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Groups with all subgroups subnormal

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Abstract. An updated survey on the theory of groups with all subgroups subnormal, including a general introduction on locally nilpotent groups, full proofs of most results, and a review of the possible generalizations of the theory.

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1 Locally nilpotent groups

In this chapter we review part of the basic theory of locally nilpotent groups. This will mainly serve to fix the notations and recall some definitions, together with some important results whose proofs will not be included in these notes. Also, we hope to provide some motivation for the study of groups with all subgroups subnormal (for short $\mathcal{N}_1$-groups) by setting them into a wider frame. In fact, we will perhaps include more material then what strictly needed to understand $\mathcal{N}_1$-groups.

Thus, the first sections of this chapter may be intended both as an unfaithful list of prerequisites and a quick reference: as such, most of the readers might well skip them. As said, we will not give those proofs that are too complicate or, conversely, may be found in any introductory text on groups which includes some infinite groups (e.g. [97] or [52], for nilpotent groups we may suggest, among many, [56]). For the theory of generalized nilpotent groups and that of subnormal subgroups, our standard references will be, respectively, D. Robinson's classical monography [96] and the book by Lennox and Stonehewer [64].

In the last section we begin the study of $\mathcal{N}_1$-groups, starting with the first basic facts, which are not difficult but are fundamental to understand the rest of these notes.

1.1 Commutators and related subgroups

Let x, y be elements of a group G. As customary, we denote by $x^y = y^{-1}xy$ the conjugate of x by y. The *commutator* of x and y is defined in the usual way as

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y.$$

Then, for $n \in \mathbb{N}$, the iterated commutator $[x,_n y]$ is recursively defined as follows

$$[x,_0 y] = x, \quad [x,_1 y] = [x, y]$$

and, for $1 \leq i \in \mathbb{N}$,

$$[x,_{i+1} y] = [[x,_i y], y].$$

Similarly, if $x_1, x_2, \ldots x_n$ are elements of G, the *simple* commutator of weight n is defined recursively by

$$[x_1, x_2, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

We list some elementary but important facts of commutator manipulations. They all follow easily from the definitions, and can be found in any introductory text in group theory.

1 Lemma. *Let G be a group, and $x, y, z \in G$. Then*

(1) $[x, y]^{-1} = [y, x]$;

(2) $[xy, z] = [x, z]^y [y, z] = [x, z][x, z, y][y, z]$;

(3) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$;

(4) *(Hall-Witt identity)* $[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1$.

2 Lemma. *Let G be a group, $x, y \in G$, $n \in \mathbb{N}$, and suppose that $[x, y, y] = 1$; then $[x, y]^n = [x, y^n]$. If further $[x, y, x] = 1$, then*

$$(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}.$$

If X is a subset of a group G then $\langle X \rangle$ denotes the subgroup generated by X. If U and V are non-empty subsets of the group G, we set

$$[U, V] = \langle [x, y] \mid x \in U, \ y \in V \rangle$$

and define inductively in the obvious way $[U,_n V]$, for $n \in \mathbb{N}$. Finally, if $A \leq G$, and $x \in G$, we let, for all $n \in \mathbb{N}$, $[A,_n x] = \langle [a,_n x] \mid a \in A \rangle$.

If $H \leq G$, H_G denotes the largest normal subgroup of G contained in H, and H^G the *normal closure* of H in G, i.e. the smallest normal subgroup of G containing H. Clearly,

$$H_G = \bigcap_{g \in G} H^g \qquad \text{and} \qquad H^G = \langle H^g \mid g \in G \rangle.$$

More generally, if X and Y are non-empty subsets of the group G, we denote by X^Y the subgroup $\langle x^y \mid x \in X, \ y \in Y \rangle$.

The following are easy consequences of the definitions.

3 Lemma. *Let H and K be subgroups of a group. Then $[H,K] \trianglelefteq \langle H,K\rangle$.*

4 Lemma. *Let X, Y be subsets of the group G. Then*

$$[\langle X\rangle, \langle Y\rangle] = [X,Y]^{\langle X\rangle\langle Y\rangle}.$$

If $N \trianglelefteq G$, then $[N, \langle X\rangle] = [N, X]$.

The next, very useful Lemma follows from the Hall-Witt identity.

5 Lemma. [Three Subgroup Lemma]. *Let A, B, C be subgroups of the group G, and let N be a normal subgroup such that $[A,B,C]$ and $[B,C,A]$ are contained in N. Then also $[C,A,B]$ is contained in N.*

The rules in Lemma 1, as well as others derived from those, may be applied to get sorts of handy analogues for subgroups. For instance, if A, B, C are subgroups of G and $[A,C]$ is a normal subgroup, then $[AB, C] = [A,C][B,C]$. More generally, we have

6 Lemma. *Let N, $H_1, \dots, H_n$ be subgroups of the group G, with $N \trianglelefteq G$, and put $Y = \langle H_1, \dots, H_n\rangle$. Then*

$$[N, Y] = [N, H_1] \dots [N, H_n].$$

The same commutator notation we adopt for groups actions: let the group G act on the group A. For all $g \in G$ and $a \in A$, we set $[a,g] = a^{-1}a^g$, and $[A,G] = \langle [a,g] \mid a \in A, g \in G\rangle$. With the obvious interpretations, the properties listed above for standard group commutators continue to hold.

For a group G, the subgroup $G' = [G,G]$ is called the *derived* subgroup of G, and is the smallest normal subgroup N of G such that the quotient G/N is abelian. The terms $G^{(d)}$ $(1 \leq d \in \mathbb{N})$ of the *derived series* of G are the characteristic subgroups defined by $G^{(1)} = G'$ and, inductively ,

$$G^{(n+1)} = (G^{(n)})' = [G^{(n)}, G^{(n)}]$$

(the second derived subgroup $G^{(2)}$ is often denote by G''). The group G is *soluble* if there exists an n such that $G^{(n)} = 1$; in such a case the smallest integer n for which this occurs is called the *derived length* of the soluble group G. Of course, subgroups and homomorphic images of a soluble group of derived length d are soluble with derived length at most d.

A group is said to be *perfect* if it has no non-trivial abelian quotients; thus, G is perfect if and only if $G = G'$.

By means of commutators are also defined the terms $\gamma_d(G)$ of the *lower central series* of a group G: set $\gamma_1(G) = G$, and inductively, for $d \geq 1$,

$$\gamma_{d+1}(G) = [\gamma_d(G), G] = [G,_d G].$$

These are also characteristic subgroups of G. A group G is *nilpotent* if, for some $c \in \mathbb{N}$, $\gamma_{c+1}(G) = 1$. The *nilpotency class* (or, simply, the class) of a nilpotent group G is the smallest integer c such that $\gamma_{c+1}(G) = 1$.

7 Lemma. *Let G be a group, and $m, n \in \mathbb{N} \setminus \{0\}$. Then*

(1) $[\gamma_n(G), \gamma_m(G)] \leq \gamma_{n+m}(G)$;.

(2) $\gamma_m(\gamma_n(G)) \leq \gamma_{mn}(G)$;

From (1), and induction on n, we have

8 Corollary. *For any group G and any $1 \leq n \in \mathbb{N}$, $G^{(n)} \leq \gamma_{2^n}(G)$. In particular a nilpotent group of class c has derived length at most $[\log_2 c] + 1$.*

Also, by using (1) and induction, one easily proves the first point of the following Lemma, while the second one follows by induction and use of the commutator identities of 1,

9 Lemma. *Let G be a group, and $1 \leq n \in \mathbb{N}$. Then*

(1) $\gamma_n(G) = \langle [g_1, g_2, \ldots, g_n] \mid g_i \in G,\ i = 1, 2, \ldots, n \rangle$.

(2) *If S is a generating set for G, then $\gamma_n(G)$ is generated by the simple commutators of weight at least n in the elements of $S \cup S^{-1}$.*

The *upper central series* of a group G is the series whose terms $\zeta_i(G)$ are defined in the familiar way: $\zeta_1(G) = Z(G) = \{x \in G \mid xg = gx \ \forall g \in G\}$ is the centre of G, and for all $n \geq 2$, $\zeta_n(G)$ is defined by

$$\zeta_n(G)/\zeta_{n-1}(G) = Z(G/\zeta_{n-1}(G)).$$

A basic observation is that, for $n \geq 1$, $\zeta_n(G) = G$ if and only if $\gamma_{n+1}(G) = 1$, and so G is nilpotent of class c if and only if $G = \zeta_c(G)$ and c is the smallest such positive integer. This follows at once from the following property.

10 Lemma. *Let G be a group, and $1 \leq n \in \mathbb{N}$. Then $[\gamma_n(G), \zeta_n(G)] = 1$.*

The next remark is often referred to as *Grün's Lemma.*

11 Lemma. *Let G be a group. If $\zeta_2(G) > \zeta_1(G)$ then $G' < G$.*

Let us recall here some elementary but more technical facts, which we will frequently use, about commutators in actions on an abelian groups.

Thus, let A be a normal abelian subgroup of a group G, $F \leq A$, and let $x \in G$. It is then easy to see that, for all $i \in \mathbb{N}$,

$$[F,_i x] = \{ \ [a,_i x] \mid a \in F \ \} \quad \text{and} \quad F^{\langle x \rangle} = \langle \ [F,_i x] \mid i \in \mathbb{N} \ \rangle.$$

12 Lemma. *Let A be a normal abelian subgroup of the group G, and $H \leq G$. Suppose that $H/C_H(A)$ is abelian. Then, for all $a \in A$, $x, y \in H$:*

$$[a, x, y] = [a, y, x].$$

PROOF. Since $H/C_H(A)$ is abelian, $[a, xy] = [a, yx]$, and, by expanding the commutators using Lemma 1, $[a, y][a, x]^y = [a, x][a, y]^x$. Since A is abelian, we get the desired equality $[a, x]^{-1}[a, x]^y = [a, y]^{-1}[a, y]^x$. $\boxed{QED}$

13 Corollary. *Let A be a normal abelian subgroup of the group G, such that $G/C_G(A)$ is abelian. Then, for all $X, Y \leq G$: $[A, X, Y] = [A, Y, X]$.*

14 Lemma. *Let A be a a normal elementary abelian p-subgroup of a group G. Then, for all $x \in G$, $[A,_{p^m} x] = [A, x^{p^m}]$ for all $m \in \mathbb{N}$.*

PROOF. It is convenient to look at x as to an endomorphism, via conjugation, of the abelian group A. Then, for all $a \in A$, $[a, x] = a^{-1}a^x = a^{x-1}$, whence, as A has exponent p,

$$[a,_p x] = a^{(x-1)^p} = a^{x^p-1} = [a, x^p]$$

and the inductive extension to any power x^{p^m} is immediate. $\boxed{QED}$

15 Corollary. *Let $1 \neq A$ be a normal elementary abelian p-subgroup of the group G. If $G/C_G(A)$ is a finite p-group, then there exists $n \geq 1$ such that $A \leq \zeta_n(G)$.*

PROOF. Let $C = C_G(A)$. We argue by induction on m, where $|G/C| = p^m$. If $m = 0$, A is central in G. Thus, let $m \geq 1$, N/C a maximal subgroup of G/C, and $x \in G \setminus N$. Then, by inductive assumption, $A \leq \zeta_k(N)$, for some $k \geq 1$. Let $A_0 = \zeta(N) \cap A$; then $A_0 \neq 1$ and $C_G(A_0) \geq N$. Now, $x^p \in N$, and by Lemma 14

$$[A_0,_p x] = [A_0, x^p] \leq [A_0, N] = 1.$$

This means that $A_0 \leq \zeta_p(G)$. Ny repeating this same argument for all the central N-factors contained in A, we get $[A,_{pk} G] = 1$, whence $A \leq \zeta_{pk}(G)$. $\boxed{QED}$

16 Lemma. *Let A be an abelian group, and x an automorphism of A such that $[A,_n x] = 1$, for $n \geq 1$.*

(i) *If x has finite order q, then $[A^{q^{n-1}}, x] = 1$.*

(ii) *If A has finite exponent $e \geq 2$, then $[A, x^{e^{n-1}}] = 1$.*

(iii) *Let the group H act on A with $[A,_n H] = 1$ ($n \geq 1$); then $\gamma_n(H) \leq C_H(A)$.*

Proof. (i) By induction on n. If $n = 1$ we have nothing to prove. Thus, let $n \geq 2$, and set $B = [A, x]$. Then $[B,_{n-1} x] = 1$, whence, by inductive assumption,

$$[A^{q^{n-2}}, x, x] = [[A, x]^{q^{n-2}}, x] = [B^{q^{n-2}}, x] = 1.$$

Now, let $b \in A^{q^{n-2}}$. Then, since $[b, x, x] = 1 = [b, x, b]$, by Lemma 2 we have $[b^q, x] = [b, x]^q = [b, x^q] = 1$. Hence, $[A^{q^{n-1}}, x] = [(A^{q^{n-2}})^q, x] = 1$, as wanted.

(ii) By induction on n. If $n = 1$, then $1 = [A, x] = [A, x^{e^0}]$. Let $n \geq 2$, and set $B = [A, x^{e^{n-2}}] \leq [A, x]$. Then, by inductive hypothesis,

$$[A, x^{e^{n-2}}, x^{e^{n-2}}] = [B, x^{e^{n-2}}] = 1 \ .$$

By Lemma 2, we then have $[A, x^{e^{n-1}}] = [A, x^{e^{n-2}e}] = [A, x^{e^{n-2}}]^e = 1$.

(iii) By induction on n, being the case $n = 1$ trivial. Let $n > 1$. Then H acts on $[A, H]$ and $[A, H,_{n-1} H] = 1$, hence, by inductive assumption

$$[A, H, \gamma_{n-1}(H)] = 1. \tag{1}$$

Let $A_0 = [A,_{n-1} H]$ and $\overline{A} = A/A_0$. Then H acts on $\overline{A}$ and $[\overline{A},_{n-1} H] = 1$. By inductive assumption we have $[\overline{A}, \gamma_{n-1}(H)] = 1$, which means $[\gamma_{n-1}, A] \leq A_0$. Since $[A_0, H] = 1$, we get $[\gamma_{n-1}(H), A, H] = 1$, which, together with (1 and the Three Subgroup Lemma, yields $\gamma_n(H), A] = [H, \gamma_{n-1}(H), A] = 1$. $\boxed{QED}$

Point (iii) of Lemma 16 is a particular case of a theorem of Kalužnin, which we will state later, together with an important generalization due to P. Hall.

It is not difficult to extend similar remarks to the case when A is nilpotent. in which case it is to be expected that the numerical values will depend also on the nilpotency class of A. We show only one of these possible generalizations.

17 Lemma. *Let A be a nilpotent group of class c, and x an automorphism of A such that $|x| = q$ and $[A,_n x] = 1$, for $n \geq 1$. Then $[A^{q^{cn-1}}, x] = 1$.*

Proof. We argue by induction on the class c of A. The case $c = 1$ is just point (i) of the previous Lemma. Thus, we assume $c \geq 2$ and write $B = A^{q^{(c-1)n-1}}$. Then, by inductive assumption,

$$[B, x] \leq \gamma_c(A) \leq Z(A).$$

In particular, $[B, x, B] = 1$, and so by Lemma 2, $[B^{q^{n-1}}, x] = [B, x]^{q^{n-1}}$. Also, $[B, x]$ is abelian and so $[[B, x], x] = [B, x, x]$. Thus, by case $c = 1$, $[[B, x]^{q^{n-1}}, x] = 1$. Hence $[B^{q^{n-1}}, x, x] = 1$. Thus

$$[B^{q^n}, x] = [B^{q^{n-1}}, x]^q = [B^{q^{n-1}}, x^q] = 1.$$

Therefore, $A^{q^{cn-1}} = B^{q^n} \leq C_A(x)$, as wanted. $\boxed{QED}$

Let us state a handy corollary, for which we need to fix the following notation. Given a group G, and an integer $n \geq 1$, we denote by G^n the subgroup of G generated by the n-th powers of all the elements of G, and set $G^\omega = \bigcap_{n\in\mathbb{N}} G^n$.

18 Corollary. *Let G be a periodic nilpotent group. Then $G^\omega \leq Z(G)$.*

Now a technical result (Lemma 21) which will be very useful. For the proof we first need the following observation

19 Lemma. *Let A be a nilpotent group of class $c > 0$, and let x be an automorphism of A. Then, for every $q \geq 1$,*

$$[A^{q^c}, \langle x\rangle] \leq [A, \langle x\rangle]^q.$$

Proof. By induction on c. If $c = 1$ we have equality $[A^q, \langle x\rangle] = [A, \langle x\rangle]^q$. Thus, let $c \geq 2$, $T = \gamma_c(A)$, and set $D = [A, \langle x\rangle]^q$. Then, D is normal in A and $\langle x\rangle$-invariant. By inductive assumption, $[A^{q^{c-1}}, \langle x\rangle] \leq DT$; i.e., setting $\overline{A} = A/D$,

$$[\overline{A}^{q^{c-1}}, \langle x\rangle] \leq \overline{T} \leq Z(\overline{A}).$$

If $a \in A$ and $u = a^{q^{c-1}}$, we have $[Du, \langle x\rangle] \leq \overline{T}$, and so $[Du^q, x] = [Du, x]^q = 1$, which is to say that

$$[a^{q^c}, \langle x\rangle] = [u^q, \langle x\rangle] \subseteq D = [A, \langle x\rangle]^q,$$

thus completing the proof. $\boxed{QED}$

20 Corollary. *Let A be a nilpotent group of class $c > 0$, and let $x_1, \dots, x_d$ be automorphisms of A. Then, for every $q \geq 1$,*

$$[A^{q^{c^d}}, \langle x_1\rangle, \dots \langle x_d\rangle] \leq [A, \langle x_1\rangle, \dots, \langle x_d\rangle]^q.$$

21 Lemma. *Let A be a nilpotent group of class c, let $x_1, x_2, \dots, x_d$ be automorphisms of A such that $[A,_n \langle x_i\rangle] = 1$ for all $i = 1, \dots, d$. Let $q_1, \dots, q_d$ be integers ≥ 1, and $q = q_1 \cdots q_d$. Then*

$$[A^{q^{nc^d}}, \langle x_1\rangle, \dots, \langle x_d\rangle] \leq [A, \langle x_1^{q_1}\rangle, \dots, \langle x_d^{q_d}\rangle].$$

Proof. We argue by induction on $d \geq 1$. If $d = 1$, $q = q_1$, write $R = [A, \langle x^q\rangle]$. Then $R \trianglelefteq \langle A, x\rangle$, and by applying Lemma 17 to the action of x on A/R, we have (since x^q centralizes A/R)

$$[A^{q^{cn}}, \langle x\rangle] \leq R$$

which is what we want.

Let then $d \geq 2$. Write $s = q_1 \dots q_{d-1}$ and $B = [A^{s^{nc^{d-1}}}, \langle x_1 \rangle, \dots, \langle x_{d-1} \rangle]$. By inductive assumption

$$B \leq [A, \langle x_1^{q_1} \rangle, \dots, \langle x_{d-1}^{q_{d-1}} \rangle]. \tag{2}$$

Now, $q^{nc^d} = s^{nc^d} q_d^{nc^d}$; thus, using Corollary 20,

$$[A^{q^{nc^d}}, \langle x_1 \rangle, \dots, \langle x_d \rangle] \leq [[A^{s^{nc^d}}, \langle x_1 \rangle, \dots, \langle x_{d-1} \rangle]^{q_d^{nc}}, \langle x_d \rangle] \leq [B^{q_d^{nc}}, \langle x \rangle].$$

By the case $d = 1$ we then have

$$[A^{q^{nc^d}}, \langle x_1 \rangle, \dots, \langle x_d \rangle] \leq [B^{\langle x_q \rangle}, \langle x_d^{q_d} \rangle] = [B, \langle x_d^{q_d} \rangle],$$

from which, applying (2), we get the desidered inclusion. $\boxed{QED}$

1.2 Subnormal subgroups and generalizations

A subgroup H of the group G is said to be *subnormal* (written $H \lhd\lhd G$) if H is a term of a finite series of G; i.e. if there exists $d \in \mathbb{N}$ and a series of subgroups, such that

$$H = H_d \trianglelefteq H_{d-1} \trianglelefteq \dots \trianglelefteq H_0 = G.$$

If $H \lhd\lhd G$, then the *defect* of H in G is the shortest lenght of such a series; it will be denoted by $d(H, G)$. We shall say that a subgroup H of G is *n-subnormal* if $H \lhd\lhd G$ and $d(H, G) \leq n$.

