g)_{yzt} &= (\Phi_\varphi g)_{ytz} = \frac{1}{2}(b_y - a_y) - c_x,\ (\Phi_\varphi g)_{ytt} = -b_x - 2c_y, \\
(\Phi_\varphi g)_{zxz} &= (\Phi_\varphi g)_{zzx} = (\Phi_\varphi g)_{txt} = (\Phi_\varphi g)_{ttx} = c_x, \\
(\Phi_\varphi g)_{zxt} &= (\Phi_\varphi g)_{ztx} = -(\Phi_\varphi g)_{txz} = -(\Phi_\varphi g)_{tzx} = \frac{1}{2}(a_x + b_x), \\
(\Phi_\varphi g)_{zyz} &= (\Phi_\varphi g)_{zzy} = (\Phi_\varphi g)_{tyt} = (\Phi_\varphi g)_{tty} = c_y, \\
(\Phi_\varphi g)_{zyt} &= (\Phi_\varphi g)_{zty} = -(\Phi_\varphi g)_{tyz} = -(\Phi_\varphi g)_{tzy} = \frac{1}{2}(a_y + b_y), \\
(\Phi_\varphi g)_{zzz} &= ca_x - a_t + 2c_z + \frac{1}{2}(a+b)a_y, \\
(\Phi_\varphi g)_{zzt} &= (\Phi_\varphi g)_{ztz} = cc_x + b_z + \frac{1}{2}(a+b)c_y, \\
(\Phi_\varphi g)_{ztt} &= cb_x + a_t - 2c_z + \frac{1}{2}(a+b)b_y,\ (\Phi_\varphi g)_{tzz} = ca_y - b_z - \frac{1}{2}(a+b)a_x, \\
(\Phi_\varphi g)_{tzt} &= (\Phi_\varphi g)_{ttz} = cc_y - a_t + 2c_z - \frac{1}{2}(a+b)c_x, \\
(\Phi_\varphi g)_{ttt} &= cb_y + b_z - \frac{1}{2}(a+b)b_x.
\end{aligned} \tag{11}
$$

From the above equations, we have

Theorem 5. *A triple (M_4, φ, g) is a Kähler-Norden-Walker manifold if and only if the following PDEs hold:*

$$
a_x = a_y = b_x = b_y = b_z = c_x = c_y = 0, \qquad a_t - 2c_z = 0. \tag{12}
$$

A Norden-Walker manifold (M_4, φ, g) satisfying the condition $\Phi_k g_{ij} + 2\nabla_k G_{ij}$ to be zero is called a quasi-Kähler manifold, where G is defined by $G_{ij} = \varphi_i^m g_{mj}$.

Remark 2. From (2) and (3) we easily see that, *the twin Norden metric G is non-Walker.*

For the covariant derivative ∇G of the associated metric G put $(\nabla G)_{ijk} = \nabla_i G_{jk}$. The non-vanishing components of $\nabla_i G_{jk}$ are

$$
\begin{aligned}
\nabla_x G_{zz} &= \nabla_x G_{tt} = c_x,\ \nabla_y G_{zz} = \nabla_y G_{tt} = c_y, \\
\nabla_z G_{xz} &= \nabla_z G_{zx} = -\nabla_z G_{yt} = -\nabla_z G_{ty} = \frac{1}{2}(a_y + c_x), \\
\nabla_z G_{xt} &= \nabla_z G_{tx} = \nabla_z G_{yz} = \nabla_z G_{zy} = \frac{1}{2}(c_y - a_x), \\
\nabla_z G_{zz} &= 2c_z - a_t + \frac{1}{2}a_y(a+b) + ca_x, \\
\nabla_z G_{zt} &= \nabla_z G_{tz} = \frac{1}{2}(ca_y + cc_x) - \frac{1}{4}((a+b)(a_x - c_y)), \\
\nabla_z G_{tt} &= 2c_z - a_t - \frac{1}{2}c_x(a+b) + cc_y, \\
\nabla_t G_{xz} &= \nabla_t G_{zx} = -\nabla_t G_{yt} = -\nabla_t G_{ty} = \frac{1}{2}(b_x + c_y), \\
\nabla_t G_{xt} &= \nabla_t G_{tx} = \nabla_t G_{yz} = \nabla_t G_{zy} = \frac{1}{2}(b_y - c_x),
\end{aligned} \tag{13}
$$

$$\begin{aligned}
\nabla_t G_{zz} &= b_z + cc_x + \frac{1}{2}c_y(a+b),\\
\nabla_t G_{zt} &= \nabla_t G_{tz} = \frac{1}{2}c(b_x + c_y) - \frac{1}{4}((c_x - b_y)(a+b)),\\
\nabla_t G_{tt} &= b_z + cb_y - \frac{1}{2}b_x(a+b).
\end{aligned}$$

From (11) and (13) we have

Theorem 6. *A triple (M_4, φ, g) is a quasi-Kähler Norden-Walker manifold if and only if the following PDEs hold:*

$$b_x = b_y = b_z = 0,\ a_y - 2c_x = 0,\ a_x - 2c_y = 0,\ ca_x - a_t + 2c_z - (a+b)c_x = 0.$$

