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Common Fixed Point Theorems in Uniform Spaces

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Abstract. In this paper we prove some fixed point theorems for weakly compatible mappings with the notation of A-distance and E-distance in uniform space.

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MSC 2000 classification: 49H10, 54H25

1 Introduction and Preliminaries

The concept of weakly compatible is defined by Jungck and Rhoades [3]. In this paper we take weakly compatible to prove common fixed point theorems. Recently, Aamri and Moutawakil [1] introduce the concept of A-distance and E-distance in uniform space. With the help of these A-distance and E-distance we prove common fixed point for weakly compatible.

Definition 1. Two self maps T and S of a metric space X are said to be weakly compatible if they commute at there coincidence points, i.e. if $Tu = Su$ for u in X, then $TSu = STu$.

By Bourbaki [2], we call uniform space (X, ϑ) a non empty set X endowed of an uniformity ϑ, the latter being a special kind of filter on $X \times X$, for all whose elements contain the diagonal $\Delta = \{(x, x)|x \in X\}$. if $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$, x and y are said to be V-close and a sequence (x^n) in X is a Cauchy sequence for ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x^n and x^m are V-close for $n, m \geq N$. An uniformly ϑ defines a unique topology $T(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X|(x, y) \in V\}$ when V runs over ϑ.

A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all $V \in \vartheta$ redices to the diagonal Δ of X i.e. if $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \vartheta$ is said to be symmetrical if $V = V^{-1} = \{(y, x)|(x, y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x, y) \in W$ then x and y are both W and V-còlose, then for our purpose, we assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X, ϑ), they always refer to the topological space $(X, T(\vartheta))$.

Definition 2. Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow \mathbb{R}^+$ is said to be an A-distance if for any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

Definition 3. Let (X, ϑ) be uniform space. A function $p : X \times X \longrightarrow \mathbb{R}^+$ is said to be an E-distance if p is an A-distance and $p(x, y) \leq p(x, z) + p(z, y)$, for every $x, y, z \in X$.

Definition 4. Let (X, ϑ) be uniform space and p be an A-distance on X.
(I) X in S complete if for every p-Cauchy sequences $\{x_n\}$ there exists $x \in X$ such that $limp(x_n, x) = 0$.
(II) X is p-Cauchy complete if for every p-Cauchy sequences $\{x_n\}$ there exists $x \in X$ such that $limx_n = x$ with respect to $\tau(\vartheta)$.
(III) $f : X \longrightarrow X$ is p-continuous if $limp(x_n, x) = 0$ implies $limp(f(x_n), f(x)) = 0$.
(IV) $f : X \longrightarrow X$ is $T(\vartheta)$-continuous if $limx_n = x$ with respect to $T(\vartheta)$ implies $limf(x_n) = f(x)$ with respect to $\tau(\vartheta)$.
(V) X is said to be p-bounded if $\delta_p(X) = sup\{p(x, y)|x, y \in X\} < \infty$.

Lemma 1. *Let (X, ϑ) be uniform space and p be an A-distance on X. Let $\{x_n\}, \{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $\mathbb{R}^+$ and converging to 0. Then, for $x, y, z \in X$, the following holds*
(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.
(b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.
(c) If $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequences in (X, ϑ).

Let $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be continuous and satisfying the conditions
(i) ψ is nondecreasing on $\mathbb{R}^+$,
(ii) $0 < \psi(t) < t$, for each $t \in (0, \infty)$.

Theorem 1. *Let (X, ϑ) be a Hausdorff uniform space and p be an A-distance on X. Let f and g are two weakly compatible defined on X such that*
(I) $f(X) \subseteq g(X)$

$$(II) \qquad p(f(x), f(y))) \leq \psi[max\{p(g(x), g(y))\}, \\ 1/2[p(g(x), f(x)) + p(g(x), f(y))], 1/2[p(g(y), f(y)) + p(g(y), f(x))]\}]$$

If $f(X)$ or $g(X)$ is a S complete subspaces of X, then f and g have a common fixed point.