Clearly, subnormality is a transitive relation, in the sense that if $S \lhd\lhd H$ and $H \lhd\lhd G$, then $S \lhd\lhd G$. Moreover, if $S \lhd\lhd G$, then $S \cap H \lhd\lhd H$ for every $H \leq G$, and $SN/N \lhd\lhd G/N$ for every $N \trianglelefteq G$. Also, the intersection of a finite set of subnormal subgroups is subnormal; but this is not in general true for the intersection of an infinite family of subnormal subgroups. The join $\langle S_1, S_2 \rangle$ of two subnormal subgroups S_1 and S_2 is not in general a subnormal subgroup (see [64] for a full discussion of this point).

The reason why groups with all subgroups subnormal became a subject of investigation lies in the following elementary facts.

22 Proposition. *(1) In a nilpotent group of class c every subgroup is subnormal of defect at most c.*

(2) A finitely generated group in which every subgroup is subnormal is nilpotent.

Let $H \leq G$; the *normal closure series* $(H^{G,n})_{n \in \mathbb{N}}$ of H in G is defined recursively by

$$H^{G,0} = G, \quad H^{G,1} = H^G, \quad \text{and} \quad H^{G,n+1} = H^{H^{G,n}}.$$

By definition, $H^{G,n+1} \trianglelefteq H^{G,n}$, and it is immediate to show that if $H \lhd\lhd G$ and $H = H_d \trianglelefteq H_{d-1} \trianglelefteq \ldots \trianglelefteq H_0 = G$ is a series from H to G, then, for all $0 \leq n \leq d$, $H^{G,n} \leq H_n$. Thus, a subgroup H is subnormal in G if and only if $H^{G,d} \leq H$ for some $d \geq 0$, and the small such d is the defect of H. The following is easily proved by induction on n.

23 Lemma. *Let G be a group, and $H \leq G$. Then*

(1) *$H^{G,n} = H[G,_n H]$ for all $n \in \mathbb{N}$.*

(2) *For $d \geq 1$, H is d-subnormal if and only if $[G,_d H] \leq H$.*

For our purposes it is convenient to explicitely state also the following easy observation.

24 Lemma. *Let H be a subgroup of the group G and suppose that, for some $n \geq 1$, $H^{G,n} \neq H$. Then there exist finitely generated subgroups G_0 and H_0 of G and H, respectively, such that $[G_0,_n H_0] \not\leq H$.*

We recall another elementary and useful fact (for a proof see [64]).

25 Lemma. *Let H and K be subnormal subgroups of the group G. If $\langle H, K \rangle = HK$, then $\langle H, K \rangle$ is subnormal in G.*

Series. Although we will not be directly interested in generalizations of subnormality, we will sometimes refer to them, notably to ascendancy; also, when working with subnormal subgroups in infinite groups, in order to have a better understanding of what is going on, or to think to feasible extensions of our results, it may be useful to be aware of them.

Our definition of a (general) subgroup series in a group is the standard one proposed by P. Hall (which in turn includes the earlier Mal'cev's definition). We give only a brief resume of the principal features of this basic notion, by essentially reproducing part of §1.2 of [96], to which we refer for a fuller account.

Let Γ be a totally ordered set; a **series** of type Γ of a group G is a set

$$\{(V_\gamma, \Lambda_\gamma) \mid \gamma \in \Gamma\}$$

of pair of subgroups V_γ, Λ_γ of G such that

(i) $V_\gamma \trianglelefteq \Lambda_\gamma$ for all $\gamma \in \Gamma$;

(ii) $\Lambda_\alpha \leq V_\beta$ for all $\alpha < \beta$ $(\alpha, \beta \in \Gamma)$;

(iii) $G \setminus \{1\} = \bigcup_{\gamma \in \Gamma} (\Lambda_\gamma \setminus V_\gamma)$.

Each $1 \neq x \in G$ lies in one and only one of the difference sets $\Lambda_\gamma \setminus V_\gamma$. Moreover, for each $\gamma \in \Gamma$,

$$V_\gamma = \bigcup_{\beta < \gamma} \Lambda_\beta \qquad \Lambda_\gamma = \bigcap_{\beta > \gamma} V_\beta \tag{3}$$

unless γ is the least element (if it exists) of Γ, in which case $V_\gamma = 1$, or the greatest element, for which $\Lambda_\gamma = G$. The subgroups V_γ, Λ_γ are called the *terms* of the series, and the quotient groups Λ_γ/V_γ the *factors* of the series.

A series of a group G is called *normal* if every term is a normal subgroup of G, and *central* if every factor is a central factor of G (i.e. $[\Lambda_\gamma, G] \leq V_\gamma$ for all $\gamma \in \Gamma$). Clearly, every central series is also a normal series.

Let $\mathcal{S}$ and $\mathcal{S}'$ be two series of the same group G. We say that $\mathcal{S}'$ is a *refinement* of $\mathcal{S}$ if every term of $\mathcal{S}$ is also a term of $\mathcal{S}'$. This relation clearly defines a partial order relation on the set of all series of the group G, which it is easily seen to satisfy the chain condition, in the sense that every chain of series of G (with respect to the refinement relation) admits an upper bound. Thus, we may apply Zorn's Lemma to the set of all series of G to get series that are not refinable. These unrefinable series of G are called *composition series.* Thus,

26 Proposition. *For every series $\mathcal{S}$ of the group G there exists a composition series which is a refinement of $\mathcal{S}$.*

Clearly, a series $\mathcal{S}$ of G is a composition series if and only if all factors of $\mathcal{S}$ are non-trivial simple groups. If we restrict attention to normal series of G (or, more generally, to series all of whose terms are invariant under the action of a given operator group A), we can still apply Zorn's Lemma, and obtain maximal, that is unrefinable, normal series (or A-invariant series) of G; these are called *chief series*, or principal series, of G, and their factors are *chief factors* of G. Every group G admits composition series and chief series, but there is no analogue of the Jordan-Holder Theorem for finite groups (even the infinite cyclic group violates it).

A series of finite type is obviously called a *finite* series. If Γ is a well-ordered set then a series of type Γ is called an *ascending* series. Now, a well-ordered set is isomorphic (as an ordered set) to a set of ordinal numbers $\{\gamma \mid \gamma < \alpha\}$ for a suitable ordinal α; we then say that the series has type α. If $\{(V_\gamma, \Lambda_\gamma) \mid \gamma < \alpha\}$ is an ascending series of G of type α for some ordinal α, then for every $\gamma < \alpha$, there is a smallest ordinal $\beta = \gamma+1$ such that $\beta > \gamma$; thus the second equality in identity (3) implies $\Lambda_\gamma = V_{\gamma+1}$, and so the terms Λ_γ are superfluous in defining the ascending series. For such a series it is customary to add the term $V_\alpha = G$ if α is a limit ordinal. Hence, given an ordinal α, an ascending series of type α of G is a set of subgroups $\{V_\gamma \mid \gamma \leq \alpha\}$ such that $V_0 = 1,\ V_\gamma \trianglelefteq V_{\gamma+1}$ for $\gamma < \alpha$, $V_\alpha = G$ and

$$V_\gamma = \bigcup_{\beta<\gamma} V_\beta$$

for every limit ordinal $\gamma \leq \alpha$.

Analogous remarks apply to *descending* series. These are defined as those series whose order type is the opposite Γ^{op} of a well-ordered set Γ. It will be more convenient to set the definition by referring again to the ordinal of Γ. Thus, for a given ordinal α, a descending series of type α^{op} of G is a set of subgroups $\{\Lambda_\gamma \mid \gamma \leq \alpha\}$ such that $\Lambda_0 = G$, $\Lambda_{\gamma+1} \trianglelefteq \Lambda_{\gamma+1}$ for $\gamma < \alpha$, $\Lambda_\alpha = 1$ and $\Lambda_\beta = \bigcap_{\gamma<\beta} \Lambda_\gamma$ for every limit ordinal $\beta \leq \alpha$.

A subgroup H of the group G is said to be *serial* if H is a term in some series of G; H is called *ascendant* (resp: *descendant*) if H is a term of a suitable ascending (descending) series of G.

27 Example. Let $\mathbb{Q}_2$ be the additive group of all rationals whose denominator is a power of 2, and let x be the automorphism of $\mathbb{Q}_2$ mapping every element into its opposite. Form the semidirect product $G = \mathbb{Q}_2 \rtimes \langle x \rangle$. For each $z \in \mathbb{Z}$ ($\mathbb{Z}$ viewed as a totally ordered set) let $V_\zeta = \Lambda_{z-1} = \langle 2^{-z}, x \rangle$. By adding $\Lambda_{-\infty} = \langle x \rangle$, $V_{-\infty} = 1$, $V_\infty = \Lambda_\infty = G$, we have a series of G, and thus $\langle x \rangle$ is a serial subgroup of G. However, $\langle x \rangle$ is not ascendant in G since it coincides with its normalizer, neither is descendant, for a proper descendant subgroup must be contained in a proper normal subgroup, while $\langle x \rangle^G = G$. By mans of the same series, one also sees that

(1) $\langle x \rangle$ is ascendant in $\langle 1, x \rangle = \mathbb{Z}\langle x \rangle$;

(2) $\mathbb{Z}\langle x \rangle / \mathbb{Z}$ is descendant in $Q = G/\mathbb{Z}$.

The group $\mathbb{Z}\langle x \rangle$ in (1) is called the *infinite dihedral* group (and denoted by D_∞), while the group Q in (2) is called the *locally dihedral* 2-group.

28 Remark. If S is a serial subgroup of the group G and $H \leq G$, then $H \cap S$ is a serial subgroup of H (and it is ascendant, descendant or subnormal if such is S in G). Ascendant (and subnormal) subgroup behave well also with respect to quotients (or, equivalently, homomorphic images): if S is an ascendant (subnormal) subgroup of the group G, then also SN/N is ascendant (subnormal) in G/N for all normal subgroups N of G. This is not true for serial and descendant subgroups: let, for example, $G = \left\langle x, y \mid y^x = y^{-1},\ x^2 = 1 \right\rangle$ be the infinite dihedral group; then $G \trianglerighteq \langle y^2, x \rangle \trianglerighteq \langle y^4, x \rangle \trianglerighteq \ldots$ is a descending series from G to $\langle x \rangle = X$, while, if n is not a power of 2, $\langle y^n \rangle X / \langle y^n \rangle$ is not even serial in $G/\langle y^n \rangle$.

Every group G admits a couple of standard normal series that will be of interest for us. They are natural extensions of the upper and lower central series defined in section 1.1.

Given the group G, the *(extended) upper central series* of G is the series whose factors $\zeta_\alpha(G)$ (α an ordinal number)are recursively defined by setting:

$$\zeta_0(G) = 1 \qquad \zeta_{\alpha+1}(G)/\zeta_\alpha(G) = \zeta(G/\zeta_\alpha(G)),$$

for any ordinal α, and

$$\zeta_\alpha(G) = \bigcup_{\lambda<\alpha} \zeta_\lambda(G)$$

if α is a limit ordinal (to be strictly adherent to our conventions on series we should add the group G as last term, but this omission will not cause any troubles, and will keep the exposition more linear). Clearly, it is a central series of G. The union of the terms of this series is a fully invariant subgroup of G called the *hypercentre* of G. Thus, the hypercentre is the term $\zeta_\alpha(G)$ of the upper central series of G corresponding to the smallest ordinal α such that $\zeta_\alpha(G) = \zeta_{\alpha+1}(G)$. The group G is called *hypercentral* if G is a term of the upper central series of G.

Similarly, we talk also of the *extended lower central series* of a group G. Its terms are inductively defined for every ordinal α, in the natural way, by setting $\gamma_0(G) = G$, $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), \gamma_\alpha(G)]$ for every ordinal α, and

$$\gamma_\beta(G) = \bigcap_{\alpha<\beta} \gamma_\alpha(G)$$

if β is a limit ordinal. The series of the $\gamma_\alpha(G)$ is clearly a descending series whose factors $\gamma_\alpha(G)/\gamma_{\alpha+1}(G)$ are central. As for the upper central series, given a group G there is a least ordinal α such that $\gamma_\alpha = \gamma_{\alpha+1}$; the $\gamma_\alpha(G)$ is called the *hypocentre* of G.

1.3 Classes of groups

By a *class* of groups we mean a family of groups that is closed under isomorphism and contains the trivial group. We will adopt the symbols $\mathfrak{F}$, $\mathfrak{A}$, $\mathfrak{N}$ to denote, respectively, the class of all finite, abelian and nilpotent groups. We will denote by $\mathcal{N}_1$ the class which is the principal object of these notes, namely that of all groups in which every subgroup is subnormal.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are group classes, $\mathfrak{X}\mathfrak{Y}$ denotes the class of all groups G which admit a normal subgroup N such that N belongs to $\mathfrak{X}$ and G/N belongs to $\mathfrak{Y}$. For instance, $\mathfrak{N}\mathfrak{A}$ is the class of nilpotent by abelian groups, i.e. those groups whose derived subgroup is nilpotent.

If $\mathfrak{X}$ is a class of groups, then $\mathrm{S}\mathfrak{X}$ and $\mathrm{Q}\mathfrak{X}$ denote, respectively, the class of all groups that are isomorphic to a subgroup of a group in $\mathfrak{X}$, and the class of all groups that are a homomorphic image of a group in $\mathfrak{X}$. A class $\mathfrak{X}$ is *subgroup closed* (respectively *quotient closed*) if $\mathfrak{X} = \mathrm{S}\mathfrak{X}$ ($\mathfrak{X} = \mathrm{Q}\mathfrak{X}$). It is plain that $\mathrm{S}(\mathrm{S}\mathfrak{X}) = \mathrm{S}\mathfrak{X}$, and that $\mathrm{Q}(\mathrm{Q}\mathfrak{X}) = \mathrm{Q}\mathfrak{X}$ for any class $\mathfrak{X}$.

Let $\mathfrak{X}$ be a class of groups. We say that a group G is *locally–*$\mathfrak{X}$ if every finite subset of G is contained in a subgroup of G belonging to $\mathfrak{X}$. The class of all locally–$\mathfrak{X}$ groups is denoted by $\mathrm{L}\mathfrak{X}$, and a class (or a group property that defines a class) $\mathfrak{X}$ is called *local* if $\mathrm{L}\mathfrak{X} = \mathfrak{X}$. An obvious example of a local class is the class $\mathfrak{A}$ of abelian groups. Like S- and Q-, L- is a *closure operator* in the sense that $\mathfrak{X} \subseteq \mathrm{L}\mathfrak{X}$ and $\mathrm{L}(\mathrm{L}\mathfrak{X}) = \mathrm{L}\mathfrak{X}$ for any class $\mathfrak{X}$. Observe that if the class $\mathfrak{X}$ is closed by subgroups, then a group G is locally-$\mathfrak{X}$ if and only if every finitely generated subgroup of G belongs to $\mathfrak{X}$. Thus, a locally finite group is a group in which every finitely generated subgroup is finite, and a *locally nilpotent* group is a group in which every finitely generated subgroup is nilpotent. A group G admitting a normal ascending series all of whose factors belong to $\mathfrak{X}$ is called a hyper–$\mathfrak{X}$–group. We will often refer in particular to *hyperabelian* groups; that is, groups admitting a normal ascending series with abelian factors. Similarly, a group G is said to be a hypo-$\mathfrak{X}$-group if G admits a descending normal series all of whose factors are $\mathfrak{X}$-groups. Thus, a *hypoabelian* group is a group admitting a normal descending series with all factors abelian. (Of course, we may define an extended derived series of the group G, by setting: $G^{(1)} = G' = [G, G]$, $G^{(\alpha+1)} = [G^{(\alpha)}, G^{(\alpha)}]$ and $G^{(\beta)} = \bigcap_{\alpha<\beta} G^{(\alpha)}$ for every ordinal α and every limit ordinal β. Thus, a group G is hypoabelian if and only if $G^{(\alpha)} = 1$ for some ordinal α).

Residuality. Let $\mathcal{P}$ be a class of groups. A group G is *residually–$\mathcal{P}$* if for every $1 \neq x \in G$ there exists $N \trianglelefteq G$ such that $G/N \in \mathcal{P}$ and $g \notin N$. This is equivalent to saying that the trivial subgroup of G is the intersection of all normal subgroups N of G such that $G/N \in \mathcal{P}$. The class of all residually–$\mathcal{P}$ groups is denoted by $\mathrm{R}\mathcal{P}$.

Let $\mathcal{R}$ be a set of normal subgroups of the group G. It is not difficult to see that if $\bigcap_{N\in\mathcal{R}} N = 1$ then G embeds in the cartesian product $Car_{N\in\mathcal{R}}(G/N)$, and it projects surjectively onto every factor. Conversely, in a cartesian product the kernels of the projections intersect in the trivial subgroup. Thus a group G is residually–$\mathcal{P}$ if and only if it is isomorphic to a subgroup $\overline{G}$ of a cartesian product of $\mathcal{P}$-groups such that the restrictions to $\overline{G}$ of the projections on the factors are surjective. If the class $\mathcal{P}$ is S-closed, we have that the residually–$\mathcal{P}$ groups are precisely the subgroups of cartesian products of $\mathcal{P}$-groups.

The two cases that are more relevant in our contest are those of *residually finite* and of *residually nilpotent* groups. Thus, a group G is residually finite if for each $1 \neq x \in G$ there exists a $H \leq G$ such that $|G : H|$ is finite G and $x \notin H$, while G is residually nilpotent if $\bigcap_{n\in\mathbb{N}} \gamma_n(G) = 1$. We recall that, by a result of Magnus, every free group is residually–(finite and nilpotent).

We now make some technical observations of elementary character that will be used later on.

29 Lemma. *Let N_n $(n \in \mathbb{N})$ be a family of normal subgroups of the group*

G, such that $N_i \geq N_{i+1}$ for all $i \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} N_n = 1$. Let F be a finite subgroup of G. Then $F = \bigcap_{n\in\mathbb{N}} FN_n$.