4 Curvature properties of Norden-Walker manifolds

If R and r are respectively the curvature and the scalar curvature of the Walker metric, then the components of R and r have, respectively, expressions (see [15], Appendix A and C)

$$\begin{aligned}
R_{xzxz} &= -\tfrac{1}{2}a_{xx},\ R_{xzxt} = -\tfrac{1}{2}c_{xx},\ R_{xzyz} = -\tfrac{1}{2}a_{xy},\ R_{xzyt} = -\tfrac{1}{2}c_{xy}, \qquad (14)\\
R_{xzzt} &= \tfrac{1}{2}a_{xt} - \tfrac{1}{2}c_{xz} - \tfrac{1}{4}a_y b_x + \tfrac{1}{4}c_x c_y,\ R_{xtxt} = -\tfrac{1}{2}b_{xx},\ R_{xtyz} = -\tfrac{1}{2}c_{xy},\\
R_{xtyt} &= -\tfrac{1}{2}b_{xy},\ R_{xtzt} = \tfrac{1}{2}c_{xt} - \tfrac{1}{2}b_{xz} - \tfrac{1}{4}(c_x)^2 + \tfrac{1}{4}a_x b_x - \tfrac{1}{4}b_x c_y + \tfrac{1}{4}b_y c_x,\\
R_{yzyz} &= -\tfrac{1}{2}a_{yy},\ R_{yzyt} = -\tfrac{1}{2}c_{yy},\\
R_{yzzt} &= \tfrac{1}{2}a_{yt} - \tfrac{1}{2}c_{yz} - \tfrac{1}{4}a_x c_y + \tfrac{1}{4}a_y c_x - \tfrac{1}{4}a_y b_y + \tfrac{1}{4}(c_y)^2,\ R_{ytyt} = -\tfrac{1}{2}b_{yy},\\
R_{ytzt} &= \tfrac{1}{2}c_{yt} - \tfrac{1}{2}b_{yz} - \tfrac{1}{4}c_x c_y + \tfrac{1}{4}a_y b_x,\\
R_{ztzt} &= c_{zt} - \tfrac{1}{2}a_{tt} - \tfrac{1}{2}b_{zz} - \tfrac{1}{4}a(c_x)^2 + \tfrac{1}{4}aa_x b_x + \tfrac{1}{4}ca_x b_y - \tfrac{1}{2}cc_x c_y - \tfrac{1}{2}a_t c_x\\
&\quad + \tfrac{1}{2}a_x c_t - \tfrac{1}{4}a_x b_z + \tfrac{1}{4}ca_y b_x + \tfrac{1}{4}ba_y b_y - \tfrac{1}{4}b(c_y)^2 - \tfrac{1}{2}b_z c_y\\
&\quad + \tfrac{1}{4}a_y b_t + \tfrac{1}{4}a_z b_x + \tfrac{1}{2}b_y c_z - \tfrac{1}{4}a_t b_y.
\end{aligned}$$

and

$$r = a_{xx} + 2c_{xy} + b_{yy}. \qquad (15)$$

Suppose that the triple (M_4, φ, g) is Kähler-Norden-Walker. Then from the last equation in (12) and (14), we see that

$$R_{ztzt} = c_{zt} - \frac{1}{2}a_{tt} = -\frac{1}{2}(a_t - 2c_z)_t = 0.$$

From (12) we easily we see that the another components of R in (14) directly all vanish. Thus we have

Theorem 7. *If a Norden-Walker manifold (M_4, φ, g) is Kähler-Norden-Walker, then M_4 is flat.*

Remark 3. We note that a Kähler-Norden manifold is non-flat, in such manifold curvature tensor pure and holomorphic [8].

Let (M_4, φ, g) be a Norden-Walker manifold with the integrable proper structure φ, i.e., $N_\varphi = 0$. If $a = b$, then from proof of the Theorem 4 we see that the equation (8) hold. If $c = c(y, z, t)$ and $c = c(x, z, t)$, then $c_{xy} = (c_x)_y = (c_y)_x = 0$. In these cases, by virtue of (8) we find $a = a(x, z, t)$ and $a = (y, z, t)$ respectively. Using of $c_{xy} = 0$ and $a_{xx} + b_{yy} = 0$ (see (9)), we from (15) obtain $r = 0$. Thus we have

Theorem 8. *If (M_4, φ, g) is a Norden-Walker non-Kähler manifold with metrics*

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x,z,t) & c(y,z,t) \\ 0 & 1 & c(y,z,t) & a(x,z,t) \end{pmatrix}, \ \tilde{g} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(y,z,t) & c(x,z,t) \\ 0 & 1 & c(x,z,t) & a(y,z,t) \end{pmatrix},$$

then M_4 is scalar flat.

5 On the Goldberg conjecture

Let (M_{2n}, J, g) be an almost Hermitian manifold. Then, Goldberg's conjecture states that an almost Hermitian manifold must be Kähler if the following three conditions are imposed: (G_1) the manifold M_{2n} is compact; (G_2) the Riemannian metric g is Einstein; (G_3) the fundamental 2-form Ω defined by $\Omega(X,Y) = g(JX,Y)$ is closed ($d\Omega = 0$).

It should be noted that no progress has been made on the Goldberg conjecture, and the orginal conjecture is stil an open problem.