Proof. Let $x_0 \in X$ and choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. Choose $x_2 \in X$ such that $f(x_1) = g(x_2)$. In general $f(x_n) = g(x_{n+1})$. Then let the sequence y_{n+1} such that

$$y_{n+1} = f(x_n) = g(x_{n+1}) \dots \tag{1}$$

Now there arise two cases:
Case 1 If $y_n = y_{n+p}$ for $n \in \mathbb{N}$, we have $z = y_n = g(x_n) = f(x_n) = g(x_{n+1} = y_{n+1}$. Now taking $u = x_n$, then $f(u) = g(u)$ and by weakly compatibility $fg(u) = gf(u)$. Now

$$\begin{aligned} d(f(z), z) &= d(f(z), f(u)) \\ &\leq \psi[max\{p(g(z), g(u))\}, 1/2[p(g(z), f(z)) + p(g(z), f(u))], \\ &\quad 1/2[p(g(u), f(u)) + p(g(u), f(z))]\}] \\ &\leq \psi[max\{p(z, f(z))\}, 1/2[p(gf(u), fg(u)) + p(gf(u), fg(u))], \\ &\quad 1/2[p(f(u), f(u)) + p(z, f(z))]\}] \\ &\leq \psi\{p(z, f(z))\} < p(z, f(z))) \end{aligned}$$

which is contradiction. It implies $f(z) = z$. Again, $z = f(z) = fg(u) = gf(u) = g(z)$. So, z is common fixed point of f and g.
Case 2. Let $y_n \neq y_{n+p}$ for all $n \in \mathbb{N}$. We have that

$$lim y_n = lim f x_n = lim g x_{n+1} = z$$

For this $z \in g(X)$ there exist ω in X such that $z = g(\omega)$. Now by condition (II) of theorem we have

$$\begin{aligned} p(f(\omega), g(\omega)) &\leq p(f(\omega), f(x_n)) + p(f(x_n), g(\omega)) \\ &\leq \psi[max\{p(g(\omega), g(x_n))\}, 1/2[p(g(\omega), f(\omega)) + p(g(\omega), f(x_n))], \\ &\quad 1/2[p(g(x_n), f(x_n)) + p(g(x_n), f(\omega))]\}] + p(f(x_n), g(\omega)) \\ &\leq \psi[max\{p(z,z)\}, 1/2[p(g(\omega), f(\omega)) + p(g(\omega), g(\omega))], \\ &\quad 1/2[p(z,z) + p(g(\omega), f(\omega))]\}] + p(z,z), \quad \text{as} \quad n \to \infty \\ &\leq \psi\{p(g(\omega), f(\omega))\} \leq p(g(\omega), f(\omega)) \end{aligned}$$

It implies that $f\omega = g\omega$. The assumption that f and g are weakly compatible implies $fg(\omega) = gf(\omega)$. Also $ff(\omega) = fg(\omega) = gf(\omega) = gg(\omega)$. Suppose that $p(f(\omega), ff(\omega)) \neq 0$. From (II), it follows

$$\begin{aligned} p(f(\omega), f(\omega)) &\leq \psi\ max[\{p(g(\omega), gf(\omega)), 1/2[p(g(\omega), f(\omega)) + p(g(\omega), ff(\omega))], \\ &\quad 1/2[p(gf(\omega), ff(\omega)) + p(gf(\omega), f(\omega))]\}] \\ &\leq \psi\{p(f(\omega), ff(\omega))\} < p(f(\omega), ff(\omega)) \end{aligned}$$

which is a contradiction. Thus $p(f(\omega), ff(\omega)) = 0$.
Suppose that $p(f(\omega), f(\omega)) \neq 0$, then also by (II)

$$\begin{aligned} p(f(\omega), f(\omega)) &\leq \psi\ max[\{p(g(\omega), g(\omega)), 1/2[p(g(\omega), f(\omega)) + p(g(\omega), f(\omega))], \\ &\quad 1/2[p(gf(\omega), f(\omega)) + p(g(\omega), f(\omega))]\}] \\ &\leq \psi\{p(f(\omega), f(\omega))\} < p(f(\omega), f(\omega)) \end{aligned}$$

which is a contradiction. Thus $p(f(\omega), f(\omega)) = 0$. Since $p(f(\omega), f(\omega)) = 0$ and $p(f(\omega), ff(\omega)) = 0$, lemma 1(a) gives $ff(\omega) = f(\omega)$.
Hence $g(f(\omega)) = f(f(\omega)) = f(\omega)$ and $z = f(\omega)$ is common fixed point of f and g.