PROOF. Clearly, $F \leq \bigcap_{n\in\mathbb{N}} FN_n$. Let $u \in \bigcap_{n\in\mathbb{N}} FN_n$. Then, for every $n \in \mathbb{N}$, there exist $x_n \in F$ and $y_n \in N_n$, such that $u = x_n y_n$. Now, as F is finite, there exists an infinite subset Γ of $\mathbb{N}$ such that $x_i = x_j = x$ for all $i, j \in \Gamma$. Hence, for all $i \in \Gamma$, $y_i = x^{-1}y$, and so $x^{-1}y \in \bigcap_{i\in\Gamma} N_i = 1$. Thus, $u = x \in F$, proving the equality. $\boxed{QED}$

30 Proposition. *Let G be a countable residually finite group. Then every finite subgroup of G is the intersection of subgroups of finite index.*

PROOF. Let $G = \{x_0, x_1, x_2, \ldots\}$, and for each $i \in \mathbb{N}$, let H_i be a subgroup of finite index that does not contain x_i. By replacing H_i with its normal core $(H_i)_G$, we may take all H_i to be normal. Now, for all $n \in \mathbb{N}$, we set $N_n = H_0 \cap H_1 \cap \ldots \cap H_n$. Hence, for all $n \in \mathbb{N}$, N_n is a normal subgroup of finite index, $N_{n+1} \leq N_n$, and $\bigcap_{n\in\mathbb{N}} N_n = 1$. Our claim is now an immediate application of Lemma 29. $\boxed{QED}$

31 Lemma. *Let $(N_\lambda)_{\lambda\in\Lambda}$ be a family of normal subgroups of the group G, such that $\bigcap_{\lambda\in\Lambda} N_\lambda = 1$. Let $H \leq G$, and $Z = C_G(H)$. Then $Z = \bigcap_{\lambda\in\Lambda} ZN_\lambda$.*

PROOF. Let $g \in \bigcap_{\lambda\in\Lambda} ZN_\lambda$. Then, for all $a \in H$, and all $\lambda \in \Lambda$,

$$[a, g] \in [H, ZN_\lambda] \leq [H, N_\lambda] \leq N_\lambda.$$

Thus $[a, g] = 1$ and so $g \in C_G(H) = Z$. $\boxed{QED}$

Varieties. Let W be a subset of the free group F on a countable set of free generators X. The *variety* $\mathfrak{V}(W)$ defined by W is the class of all groups G such that $\phi(w) = 1$ for every homomorphism $\phi : F \to G$ and every $w \in W$. A convenient way to look at this is to consider every element $w = w(x_1, \ldots, x_n)$ of W (with $\{x_1, \ldots, x_n\}$ a subset of X) as a *law* that has to be satisfied by the groups in the variety $\mathfrak{V}(W)$; in the sense that $G \in \mathfrak{V}(W)$ if and only if, in G, $w(g_1, \ldots, g_n) = 1$ for every substitution $x_i \leftrightarrow g_i$ by elements $g_i \in G$. For example, the class of abelian groups is the variety defined by the single word $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$.

It is clear that every variety $\mathfrak{V}(W)$ is a group class which is closed by subgroups, quotients and cartesian products (and thus it is R-closed too). The converse of this fact is also true (for a proof ee e.g. [97], 2.3.5; or [52], 15.2.1).

32 Theorem. [Birkhoff] *A class of groups is a variety if and only if it is closed by subgroups, quotients and cartesian products.*

In general, given a set $W \subseteq F$ and a group G, the subgroup $W(G)$ generated by all possible substitutions by elements of G in the words $w = w(x_1, \ldots, x_n)$

of W, is called the *verbal subgroup* of G defined by W. Thus

$$W(G) = \langle w(g_1, \dots, g_n) \mid w(x_1, \dots, x_n) \in W,\ g_i \in G \rangle.$$

Hence $G \in \mathfrak{V}(W)$ if and only if the W-verbal subgroup of G is trivial. For instance, if $n \geq 2$, in any group G, the n-th term of the lower central series $\gamma_n(G)$ is the verbal subgroup defined by the single law $[x_1, x_2, \dots, x_n]$.

It is obvious that for every group homomorphism $\phi : G \to H$ we have $\phi(W(G)) \leq W(H)$. Therefore verbal subgroups are fully characteristic, and in particular if $N \trianglelefteq G$ then $W(G/N) = W(G)N/N$.

Locally graded groups. A group G is said to be *locally graded* if every non-trivial finitely generated subgroup of G has a non-trivial finite homomorphic image. This is a rather large class of groups, containing for instance all residually finite groups and all locally–(soluble by finite) groups. It is often considered in order to avoid finitely generated simple groups, and in particular the so-called Tarski monsters, i. e. infinite groups in which all proper subgroups are cyclic of the same order. Tarski monsters do exist and have been constructed by Ol'shnskii (see [88] for the periodic case, and [87] for the torsion–free case) and Rips (unpublished).

Groups like the Golod-Shafarevic finitely generated infinite p-groups (for a simple approach see Ol'shnskii [88]) are locally graded (in fact they are even residually finite). In the theory of locally nilpotent groups, to avoid such groups too, it is sometimes convenient to restrict to a proper, but still large, subclass of the class of locally graded groups, which is denoted by $\mathfrak{W}$ and was introduced by Phillips and Wilson in [92]: a group G is in $\mathfrak{W}$ if every non-nilpotent finitely generated subgroup of G has a non-nilpotent finite image. A theorem of Robinson [95] ensures that $\mathfrak{W}$ *contains all locally (hyperabelian by finite) groups.*

Observe that the class of locally graded groups and the class $\mathfrak{W}$ are clearly local and closed by subgroups, but are not closed by quotients, as consideration of free groups of rank at least 2 shows.

Countable recognition. In many situations, it is convenient to be able to deal just with countable groups in a certain class. Thus, the following concept is of importance. We say that a class of groups $\mathcal{P}$ is *countably recognizable* if a group G belongs to $\mathcal{P}$ provided that all countable subgroups of G belong to $\mathcal{P}$. Observe that a finitely generated group is countable.

33 Theorem. *Let $1 \leq c \in \mathbb{N}$. The following classes of groups are countably recognizable: nilpotent groups, nilpotent groups of class at most c, soluble groups, soluble groups of derived length at most c,*

PROOF. The claim is clearly true for the classes of nilpotent groups of class at most c, and of soluble groups of derived length at most c. In fact for these cases it is enough to make the assumption on finitely generated subgroups.

Now, suppose that all countable subgroups of the group G are nilpotent. In particular all finitely generated subgroups of G are nilpotent. Suppose that, for all $i \geq 1$ there exists a finitely generated subgroup U_i of G whose nilpotency class is greater that i. Then the subgroup $\langle U_i \ ; \ i \in \mathbb{N}\rangle$ is countable and not nilpotent, which contradicts our assumption. Hence there exists a bound on the nilpotency class of finitely generated subgroups of G, and so G is nilpotent. The proof for the class of soluble groups is similar. $\boxed{QED}$

In fact, it is not difficult to prove that a countable union of countably recognizable group classes is countably recognizable. For this and more general result on this subject see section 8.3 in [96].

Radicable groups. A property which is somehow opposite from being a finiteness condition, in the sense that the trivial group is the only finite group that satisfies it, is radicability. A group G is *radicable* if for every $1 \neq g \in G$ and every $0 \neq d \in \mathbb{N}$, there exists in G a d-rooth of g, i.e. an element $h \in G$ such that $h^d = g$. The most obvious example of a radicable group is the additive group $\mathbb{Q}$ of the rationals.

Radicable abelian groups are called *divisible* groups. Besides the group $\mathbb{Q}$, the fundamental divisible groups are the groups of Prüfer type C_{p^∞} (often called *quasicyclic* groups); these latter ones are defined for every prime number p: C_{p^∞} is isomorphic to the multiplicative group of all p^n-th complex roots of unity for all $n \in \mathbb{N}$. The Prüfer group C_{p^∞} has the following presentation

$$C_{p^\infty} = \langle u_0, u_1, u_2, \dots \mid u_0 = 1,\ u_{i+1}^p = u_i \text{ for } i \in \mathbb{N}\rangle,$$

and the property that every proper subgroup is one of the $\langle u_i\rangle$ (and thus a cyclic p-group). An abelian group is divisible if and only if it is isomorphic to a direct product of copies of $\mathbb{Q}$ and groups of Prüfer type (see for instance [97] 4.1.5).

A group G is *semi-radicable* if, with the notation introduced in 18, $G^\omega = G$. It is rather straightforward to see that a semi-radicable abelian group is divisible. This is true for nilpotent groups also, but observe that Corollary 18 implies that a periodic semi-radicable nilpotent group is abelian; on the other hand the groups of upper unitriangular rational matrices $UT(n, \mathbb{Q})$ are examples of torsion-free radicable nilpotent groups that are not abelian.

The following Lemma is a sort of refinement of 16(i).

34 Lemma. *Let A be a normal abelian divisible subgroup of G, and $H \leq G$ such that $[A, {}_nH] = 1$ for some positive integer n. If H/H' is periodic, then $[A, H] = 1$.*

Proof. Let $B = [A, H, H]$. Then B is normal in $\langle A, H\rangle$ and, by the Three-Subgroup Lemma 5, $[H', A] = [H, H, A] \leq B$. Thus $H' \leq C_H(A/B)$, and, since A/B is divisible, it follows from Lemma 16 (i) that $[A/B, H] = 1$ or, in other words, $[A, H] = B$. From this, the result follows. $\boxed{QED}$

Next, an interesting property of subnormal (more generally, ascendant) divisible subgroups.

35 Lemma. *Let A be a periodic abelian divisible subgroup of the group G. If A is ascendant, then A^G is abelian and divisible.*

PROOF. Let A be an ascendant periodic divisible abelian subgroup of G and let $A = A_0 \trianglelefteq A_1 \trianglelefteq \ldots \trianglelefteq A_\alpha = G$ be an ascending series from A to G. For every ordinal $\beta \leq \alpha$ let $U_\beta = A^{A_\beta}$; then $U_{\beta+1} \leq A_\beta^{A_{\beta+1}} = A_\beta$. Hence $U_{\beta+1}$ normalizes U_β and so the U_β $(1 \leq \beta \leq \alpha)$ are the terms of an ascending series from A to $U_\alpha = A^G$. Suppose, by contradiction, that A^G is not abelian, and let β be the least ordinal such that U_β is not abelian. Then, clearly, $1 < \beta$ cannot be a limit ordinal, so $U_\beta = A^{A_\beta} = U_{\beta-1}^{A_\beta}$. Let $g \in A_\beta$. Then the abelian subgroups $U_{\beta-1}$ and $U_{\beta-1}^g$ are both normal in U_β, hence $[U_{\beta-1}, U_{\beta-1}^g, U_{\beta-1}^g] = 1$, and so, by Lemma 34, $[U_{\beta-1}, U_{\beta-1}^g] = 1$. This shows that $A^{A_\beta} = U_\beta$ is abelian, against our choice. Thus A^G is abelian, and it is then clear that it is divisible. $\boxed{QED}$

With the same arguments it is easy to see that *two ascendant periodic divisible abelian subgroups of a group generate an abelian (ascendant) subgroup.* For non-periodic groups the situation can be very different: see, for instance, [64] 2.1.7 for an example of a group generated by two subnormal torsion-free abelian divisible group which is not hypercentral.

We continue by mentioning some important classes of groups defined by *finiteness conditions*. We recall that a finiteness condition is a property which is satisfied by all finite groups (often for trivial reasons). So, for instance, the properties of being periodic, finitely generated, locally finite or linear (i.e. isomorphic to a subgroup of some matrix group $GL(n, \mathbb{K})$ for some field $\mathbb{K}$) all are finiteness conditions

Periodic, locally finite, and groups with finite exponent. A group G is periodic if it does not contain elements of infinite order, while G is locally finite if every finitely generated subgroup of G is finite. The class of locally finite group is strictly contained in the class of periodic group. In particular, there exist finitely generated p-groups that are not finite. Since a finitely generated nilpotent periodic group is finite, we infer that p-groups need not be locally nilpotent. The first examples of groups of this kind were constructed by Golod and Shafarevic (see [31]).

The *exponent* of a group G is, if it exists, the smallest integer $n \geq 1$ such that $g^n = 1$ for all $g \in G$. Otherwise we say that the group G has infinite exponent. Clearly, if G has finite exponent then G is periodic and its exponent is the least common multiple of the orders of its elements. The Golod-Shafarevic groups have infinite exponent.

The question as to whether a finitely generated group with finite exponent can be infinite is known as "Burnside Problem' (after W. Burnside who proposed it back in 1902). To set it more properly, let r, n be positive integers: the r-generator Burnside group of exponent n is defined as $B(r,n) = F_r/N$, where F_r is the free group with r generators and N is the normal subgroup of F_r generated by $\{x^n \mid x \in F_r\}$. Burnside's question is then for which pairs (r,n) is $B(r,n)$ finite. A part the trivial case $r = 1$ (for $B(1,n)$ is obviously cyclic of order n), $B(r,n)$ is known to be finite for arbitrary r and $n = 2, 3, 4, 6$. Case $n = 2$ is easy (a group of exponent 2 is elementary abelian), while the cases $n = 3, 4, 6$ are due, respectively, to Burnside himself, to Sanov and to M. Hall. In 1968 Novikov and Adjan proved that, for $r > 1$ and n a large enough odd number, $B(r,n)$ is infinite. Subsequently Adjan improved the previous lower bound for n by showing that $B(r,n)$ is infinite for every $r > 1$ and every odd $n \geq 665$. Later, Ol'shanskii proved that for every prime $p > 10^{40}$ there exists an infinite p-group all of whose proper subgroups are cyclic of order p. As far as I know, it is still undecided whether $B(2,5)$ and $B(2,8)$ are infinite.

Since $B(r,n)$ need not be finite, even more important it appears the so-called *restricted Burnside problem*. This asks if there is a bound for the orders of finite r-generated groups of exponent n. That is, if the finite residual K of $B(r,n)$ has finite index, or, in other words, if $R(r,n) = B(r,n)/K$ is finite. In 1956 P. Hall and G. Higman [39] established a reduction theorem to prime powers, by showing that $R(r,n)$ is finite if and only if $R(r,q)$ is finite for every prime power q dividing n. Meanwhile, Kostrikin [55] proved that $R(r,p)$ is finite for all r and p a prime. It took many years before Zel'manov ([124], [125]) was able to prove that $R(r,p^k)$ is finite for every prime power p^k, thus completing the proof that $R(r,n)$ is finite for every r and n.

Zel'manov results, whose proofs are far beyond the scope of this survey, have important consequences for the theory of locally nilpotent groups. We report two of the more immediate in the following statement.

36 Theorem. [Zel'manov]

(1) *For every $n \geq 1$ the class of locally nilpotent groups of exponent dividing n is a variety.*

(2) *A residually nilpotent group of finite exponent is locally nilpotent.*

In fact, modulo the Hall-Higman reduction, both these statements are equivalent to the finiteness of $R(r,n)$ for all r, n.

For a good account of the questions and results related to the Burnside Problems we refer to the the book of Vaughan-Lee [120].

Max and Min. Among the most natural and important finiteness conditions are Max and Min: respectively, the maximal and the minimal condition on chain of subgroups. The easiest examples of infinite groups satisfying Max and Min are, respectively, the infinite cyclic group $(\mathbb{Z},+)$ and the Prüfer groups C_{p^∞}. We recall that, besides being clearly subgroup and quotient closed, both classes of all groups satisfying Max and of those satisfying Min are extension closed: in the sense that if N is a normal subgroup of the group G and both N and G/N satisfy Max (respectively, Min), then G satisfies Max (Min). In general, if $\mathcal{P}$ be a family of subgroups of the group G, then G is said to satisfy the *minimal (maximal) condition* on $\mathcal{P}$-subgroups if every descending (ascending) chain of $\mathcal{P}$-subgroups of G is finite.

Černikov groups. We will often encounter groups belonging to this class, which are defined as follows. A group is a *Černikov group* if it admits a normal subgroup of finite index which is the direct product of a finite number of groups of Prüfer type. Thus, a Černikov group is (abelian divisible)-by-finite. The classical result of Černikov is

37 Theorem. *A soluble group satisfies Min if and only if it is a soluble Černikov group.*

Since Černikov groups are locally finite, a locally nilpotent such group is the direct product of Černikov p-groups. These may be described as follows.

38 Proposition. *Let p be a prime, and G a Černikov p-group. Then G is isomorphic to a subgroup of the wreath product $C_{p^\infty} \wr P$, where P is a suitable finite p-group.*

From this, it easily follows that a nilpotent Černikov group is central-by-finite. We recall also a deep result due to Šunkov [116], and independently to Kegel and Wehrfritz [53].

39 Theorem. *A locally finite group which satisfies the minimal condition on abelian subgroups is a Černikov group.*

Polycyclic groups. A group is *polycyclic* if it admits a finite series with cyclic factors. For soluble groups satisfying Max the basic remark is

40 Theorem. *A group is a soluble group satisfying Max if and only if it is a polycyclic group.*

When G is nilpotent, we may say more.

41 Proposition. *Let G be a nilpotent group. Then the following conditions are equivalent.*

(i) *G is finitely generated;*

(ii) *G/G' is finitely generated;*

(iii) *G is polycyclic;*

(iv) *G satisfies Max.*

We also recall a couple of well known and important results. The first is due to Mal'cev, and the second to Hirsch (for proofs, see [97], or Segal [99] which is the standard reference for polycyclic groups).

42 Theorem. *Let G be a polycyclic group. Then*

(i) *Every subgroup of G is the intersection of subgroups of finite index of G;*

(ii) *G is nilpotent if and only if every finite quotient of G is nilpotent.*

A further related and useful result is the following one.

43 Theorem. *A finitely generated torsion-free nilpotent group is residually a finite p-group for every prime p.*

An important feature of polycyclic groups is the fact that if G is polycyclic, then in any finite series with cyclic factors of G the number of infinite factors is an invariant, called the Hirsch length of G (and denoted by $h(G)$).

Finite rank. There are several notions of rank of a group. When not otherwise specified, by a group of finite rank we will always mean a group of *finite Prüfer rank*, i.e. a group G with the property that there exists a $d \in \mathbb{N}$ such that every finitely generated subgroup of G can be generated by d elements. If this happens, the smallest such d is called the (Prüfer) rank of G. This is clearly a finiteness condition. For example the additive groups $\mathbb{Z}$, $\mathbb{Q}$ and the groups C_{p^∞} all are abelian groups of rank 1.

Among others, a much weaker condition is that of *finite abelian subgroup rank*: a group G has finite abelian subgroup rank if every abelian subgroup of G which is either free abelian or elementary abelian is finitely generated.

FC-groups. An FC-group is a group in which every element has a finite number of conjugates. Thus, G is an FC-group if and only if $|G : C_G(g)|$ is finite for every $g \in G$.

44 Proposition. [R. Baer, B. Neumann] *Let G be an FC-group. Then*

(i) $G/Z(G)$ is periodic and residually finite;

(ii) if g is an element of finite order of G, then $\langle g\rangle^G$ is finite;

(iii) the set $T(G)$ of all elements of finite order of G is a characteristic subgroup of G, and $G' \leq T(G)$.

Strictly related to elements with a finite number of conjugates is Dic'man Lemma.

45 Lemma. *Let U be a normal subset of the group G (i.e. $x^g \in U$ for every $x \in U$ and $g \in G$). If U is finite and consists of elements of finite order, then $\langle U\rangle$ is a finite normal subgroup of G.*

1.4 Nilpotent groups and their generalizations

In nilpotent groups the nature of the lower central factors, and sometimes that of the whole group, is strictly related to the properties of the first of them. This is elucidated by the following result.

46 Theorem. [Robinson see [94]] *Let H be a group and $A = H/H'$. Then, for every $c \geq 1$, there is an epimorphism:*

$$\underbrace{A \otimes A \otimes \cdots \otimes A}_{c\ times} \longrightarrow \gamma_c(H)/\gamma_{c+1}(H)$$

From this, a number of facts connecting the properties of a nilpotent group H to that of its abelianization H/H', follow more or less easily; for instance, the following elementary but basic observation.

47 Proposition. *Let G be a nilpotent group, and π a set of primes. If G admits set S of generators all of whose elements have finite bounded π-order, then G is a π-group of finite exponent; if, further. S is finite, then G is finite.*

The following useful property may be also deduced rather easily from 46.

48 Proposition. *The rank of a finitely generated nilpotent group G does not exceed a value which depends on the number of generators and the nilpotency class of G.*

Another handy fact that can be proved using 46 is

49 Lemma. *Let H be a nilpotent group of class $c \geq 1$; then the map*

$$\begin{array}{rcl} H \times \ldots \times H & \rightarrow & \gamma_c(H) \\ (x_1, \ldots, x_c) & \mapsto & [x_1, \ldots, x_c] \end{array}$$

is a homomorphism in every variable.