Let (M_{2n}, φ, g) be an almost Norden manifold. Given an almost complex structure φ on M_{2n}, take any Riemannian metric $\tilde{g}$, which exists provided M_{2n} is compact (paracompact) [9, p. 60]. We obtain a Hermitian metric h by setting

$$h(X,Y) = \tilde{g}(X,Y) + \tilde{g}(\varphi X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_{2n})$. The pair $(\varphi, \tilde{g})$ defines a fundamental 2-form Ω_φ by

$$\Omega_\varphi(X,Y) = h(\varphi X, Y).$$

We call it a φ-compatible 2-form.

Let (M_{2n}, φ, g) be an almost Norden manifold, and choose a φ-compatible 2-form Ω_φ on M_{2n}. Then we can propose an almost Norden version of Goldberg conjecture as follows [16]: if (G_1) M_{2n} is compact, (G_2) g is Einstein, and if (G_3') a φ-compatible 2-form Ω_φ is closed, then φ must be integrable.

Let now (M_4, φ, g) be an almost Norden-Walker 4-manifold. The pair (φ, g) defines as usual, a rank two tensor $G(X,Y) = g(\varphi X, Y)$, but G is symmetric (in fact another neutral metric) and pure, rather than a 2-form. We call it a twin Norden metric, which plays a role similar to the fundamental 2-form Ω in Hermitian geometry. If we define an operator Φ_φ applied to a pure twin metric G, then we have

$$(\Phi_\varphi G)(X,Y,Z) = (\Phi_\varphi g)(\varphi X, Y, Z) + g(N_\varphi(X,Y), Z).$$

If $G \in Ker\Phi_\varphi$, then by virtue of Theorem 1, we have $\nabla_G \varphi = 0$, where ∇_G is the Levi-Civita connection of the twin Norden metric G, which coincides with the Levi-Civita connection of the orginal Norden metric g in Kähler-Norden-Walker manifolds. Since ∇_G is a torsion-free connection, then φ must be integrable. Thus, we can propose a result concerning the Norden version of Goldberg conjecture as follows: (NG) if $G \in Ker\Phi_\varphi$, then φ must be integrable.

6 Opposite almost complex structure φ'

It is known that an oriented 4-manifold with a field of 2-planes, or equivalently endowed with a neutral indefinite metric, admits a pair of almost comlex structure φ and an opposite almost complex structure φ', which satisfy the following properties ([11]-[13], [15]):

i) $\varphi^2 = \varphi'^2 = -1$,

ii) $g(\varphi X, \varphi Y) = g(\varphi' X, \varphi' Y) = g(X, Y)$,

iii) $\varphi\varphi' = \varphi'\varphi$,

iv) the preferred orientation of φ coincides with that of M_4,

v) the preferred orientation of φ' is opposite to that of M_4.

Let (M_4, φ, g) be an almost Norden-Walker manifolds. For a Walker manifold M_4, with the proper almost complex structure φ, the g-orthogonal opposite almost complex structure φ' takes the form

$$\begin{aligned}
&\varphi'\partial_1 = -(\theta_1 c + \tfrac{\theta_2}{2}a)\partial_1 - \tfrac{\theta_1}{2}b\partial_2 + \theta_2\partial_3 + \theta_1\partial_4,\\
&\varphi'\partial_2 = (-\tfrac{\theta_1}{2}a + \theta_2 c)\partial_1 + \tfrac{\theta_2}{2}b\partial_2 + \theta_1\partial_3 - \theta_2\partial_4,\\
&\varphi'\partial_3 = -(\tfrac{\theta_1}{2}ac + \tfrac{\theta_2}{4}a^2 + \tfrac{\theta_2}{\theta_1^2+\theta_2^2})\partial_1 - (\tfrac{\theta_1}{4}ab + \tfrac{\theta_1}{\theta_1^2+\theta_2^2})\partial_2 + \tfrac{\theta_2}{2}a\partial_3 + \tfrac{\theta_1}{2}a\partial_4,\\
&\varphi'\partial_4 = -(\theta_1 c^2 + \tfrac{\theta_1}{4}ab + \tfrac{\theta_1}{\theta_1^2+\theta_2^2} + \tfrac{\theta_2}{2}(ac - bc))\partial_1 + (-\tfrac{\theta_1}{2}bc + \tfrac{\theta_2}{4}b^2 + \tfrac{\theta_2}{\theta_1^2+\theta_2^2})\partial_2\\
&\qquad + (\tfrac{\theta_1}{2}b + \theta_2 c)\partial_3 + (\theta_1 c - \tfrac{\theta_2}{2}b)\partial_4,
\end{aligned}$$

where θ_1 and θ_2 are two parameters.

In the present paper, we shall focus our attention to one of explicit forms of φ', obtained by fixing two parameters as $\theta_1 = 1$ and $\theta_2 = 0$ (only for simplicity), as follows:

$$\begin{aligned}
&\varphi'\partial_1 = -c\partial_1 - \tfrac{1}{2}b\partial_2 + \partial_4, \qquad \varphi'\partial_2 = -\tfrac{1}{2}a\partial_1 + \partial_3,\\
&\varphi'\partial_3 = -\tfrac{1}{2}ac\partial_1 - (\tfrac{1}{4}ab + 1)\partial_2 + \tfrac{1}{2}a\partial_4,\\
&\varphi'\partial_4 = -(c^2 + \tfrac{1}{4}ab + 1)\partial_1 - \tfrac{1}{2}bc\partial_2 + \tfrac{1}{2}b\partial_3 + c\partial_4,
\end{aligned} \tag{16}$$

and φ' has the local components

$$\varphi' = (\varphi'^{\,i}_j) = \begin{pmatrix} -c & -\tfrac{1}{2}a & -\tfrac{1}{2}ac & -(c^2 + \tfrac{1}{4}ab + 1)\\ -\tfrac{1}{2}b & 0 & -(\tfrac{1}{4}ab + 1) & -\tfrac{1}{2}bc\\ 0 & 1 & 0 & \tfrac{1}{2}b\\ 1 & 0 & \tfrac{1}{2}a & c \end{pmatrix}. \tag{17}$$