QED

Theorem 2. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let f and g are two weakly compatible defined on X such that*
(I) $f(X) \subseteq g(X)$

$$(II) \qquad p(f(x), f(y))) \leq \psi[max\{p(g(x), g(y))\}, 1/2[p(g(x), f(x)) + p(g(x), f(y))], 1/2[p(g(y), f(y)) + p(g(y), f(x))]\}]$$

If $f(X)$ or $g(X)$ is a S complete subspaces of X, then f and g have a unique common fixed point.

Proof. Since E distance in A distance therefore f and g have a common point. Suppose z_1 and z_2 are common fixed points of f and g, then $f(z_1) = g(z_1) = z_1$ and $f(z_2) = g(z_2) = z_2$.

If $p(z_1, z_2) \neq 0$, then by (II)

$$\begin{aligned} p(z_1, z_2) &= p(f(z_1), f(z_2)) \leq \psi\ max[\{p(g(z_1), g(z_2)), \\ &\quad 1/2[p(g(z_1), f(z_1)) + p(g(z_1), f(z_2))], \\ &\quad 1/2[p(g(z_2), f(z_2)) + p(g(z_2), f(z_1))]\}] \\ &= \psi\ \{p(z_1, z_2)\} < p(z_1, z_2) \Rightarrow p(z_1, z_2) = 0. \end{aligned}$$

Consequently by $(p2)$ we have $p(z_1, z_1) \leq p(z_1, z_2) + p(z_2, z_1) \Rightarrow p(z_1, z_2) = 0$. Now, we have $p(z_1, z_1) = 0$ and $p(z-1, z_2) = 0$ therefore $z_1 = z_2$.

QED

Theorem 3. *Let (X, ϑ) be a Hausdorff uniform space and p be an A-distance on X. Let f and g are two weakly compatible defined on X such that*
(I) $f^r(X) \subseteq g^s(X)$

$$(II) \qquad p(f^r(x), f^r(y))) \leq \psi[max\{p(g^s(x), g^s(y))\}, \\ 1/2[p(g^s(x), f^r(x)) + p(g^s(x), f^r(y))], 1/2[p(g^s(y), f^r(y)) + p(g^s(y), f^r(x))]\}]$$

where r and s are positive integers. If $f(X)$ or $g(X)$ is a S complete subspaces of X, then f and g have a common fixed point.

Proof. Same as theorem 1. QED

Theorem 4. *Let (X, ϑ) be a Hausdorff uniform space and p be an A-distance on X. Let f and g are two weakly compatible defined on X such that*
(I) $f^r(X) \subseteq g^s(X)$

$$(II) \qquad p(f^r(x), f^r(y))) \leq \psi[max\{p(g^s(x), g^s(y))\}, \\ 1/2[p(g^s(x), f^r(x)) + p(g^s(x), f^r(y))], 1/2[p(g^s(y), f^r(y)) + p(g^s(y), f^r(x))]\}]$$

where r and s are positive integers. If $f(X)$ or $g(X)$ is a S complete subspaces of X, then f and g have a unique common fixed point.

Proof. Same as theorem 2. QED

Example. Let $X = [0, 1]$ and $d(x, y) = |x - y|$. Self mappings f and g are defined as $fx = x^2$ if $x \in [0, 1/2)$ & $= 1/2$ if $x \in [1/2, 1]$ and $gx = 0$ if $x \in [0, 1/2)$ & $= x$ if $x \in [1/2, 1]$. Now, consider the functions p and ψ as: $\psi(x) = x^2$ and $p(x, y) = 0$ if $y \in [0, 1/2)$ & $= y$ if $y \in [1/2, 1]$. All conditions of Theorem 2 are satisfied and $1/2$ is common point of f and g.

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