Thus, we have in particular,

50 Corollary. *Let S be a generating set for the group G. Then G is nilpotent of class at most c if and only if $[x_1, x_2, \ldots, x_{c+1}] = 1$ for any elements $x_1, x_2, \ldots, x_{c+1} \in S$.*

The centre of a nilpotent group (i.e. the first factor of the upper central series) has also a certain influence on the whole group. Here is an important instance of this.

51 Proposition. *Let G be a nilpotent group of class c and suppose that the centre of G has finite exponent e. Then G has exponent dividing e^c.*

Proof. We let $Z = Z(G)$, and proceed by induction on the nilpotency class c of G. If $c = 1$ then $G = Z$ and there is nothing to prove. Let $c \geq 2$ and let $y \in \zeta_2(G)$, $g \in G$. Then $[g, y] \in Z$ and so, by Lemma 2, $1 = [g, y]^e = [g, y^e]$. Therefore

$y^e \in Z$, showing that $\zeta_2(G)/Z$ has exponent dividing e. By inductive assumption G/Z has exponent dividing e^{c-1} and from this the conclusion follows. $\boxed{QED}$

There are two more properties relative to the mutual behavior of the terms of the lower and upper central series of a group, which are often useful, that we like to recall. Their proofs may be found, for instance, in §14.5 of [97].

52 Proposition. [Baer] *Let G be a group such that, for some $i \geq 1$, $G/\zeta_i(G)$ is finite, then $\gamma_{i+1}(G)$ is finite.*

53 Proposition. [P. Hall] *Let G be a group such that, for some $i \geq 1$, $\gamma_{i+1}(G)$ is finite, then $G/\zeta_{2i}(G)$ is finite.*

The class of all nilpotent groups whose nilpotency class does not exceed a certain integer $c \geq 1$ will be denoted by $\mathfrak{N}_c$. Needless to say, for every $c \geq 1$, the class $\mathfrak{N}_c$ is closed by subgroups, quotients and cartesian products (in fact, it forms a variety). The class of nilpotent groups $\mathfrak{N} = \bigcup_{c\in\mathbb{N}} \mathfrak{N}_c$ is closed by subgroups and quotients, but it is not closed under direct products (indeed, the smallest variety contag $\mathfrak{N}$ is the class of all groups). However, a fundamental result (due in its generality to Fitting), ensures that the subgroup generated by two normal nilpotent subgroups is still nilpotent (i.e. $\mathfrak{N} =_{\mathrm{N}_0}\mathfrak{N}$); for the proof see for instance [97] or [52].

54 Theorem. [Fitting's Thorem]. *Let H and K be nilpotent normal subgroups of a group G, of nilpotency class c and d, respectively. Then their join HK is a nilpotent normal subgroup of G of nilpotency class at most $c + d$.*

Nilpotency criteria. For finite groups there are a number of conditions each of those is equivalent to nilpotency. The next theorem lists some of the most relevant and simple of them.

55 Theorem. *Let G be a finite group. Then the following conditions are equivalent to nilpotency.*

(i) *every chief factor of G is central;*

(ii) *every maximal subgroup of G is normal;*

(iii) *G is the direct product of its primary (i.e. Sylow) subgroups;*

(iv) *for every proper subgroup H of G, $N_G(H) > H$.*

For infinite groups all these conditions are weaker than nilpotence, and imposing any of them gives rise to different classes of so-called *generalized nilpotent groups.* In fact there are many other ways to define classes of generalized nilpotent groups, and those obtained by imposing any of the conditions (i) - (iii) of the theorem determine classes of groups that are rather far even from being locally nilpotent.

However, there are relevant nilpotency criteria which work for arbitrary groups. The following is one of the most useful, specially when dealing with groups that are known to be soluble.

56 Theorem. [P. Hall see [36]] *Let N be a normal subgroup of the group G. If N is nilpotent of class c and G/N' is nilpotent of class d, then G is nilpotent of class at most $\binom{c+1}{2}d - \binom{c}{2}$.*

PROOF. See [97], 5.2.10. $\boxed{QED}$

As a consequence, we have that if the group class $\mathfrak{X}$ is closed by quotients and normal subgroups, and has the property that all metabelian groups in $\mathfrak{X}$ are nilpotent (of class bounded by c), then all soluble groups in $\mathfrak{X}$ are nilpotent (of class bounded by a function of c and the derived length of the group).

Another nilpotency criterion (also due to P. Hall) arises from the idea of a *series stanbilizer*. Let $\mathcal{S} = \{(V_\gamma, \Lambda_\gamma) \mid \gamma \in \Gamma\}$ be a series of the group G. The *stability group* of $\mathcal{S}$ is the set of all automorphisms ϕ of G that centralize every factor of $\mathcal{S}$; that is $[\Lambda_\gamma, \phi] \leq V_\gamma$ for all $\gamma \in \Gamma$. It is readily seen that the stability group of a series is a subgroup of $Aut(G)$. Obviously, a group $H \leq Aut(G)$ is said to stabilize the series $\mathcal{S}$ of G if H is contained in the stability group of $\mathcal{S}$.

57 Theorem. [P. Hall see [36]] *Let $H \leq Aut(G)$ stabilize a finite series of length n of the group G. Then H is nilpotent of class at most $\binom{n}{2}$.*

PROOF. Let $G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n = 1$ be a series of length n of G stabilized by H. We argue by induction on n, being the case $n = 1$ trivial. Let $n \geq 2$ and $Y_0 = Y = C_H(G_1)$. By inductive hypothesis, H/Y is nilpotent of class at most $\binom{n-1}{2}$. For $i \geq 1$, write $Y_i = [Y,_i H] = [Y_{i-1}, H]$; we show, by induction on i, that $[G, Y_i] \leq G_{i+1}$. This is clear for $i = 0$; let $i \geq 1$, then $[Y_i, G] = [Y_{i-1}, H, G]$. Let $h \in H$, $y \in Y_{i-1}$, $g \in G$, then, taking into account that $[H, G, Y_{i-1}] \leq [G_i, Y] = 1$, by the Hall-Witt identity 1 we have

$$[y, h^{-1}, g]^h [g, y^{-1}, h]^y = 1.$$

So $[y, h^{-1}, g] \in [G, Y_{i-1}, H]^H$ and, applying the inductive assumption

$$[Y_i, G] = [H, Y_{i-1}, G] \leq [G_i, H] \leq G_{i+1}.$$

For $i = n - 1$ we get $[Y_{n-1}, G] \leq G_n = 1$, that is $Y_{n-1} = [Y,_{n-1} H] = 1$. Thus, $Y \leq \zeta_{n-1}(H)$. Since H/Y has class at most $\binom{n-1}{2}$, this completes the proof. $\boxed{QED}$

The bound $\binom{n}{2}$ on the nilpotency class of the stability group H has been improved by Hurley in [49]. For stabilizers of finite normal series, it is in fact much stricter.

58 Theorem. [Kalužnin] *The stability group of a finite normal series of length n of a group is nilpotent of class at most $n-1$.*

PROOF. Essentially the same of that of point (iii) of Lemma 16. $\boxed{QED}$

A class of groups is a class of *generalized nilpotent groups* if it contains $\mathfrak{N}$ and every finite member of it is nilpotent (see chapter 6 in [96]).

Local nilpotency. That of locally nilpotent groups is perhaps the most obvious class of generalized nilpotent groups. We remind from section 1.3 that a group G is *locally nilpotent* if every finitely generated subgroup of G is nilpotent. The locally dihedral 2-group is among the simplest examples of non-nilpotent locally nilpotent groups.

Although it will not play a great role in the rest of these notes, the Hirsch–Plotkin Theorem is one of the basic results in the theory of infinite groups.

59 Theorem. *In any group G the product of two normal locally nilpotent subgroups is locally nilpotent. Thus G has a unique maximal locally nilpotent normal subgroup, which is called the* Hirsch-Plotkin radical *of G, and contains all locally nilpotent ascendant subgroups of G.*

PROOF. See [97], 12.1.3; or [52], 18.1.2. $\boxed{QED}$

As we assume knowledge of the basic theory of nilpotent groups, we will not in general provide proofs for those results on locally nilpotent groups that are easy consequences of the corresponding results for the nilpotent case, and can be found in most textbooks (e.g. Chapter 12 of [97]). Among these, the following one is fundamental.

60 Theorem. *Let G be a locally nilpotent group. Then the set of all elements of finite order of G is a fully invariant subgroup, called the* ***torsion*** *subgroup of G, and denoted by $T(G)$. Moreover, $T(G)$ is a direct product of locally finite p-groups.*

We call the unique maximal normal p-subgroup (which may well be trivial) of a periodic locally nilpotent group G, the *p-component* of G. Let us stress the fact that a periodic locally nilpotent group is locally finite and the direct product of its non-trivial primary components. Conversely, a direct product of locally finite p-groups (for various primes p) is a locally nilpotent group.

Our next observation is an easy generalization of Fitting's Theorem.

61 Lemma. *Let N, H be nilpotent subgroups of the group G, of nilpotency class c and d, respectively. If $N \trianglelefteq G$, and H is subnormal of defect n, then NH is nilpotent of class at most $nc + d$.*

PROOF. We can assume $G = NH$, and proceed by induction on the defect n of H in G. If $n = 0$, then $H = G = NH$ is nilpotent of class d. Thus, let $n \geq 1$.

Then H has defect $n-1$ in H^G, and $H^G = H^G \cap NH = (H^G \cap N)H$. By inductive assumption, H^G is nilpotent of class at most $(n-1)c + d$. Hence, by Fitting's Theorem, $G = NH^G$ is nilpotent of class at most $c+(n-1)c+d = nc+d$. $\boxed{QED}$

62 Lemma. *Let H be a non-trivial finitely generated subgroup of the locally nilpotent group G. Then $H \not\leq [G, H]$.*

Proof. Suppose, by contradiction, that $H \leq [G, H]$. Then, since H is finitely generated, there exists another finitely generated subgroup F of G such that $H \leq [F, H]$, and we may clearly assume $H \leq F$. Now, F is nilpotent, and so there exists a least term $\gamma_n(F)$ of the lower central series of F which does not contain H. Then, $[F, H] \leq [F, \gamma_{n-1}(F)] = \gamma_n(F)$ does not contain H, a contradiction. $\boxed{QED}$

Although this Lemma suggests that a locally nilpotent group is rich in normal subgroups, it should be noted that the property stated in it is a rather weak one. In fact the same argument in the proof of 62 shows that if G is a residually nilpotent group (for instance, a free group), then $H \not\leq [G, H]$ for all non-trivial subgroups H of G (see [96] §6.2, for a thorough discussion of this and related properties).

63 Theorem. *Let G be a locally nilpotent group. Then*

(a) (Baer [3]) *Every maximal subgroup of G is normal.*

(b) (Mal'cev, McLain [71]) *Every chief factor of G is central. Thus, every chief series of a locally nilpotent group is central and it is a composition series.*

Proof. (a) Let G be locally nilpotent and suppose by contradiction that M is a maximal subgroup of G which is not normal. Then $N \not\geq G'$, and so there exists $g \in G' \setminus M$. Since $G = \langle M, g\rangle$, there exists a finitely generated subgroup X of M such that $g \in \langle X, g\rangle'$. Let $H = \langle X, g\rangle$; then $X \leq M \cap H$, and $H = (M \cap H)H'$. Since H is nilpotent, this forces $M \cap H = H$ and the contradiction $g \in M \cap H$.

(b) It is enough to show that a minimal normal subgroup A of the locally nilpotent group G is central. If this is not the case there exist $a \in A$ and $g \in G$ such that $b = [a, g] \neq 1$. Since, by minimality of A, $A = \langle b\rangle^G$, we get

$$\langle a\rangle \subseteq \langle b\rangle^G \subseteq [\langle a\rangle, G]^G = [\langle a\rangle, G],$$

thus contradicting Lemma 62. $\boxed{QED}$

Note that locally nilpotent groups need not admit maximal subgroups: for example, the wreath product $C_{p^\infty} \wr C_{p^\infty}$ is a locally finite p-group with no maximal subgroups.

64 Lemma. *Let G be a group. Then all subgroups of G are serial if and only if for every $H \leq G$ all maximal subgroups of H are normal.*

PROOF. Suppose that every subgroup of G is serial; let $H \leq G$ and M a maximal subgroup of H. By intersecting with H every term of a series of G containing M, we get a series of H containing M. But M is maximal in H, so $M \trianglelefteq H$.

Conversely, suppose that for every $H \leq G$ all maximal subgroups of H are normal. Let $L \leq G$ and let $\mathcal{C}$ be the family of all chains of subgroups of G that contain L as a term and satisfy conditions (ii) and (iii) (but not necessarily (i)) of the definition of a series. By standard application of Zorn's Lemma, $\mathcal{C}$ has a maximal element, which, because of the assumption on G, must also satisfy normality condition (i), and it is therefore a series of G with L as a term. $\boxed{QED}$

Now, by point (a) of Theorem 63, we have:

65 Corollary. [Baer see [3]] *In a locally nilpotent group every subgroup is serial.*

This Corollary, as well as Theorem 63, follows also as an application of Mal'cev's general (and by now classical) method for proving local theorems in algebraic systems. For this important method we refer to the Appendix of [52] or Section 8.2 of [96].

Seriality of all subgroups does not imply local nilpotence. In fact, in [121] J. Wilson constructs finitely generated infinite p-groups - hence not (locally) nilpotent - in which every subgroup is serial (and every chief factor is central). On the other hand groups in which every subgroup is ascendant (called N-groups) are locally nilpotent and, as such, will be considered more at length in the next section.

Engel conditions. Along with that of locally nilpotent groups, the class of Engel groups is the class of generalized nilpotent groups that have received most attention through the years.

An element g of the group G is said to be *left Engel* if, for any $x \in G$, there exists a positive integer $n = n(g, x)$ such that $[x,_n g] = 1$. If further such an integer n does not depend on x, then g is called a left n-Engel element. A group G is called an *Engel group* if every element of G is left Engel, and it is called an n-Engel group if every element of G is left n-Engel, for a fixed n. A group which is n-Engel for some n is called a *bounded Engel group*. For a given $n \geq 1$, the class of all n-Engel groups is a variety.

A classical result of Zorn (see [97], 12.3.4) ensures that finite Engel groups are nilpotent. Observe that every locally nilpotent group is an Engel group. In fact, if G is locally nilpotent, and $x, g \in G$, then $\langle x, g\rangle$ is nilpotent, and this implies that, for some $n \in \mathbb{N}$, $[x,_n g] = 1$. On the other hand, the celebrated examples due to Golod are finitely generated Engel groups that are not nilpotent (in fact, for every $d \geq 2$ and any prime p, Golod constructs d-generated infinite p-groups in which all $(d-1)$–generated subgroup are nilpotent - an thus finite). However, it appears to be still an open question whether bounded Engel groups are locally nilpotent. Although no counterexample is known, and there are important recent results that prove this in some relevant cases, it seems unlikely that to be true in general.

The general theory of Engel groups is well beyond the scope of these notes. In fact, we will restrict to a few facts, that are more closely connected to our subject. A few results on n-Engel groups with small n will be recalled in Section 4.1, while, moving to general bounded Engel conditions, we like to mention here a couple of recent and deep theorems.

66 Theorem. [J. Wilson see [122]] *A residually finite bounded Engel group is locally finite.*

67 Theorem. [Zel'manov see [123]] *A torsion–free locally nilpotent n-Engel group is nilpotent of nilpotency class depending only on n.*

By using these and Zel'manov solution of the restricted Burnside problem, it is possible to show that locally graded bounded Engel groups are locally nilpotent (see [54]), and then the following general statement (see, for instance, [12]).

68 Theorem. *For every $n \geq 1$ there exist integers $e(n)$ and $c(n)$ such that if G is a locally graded n-Engel group then $\gamma_{c(n)}(G)^{e(n)} = 1$.*

These represent the reaching point of the work of many authors, and the proofs cannot be included here; what will be enough for most of our purposes is a much earlier version, first due to Gruenberg (and whose proof can be found in [96], 7.36).

69 Proposition. *For $n, d \geq 1$ there exist integers $e = e(n, d)$ and $c = c(nd,)$ such that if G is a soluble n-Engel group of derived length d, then $\gamma_c(G)^e = 1$.*

$e(n, d)$ and $c(n, d)$ may be given explicit upper bounds; in particular one has

70 Corollary. *A torsion-free soluble n-Engel group of derived length d is nilpotent of class bounded by n^{d-1}.*

1.5 Classes of locally nilpotent groups

In this section we give a brief account of some relevant classes of locally nilpotent groups. Our approach follows that of D. Robinson in the second volume of [96], in the sense that most of the classes that we will point out are defined in terms of embedding properties of all their (finitely generated or arbitrary) subgroups.

Baer and Gruenberg groups. The following basic result is due to Baer [4] for the case of subnormal subgroups and to Gruenberg [32] for that of ascendant ones.

71 Theorem. *Let H and K be finitely generated nilpotent subgroups of the group G. If H and K are subnormal (ascendant), then $J = \langle H, K\rangle$ is a subnormal (ascendant) nilpotent subgroup of G.*

For the proof we need a Lemma which will be useful on other occasions.

72 Lemma. [Gruenberg see [32]] *Let G be a locally nilpotent group and X a finitely generated subgroup of G. If A is an ascendant (subnormal) subgroup of G normalized by X then there exists an ascending (finite) series containing A all of whose terms are normalized by X.*

Proof. Let $A = A_0 \trianglelefteq A_1 \trianglelefteq \ldots \trianglelefteq A_\alpha = G$ be an ascending series from A to G. For each ordinal $\beta \le \alpha$ put

$$B_\beta = \bigcap_{x\in X} A_\beta^x.$$

Clearly the B_β ($\beta \le \alpha$) are normalized by X and are the terms of a chain of subgroups of G with $B_0 = A_0 = A$ and $B_\alpha = A_\alpha = G$; we show that they form an ascending series. For every $\beta < \alpha$ it is clear that $B_\beta \trianglelefteq B_{\beta+1}$, so what we have to prove is that for every limit ordinal $\beta \le \alpha$, $\bigcup_{\lambda<\beta} B_\lambda = B_\beta$. Inclusion $\bigcup_{\lambda<\beta} B_\lambda \le B_\beta$ is obvious. Conversely, let $g \in B_\beta$; then $\langle g, X\rangle$ is finitely generated and thus nilpotent. It follows that $\langle g\rangle^X$ is finitely generated; but

$$\langle g\rangle^X \le B_\beta^X = B_\beta \le A_\beta = \bigcup_{\lambda<\beta} A_\lambda,$$

whence $\langle g\rangle^X \le A_\mu$ for some $\mu < \beta$. Therefore $\langle g\rangle^X \le \bigcup_{x\in X} A_\mu^x = B_\mu$. This proves the equality $\bigcup_{\lambda<\beta} B_\lambda = B_\beta$ and thus completes the proof. $\boxed{QED}$

Proof. of Theorem 71. Let H, K be finitely generated nilpotent ascendant subgroups of the group G. Then, by Theorem 59, $J = \langle H, K\rangle$ is contained in the Hirsch-Plotkin radical of G and so, being finitely generated, it is nilpotent. We have then to show that J is ascendant in G (the subnormal case is proved with the same arguments and it is easier). Now, since J is nilpotent, H is subnormal in it; we proceed by induction on the defect d of H in J. If $d = 0$ then $H = J$ and there is nothing to prove. Let $d \ge 1$; then H^J is finitely generated and so it is generated by a finite number of conjugates of H. Let H^x be such a conjugate; then, like H, H^x is ascendant and the defect of H in

$\langle H, H^x \rangle$ is at most $d-1$, so that $\langle H, H^x \rangle$ is ascendant in G by inductive assumption. Repeating this argument a finite number of times, we conclude that H^J is ascendant in G. Since K normalizes H^J, by Lemma 72 there exists an ascending series $H^J = T_0 \trianglelefteq T_1 \trianglelefteq \ldots \trianglelefteq T_\alpha = G$ all of whose terms are normalized by K. For each ordinal $\beta \leq \alpha$, let $J_\beta = T_\beta K$. As, clearly, $\bigcup_{\lambda<\beta} J_\lambda = J_\beta$ for every limit ordinal β, these are terms of an ascending chain of subgroups of G. Now, since K is ascendant in G, for $\beta < \alpha$ we have that $J_\beta = T_\beta K$ is ascendant in $J_{\beta+1} = T_{\beta+1}K$. Hence the chain of $J_\beta(\beta \leq \alpha)$ may be refined to an ascending series from $J_0 = H^J K = \langle H, K \rangle$ to $J_\alpha = G$, and this completes the proof. $\boxed{QED}$

A *Baer group* is a group all of whose cyclic subgroups are subnormal. A *Gruenberg group* is a group all of whose cyclic subgroups are ascendant.