For the covariant derivative $\nabla\varphi'$ of the opposite almost complex structure φ', the non-vanishing components of which are

$$\begin{aligned}
\nabla_x\varphi'^{\,x}_x &= -\nabla_x\varphi'^{\,y}_y = \nabla_x\varphi'^{\,z}_z = -\nabla_x\varphi'^{\,t}_t = \frac{1}{2}\nabla_z\varphi'^{\,z}_x = -\frac{1}{2}c_x, \qquad (18)\\
\nabla_y\varphi'^{\,x}_x &= -\nabla_y\varphi'^{\,y}_y = \nabla_y\varphi'^{\,z}_z = -\nabla_y\varphi'^{\,t}_t = \frac{1}{2}\nabla_t\varphi'^{\,t}_y = -\frac{1}{2}c_y,\\
\nabla_x\varphi'^{\,x}_t &= -cc_x, \nabla_y\varphi'^{\,x}_t = -cc_y,\ \nabla_z\varphi'^{\,x}_x = -c_z - \frac{1}{4}ba_y + \frac{1}{2}a_t + \frac{1}{2}cc_y + \frac{3}{4}ac_x,\\
\nabla_z\varphi'^{\,y}_x &= \frac{1}{4}bc_y + \frac{1}{4}ba_x,\ \nabla_z\varphi'^{\,t}_x = \nabla_z\varphi'^{\,z}_y = -\frac{1}{2}c_y - \frac{1}{2}a_x,\\
\nabla_z\varphi'^{\,x}_y &= \frac{1}{4}aa_x + ca_y + \frac{1}{4}ac_y,\ \nabla_z\varphi'^{\,y}_y = c_z - \frac{1}{4}ac_x - \frac{1}{2}a_t + \frac{1}{2}ca_x + \frac{3}{4}ba_y,\\
\nabla_z\varphi'^{\,t}_y &= -a_y,\ \nabla_z\varphi'^{\,x}_z = \frac{1}{4}aca_x - a_y + \frac{1}{4}acc_y + \frac{1}{4}a^2c_x,\\
\nabla_z\varphi'^{\,y}_z &= \frac{1}{8}abc_y + \frac{1}{8}aba_x - \frac{1}{2}c_y - \frac{1}{2}a_x,\\
\nabla_z\varphi'^{\,z}_z &= -c_z - \frac{1}{4}ac_x + \frac{1}{2}a_t - \frac{1}{2}ca_x - \frac{1}{4}ba_y,\ \nabla_z\varphi'^{\,t}_z = -\frac{1}{4}ac_y - \frac{1}{4}aa_x,\\
\nabla_z\varphi'^{\,x}_t &= -2cc_z + (\frac{1}{8}ab - \frac{1}{2}c^2 + ac - \frac{1}{2})a_x + ca_t + (\frac{1}{8}ab + \frac{1}{2}c^2 - \frac{1}{2})c_y,
\end{aligned}$$

$$
\begin{aligned}
\nabla_z\varphi'^{\,y}_t &= -c_x+\frac{1}{4}b^2a_y+\frac{1}{4}bcc_y+\frac{1}{4}bca_x,\ \nabla_z\varphi'^{\,z}_t=-cc_x-\frac{1}{4}ba_x-\frac{1}{4}bc_y,\\
\nabla_z\varphi'^{\,t}_t &= c_z-\frac{1}{4}ba_y-\frac{1}{2}a_t-\frac{1}{2}cc_y-\frac{1}{4}ac_x,\ \nabla_t\varphi'^{\,x}_x=-\frac{1}{2}b_z-\frac{1}{4}bc_y+\frac{1}{2}cb_y+\frac{3}{4}ab_x,\\
\nabla_t\varphi'^{\,y}_x &= \frac{1}{4}bb_y+\frac{1}{4}bc_x,\ \nabla_t\varphi'^{\,z}_x=-b_x,\ \nabla_t\varphi'^{\,t}_x=\nabla_t\varphi'^{\,z}_y=-\frac{1}{2}b_y-\frac{1}{2}c_x,\\
\nabla_t\varphi'^{\,x}_y &= \frac{1}{4}ac_x+cc_y+\frac{1}{4}ab_y,\ \nabla_t\varphi'^{\,y}_y=-\frac{1}{4}ab_x+\frac{1}{2}b_z+\frac{1}{2}cc_x+\frac{3}{4}bc_y,\\
\nabla_t\varphi'^{\,x}_z &= \frac{1}{4}acb_y-c_y+\frac{1}{4}acc_x+\frac{1}{4}a^2b_x,\ \nabla_t\varphi'^{\,y}_z=\frac{1}{8}abb_y+\frac{1}{8}abc_x-\frac{1}{2}b_y-\frac{1}{2}c_x,\\
\nabla_t\varphi'^{\,z}_z &= -\frac{1}{4}ab_x-\frac{1}{2}b_z-\frac{1}{4}bc_y-\frac{1}{2}cc_x,\ \nabla_t\varphi'^{\,t}_z=-\frac{1}{4}ab_y-\frac{1}{4}ac_x,\\
\nabla_t\varphi'^{\,x}_t &= -cb_z+(\frac{1}{8}ab-\frac{1}{2}c^2-\frac{1}{2})c_x+acb_x+(\frac{1}{8}ab+\frac{1}{2}c^2-\frac{1}{2})b_y,\\
\nabla_t\varphi'^{\,y}_t &= -b_x+\frac{1}{4}b^2c_y+\frac{1}{4}bcc_x+\frac{1}{4}bcb_y,\ \nabla_z\varphi'^{\,z}_t=-cb_x-\frac{1}{4}bc_x-\frac{1}{4}bb_y,\\
\nabla_t\varphi'^{\,t}_t &= -\frac{1}{4}bc_y-\frac{1}{2}cb_y+\frac{1}{2}b_z-\frac{1}{4}ab_x.
\end{aligned}
$$