The classes of Baer and Gruenberg groups are closed by subgroups and homomorphic images. The next theorem implies in particular that they are closed by normal products (a group G is said to be a normal product of its subgroups H and K if H, K are both normal in G and $G = HK$).

73 Theorem. *Let G be a group. The following conditions are equivalent.*

i) *G is a Baer (Gruenberg) group;*

ii) *Every finitely generated subgroup of G is subnormal (ascendant);*

iii) *Every finitely generated subgroup of G is subnormal (ascendant) and nilpotent;*

iv) *G is generated by cyclic subnormal (ascendant) subgroups.*

Proof. The only implication that needs to be proved is iv) $\Rightarrow$ iii). Thus, let S be a generating set of the group G such that $\langle x \rangle$ is subnormal (ascendant) in G for all $x \in S$. Let F be a finitely generated subgroup of G. Then $F \leq \langle S_0 \rangle$ for some finite subset S_0 of S. By Theorem 71 and an obvious induction $\langle S_0 \rangle$ is nilpotent and subnormal (ascendant), whence F is nilpotent and subnormal (ascendant). $\boxed{QED}$

In particular, Gruenberg (and Baer) groups are locally nilpotent.

Clearly, every Baer group is a Gruenberg group. The simplest example of a Gruenberg group which is not a Baer group is the *locally dihedral* 2*-group.* This is defined as the semidirect product $G = A \rtimes \langle x \rangle$, where A is a Prüfer group C_{2^∞} and x the automorphism of A which maps every element in its inverse; it is easy to check that $[G, x] = [A, x] = A$, and so $\langle x \rangle$ cannot be subnormal in G; on the other hand, if, for all $n \in \mathbb{N}$, A_n is the unique subgroup of order 2^n of A, then $A_n H \trianglelefteq A_{n+1} H$ for any subgroup H of G, and from this it follows that every subgroup of G is ascendant. Now a torsion-free example.

Example. For each $n \geq 1$ let $A_n = \mathbb{Z}^n$ be a free abelian group of rank n, with set of free generators $\{e_{1,n}, \ldots, e_{n,n}\}$, and let A be the direct product of all A_n $(n \geq 1)$. Let g be the automorphism of A that fixes every direct summand A_n and acts on it as a unitriangular matrix whose non-diagonal entries are 1 over the main diagonal and 0 everywhere else (thus, g is the linear extension of the map $e_{1,n}^g = e_{1,n}$ and $e_{i,n}^g = e_{i,n} + e_{i-1,n}$ if $0 < i \leq n$). Let $G = A \rtimes \langle g \rangle$ be the semidirect product defined by this action. Then G is clearly torsion-free. To prove that G is a Gruenberg group it is enough to show (by Theorem 73) that $\langle g \rangle$ is ascendant in G. But this is clear: for every $n \geq 1$, let $B_n = A_1 \times \ldots \times A_n$ and $B_0 = 1$; then $B_{n+1}\langle g \rangle / B_n \simeq A_{n+1}\langle g \rangle$ and so $B_n\langle g \rangle$ is subnormal in $B_{n+1}\langle g \rangle$ for all $n \geq 0$. By refining each these intermediate finite series we get an ascending series (of type ω) from $\langle g \rangle$ to G (more formally, assign the inverse lexicographic order to the base $\{e_{i,n} \mid 1 \leq i \leq n,\ 0 \neq n \in \mathbb{N}\}$ of A, and for each (i,n) let $B_{(i,n)} = \langle e_{j,k} \mid (j,k) \leq (i,n) \rangle$; then the subgroups $H_{(0,0)} = \langle g \rangle$ and $H_{(i,n)} = B_{(i,n)}\langle g \rangle$, for $1 \leq i \leq n$, are the terms of an ascending series). However, G is not a Baer group. In fact, for each $n \geq 2$, $[A_n,\, _{n-1}\langle g \rangle] \neq 1$, and so $\langle g \rangle$ cannot be subnormal in G.

Not all locally nilpotent groups are Gruenberg groups (see [96] for an example). On the other hand, by observing that a countable locally nilpotent group is the union of an ascending chain of (finitely generated) nilpotent groups, one easily proves that *every countable locally nilpotent group is a Gruenberg group.* Thus, in particular, the class of Gruenberg groups is not countably recognizable; while it easily follows from Lemma 24 that the class of Baer groups is countably recognizable.

We now give another characterization of Gruenberg groups inside the class of locally nilpotent groups. Following Mal'cev we say that a group G is a SN^*-group if G admits an ascending series with abelian factors. Since subgroups and quotients of abelian groups are abelian, it is easy to see that every subgroup and every quotient of a SN^*-group is an SN^*-group.

74 Lemma. *A group G has a unique maximal normal SN^*-subgroup, which contains every ascendant SN^*-subgroup of G.*

Proof. Suppose that $N_1 \trianglelefteq N_2 \trianglelefteq N_3 \trianglelefteq \ldots$ is a chain of SN^*-subgroups of the group G. Then, for every $n \geq 1$, N_n/N_{n-1} is a SN^*-group. So, if we start from the terms of an abelian ascending series of N_1 and successively add the inverse images modulo N_{n-1} of the terms of an abelian ascending series of N_n/N_{n-1}, we eventually get an abelian ascending series of $N = \bigcup_{n \in \mathbb{N}} N_n$; therefore, N is a SN^*-subgroup of G. If we further assume that all the subgroups N_n are normal in G, we get that $\bigcup_{n \in \mathbb{N}} N_n$ is a normal SN^*-subgroup of G. Thus, by Zorn's Lemma every group G admits maximal normal SN^*-subgroups. A similar argument shows that if N and K are normal SN^*-subgroups of G, then

NK is a normal SN^*-subgroup of G. This proves that G has a unique maximal normal SN^*-subgroup, which we may call the SN^*-radical of G.

Just for this proof, let us denote by $\Theta(G)$ the SN^*-radical of a group G. Let H be an ascending SN^*-subgroup of G, and $H = H_0 \leq H_1 \leq \ldots \leq H_\alpha = G$ an ascending series from H to G. We prove that $H \leq \Theta(G)$ by induction on the ordinal α. Let $\alpha = \beta + 1$; since $\Theta(H_\beta)$ is characteristic in H_β, $\Theta(H_\beta)$ is a normal SN^*-subgroup of G, and so it is contained in $\Theta(G)$. Now, $H \leq \Theta(H_\beta)$ by inductive assumption, and we are done. Thus, let α be a limit ordinal. Then the inductive assumption ensures that $\Theta(H_\lambda) \leq \Theta(H_\mu)$, for all $\lambda \leq \mu < \alpha$. Hence $S = \bigcup_{\beta<\alpha} \Theta(H_\beta)$ is a normal subgroup of G, and, by the observation at the beginning of the proof, S is a SN^*-group. Thus $S \leq \Theta(G)$. Since $H \leq S$, this completes the proof. $\boxed{QED}$

We are ready to give the announced characterization of Gruenberg groups.

75 Theorem. [Gruenberg see [32]] *A locally nilpotent group is a Gruenberg group if and only if it is a SN^*-group.*

Proof. In one direction, the Theorem is an immediate corollary of 74.

For the converse, let us first assume that $G = A\langle x\rangle$ is a locally nilpotent group with A a normal abelian subgroup and x an element of G. For all $n \in \mathbb{N}$, let $X_n = (\zeta_n(G) \cap A)\langle x\rangle$. Then, clearly, $\langle x\rangle = X_0 \trianglelefteq X_1 \trianglelefteq X_2 \trianglelefteq \ldots$ Now, let $a \in A$; then $\langle a, x\rangle$ is a nilpotent group of class, say, c, and observe that, since A is abelian, $\zeta_d(\langle a, x\rangle) \cap A \leq \zeta_d(G)$ for every $1 \leq d \leq c$. Thus, $a \in X_c$. Hence $\bigcup_{n\in\mathbb{N}} X_n = G$ and $\langle x\rangle$ is ascendant in G.

Let now G be a locally nilpotent group admitting an ascending series with abelian factors, and let $g \in G$. By Lemma 72 there exists an ascending series with abelian factors $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_\alpha = G$ whose terms are all normalized by g. For each ordinal $\beta \leq \alpha$ we set $H_\beta = \langle G_\beta, g\rangle = G_\beta\langle g\rangle$. The H_β ($\beta \leq \alpha$) are the elements of an ascending chain of subgroups of G, and clearly $H_\beta = \bigcup_{\lambda<\beta} H_\lambda$ if β is a limit ordinal. If $\lambda + 1 \leq \alpha$, then $H_{\lambda+1}$ normalizes G_λ. Now, the group $H_{\lambda+1}/G_\lambda = \langle G_{\lambda+1}, g\rangle/G_\lambda$ is abelian by cyclic and so, by what observed before, H_λ/G_λ is ascendant in $H_{\lambda+1}/G_\lambda$, i.e. H_λ is ascendant in $H_{\lambda+1}$. Thus the chain of subgroups of G whose terms are the H_β ($\beta \leq \alpha$) may be refined to an ascending series of G. Since the first term is $H_0 = \langle g\rangle$, we have that $\langle g\rangle$ is ascendant in G, thus proving that G is a Gruenberg group. $\boxed{QED}$

76 Corollary. *A soluble locally nilpotent group is a Gruenberg group.*

After these general facts, let us mention a couple of useful properties of Baer groups. For the second one (78), observe that if p is a prime and G is a soluble p-group of finite exponent, then G has a finite normal series with elementary abelian factors.

77 Lemma. *Let G be a Baer group, and N a normal nilpotent subgroup of G. If G/N is finitely generated, then G is nilpotent. In particular, if G has a nilpotent subgroup of finite index, then G is nilpotent.*

PROOF. Let G be a Baer group, and let N be a normal nilpotent subgroup such that G/N is finitely generated. Let $x_1N, x_2N, \ldots, x_nN$ be a set of generators of G/N. Then, since G is a Baer group, $H = \langle x_1, x_2, \ldots, x_n \rangle$ is a nilpotent subnormal subgroup of G. It now follows from Lemma 61 that $G = NH$ is nilpotent. Now, suppose that G has a nilpotent subgroup H of finite index. Then H has only a finite number of conjugates in G, and so G/H_G is finite. By the previous fact it follows that G is nilpotent. $\boxed{QED}$

78 Proposition. *Let p be a prime and $1 \neq G$ a soluble p-group of finite exponent. Let n be the length of a shortest normal series of G with elementary abelian factors, and let $d = 1+p+\ldots+p^{n-1}$. Then for every $g \in G$, $[G,_d \langle g \rangle] = 1$. Thus, a soluble p-group of finite exponent is a Baer group and a bounded Engel group.*

PROOF. Fixed a prime p, for $n \geq 1$ we write $d(n) = 1 + p + \ldots + p^{n-1}$. We then argue by induction on n. If $n = 1$ our claim is trivial, so let $n \geq 2$, $g \in G$ and let A by the first non trivial term of a normal series of G with elementary abelian factors. Then G/A has a series of this kind with $n-1$ factors, therefore, by inductive assumption $[G,_{d(n-1)} \langle g \rangle] \leq A$. Now, observe that certainly $g^{p^{n-1}} \in A$. Thus, since A is an elementary abelian p-group, by Lemma 14 we have

$$[A,_{p^{n-1}} \langle g \rangle] = [A,_{p^{n-1}} g] = [A, g^{p^{n-1}}] = 1.$$

Therefore $1 = [G,_d \langle g \rangle]$, where $d = d(n-1) + p^{n-1} = d(n)$. $\boxed{QED}$

79 Corollary. *A p-group with a normal nilpotent subgroup of finite index and finite exponent is nilpotent.*

PROOF. A group satisfying the assumptions in the statement is certainly soluble, so the result follows immediately from 78 and 77. $\boxed{QED}$

As we are interested in subnormal subgroups, let us also mention the following.

80 Lemma. *A perfect subnormal subgroup of a Baer group is normal.*

PROOF. See [64], 2.5.10. $\boxed{QED}$

Let G be a group. Then the subgroup $B(G)$ generated by all cyclic subnormal subgroups of G is called the *Baer radical* of G; the subgroup $\Gamma(G)$ generated by all cyclic ascendant subgroups of G is called the *Gruenberg radical* of G.

Clearly, Baer and Gruenberg radicals are characteristic subgroups contained in the Hirsch-Plotkin radical, and, by Theorem 73 they contain, respectively, every subnormal (ascendant) Baer (Gruenberg) subgroup of the group. We remark that, even in a locally nilpotent group, the subgroup generated by two subnormal nilpotent subgroups need not be nilpotent (see [64] for more details).

Finally, observe the following consequence of Theorem 75.

81 Corollary. *Let G be a locally nilpotent group and let $\Gamma(G)$ be its Gruenberg radical. Then $\Gamma(G/\Gamma(G)) = 1$.*

Fitting groups. A group G is called a *Fitting group* if every finitely generated subgroup of G is contained in a normal nilpotent subgroup

By Fitting's Theorem, a group G is a Fitting group if and only if, for every element x of G, the normal closure $\langle x\rangle^G = \langle\{x^g | g \in G\}\rangle$ is nilpotent. The class of Fitting groups is contained in the class of Baer groups, and it is closed by subgroups and homomorphic images, but not by normal products (see Theorem 2.1.2 in [64]). The following remark is easy to prove.

82 Proposition. *A Fitting group is hyperabelian.*

The *Fitting radical* $F(G)$ of a group G is the subgroup generated by all $x \in G$ such that $\langle x\rangle^G$ is nilpotent.

In other terms, the Fitting radical of G is the subgroup generated by all normal nilpotent subgroups of G. Clearly, $F(G)$ is a Fitting group and is contained in the Baer radical $B(G)$, but in general it does not contain all normal Fitting subgroups of G. On the other hand, examples constructed by Dark show that there exist Baer groups with no non-trivial normal abelian subgroup.

83 Lemma. *A nilpotent by abelian Baer group is a Fitting group.*

Proof. Let G be a Baer group such that G' is nilpotent, and let $x \in G$. Then $\langle x\rangle^G = \langle x\rangle[G, \langle x\rangle] \leq \langle x\rangle G'$, which is nilpotent by Lemma 77. $\boxed{QED}$

Wreath products (and wreath powers) are a very useful tool when constructing groups with particular features. The next example is a simple issue of that. Before, let us recall an easy property of standard restricted wreath products.

84 Lemma. *Let A, H be groups and $G = A \wr H$ the standard wreath product. We look at H as a subgroup of G (complementing the base group). Let K be an infinite subgroup of H. Then $N_G(K) = N_H(K)$ (in particular, if H is infinite, $N_G(H) = H$).*

Proof. Let A, H, G and K be as in the statement, and let $G = BH$, where B is the base group. Then, $N_G(K) = N_B(K)N_H(K)$. Now $[N_B(K), K] \leq B \cap K = 1$ and so $N_B(H) = C_B(H)$. Now, an element $f \in B$ centralizes K if and only if f is constant on all orbits of K. Since H is taken in its regular permutation representation and K is infinite, such orbits are all infinite, and so (being our product the restricted one) $C_B(K) = 1$. Thus $N_G(K) = N_H(K)$. QED

85 Example. An abelian by nilpotent Baer group that is not Fitting. Let p be a fixed prime and let A be a vector space over the field $GF(p)$ with base indexed on $\mathbb{N}$, $\{a_i \mid i \in \mathbb{N}\}$. Let x be the automorphism of A defined by

$$a_i^x = a_i + a_{i-1} \text{ if } i \not\equiv 0 \pmod{p} \quad \text{and} \quad a_{np}^x = a_{np} \ (\forall n \in \mathbb{N}).$$

Observe that x has order p. We look at x as an automorphism of the additive group A (which is an elementary abelian p-group) and consider the semidirect product $H = A \rtimes \langle x \rangle$. Then, being A abelian, an easy computation shows that $\zeta(H) = C_A(x) = \langle a_i \mid i \equiv 0 \pmod{p} \rangle$, and for $1 \le n \le p-1$,

$$\zeta_n(H) = \langle a_i \mid i \equiv 0, 1, \ldots, n-1 \pmod{p} \rangle.$$

So, $H/\zeta_{p-1}(H)$ is abelian, and thus H is nilpotent of class p (and exponent p^2). Consider now the wreath product $G = C_p \wr H = BH$, where C_p is a cyclic group of order p. Then G is soluble of exponent p^3 and so, by Proposition 78, it is a Baer group. On the other hand, $\langle x \rangle^H$ contains all elements $[a_i, x]$ and so $\langle x \rangle^H = H' \langle x \rangle$ is an infinite subgroup of H. By 84, $N_G(\langle x \rangle^H) = H$. But $\langle x \rangle^G \cap H = \langle x \rangle^H$, and so $\langle x \rangle^H$ is self-normalizing in $\langle x \rangle^G$ which therefore is not nilpotent (for, clearly, $\langle x \rangle^G > \langle x \rangle^H$). Thus G is not a Fitting group.

Hypercentral groups. Hypercentral groups, often called ZA-groups, are a natural generalization of nilpotent groups. We recall their definition.

A group G is *hypercentral* if it admits an ascending central series.

Arguing as in the finite case, it is easy to show that *a group G is hypercentral if and only if G coincides with its hypercentre*, or, in other words, if there exists an ordinal α such that $\zeta_\alpha(G) = G$. If G is hypercentral, then the least ordinal α such that $\zeta_\alpha(G) = G$ is called the *(hypercentral) length* of G. One has the following easy characterization of hypercentral groups.

86 Proposition. *A group G is hypercentral if and only if every non-trivial homomorphic image of G has non-trivial centre.*

Proof. Since the quotient of any group modulo its hypercentre has obviously trivial center, one implication is clear. Conversely, let G be hypercentral of length α, and let N be a proper normal subgroup of G. Then there exists a smallest ordinal $\beta < \alpha$ such that N does not contain $\zeta_\beta(G)$. Clearly β is not a limit ordinal, and is not 0. Thus, $\zeta_{\beta-1}(G) \le N$, and so it follows that $\zeta_\beta(G)N/N$ is a non-trivial central subgroup of G/N. QED

Thus, the class of hypercentral groups is closed by subgroups and quotients; we leave to the reader the exercise of proving that it is countably recognizable.

A simple way to contract hypercentral groups of length ω is to take direct products of nilpotent groups with unbounded nilpotency class. The locally dihedral 2-group is hypercentral of length $\omega+1$, and, similarly, all Černikov p-groups are hypercentral. The group in the example above is a torsion-free hypercentral group of length $\omega + 1$.

In fact, for every ordinal α there exist hypercentral groups of length α.

It is not difficult to show directly that every hypercentral group is locally nilpotent. However, we take a different approach.

87 Proposition. *Every subgroup of a hypercentral group is ascendant.*

PROOF. Let H be a subgroup of the hypercentral group G, and suppose that G has hypercentral length α. Then, by setting $H_\lambda = \zeta_\lambda(G)H$, for all ordinals $\lambda \leq \alpha$, one clearly obtains an ascending series from H to G. $\boxed{QED}$

Therefore *a hypercentral group is a Gruenberg group* and so it is locally nilpotent. The locally dihedral 2-group is the simplest example of a hypercentral group which is not a Baer group (on the other hand it is clear that a hypercentral group of length ω is a Fitting group). Hypercentral groups share with nilpotent groups a number of useful properties, that may be proved by adjusting in an easy way the proof for the nilpotent case;.

88 Lemma. *Let G be a hypercentral group. Then*

(i) If $1 \neq N \trianglelefteq G$ *then* $N \cap \zeta(G) \neq 1$*;*

(ii) if A *is a maximal normal abelian subgroup of* G*, then* $A = C_G(A)$*.*

With the aid of 88 it is easy to prove that the class of hypercentral groups is closed by normal products.

89 Proposition. [P. Hall [37]] *Let* H, K *be two normal hypercentral subgroups of a group; then* HK *is hypercentral.*

PROOF. If a group G is the product of two normal hypercentral subgroups, and W is the hypercentre of G, then G/W is also a product of two normal hypercentral subgroups and, by definition of hypercentre, it has trivial centre. Thus, in order to prove that $W = G$, it will suffice to show that a product $1 \neq G = HK$ of two normal hypercentral subgroups H and K necessarily has non-trivial centre. Thus, we may assume $H \neq 1$, and let $Z = \zeta(H)$. Clearly, $Z \cap \zeta(K) \leq \zeta(G)$, so we suppose $Z \cap \zeta(K) = 1$. But then, since $Z \cap K \trianglelefteq K$, Lemma 88 forces $Z \cap K = 1$. Hence $[Z, K] \leq Z \cap K = 1$, and so $Z \in \zeta(G)$. $\boxed{QED}$

For further reference, we also observe the following fact.

90 Lemma. *Let* N *be a normal subgroup of the locally nilpotent group* G*. If* N *is hypercentral and* G/N *is finitely generated, then* G *is hypercentral.*

Proof. Let S be a finite subset of G that generates G modulo N, and let H be the hypercentre of G. Assume, by contradiction, that $H \neq G$. Then, since G/N is nilpotent, $H \not\geq N$, and so $K = H \cap N < N$. Clearly $K \trianglelefteq G$ and, by Proposition 86, $A/K = \zeta(N/K) \neq 1$. Let $a \in A \setminus K$, and $U = \langle a, S\rangle K$. Then, being finitely generated, U/K is nilpotent and $(A \cap U)/K$ is a non-trivial subgroup of it. Hence $V/K = \zeta(U/K) \cap (A \cap U)/K$ is not trivial. Now $[V, N] \leq K$ because $V \leq A$, and $[V, \langle S\rangle] \leq K$ because $V/K \leq \zeta(U/K)$. Thus, since $G = N\langle S\rangle$, we get $V/K \leq \zeta(G/K)$ and the contradiction $V \leq H \cap N = K$. QED

We add some considerations about the dual and much more intricate case of groups with a lower (i.e. descendant) central series. Such groups are called *hypocentral*. A simple example of a hypocentral group which is not locally nilpotent is the infinite dihedral group D_∞. If G is hypocentral and α is the smallest ordinal such that $\gamma_\alpha(G) = 1$, we say that G has hypocentral type length α. For instance, the infinite dihedral group has hypocentral type length ω. Hypocentral groups form a class of generalized nilpotent groups; however, this class is too large to be considered in general. For instance, by a famous theorem of Magnus, it includes every free group.

In fact, free groups (and the infinite dihedral group as well) belong to the narrower class of *residually nilpotent* groups. A group G is residually nilpotent if for each $1 \neq x \in G$ there exists a normal subgroup N of G such that G/N is nilpotent and $x \notin N$. It is immediate to prove that G is residually nilpotent if and only if $\gamma_\omega(G) = \bigcap_{n\in\mathbb{N}} \gamma_n(G) = 1$. Notice also that a locally nilpotent group which is residually finite is residually nilpotent (but not the converse).

Golod examples prove the existence of finitely generated residually finite p-groups that are not finite (observe that a residually finite p-group is residually nilpotent). Also, we have already mentioned (Theorem 36) that the recent solution by Zelmanov of the Restricted Burnside Problem implies that *a residually finite p-group of finite exponent is locally nilpotent.*

Let us give a simple example of a locally nilpotent hypocentral group that is not residually nilpotent.

91 Example. Let $\mathbb{K}$ be a field and, for every $1 \leq n \in \mathbb{N}$, let T_n be the unitriangular matrix group $UT(n, \mathbb{K})$. Let $W = \mathrm{Dir}_{n\geq 1} T_n$; then $\gamma_k(W) = \mathrm{Dir}_{n\geq 1}\gamma_k(T_n)$ for all $k \in \mathbb{N}$. Thus, $\gamma_\omega(W) = 1$ and W is residually nilpotent (and hypercentral of length ω). Let $Z = \zeta(W) = \mathrm{Dir}_{n\geq 1}\zeta(T_n)$. Now, for all $n \geq 1$, $\zeta(T_n)$ is isomorphic to the additive group of $\mathbb{K}$ via, say, the isomorphism ϕ_n. Let N be the kernel of the homomorphism

$$\begin{array}{rcl} Z & \to & (\mathbb{K}, +) \\ (x_1, x_2, \ldots) & \mapsto & \sum_{n\geq 1} \phi_n(x_n). \end{array}$$

Then $N \trianglelefteq W$, $Z/N \simeq (\mathbb{K}, +)$, and $N\zeta(T_n) = Z$ for all $n \geq 1$. Finally, let $G = W/N$. Since $G/Z \simeq W/Z \simeq \mathrm{Dir}_{n\geq 1}(T_n/\zeta(T_n))$, we clearly have that G/Z is residually nilpotent, i.e. $\gamma_\omega(G) \leq Z/N$. On the other hand, for all $k \in \mathbb{N}$,

$$\gamma_k(G) = \frac{\gamma_k(W)N}{N} \geq \frac{\zeta_k(T_{k+1})N}{N} = \frac{\zeta(T_{k+1})N}{N} = \frac{Z}{N}.$$

Thus $\gamma_\omega(G) = Z/N$ and G is not residually nilpotent (but $\gamma_{\omega+1}(G) = 1$.)

Observe that this example incidentally shows that, even in the class of locally nilpotent groups, homomorphic images of residually nilpotent groups need not be residually nilpotent. In fact, we will show in section 3.5 that every locally nilpotent group is a homomorphic image of a suitable residually finite locally nilpotent group.

The normalizer condition. A group G is said to satisfy the *normalizer condition* if $H \neq N_G(H)$ for all proper subgroups H of G. Following [96], we denote by N the class of all groups satisfying the normalizer condition.

92 Proposition. *A group G satisfies the normalizer condition if and only if every subgroup of G is ascendant. Thus N-groups are Gruenberg groups.*

PROOF. Since, clearly, a proper ascendant subgroup of a group cannot be self-normalizing, in one direction the implication is obvious. Conversely, let G be an N-group, and H a proper subgroup of it. Then one defines an ascending series of successive normalizers by setting $N^0(H) = H$, $N^{\alpha+1}(H) = N_G(N^\alpha(H))$ for any ordinal α, and $N^\beta(H) = \bigcup_{\alpha<\beta} N^\alpha(H)$, for any limit ordinal β. Since G satisfies the normalizer condition, this series will eventually reach G, thus showing that H is ascendant. $\boxed{QED}$

This shows, in particular that the class of N-groups is closed by subgroups (a fact which is not immediately obvious). The class N is also clearly closed by quotients; it will be observed that it is not closed by direct products and that it is countably recognizable.

By Propositions 87 and 92, every hypercentral group is an N-group, and we have the following chain of proper inclusions for group classes:

nilpotent $\subset$ *hpercentral* $\subset$ *N-groups* $\subset$ *Gruenberg*;

and we have another chain of proper inclusions for group classes:

nilpotent $\subset$ *Fitting* $\subset$ *Baer* $\subset$ *Gruenberg.*

To prove that the class of hypercentral groups is properly contained in N is not that easy. The first examples of N-groups with trivial centre are due to Heineken and Mohamed [47] and appeared in 1968. These groups, whose construction we will report in chapter 3, are extensions of an elementary abelian p-group by a Prùfer group C_{p^∞} (for any fixed prime p), and have the property that all of their proper subgroups are nilpotent and subnormal.

93 Example. Let $G = C_p\, wr\, C_{p^\infty}$, and let B be the base group of G. If H is a subgroup of G such that $BH \neq G$, then BH is nilpotent and normal in G (in particular H is nilpotent and subnormal of G). On the other hand, if $H = C_{p^\infty}$, by Lemma 84 we have $H = N_G(H)$, and so $\zeta(G) = 1$. Thus, G is a Fitting group but it is not hypercentral.

94 Example. The group of the example at page 32 is a Baer group that is not Fitting, and that also does not satisfy the normalizer condition. Another example with these properties is the group $G = C_p \wr (C_p \wr A)$, where A is an infinite elementary abelian p-group. G is a soluble group of exponent p^3, and so it is a Baer group by Proposition 78. But G is not a Fitting group, and does not satisfy the normalizer condition.

Let us add some more comments on radicable groups. For torsion-free groups this property is not very decisive: a theorem of Mal'cev ensures that every nilpotent torsion-free group N may be embedded in torsion-free radicable group which is still nilpotent of the same nilpotency class of N. For periodic groups the situation is different: a periodic hypercentral semi-radicable group is abelian (and radicable). Nevertheless, radicable locally finite p-groups may also be rather complicated: in fact, a consequence of a result of Baumslag [6] is that every p-group may be embedded in a radicable p-group. Here is a sketch of the argument. Let P be any group, C_n a cyclic group of order n, and embed P as the diagonal subgroup $\delta(P)$ in the base group of the standard wreath product $P_1 = P \wr C_n$; then every element of $\delta(P)$ has a n-th root in P_1. If we start from a p-group $P = P_0$, take $n = p$, and iterate the process (the embedding is in the diagonal subgroup $P_i \mapsto \delta(P_i) \leq P_i \wr C_p = P_{i+1}$), we get a direct limit group $\overline{P}$, which is a radicable p-group, and contains a copy of the original P as a subgroup. Observe that if P is locally finite then such is $\overline{P}$; moreover if P is nilpotent then P is subnormal in $\overline{P}$ (of defect equal to its nilpotency class) and $\overline{P}$ is a Fitting group. On the other hand we have:

95 Proposition. *A radicable periodic group satisfying the normalizer condition is abelian.*

Proof. Let G be a radicable periodic N-group. We may clearly assume that G is a p-group for a prime p. Let $x = x_0 \in G$. Then there exists $x_1 \in G$ such that $x_1^p = x_0$, and for $i \geq 2$, inductively we find $x_i \in G$ with the property that $x_i^p = x_{i-1}$. Let $U = \langle x_i \mid i \in \mathbb{N}\rangle$; then $U \simeq C_{p^\infty}$. Since G is a N-group, U is ascendant in G and so, by Lemma 35, U^G is abelian. This means that x commutes with all of its conjugates. Hence $[y, x, x] = 1$ for all $x, y \in G$. Now, let $x, y \in G$ with $m = |x|$, and let $t \in G$ such that $t^m = y$; then by Lemma 2 we have $[x, y] = [x, t^m] = [x^m, t] = 1$, thus proving that G is abelian. $\boxed{QED}$

This does not hold for semi-radicable groups: in fact, let U be one of the p-

groups constructed by Heineken and Mohamed. Then $U/U' \simeq C_{p^\infty}$ (see, in fact, Chapter 3), and it is not difficult to see that U is semi-radicable not radicable.

Finiteness conditions. Locally nilpotent groups satisfying various finiteness conditions have been largely studied in the past, and much is known about them. While refering to the first volume of Robinson's monograph [96] for a full account, we restrict to mentioning, for further reference, just a special case of a result of Plotkin

96 Theorem. *Let G be a locally nilpotent group. Then*

(1) *G satisfies the maximal condition on abelian subgroups if and only if G is a finitely generated nilpotent group;*

(2) *G satisfies the minimal condition on abelian subgroups if and only if G is a direct product of finitely many Černikov p-groups.*

1.6 Preliminaries on $\mathcal{N}_1$

The class of groups in which every subgroup is subnormal, which we denote by $\mathcal{N}_1$, represents a case for which it is difficult to make any immediate but not trivial observation.

Among the natural classes of generalized nilpotent groups, $\mathcal{N}_1$ is perhaps the closest to nilpotency, as it will be seen in these notes. Indeed, it may be useful to warn that, although Lemma 62 seems to confirm the idea that locally nilpotent groups are plenty of normal (and subnormal) subgroups, this is not quite true in general. For instance, F. Leinen (see [62]) has shown that, given a prime p, in the unique countable existentially closed locally finite p-group (which was discovered by P. Hall, and contains as a subgroup every finite p-group) all subnormal subgroups are normal and form a unique chain of subgroups.

Besides groups of Heineken–Mohamed type (non-nilpotent groups with all of their proper subgroups nilpotent and subnormal), another way of explicitely constructing non–nilpotent $\mathcal{N}_1$-groups (which we treat in Chapter 6), was discovered by H. Smith. It produces in particular hypercentral, residually finite, $\mathcal{N}_1$-groups of finite rank. Thus, none of these properties: hypercentrality, finite rank, residual finiteness, associated to $\mathcal{N}_1$ is enough to ensure nilpotency.

Clearly, a $\mathcal{N}_1$-group is a Baer group satisfying the normalizer condition, but not viceversa, as the direct product of infinitely many nilpotent groups with unbounded classes shows. It is also obvious that the class $\mathcal{N}_1$ is closed by subgroups and quotients; but it is not closed by direct products. In fact, taking for granted the existence of a $\mathcal{N}_1$-group H with trivial centre, then the diagonal subgroup $D = \{(x, x) \mid x \in H\}$ of the direct power $H \times H$, is self-normalizing. Indeed, it is not difficult to prove the following fact.

97 Lemma. *Let H be a group and let D be the diagonal subgroup of $H \times H$. Then:*

(a) $D \neq N_{H\times H}(D)$ if and only if $\zeta(H) \neq 1$;

(b) D is subnormal in $H \times H$ if and only if H is nilpotent;

(c) D is ascendant in $H \times H$ if and only if H is hypercentral;

Observe that point (c) gives a sort of 'outer' characterization of hypercentral groups inside the class N: a group G is hypercentral if and only if the direct product $G \times G$ satisfies the normalizer condition.

Let us repeat another elementary but basic fact (in fact, Lemma 24). Let H be a subgroup of the group G. Then H is subnormal of defect at most n if and only if $[G,_n U] \leq H$ for any finitely generated subgroup U of H. In particular, if all finitely generated subgroups of H are subnormal of defect at most n, then H is subnormal of defect at most n.

We now start proving something. The first result is indeed one of the most useful arguments in studying $\mathcal{N}_1$-groups. In essence it was firstly observed by C. Brookes in [7].

98 Theorem. [Brookes]. *Let G be a group in $\mathcal{N}_1$, and let Θ be a family of subgroups of G such that $G \in \Theta$. Then there exists a subgroup $H \in \Theta$, a finitely generated subgroup F of H, and a positive integer d, such that every $F \leq K \leq H$, with $K \in \Theta$, has defect at most d in H.*

Proof. Let G be a counterexample. By an inductive procedure we construct two chains of subgroups

$$\{1\} = F_0 \leq F_1 \leq \ldots \leq F_i \leq F_{i+1} \leq \ldots$$

$$G = H_0 \geq H_1 \geq \ldots \geq H_i \geq H_{i+1} \geq \ldots$$

such that, for each $i, j \in \mathbb{N}$, F_i is finitely generated, $H_i \in \Theta$, $F_i \leq H_j$ and $[H_i, {}_iF_{i+1}] \not\leq H_{i+1}$.

Set $F_0 = \{1\}, H_0 = G$, and suppose we have already defined $F_0, \ldots, F_i$ and $H_0, \ldots, H_i$. Since $F_i \leq H_i \in \Theta$, and G is a counterexample, there exists a subgroup $\Theta \ni H_{i+1} \leq H_i$ with $F_i \leq H_{i+1}$, and $d(H_{i+1}, H_i) = i + 1$. This implies that there exists a finitely generated subgroup K of H_{i+1} such that $[H_i, {}_iK] \not\leq H_{i+1}$. We put $F_{i+1} = \langle F_i, K \rangle$. Then F_{i+1} is finitely generated, $F_i \leq F_{i+1} \leq H_{i+1}$, and $[H_i, {}_iF_{i+1}] \not\leq H_{i+1}$.

By induction, we thus construct the two chains $\{F_i\}_{i\in\mathbb{N}}$, $\{H_i\}_{i\in\mathbb{N}}$ with the desired properties. We then put

$$F = \bigcup_{i\in\mathbb{N}} F_i \ .$$

Then $F \leq \bigcap_{i\in\mathbb{N}} H_i$ is subnormal in G. So there exists an integer k such that $[G,_k F] \leq F$. In particular we have

$$[G,\ _kF_{k+1}] \leq [G,\ _kF] \leq F \leq H_{k+1}$$

which contradicts the choice of F_{k+1}. $\boxed{QED}$

Next proposition generalizes a result appearing in [101], where its proof is credited to D. Robinson.

99 Proposition. *Let $G \in \mathcal{N}_1$, and A a normal nilpotent periodic subgroup of G. Let $A^\omega = \bigcap_{n\in\mathbb{N}} A^n$. Then there exists $d \geq 1$ such that $A^\omega \leq \zeta_d(G)$.*

Proof. Write $D = A^\omega$. We may clearly suppose that G/D is countable; thus let $G/D = \{a_1D, a_2D, a_3D, \ldots\}$. Assume that, for a $1 \leq n \in \mathbb{N}$ we have integers $m_1, m_2, \ldots, m_n$ such that, if $U_n = \langle a_1^{m_1}, a_2^{m_2}, \ldots, a_n^{m_n}\rangle$, then $A \cap U_n = 1$. Now, U_n is a subgroup of the finitely generated nilpotent group $\langle U_n, a_{n+1}\rangle$. Also, since A is periodic, $A \cap \langle U_n, a_{n+1}\rangle$ is finite. then, by Theorem 42, there exists a subgroup of finite index of $\langle U_n, a_{n+1}\rangle$ that contains U_n and has trivial intersection with A. In particular, there exists a $m_{n+1} \geq 1$ such that $U_{n+1} = \langle U_n, a_{n+1}^{m_{n+1}}\rangle$ has trivial intersection with A. In this way we get, by induction, a sequence $(m_n)_{n\geq 1}$ of integers such that, for all n, $A \cap \langle a_1^{m_1}, \ldots, a_n^{m_n}\rangle = 1$. We now set $U = \langle a_n^{m_n} \mid 1 \leq n \in \mathbb{N}\rangle$. Then $A \cap U = 1$, and for each $x \in G$ there exists $1 \leq k \in \mathbb{N}$ such that $x^k \in U$.

Now, $G \in \mathcal{N}_1$, so U is a subnormal subgroup; let d be the defect of U in G. Then $[A,_d U] \leq A \cap U = 1$. Let $x_1, \ldots, x_d \in G$, and let $m_1, \ldots, m_d \in \mathbb{N}$ with $x_i^{m_i} \in U$. Then Lemma 21 yields

$$[D, x_1, \ldots, x_d] \leq [A, \langle x_1^{m_1}\rangle, \ldots, \langle x_d^{m_d}\rangle] \leq [A,_d U] = 1.$$

This proves that $D \leq \zeta_d(G)$. $\boxed{QED}$

It is convenient to state explicitely an immediate corollary of this.