From (2), (4) and (18) we have

Theorem 9. *The opposite almost complex structure of an almost Norden-Walker manifold (M_4,φ',g) is isotropic Kähler if and only if the following PDEs hold:*

$$c_x(2ba_y-2ac_x+4c_z-2a_t+2ca_x)+c_y(2b_z-2ab_x)=0. \tag{19}$$

From (19) we have

Corollary 1. *The triple (M_4,φ',g) with metric*

$$g=(g_{ij})=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&a(x,y,z,t)&c(z,t)\\0&1&c(z,t)&b(x,y,z,t)\end{pmatrix}$$

is always isotropic Kähler.

6.1 Integrability of φ'

The opposite almost complex structure φ' is integrable if the analogue of the PDE's (6) for $\varphi'^{\,i}_j$ in (17) vanish. From some calculation, we have explicitly the following theorem.

Theorem 10. *The opposite almost complex structure φ' of an almost Norden-Walker manifold is integrable if and only if the following PDEs hold:*

$$
\begin{aligned}
&b_y=0,\quad a_x-2c_y=0,\quad ab_x-2b_z=0,\\
&ba_y-2a_t-2ac_x+4cc_y+4c_z=0.
\end{aligned}
\tag{20}
$$

Let (M_4,φ',g) be a Norden-Walker manifold with the integrable almost complex structure φ', i.e. $N_{\varphi'}=0$. If $a=0$, then from (20) $b_y=b_z=c_y=c_z=0$.

Thus we have

Theorem 11. *Let $a=0$. The triple (M_4,φ',g) with metric*

$$g=(g_{ij})=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&c(x,t)\\0&1&c(x,t)&b(x,t)\end{pmatrix}$$

is always Norden-Walker.

7 Norden-Walker-Einstein metrics

We now turn our attention to the Einstein conditions for the Norden-Walker metric g in (2).

Let R_{ij} and S denote the Ricci curvature and the scalar curvature of the metric g in (2). The Einstein tensor is defined by $G_{ij} = R_{ij} - \frac{1}{4}Sg_{ij}$ and has non zero components as follows (see [15], Appendix D):

$$\begin{array}{l}
G_{xz} = \frac{1}{4}a_{xx} - \frac{1}{4}b_{yy}, \quad G_{xt} = \frac{1}{2}c_{xx} + \frac{1}{2}b_{xy}, \\
G_{yz} = \frac{1}{2}a_{xy} + \frac{1}{2}c_{yy}, \quad G_{yt} = \frac{1}{4}b_{yy} - \frac{1}{4}a_{xx}, \\
G_{zz} = \frac{1}{4}aa_{xx} + ca_{xy} + \frac{1}{2}ba_{yy} - a_{yt} + c_{yz} - \frac{1}{2}a_y c_x + \frac{1}{2}a_x c_y \\
\qquad + \frac{1}{2}a_y b_y - \frac{1}{2}(c_y)^2 - \frac{1}{2}ac_{xy} - \frac{1}{4}ab_{yy}, \\
G_{zt} = \frac{1}{2}ac_{xx} + \frac{1}{2}cc_{xy} + \frac{1}{2}a_{xt} - \frac{1}{2}c_{xz} - \frac{1}{2}a_y b_x + \frac{1}{2}c_x c_y + \frac{1}{2}bc_{yy} \\
\qquad - \frac{1}{2}c_{yt} + \frac{1}{2}b_{yz} - \frac{1}{4}ca_{xx} - \frac{1}{4}cb_{yy}, \\
G_{tt} = \frac{1}{2}ab_{xx} + cb_{xy} + c_{xt} - b_{xz} - \frac{1}{2}(c_x)^2 + \frac{1}{2}a_x b_x - \frac{1}{2}b_x c_y + \frac{1}{2}b_y c_x \\
\qquad + \frac{1}{4}bb_{yy} - \frac{1}{4}ba_{xx} - \frac{1}{2}bc_{xy}.
\end{array} \tag{21}$$

The metric g in (2) is almost Norden-Walker-Einstein if all the above components G_{ij} vanish ($G_{ij} = 0$).