100 Corollary. *Let $G \in \mathcal{N}_1$, and D be a normal abelian divisible periodic subgroup of G. If G/D is nilpotent (hypercentral), then G is nilpotent (hypercentral).*

2 Torsion-free Groups

The proof that torsion–free $\mathcal{N}_1$-groups are nilpotent is relatively simple and does not require a lot of preparation. Thus, inverting the historical development, we present it before anything else in this short chapter. The price will be that, in order to be as self consistent as possible, we will state and prove for a special case some results that will be later (and with much more effort) shown to hold in general (notably Proposition 118 and Lemma 120); the proofs in the torsion-free case are considerably shorter, and we hope that the repetition will not annoy the reader.

2.1 Locally nilpotent torsion–free groups

Let us begin with some simple properties of the ascending central series of a torsion-free group.

101 Lemma. *Let G be an Engel group, and $a, b \in G$ with $\langle a \rangle^G$ torsion-free. Assume that there exists $1 \leq n \in \mathbb{N}$ such that $[a, b^n] = 1$. Then $[a, b] = 1$.*

Proof. Since G is an Engel group there exists an integer k such that $[a,_k b] = 1$. We make induction on k (for $k = 1$ there is nothing to prove). Our assumption implies that b^n is in the centre of $\langle a, b \rangle$, and so $[[a, b], b^n] = 1$. Now, $[a, b] \in \langle a \rangle^G$ and by inductive assumption we then have $[a, b, b] = 1$, whence, by Lemma 2, $[a, b]^n = [a, b^n] = 1$. Since $[a, b] \in \langle a \rangle^G$, which is torsion-free, we conclude that $[a, b] = 1$. $\boxed{QED}$

102 Corollary. *Let G be a locally nilpotent group, and $a, b \in G$. Suppose that $[a^n, b^m]$ has finite order, for some $n, m \geq 1$. Then $[a, b]$ has finite order.*

Proof. Apply Lemma 101 to G/T, where T is the torsion subgroup of G. $\boxed{QED}$

Another immediate application of this Lemma is the following useful fact.

103 Proposition. *Let G be a locally nilpotent group.*

(1) *If N is a normal torsion-free subgroup of G then $G/C_G(N)$ is torsion-free;*

(2) *if G is torsion-free then $G/\zeta_\alpha(G)$ is torsion-free for every ordinal α.*

Proof. (1) Let $b \in G$ and $1 \leq n \in \mathbb{N}$ be such that $b^n \in C_G(N)$. Then, by Lemma 101, $b \in C_G(N)$. This shows that $G/C_G(N)$ is torsion-free.

(2) Let G be torsion-free. Then point (1) applied to $N = G$ yields $G/\zeta_1(G)$ torsion-free; and the same argument, applied to any ordinal of type $\alpha + 1$ shows that $G/\zeta_{\alpha+1}(G)$ is torsion-free if such is $G/\zeta_\alpha(G)$. To complete the proof by induction on α, it remains to consider the case of a limit ordinal β. Thus, take $\zeta_\beta(G) = \bigcup_{\alpha<\beta} \zeta_\alpha(G)$; if $g^n \in \zeta_\beta(G)$ for $g \in G$ and $1 \leq n \in \mathbb{N}$, then $g^n \in \zeta_\alpha(G)$ for some $\alpha < \beta$. By inductive assumption we have $g \in \zeta_\alpha(G) \leq \zeta_\beta(G)$, and we are done. $\boxed{QED}$

104 Lemma. *Let A be a normal abelian torsion-free subgroup of the locally nilpotent group G. Let $a \in A$ and $x_1, \dots, x_n \in G$. If $[a, x_1, \dots, x_n] \neq 1$ then the elements of A: $a,\ [a, x_1],\ [a, x_1, x_2], \dots,\ [a, x_1, \dots, x_n]$ are independent.*

PROOF. Assume the contrary; then there exists $0 \leq s \leq n$ and $d_s, \ldots, d_n \in \mathbb{Z}$, with $d_s \neq 0$, such that

$$[a, x_1, \ldots, x_s]^{d_s}[a, x_1, \ldots, x_{s+1}]^{d_{s+1}} \ldots\ [a, x_1, \ldots, x_n]^{d_n} = 1.$$

Now, the group $X = \langle a, x_1, \ldots, x_n \rangle$ is nilpotent, and so there exists an integer $k \geq 1$ such that $b = [a, x_1, \ldots, x_s] \in \zeta_k(X) \setminus \zeta_{k-1}(X)$. Then

$$b^{-d_s} = [b, x_{s+1}]^{d_{s+1}} \ldots\ [b, x_{s+1}, \ldots, x_n]^{d_n} \in \zeta_{k-1}(X).$$

Thus, since A is normal and abelian, $1 = [b^{-d_s}, {}_{k-1}X] = [b, {}_{k-1}X]^{-d_s}$. Hence $[b, {}_{k-1}X] = 1$ because A is torsion-free. But this means $b \in \zeta_{k-1}(X)$, a contradiction. $\boxed{QED}$

105 Lemma. [Čarin]. *Let G be a locally nilpotent group, and A a normal abelian subgroup of G. If A is torsion-free of finite rank d, then $A \leq \zeta_d(G)$ and $G/C_G(A)$ is torsion-free nilpotent.*

PROOF. The first assertion follows immediately from Lemma 104 and the definition of rank of an abelian group. From Proposition 103 we have that $G/C_G(A)$ is torsion-free. Finally, $C_G(A) \geq \gamma_d(G)$ (and so $G/C_G(A)$ is nilpotent) follows from Lemma 7 (3). $\boxed{QED}$

The point in Čarin's Lemma is that the (abstract) divisible closure D of the torsion-free abelian group A (i.e. $A \otimes_{\mathbb{Z}} \mathbb{Q}$) is a direct product of d copies of the additive group of the rationals $\mathbb{Q}$, and the action of G on A can be uniquely extended to an action on D. Then local nilpotency easily yields that G acts unipotently on D and so $G/C_G(D) = G/C_G(A)$ may be embedded in the unitriangular group $UT(d, \mathbb{Q})$ which is nilpotent torsion-free of class $d-1$ and has finite rank (see [96] for a proof along these lines).

Now, recall that if H is a polycyclic group (thus, in particular, if H is a finitely generated nilpotent group), then the number of infinite cyclic factors in a polycyclic series of H is an invariant of H (see [97], 5.4.13), which is denoted by $h(H)$ and called the *Hirsch length* of H.

106 Corollary. *Let G be a locally nilpotent torsion-free group, and H a finitely generated normal subgroup of G. Then $H \leq \zeta_h(G)$, where h is the Hirsch length of H.*

PROOF. This follows easily from Lemma 105 and induction on the Hirsch length $h(H)$, keeping in mind that $1 \neq Z(H)$ is normal in G, H/Z is torsion free, and $h(H/Z(H)) + h(Z(H)) = h(H)$. $\boxed{QED}$

2.2 Isolators

The basic aspects of root extraction in locally nilpotent groups are subsumed in the elegant P. Hall's theory of isolators [38], which is a fundamental tool in what follows, and that we introduce in its simpler form.

Recall that if π is a set of primes, an integer $n \neq 0$ is a π-number if all of its prime divisors belong to π.

107 Definition. Let π be a set of primes and let H be a subgroup of a group G. The **π-isolator** of H in G is the set

$$I_G^\pi(H) = \{\ x \in G \mid x^n \in H \text{ for some } \pi\text{-number } n \geq 1\}\ .$$

If π is the set of all primes, we then omit it in the notation and speak about the *isolator* $I_G(H)$ of $H \leq G$; thus

$$I_G(H) = \{x \in G \mid x^n \in H \text{ for some } 1 \leq n \in \mathbb{N}\}.$$

The results we prove thereafter are stated for the full isolator, since it is this case that we will need, although some of them (in particular Lemmas 108 and 109) admit a 'local' version which may be proved by specializing the same arguments.

108 Lemma. *Let G be a locally nilpotent group. Then, for all $H \leq G$, $I_G(H)$ is a subgroup of G.*

Proof. Let $x, y \in I_G(H)$, where H is a subgroup of the locally nilpotent group G. Then there exists $1 \leq m \in \mathbb{N}$, such that $\langle x^m, y^m \rangle \leq H$. Now, $U = \langle x, y \rangle$ is nilpotent, of class, say, c. We prove, by induction on c, that $|U : \langle x^m, y^m \rangle|$ is finite, from which $U \subseteq I_G(H)$ clearly follows. If U is abelian, this fact is clear. Otherwise, by inductive assumption, we have that $Y = \gamma_c(U)\langle x^m, y^m \rangle$ has finite index in U. Now, $\gamma_c(U)$ is generated by the simple commutators of length c whose entries are x and y. If $w = [u_1, \ldots, u_c]$ is such a commutator, then (see Lemma 49) $w^{m^c} = [u_1^m, \ldots, u_c^m] \in \gamma_c(U) \cap \langle x^m, y^m \rangle)$. Thus, the abelian group $Y/\langle x^m, y^m \rangle) \simeq \gamma_c(U)/(\gamma_c(U) \cap \langle x^m, y^m \rangle)$ is finitely generated by elements of bounded exponent, and is therefore finite. Hence, as wanted, $|U : \langle x^m, y^m \rangle|$ is finite. $\boxed{QED}$

Needless to say, if H is a subgroup of the locally nilpotent group G, then $I_G(I_G(H)) = I_G(H)$, and $I_G(H) \trianglelefteq G$ if $H \trianglelefteq G$. We say that a subgroup H of the group G is *isolated* (respectively π-isolated) if $H = I_G(H)$ $(H = I_G^\pi(H))$.

109 Lemma. *Let G be a locally nilpotent group, and let $H, K \leq G$. Then, for every $1 \leq n \in \mathbb{N}$,*

(1) $[G, I_G(H)] \leq I_G([G, H])$*, thus if U/V is a central factor of G, then also $I_G(U)/I_G(V)$ is a central factor;*

(2) $\gamma_n(I_G(H)) \le I_G(\gamma_n(H))$;

(3) $I_G(H)^{(n)} \le I_G(H^{(n)})$.

PROOF. (1) Let $M = I_G([G,H])$. Then $M \trianglelefteq G$, since $[G,H] \trianglelefteq G$, and G/M is torsion–free. Let $b \in I_G(H)$, and $n \in \mathbb{N}$ such that $b^n \in H$. Then, for any $g \in G$, $[g,b^n] \in [G,H] \le M$. Now, G/M is torsion-free and thus from Lemma 101 it follows $[g,b] \in M$. This shows that $[G, I_G(H)] \le M$.

(2) We proceed by induction on n. If $n = 1$, then the inclusion reduces to $I_G(H) = I_G(H)$. Let now $n \ge 2$, and set $K = \gamma_n(H)$. Let $x_1, \dots, x_n \in I_G(H)$; then there exists $1 \le m \in \mathbb{N}$ such that $x_i^m \in H$ for all $1 \le i \le n$. By inductive hypothesis, $y = [x_1, \dots, x_{n-1}] \in I_G(\gamma_{n-1}(H))$, and so there exists $1 \le t \in \mathbb{N}$ such that $y^t \in \gamma_{n-1}(H)$. Hence, $[y^t, x_n^m] \in K \le I_G(K)$. By Lemma 101, this implies $[y, x_n] \in I_G(K)$, which is what we wanted.

(3) For $n = 1$, $H^{(1)} = \gamma_2(H)$ and we have proved the inclusion in point (1). Thus, let $n \ge 2$. Applying the induction hypothesis and again point (1), we get:

$$I_G(H)^{(n)} = \gamma_2(I_G(H)^{(n-1)}) \le \gamma_2(I_G(H^{(n-1)})) \le I_G(\gamma_2(H^{(n-1)})) = I_G(H^{(n)}),$$

which is our assertion. $\boxed{QED}$

Lemma 109 is an instance of a more general result established by P. Hall in [38]: if $H_1, \dots, H, n$ are subgroups of a group G and θ is any word in n variables, we define $\theta(H_1, \dots, H_n)$ to be the subgroup of G generated by all the elements of the form $\theta(h_1, \dots, h_n)$ where $h_i \in H_i$ for all $i = 1, \dots, n$. If G is locally nilpotent, then $\theta(I_G^\pi(H_1), \dots, I_G^\pi(H_n)) \le I_G^\pi(\theta(H_1, \dots, H_n))$. To prove this, we begin with a lemma.

110 Lemma. *Let $A_1, \dots, A_n$ be subgroups of the locally nilpotent group G, and for each $i = 1, \dots, n$, let $B_i \le A_i$ with $|A_i : B_i|$ finite. Let π be the set of all prime divisors of the indices $|A_i : B_i|$ and $\theta(x_1, \dots, x_n)$ a word. Then the index $|\theta(A_1, \dots, A_n) : \theta(B_1, \dots, B_n)|$ is finite and a π-number.*

PROOF. Let $H = \theta(A_1, \dots, A_n)$, $K = \theta(B_1, \dots, B_n)$, and suppose by contradiction that $|H : K|$ is not a π-number. Then, since G satisfies the maximal condition on subgroups (Proposition 41), there exists $N \trianglelefteq G$ maximal such that $|HN : KN|$ is either infinite or divided by a prime not in π. We may well assume $N = 1$. Let Z be the centre of G. Then $Z \cap K \trianglelefteq G$ and so (by our choice of N) $Z \cap K = 1$. Suppose that Z contains an infinite cyclic subgroup Y. Then $|HY : KY|$ is a π-number, and therefore $1 \ne Y \cap H$. Thus $C = Y \cap H$ is an infinite cyclic group. Let q be a prime with $q \notin \pi$. Then $1 \ne C^q \trianglelefteq G$, whence $|C^q H : C^q K|$ is a π-number. But, as $K \cap C = 1$, we have the contradiction

$$|C^q H : C^q K| = |H : C^q K| = |H : CK||CK : C^q K| = q|H : CK|.$$

Thus Z does not have any elements of infinite order. Let R by a cyclic subgroup of prime order q of Z. As before we have $R \leq H$, $R \cap K = 1$, and $|H : RK|$ a π-number. Hence

$$|H : K| = |H : RK||RK : K| = |H : RK||R : R \cap K| = |H : RK|q.$$

Since we are assuming that $|H : K|$ is not a π-number, this forces $q \notin \pi$. Therefore Z is a finite π'-group. But then, by Proposition 51, G is a π'-group, which is clearly a contradiction. $\boxed{QED}$

We may now prove Hall's result.

111 Theorem. [P. Hall] *Let $\theta(x_1, \ldots, x_n)$ be a word, π a set of primes, and $H_1, \ldots, H_n$ subgroups of a locally nilpotent group G, then*

$$\theta(I_G^\pi(H_1), \ldots, I_G^\pi(H_n)) \leq I_G^\pi(\theta(H_1, \ldots, H_n)).$$

Proof. Let $U = \theta(I_G^\pi(H_1), \ldots, I_G^\pi(H_n))$, $V = \theta(H_1, \ldots, H_n)$, and take an element $g = \theta(g_1, \ldots, g_n)$ with $g_i \in I_G^\pi(H_1)$. For any $i = 1, \ldots, n$ we then have $g_i^{m_i} \in H_i$ for some π-number m_i; we write $A_i = \langle g_i \rangle$ and $B_i = \langle g_i^{m_i} \rangle$. Since $\langle A_1, \ldots, A_n \rangle$ is nilpotent, we can apply Lemma 110 and deduce that $|\theta(A_1, \ldots, A_n) : \theta(B_1, \ldots, B_n)|$ is a π-number. As $\theta(B_1, \ldots, B_n)$ is subnormal in $\theta(A_1, \ldots, A_n)$ and $g \in \theta(A_1, \ldots, A_n)$, it follows that $g^m \in \theta(B_1, \ldots, B_n) \leq V$ for some π-number m. Thus $g \in I_G^\pi(V)$. Since the elements like g generate U, we have $U \leq I_G^\pi(V)$, as wanted. $\boxed{QED}$

112 Corollary. *Let H, K be subgroups of a locally nilpotent group G, then $[I_G(H), I_G(K)] \leq I_G([H, K])$.*

Remarks. Let H be a subgroup of a locally nilpotent group G. Observe that the Corollary implies that $I_G(N_G(H)) \leq N_G(I_G(H))$; in particular, the normalizer of an isolated subgroup is also isolated. Another immediate consequence is that if H is subnormal of defect d, then $I_G(H)$ is subnormal of defect at most d.

We now move to torsion-free groups, for which the results are stronger.

113 Lemma. *Let G be a locally nilpotent, torsion-free group, and let $H \leq G$. Then, for every ordinal α,*

$$\zeta_\alpha(I_G(H)) = I_G(\zeta_\alpha(H))$$

Proof. We make induction on α. If $\alpha = 0$, then the equality reduces to $1 = I_G(1)$ which is satisfied since G is torsion-free. Assume now that $\alpha = \beta+1$ for some ordinal β, and let $K = \zeta_\beta(H)$. Let $x \in I_G(\zeta_\alpha(H))$, and let $g \in I_G(H)$. Then there exists $1 \leq m \in \mathbb{N}$ such that $[g^m, x^m] \in K$. By Lemma 101, it follows that $[g, x] \in I_G(K)$, and this holds for all $g \in I_G(H)$. Now, by inductive assumption,

$I_G(K) = \zeta_\beta(I_G(H))$, and so $x \in \zeta_\alpha(I_G(H))$. Conversely, let $y \in \zeta_\alpha(I_G(H))$. Then $y^n \in H$ for some $1 \leq n \in \mathbb{N}$. Hence,

$$[H, y^n] \leq [I_G(H), y^n] \cap H \leq \zeta_\beta(I_G(H)) \cap H = I_G(K) \cap H = I_H(K).$$

Now, by Lemma 103, $I_H(K) = K$. Thus, $y^n \in \zeta_\alpha(H)$ and so $y \in I_G(\zeta_\alpha(H))$.

Suppose now that α is a limit ordinal, i.e. $\alpha = \bigcup_{\beta<\alpha} \beta$. Then, by definition,

$$\zeta_\alpha(I_G(H)) = \bigcup_{\beta<\alpha} \zeta_\beta(I_G(H)) = \bigcup_{\beta<\alpha} I_G(\zeta_\beta(H)) = I_G(\bigcup_{\beta<\alpha} \zeta_\beta(H)) = I_G(\zeta_\alpha(H)),$$

thus completing the proof. $\boxed{QED}$

114 Corollary. *Let G be locally nilpotent torsion-free group. If G has a subgroup H, with $I_G(H) = G$, and which is nilpotent (soluble, hypercentral) of class c (of derived length d, of length α), then G is nilpotent of class c (soluble of derived length d, hypercentral of length α).*

Proof. Since being G torsion-free is equivalent to $I_G(1) = 1$, the assertions for the three cases follow, respectively, from 109 (2) (or 113), 109 (3), and 113. $\boxed{QED}$

115 Lemma. *Let G be a locally nilpotent group G which admits a nilpotent subgroup H of finite index. If $T(H)$ has finite exponent, then G is nilpotent.*

Proof. By replacing H with its normal core, we may assume that H is normal. $T(H)$ is nilpotent of finite exponent, and it admits a characteristic finite series all of whose factors are central and elementary abelian for a finite number of primes. If U/V is a factor of this series which is a p-group, then $H \geq C_G(U/V)$ so $G/C_G(U/V)$ is finite and therefore a p-group. By Corollary 15, U/V is contained in some term $\zeta_m(G/V)$ ($m \in \mathbb{N}$) of the upper central series of G/V. By repeated application, this shows that $T(H) \leq \zeta_n(G)$ for some $n \in \mathbb{N}$. Now, $T(G)/T(H) \simeq T(G)H/H$ is a finite normal section of G and so $T(G) \leq \zeta_k(G)$ for some $k \in \mathbb{N}$. Finally, Corollary 114 ensures that $G/T(G)$ is nilpotent, thus proving that G is nilpotent. $\boxed{QED}$

A Lemma of Möhres. Möhres Lemma is a simple but very useful application of the concept of isolators in torsion–free groups.