Theorem 12. *Let (M_4, φ', g) be a Norden-Walker manifold. If*

$$a_x = b_x = c_x = c_z = 0 \quad (or\ a_x = a_y = c_x = c_z = 0), \tag{22}$$

then g is a Norden-Walker-Einstein.

Proof. Suppose that the triple (M_4, φ', g) be a Norden-Walker manifold. Then from (20) and (22), we see that the assertion is clear, i.e., $G_{ij} = 0$. QED

Corollary 2. *The triple (M_4, φ', g) with metric*

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(y,z,t) & c(t) \\ 0 & 1 & c(t) & b(t) \end{pmatrix}$$

is always Norden-Walker-Einstein.

8 Counterexamples to Goldberg's conjecture

1. Let (M_4, φ, g) be an almost Norden-Walker manifold.
Consider the metric

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x,y,z,t) & 0 \\ 0 & 1 & 0 & a(x,y,z,t) \end{pmatrix}.$$

That is the metric is defined by putting $a = b,\ c = 0$ in the generic canonical form (2). In this case, we see from (21) that the Einstein condition consist of the following PDE's:

$$a_{xx} - a_{yy} = 0, \quad a_{xy} = 0, \quad aa_{xx} - 2a_{yt} + (a_y)^2 = 0,$$
$$a_{xt} - a_x a_y + a_{yz} = 0, \quad aa_{xx} - 2a_{xz} + (a_x)^2 = 0.$$

If a is independent of y and t, and if a contains x only linearly, the first four PDE's hold trivially, and the last one reduces to: $2a_{xz} - (a_x)^2 = 0$. We see that $a = -\frac{2x}{z}$ is a solution to the PDE, and therefore the metric

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -\frac{2x}{z} & 0 \\ 0 & 1 & 0 & -\frac{2x}{z} \end{pmatrix} \tag{23}$$

is Einstein on the coordinate patch $z > 0$ (or $z < 0$). Thus, the second condition (G_2) of Goldberg conjecture holds. We know that this metric admits a proper almost complex structure as follows:

$$\varphi\partial_x = \partial_y, \quad \varphi\partial_y = -\partial_x, \quad \varphi\partial_z = a\partial_y - \partial_t, \quad \varphi\partial_t = -a\partial_x + \partial_z. \tag{24}$$

For the Einstein metric (23), the proper almost complex structure φ in (24) becomes

$$\varphi\partial_x = \partial_y, \quad \varphi\partial_y = -\partial_x, \quad \varphi\partial_z = -\frac{2x}{z}\partial_y - \partial_t, \quad \varphi\partial_t = \frac{2x}{z}\partial_x + \partial_z.$$

Then, the integrability of φ, given in Theorem 3, becomes

$$a_x + b_x + 2c_y = 2a_x = -\frac{4}{z} \neq 0, \quad a_y + b_y - 2c_x = 2a_y = 0.$$

Thus, φ cannot be integrable.

Similarly, the opposite almost complex structure φ' in (16) has the form

$$\varphi'\partial_x = -\tfrac{x}{z}\partial_y + \partial_t, \quad \varphi'\partial_y = \tfrac{x}{z}\partial_x + \partial_z,$$
$$\varphi'\partial_z = -((\tfrac{x}{z})^2 + 1)\partial_y - \tfrac{x}{z}\partial_t, \quad \varphi'\partial_t = -((\tfrac{x}{z})^2 + 1)\partial_x - \tfrac{x}{z}\partial_z.$$

The φ'- integrability condition (20) in Theorem 10 becomes

$$b_y = 0, \quad a_x - 2c_y = a_x = -\tfrac{2}{z} \neq 0, \quad ab_x - 2b_z = aa_x = \tfrac{4x}{z^2} \neq 0,$$
$$ba_y - 2a_t - 2ac_x + 4cc_y + 4c_z = 0.$$

Thus, φ' is not integrable.

2. Let (M_4, φ', g) be an almost Norden-Walker manifold. We assume that $a,\ b,\ c$ does not depend on x and y, i.e., $a = a(z,t),\ b = b(z,t),\ c = c(z,t)$. Therefore, the metric g in (2) becomes

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(z,t) & c(z,t) \\ 0 & 1 & c(z,t) & b(z,t) \end{pmatrix}.$$

In this case, we see from (21) that the metric g is Norden-Walker-Einstein, i.e., $G_{ij} = 0$. Thus, the second condition (G_2) holds.

If $a,\ b$ and c are independent of x and y, the φ'- integrability condition (20) in Theorem 10 becomes

$$b_z = 0, \quad a_t - 2c_z = 0.$$

On the other hand, since $b = b(z,t)$, we have $b_z \neq 0$. Thus, φ' is not integrable.

9 Holomorphic Norden-Walker (Kähler-Norden-Walker) metrics on (M_4, φ', g)

Let (M_4, φ', g)be an almost Norden-Walker manifold. Substituting (2) and (17) in (10), we find the following Kähler-Norden-Walker condition of (M_4, φ', g).

$$\begin{aligned}
(\Phi_{\varphi'}g)_{xxz} &= (\Phi_{\varphi'}g)_{xzx} = -c_x,\ (\Phi_{\varphi'}g)_{xxt} = (\Phi_{\varphi'}g)_{xtx} = -(\Phi_{\varphi'}g)_{txx} = -b_x, \qquad (25)\\
(\Phi_{\varphi'}g)_{xyz} &= (\Phi_{\varphi'}g)_{xzy} = -c_y - \frac{1}{2}a_x,\ (\Phi_{\varphi'}g)_{xzz} = -ca_x - 2c_z - \frac{1}{2}ba_y + a_t,\\
(\Phi_{\varphi'}g)_{xyt} &= (\Phi_{\varphi'}g)_{xty} = -(\Phi_{\varphi'}g)_{txy} = -(\Phi_{\varphi'}g)_{tyx} = -c_x - \frac{1}{2}b_y,\\
(\Phi_{\varphi'}g)_{xzt} &= (\Phi_{\varphi'}g)_{xtz} = -cc_x - \frac{1}{2}bc_y - \frac{1}{2}b_z - \frac{1}{4}ab_x - \frac{1}{4}ba_x,\\
(\Phi_{\varphi'}g)_{xtt} &= -2cb_x - bc_x - \frac{1}{2}bb_y,\ (\Phi_{\varphi'}g)_{yxt} = (\Phi_{\varphi'}g)_{ytx} = -\frac{1}{2}b_y,\\
(\Phi_{\varphi'}g)_{yxz} &= (\Phi_{\varphi'}g)_{yzx} = -(\Phi_{\varphi'}g)_{zxy} = -(\Phi_{\varphi'}g)_{zyx} = -\frac{1}{2}a_x,\\
(\Phi_{\varphi'}g)_{yyz} &= (\Phi_{\varphi'}g)_{yzy} = -(\Phi_{\varphi'}g)_{zyy} = -a_y,\\
(\Phi_{\varphi'}g)_{yyt} &= (\Phi_{\varphi'}g)_{yty} = -\frac{1}{2}(\Phi_{\varphi'}g)_{tyy} = -c_y, (\Phi_{\varphi'}g)_{yzz} = -\frac{1}{2}aa_x,\\
(\Phi_{\varphi'}g)_{yzt} &= (\Phi_{\varphi'}g)_{ytz} = -\frac{1}{2}ac_x - \frac{1}{2}a_t - \frac{1}{4}ab_y - \frac{1}{4}ba_y + c_z,\\
(\Phi_{\varphi'}g)_{ytt} &= -\frac{1}{2}ab_x + b_z - cb_y - bc_y,\ (\Phi_{\varphi'}g)_{zxz} = (\Phi_{\varphi'}g)_{zzx} = -\frac{1}{2}ac_x,\\
(\Phi_{\varphi'}g)_{zxt} &= (\Phi_{\varphi'}g)_{ztx} = \frac{1}{4}ba_x - \frac{1}{4}ab_x - \frac{1}{2}b_z,\\
(\Phi_{\varphi'}g)_{zyz} &= (\Phi_{\varphi'}g)_{zzy} = -\frac{1}{2}ac_y,\\
(\Phi_{\varphi'}g)_{zyt} &= (\Phi_{\varphi'}g)_{zty} = \frac{1}{4}ba_y - \frac{1}{4}ab_y - c_z + \frac{1}{2}a_t,\\
(\Phi_{\varphi'}g)_{zzz} &= -\frac{1}{2}aca_x - ac_z - \frac{1}{4}aba_y - a_y + \frac{1}{2}aa_t,\\
(\Phi_{\varphi'}g)_{zzt} &= (\Phi_{\varphi'}g)_{ztz} = -\frac{1}{2}acc_x - \frac{1}{4}abc_y - c_y - \frac{1}{2}ab_z,\\
(\Phi_{\varphi'}g)_{ztt} &= -\frac{1}{2}acb_x - \frac{1}{4}abb_y - b_y - cb_z + \frac{1}{2}ba_t - bc_z,\\
(\Phi_{\varphi'}g)_{txz} &= (\Phi_{\varphi'}g)_{tzx} = -cc_x + \frac{1}{4}ab_x - \frac{1}{4}ba_x + \frac{1}{2}b_z,\\
(\Phi_{\varphi'}g)_{txt} &= (\Phi_{\varphi'}g)_{ttx} = -\frac{1}{2}bc_x,\ (\Phi_{\varphi'}g)_{tyt} = (\Phi_{\varphi'}g)_{tty} = \frac{1}{2}bc_y,\\
(\Phi_{\varphi'}g)_{tyz} &= (\Phi_{\varphi'}g)_{tzy} = -cc_y + \frac{1}{4}ab_y - \frac{1}{4}ba_y - \frac{1}{2}a_t + c_z,\\
(\Phi_{\varphi'}g)_{tzz} &= -c^2a_x - \frac{1}{4}aba_x - a_x - 2cc_z - \frac{1}{2}bca_y + \frac{1}{2}ab_z + ca_t,\\
(\Phi_{\varphi'}g)_{tzt} &= (\Phi_{\varphi'}g)_{ttz} = -c^2c_x - \frac{1}{4}abc_x - c_x + bc_z - \frac{1}{2}bcc_y - \frac{1}{2}ba_t,\\
(\Phi_{\varphi'}g)_{ttt} &= -c^2b_x - \frac{1}{4}abb_x - b_x - \frac{1}{2}bcb_y + \frac{1}{2}bb_z.
\end{aligned}$$

The following theorem is same to the Theorem 5.

Theorem 13. *A triple (M_4, φ', g) is a Kähler-Norden-Walker manifold if and only if the following PDEs hold:*

$$a_x = a_y = b_x = b_y = b_z = c_x = c_y = 0, \qquad a_t - 2c_z = 0.$$

Corollary 3. *The triple (M_4, φ', g) with metric*

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(z) & 0 \\ 0 & 1 & 0 & b(t) \end{pmatrix}$$

is always Kähler-Norden-Walker.

Let (M_4, φ', g) be an almost Norden-Walker manifold. For the covariant derivative $\nabla G'$ of the twin metric G' put $(\nabla G')_{ijk} = \nabla_i G'_{jk}$, where G' is deffined by $G'_{ij} = \varphi'^{m}_{i} g_{mj}$. Then, after some calculations we obtain

$$\begin{aligned}
\nabla_x G'_{xz} &= \nabla_x G'_{zx} = -\nabla_x G'_{yt} = -\nabla_x G'_{ty} = \frac{1}{2}\nabla_z G'_{xx} = -\frac{1}{2}c_x, \qquad (26)\\
\nabla_x G'_{zz} &= -\frac{1}{2}ac_x, \nabla_x G'_{zt} = \nabla_x G'_{tz} = -\frac{1}{2}cc_x, \nabla_x G'_{tt} = \nabla_y G'_{tt} = \frac{1}{2}bc_x,\\
\nabla_y G'_{xz} &= \nabla_y G'_{zx} = -\nabla_y G'_{yt} = -\nabla_y G'_{ty} = \frac{1}{2}\nabla_t G'_{yy} = -\frac{1}{2}c_y,\\
\nabla_y G'_{zz} &= -\frac{1}{2}ac_y, \nabla_y G'_{zt} = \nabla_y G'_{tz} = -\frac{1}{2}cc_y,\\
\nabla_z G'_{xy} &= \nabla_z G'_{yx} = -\frac{1}{2}a_x - \frac{1}{2}c_y, \nabla_z G'_{xt} = \nabla_z G'_{tx} = -\frac{1}{4}bc_y - cc_x - \frac{1}{4}ba_x,\\
\nabla_z G'_{xz} &= \nabla_z G'_{zx} = -\frac{1}{4}ac_x - \frac{1}{2}ca_x - c_z + \frac{1}{2}a_t - \frac{1}{4}ba_y,\\
\nabla_z G'_{yy} &= -a_y, \nabla_z G'_{yz} = \nabla_z G'_{zy} = -\frac{1}{4}aa_x - \frac{1}{4}ac_y,\\
\nabla_z G'_{yt} &= \nabla_z G'_{ty} = c_z - \frac{1}{2}a_t - \frac{1}{4}ac_x - \frac{1}{2}cc_y - \frac{1}{4}ba_y,\\
\nabla_z G'_{zz} &= -ac_z + \frac{1}{2}aa_t - \frac{1}{4}aba_y - \frac{1}{2}aca_x - a_y,\\
\nabla_z G'_{zt} &= \nabla_z G'_{tz} = -cc_z - \frac{1}{4}bca_y + \frac{1}{2}ca_t - (\frac{1}{2}c^2 + \frac{1}{8}ab + \frac{1}{2})a_x\\
&\quad -\frac{1}{4}acc_x - (\frac{1}{4}ab - \frac{1}{8}ab + \frac{1}{2})c_y,\\
\nabla_z G'_{tt} &= bc_z - \frac{1}{2}ba_t - \frac{1}{2}bcc_y - (c^2 + \frac{1}{4}ab + 1)c_x,\\
\nabla_t G'_{xx} &= -b_x, \nabla_t G'_{xy} = \nabla_t G'_{yx} = -\frac{1}{2}c_x - \frac{1}{2}b_y,\\
\nabla_t G'_{xz} &= \nabla_t G'_{zx} = -\frac{1}{4}ab_x - \frac{1}{2}b_z - \frac{1}{4}bc_y - \frac{1}{2}cc_x,\\
\nabla_t G'_{xt} &= \nabla_t G'_{tx} = -\frac{1}{4}bb_y - cb_x - \frac{1}{4}bc_x,\ \nabla_t G'_{yz} = \nabla_t G'_{zy} = -\frac{1}{4}ab_y - \frac{1}{4}ac_x,\\
\nabla_t G'_{yt} &= \nabla_t G'_{ty} = \frac{1}{2}b_z - \frac{1}{4}ab_x - \frac{1}{4}bc_y - \frac{1}{2}cb_y,\\
\nabla_t G'_{zz} &= -\frac{1}{2}ab_z - \frac{1}{4}abc_y - \frac{1}{2}acc_x - c_y,\\
\nabla_t G'_{zt} &= \nabla_t G'_{tz} = -(\frac{1}{8}ab + \frac{1}{2})b_y - \frac{1}{4}acb_x - \frac{1}{4}bcc_y
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}cb_z - (\frac{1}{8}ab + \frac{1}{2}c^2 + \frac{1}{2})c_x, \\
\nabla_t G'_{tt} \quad = \quad & \frac{1}{2}bb_z - \frac{1}{2}bcb_y - \frac{1}{4}abb_x - c^2 b_x - b_x.
\end{aligned}
$$

From (25) and (26) we have

Theorem 14. *A triple (M_4, φ', g) is a quasi-Kähler Norden-Walker manifold if and only if the following PDE's hold:*

$$a_x = a_y = b_x = b_y = b_z = c_x = c_y = 0, \; a_t - 2c_z = 0.$$

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