116 Lemma. [Möhres [78]] *Let G be a locally nilpotent, countable group, F a finitely generated subgroup of G, and M a finite subset of G with $F \cap M = \emptyset$. Then there exists a subgroup H of G such that $I_G(H) = G$, $F \leq H$, and $H \cap M = \emptyset$.*

Proof. Let $G = \{x_i \mid i \in \mathbb{N}\}$. Suppose that for $n \in \mathbb{N}$ we are given positive integers $m_0, m_1, \ldots, m_n$ such that

$$\langle F, x_0^{m_0}, \ldots, x_n^{m_n} \rangle \cap M = \emptyset.$$

Let $H_n = \langle F, x_0^{m_0}, \ldots, x_n^{m_n} \rangle$, and $K = \langle H_n, x_{n+1} \rangle$. Then K is finitely generated and so polycyclic. By Mal'cev Theorem 42, H_n is the intersection of all subgroups of K of finite index containing it. Since M is finite, it follows that there exists a subgroup W of finite index in K which contains H_n, and such that $W \cap M = \emptyset$. Thus, there is a $0 \neq m_{n+1} \in \mathbb{N}$ such that $x_{n+1}^{m_{n+1}} \in W$. Setting $H_{n+1} = \langle H_n, x_{n+1}^{m_{n+1}} \rangle$, we have $F \leq H_{n+1}$, and $H_{n+1} \cap M = \emptyset$. We now put

$$H = \bigcup_{i \in \mathbb{N}} H_i = \langle F, x_i^{m_i} \mid i \in \mathbb{N} \rangle.$$

Then $F \leq H$, $I_G(H) = G$, and $H \cap M = \emptyset$. $\boxed{QED}$

2.3 Torsion–free $\mathcal{N}_1$-groups

In this section we show that a torsion-free group with all subgroups subnormal is nilpotent. In [78], W. Möhres proved that such a group is soluble and hypercentral, and later H. Smith [108] was able to establish nilpotency, Here, we will follow the proof given in [15], which in turn makes a heavy use of Möhres's ideas. Let us begin with a general observation.

117 Lemma. *Let H be a torsion-free nilpotent group of class c, and assume that H/H' can be generated by r elements. Then the Hirsch length of H is bounded by $r + r^2 + \ldots + r^c$.*

Proof. Let $A = H/H'$. Then, for every $1 \leq i \leq c$, there is an epimorphism:

$$\underbrace{A \otimes A \otimes \cdots \otimes A}_{i \text{ times}} \longrightarrow \gamma_i(H)/\gamma_{i+1}(H)$$

(Theorem 46). Now, A is a r-generated abelian group, and so it has Hirsch length at most r. Similarly, for each $i \geq 1$, the i-th tensor power $A \otimes \cdots \otimes A$ has Hirsch length at most r^i. Hence, for each $1 \leq i \leq c$, $\gamma_i(H)/\gamma_{i+1}(H)$ has Hirsch length at most r^i. Since $\gamma_{c+1}(H) = 1$, it plainly follows that H has Hirsch length at most $r + r^2 + \ldots + r^c$. $\boxed{QED}$

We first deal with groups with all subgroups subnormal of bounded defect. Thus, for each $1 \leq n \in \mathbb{N}$, let us denote by $\mathfrak{U}_n$ the class of groups in which every subgroup is subnormal of defect at most n. It is clear that every $\mathfrak{U}_n$ group is locally nilpotent and $(n+1)$-Engel. We observe that a torsion-free group G

in $\mathfrak{U}_n$ is in fact a n-Engel group. Let $x \in G$ and $Y = \langle x \rangle^{G,n-1}$; then $\langle x \rangle \trianglelefteq Y$. Since Y is torsion-free and locally nilpotent, it follows from Lemma 105 that $x \in Z(Y)$; in particular $[g,_n x] = [g,_{n-1} x, x] \in [Y, x] = 1$ for all $g \in G$.

The next Proposition is a special case of Roseblade's Theorem (see section 4.2), and, of course of Zel'manov theorem 67.

118 Proposition. *There exists a function $\rho_0 : \mathbb{N} \to \mathbb{N}$, such that a torsion-free group in which every subgroup is subnormal of defect at most n, is nilpotent of nilpotency class not exceeding $\rho_0(n)$.*

Proof. We will define by recursion on n a value $\rho_0(n)$, such that, if G is a torsion-free $\mathfrak{U}_n$-group, then $\gamma_{\rho_0(n)+1}(G) = 1$.

A $\mathfrak{U}_1$-group is a group in which every subgroup is normal, and it is well known since Dedekind that a torsion-free such group is abelian. Thus $\rho_0(1) = 1$.

Let $n \geq 1$, and assume we have defined $\rho_0(i)$ for $1 \leq i \leq n-1$. Let G be a torsion-free $\mathfrak{U}_n$-group. Then, for each $H \leq G$, we have a series

$$H = H^{G,n} \trianglelefteq H^{G,n-1} \trianglelefteq \ldots \trianglelefteq H^{G,1} = H^G \trianglelefteq G.$$

Now, if $H \leq K \leq H^G$, then clearly $K^G = H^G$. It follows that $H^{G,1}/H^{G,2}$ belongs to $\mathfrak{U}_{n-1}$. Similarly, we have, for all $i = 1, \ldots, n-1$,

$$\frac{H^{G,i}}{H^{G,i+1}} \in \mathfrak{U}_{n-i}.$$

For $1 \leq i \leq n-1$, we put $H_{i+1} = I_{H^{G,i}}(H^{G,i+1})$. Then $H_{i+1} \trianglelefteq H^{G,i}$, and $H^{G,i}/H_{i+1}$ is a torsion-free $\mathfrak{U}_{n-i}$-group. By inductive assumption, $H^{G,i}/H_{i+1}$ is nilpotent of class at most $\rho_0(n-i)$ and so it is solvable of derived length at most $[\log_2(\rho_0(n-i))] + 1$. Let $c(n) = \sum_{i=1}^{n-1} ([\log_2(\rho_0(i))] + 1)$. then

$$(H^G)^{(c(n))} \leq I_G(H)$$

and this holds for every $H \leq G$. Write $M = (H^G)^{(c(n))}$. Then from $M \leq I_G(H)$, we clearly get $I_{H^G}(M) \leq I_G(H)$. Now, $H^G/I_{H^G}(M)$ is a soluble torsion-free n-Engel group, hence by Corollary 70, it is nilpotent of class at most

$$\alpha(n) = n^{c(n)}.$$

Thus $\gamma_{\alpha(n)+1}(H^G) \leq I_{H^G}(M) \leq I_G(H)$, and this holds for every $H \leq G$. In particular, for all $x \in G$, $\langle x \rangle^G$ is nilpotent of class at most $\alpha(n)$.

Now, let $x_1, x_2, \ldots, x_{\alpha(n)}$ be elements of G, and let $H = \langle x_1, x_2, \ldots, x_{\alpha(n)} \rangle$. Then, by Fitting Theorem, H^G is nilpotent of class at most $\alpha(n)^2$. In particular, H has nilpotency class at most $\alpha(n)^2$. Since H is generated by $\alpha(n)$ elements, it follows from Lemma 117 that its Hirsch length is bounded by

$$g(n) = \alpha(n) + \alpha(n)^2 + \ldots + \alpha(n)^{\alpha(n)^2} \leq \alpha(n)^{\alpha(n)^2+1}.$$

Hence, $\gamma_{\alpha(n)+1}(H^G)$ has Hirsch length at most $\alpha(n)^{\alpha(n)^2+1}$, and so, by Corollary 106,

$$\gamma_{\alpha(n)+1}(H^G) \leq \zeta_{\alpha(n)^{\alpha(n)^2+1}}(G).$$

This yields that G is nilpotent of class at most $\alpha(n) + \alpha(n)^{\alpha(n)^2+1}$. $\boxed{QED}$

The exact values of $\rho_0(n)$ (in the torsion–free case) are known only for $n \leq 4$, and in these cases we have $\rho_0(n) = n$. For $n = 2$ this follows from Levi's results on 2-Engel groups, while for $n = 3, 4$ it has been established, respectively, by Traustason [118] and Smith and Traustason [114] (see also Section 4.2).

1 Question. [see [114]] Is the nilpotency class of every torsion–free group with all subgroups n-subnormal bounded by n?

We now drop the assumption of bounded defects.

119 Proposition. [Möhres [78]] *Let G be a non-nilpotent torsion-free $\mathcal{N}_1$-group. Then there exist a $n \in \mathbb{N}$, a non-nilpotent subgroup H of G and a finitely generated subgroup F of H, such that all subgroups U with $F \leq U \leq H$ have defect at most n in H. If G is countable, then H can be taken such that $I_G(H) = G$.*

Proof. Without loss of generality, we may assume that G is a countable counterexample. Set $H_0 = 1$, and suppose that, for a $1 \leq n \in \mathbb{N}$, we have found a finitely generated subgroup H_{n-1} of G, and elements $x_1, \ldots, x_{n-1}$, such that $\{x_1, \ldots, x_{n-1}\} \cap H_{n-1} = \emptyset$. Then, by Lemma 116, there exists a $K_n \leq G$, with $I_G(K_n) = G$ and such that $\{x_1, \ldots, x_{n-1}\} \cap K_n = \emptyset$¿ Since G is a counterexample to the proposition, there exists a finitely generated subgroup H_n of K_n, containing H_{n-1}, that has defect at least $n+1$ in G. Hence, there exists a $x_n \in [G,_n H_n] \setminus H_n$. Then $\{x_1, \ldots, x_{n-1}, x_n\} \cap H_n = \emptyset$. Let now $H = \bigcup_{n\in\mathbb{N}} H_n$. H is subnormal in G of defect, say, d. Thus,

$$x_d \in [G,_d H_d] \leq [G,_d H] \leq H = \bigcup_{n\in\mathbb{N}} H_n,$$

whence $x_d \in H_j$ for some $j > d$, which contradicts the choice of H_j. $\boxed{QED}$

120 Lemma. [Möhres]. *A torsion-free group in which all subgroups are subnormal is soluble.*

Proof. Let G be a torsion-free $\mathcal{N}_1$-group. Since solubility is a countably recognizable property (see 33), we may assume that G is countable and not nilpotent. Then by Proposition 119 there exist a $n \in \mathbb{N}$, a non-nilpotent subgroup H of G and a finitely generated subgroup F of H, such that $I_G(H) = G$, and all subgroups U with $F \leq U \leq H$ have defect at most n in H. We now proceed by induction on n to prove that H is soluble. If $n = 1$, then $F \trianglelefteq G$ and H/F is Hamiltonian. Hence $|(G/F)'| \leq 2$, and, as F is nilpotent, we have in particular that H is soluble. Let now $n \geq 1$, and observe that if $F \leq U \leq F^H$, then $U^H = F^H$. Hence, all subgroups of F^H containing F have defect at most $n-1$ in F^H. By inductive assumption, F^H is solvable, and by Lemma 109, $N = I_H(F^H)$ is a normal soluble subgroup of H. Finally, H/N is solvable by Proposition 118, and so H is soluble. As $G = I_G(H)$, by Lemma 109 we conclude that G is soluble. $\boxed{QED}$

The next Lemma is indeed a key argument. Given a prime p, and positive integers k, n, we define

$$f_p(k,n) = (n+2)p^{[log_p k(n+2)]+1} .$$

121 Lemma. *Let $G = A\langle x\rangle$ be a nilpotent group, where $A \trianglelefteq G$ is an elementary abelian p-group. Assume also that there exists a subgroup F of A, and a $n \in \mathbb{N}$, such that $|F| = p^k$, and every subgroup H of G with $F \leq H$ is subnormal of defect at most n in G. Then $[A,_{f_p(k,n)-1}\, x] = 1$.*

Proof. Set $s = f_p(k,n)$, and $m = [log_p k(n+2)] + 1$. Then $p^m > k(n+2)$. Assume, by contradiction, that $[A,_{s-1}\, x] \neq 1$. By obvious induction we may then assume $[A,_s\, x] = 1$. Also, the subgroups

$$A,\ [A,x],\ [A,_2\, x],\ [A,_3\, x],\ \dots\ ,\ [A,_{s-1}\, x],\ [A,_s\, x] = 1$$

are all distinct. In particular, we have

$$|[A,_{(n+1)p^m}\, x]| \geq p^{s-(n+1)p^m} = p^{(n+2)p^m-(n+1)p^m} = p^{p^m} > p^{k(n+2)} = |F|^{n+2}.$$

Now, by Lemma 14,

$$[A,_{n+2}\, x^{p^m}] = [A,_{(n+2)p^m}\, x] = [A,_s\, x] = 1$$

whence $[F,_{n+2}\, x^{p^m}] = 1$. As $F^{\langle x^{p^m}\rangle}$ is generated by the subgroups $[F,_i\, x^{p^m}]$, it there follows that

$$|F^{\langle x^{p^m}\rangle}| \leq |F|^{n+2}.$$

Let now $H = \langle A, x^{p^m}\rangle$. Since A is normal abelian and $F^{\langle x^{p^m}\rangle} \leq A$, we have $F^H = F^{\langle x^{p^m}\rangle}$. Now, $H/F^H = (A/F^H)(\langle x^{p^m}\rangle F^H/F^H)$, where A/F^H is normal abelian, and $\langle x^{p^m}\rangle F^H/F^H$ is a cyclic subgroup of defect at most n. By Lemma 61, A/F^H is nilpotent of class at most $n+1$. In particular we have

$$[A,_{n+1}\, x^{p^m}] \leq F^H = F^{\langle x^{p^m}\rangle} .$$

Since, by Lemma 14, $[A,_{(n+1)p^m}\, x] = [A,_{n+1}\, x^{p^m}]$, we finally have

$$|[A,_{(n+1)p^m}\, x]| \leq |F^{\langle x^{p^m}\rangle}| \leq |F|^{n+2}$$

contradicting what we had obtained above. $\boxed{QED}$

122 Lemma. *Let G be a torsion free locally nilpotent group. Let A be an abelian normal subgroup of G such that G/A is abelian. Suppose that there exist a finitely generated subgroup F of A and $n \in \mathbb{N}$ such that all subgroups of G containing F are subnormal of defect at most n. Then G is nilpotent (and its nilpotency class is bounded by a function of $(n, \mathrm{rk}(F))$).*

Proof. Assume the hypothesis of the Lemma, and let k be the rank of F.

Let $x \in G$ and $X = F^{\langle x \rangle}$. Then $X \leq A$ because $F \leq A \trianglelefteq G$. Since G is locally nilpotent, $\langle F, x \rangle$ is nilpotent and X is a finitely generated torsion free abelian group. Let r be the rank of X. Now set $Y = X^2$. Then $Y \trianglelefteq \langle F, x \rangle$ and X/Y is an elementary abelian group of order 2^r. Let $\overline{F} = FY/Y$. Then $|\overline{F}| = 2^k$ and all subgroups of $\langle F, x \rangle / Y$ that contain $\overline{F}$ have defect at most n. Also, $\overline{X} = X/Y = \overline{F}^{\langle x \rangle}$. By Lemma 121, $[\overline{X},_s x] = 1$, where $s = f_2(k, n) - 1$. Let 2^h the smallest power of 2 larger than s. Then $[\overline{X}, x^{2^h}] = [\overline{X},_{2^h} x] = 1$, so $\overline{F}$ has at most 2^h conjugates in $\langle F, x \rangle / Y$. Since $\overline{X}$ is an abelian group generated by the conjugates of $\overline{F}$, we get $2^r = |X/Y| \leq |\overline{F}|^{2^h} = 2^{k2^h}$ and thus $r \leq k2^h$ (observe that h does not depend on x, but only on k and n).

We have then obtained that, for all $x \in G$, $F^{\langle x \rangle}$ is a torsion free abelian group of rank at most $u = k2^h$. Since $\langle F, x \rangle$ is torsion free and nilpotent, it follows that, for all $x \in G$,

$$[\langle F, x \rangle,_u x] = 1 \ .$$

Now, $F \leq \langle F, x \rangle$, so $\langle F, x \rangle$ is subnormal of defect at most n in G. Thus we have, for all $g, x \in G$

$$[g,_{n+u} x] = [[g,_n x],_u x] \in [\langle F, x \rangle,_u x] = 1.$$

Then G is a metabelian torsion free $(n+u)$-Engel group and so by Corollary 70, G is nilpotent of class at most $n + u$. $\boxed{QED}$

The following variant of Theorem 56 appears in W. Möhres doctoral dissertation.

123 Lemma. *Let N be a nilpotent normal subgroup of the locally nilpotent torsion-free group G. If $G/I_G(N')$ is nilpotent, then G is nilpotent.*

Proof. Observe that, by Lemma 109, $I_G(I_G(N)') \geq I_G(N') = I_G(I_G(N')) \geq I_G(I_G(N)')$. Thus, $I_G(I_G(N)') = I_G(N')$. Since $I_G(N)$ is nilpotent by Lemma 113, we may assume that $N = I_G(N)$.
We now proceed by induction on the nilpotency class c of N; the case $c = 1$ being just our assumption. Let $c \geq 2$, and let $K = I_G(\gamma_c(N))$. Then $K \trianglelefteq G$, and G/K is torsion-free. Moreover, $K \leq Z(N)$, and N/K has class at most $c - 1$. Thus, by inductive hypothesis, G/K is nilpotent. Let $K/K = K_0/K \leq K_1/K \leq \ldots \leq K_d/K = N/K$ be the intersection of the upper central series of G/K with

N/K; and for $s = 0, 1, \dots, 2d$ let $T_s = \langle [K_i, K_j] \mid 0 \le i, j \le d,\ i+j = s \rangle$. Now, if $1 \le i, j \le d$, by the three subgroup Lemma 5, we have

$$[K_i, K_j, G] \le [K_j, G, K_i][G, K_i, K_j] \le [K_{j-1}, K_i][K_{i-1}, K_j] \le T_{i+j-1},$$

showing that $[T_s, G] \le T_{s-1}$ for all $s \ge 1$. In other words, G centralizes the series $1 = [K, K] = T_0 \le T_1 \le \dots \le T_{2d} = N'$. By Lemma 109, G centralizes the series of the isolators $1 = I_G(\{1\}) \le I_G(T_1) \le \dots \le I_G(N')$. As $G/I_G(N')$ is nilpotent by assumption, it follows that G is nilpotent. $\boxed{QED}$

We are finally in a position to prove the main result.

124 Theorem. [H. Smith [108]]. *A torsion-free group in which all subgroups are subnormal is nilpotent.*

Proof. Let G be a torsion free group with all subgroups subnormal. By a Lemma 120, G is soluble. We argue by induction on the derived length d of G.

Suppose first that G is metabelian and, by contradiction, that G is not nilpotent. Then, by Proposition 119 we may assume that there exists a finitely generated subgroup F of G and a $n \in \mathbb{N}$ such that all subgroups of G containing F are subnormal of defect at most n. Let $H = FG'$ and $L = I_G(H')$. H is a normal subgroup of G, so L is normal, and G/L is torsion-free. Since G' is abelian and F is finitely generated and subnormal, H is nilpotent. Since G is not nilpotent, it follows from 123 that G/L is not nilpotent. So we may assume that H is abelian. By Lemma 122, G is nilpotent.

The general case is now an immediate application of Lemma 123. Let d be the derived length of G and let $N = G'$. By inductive hypothesis, N is nilpotent. By the metabelian case $G/I_G(N')$ is nilpotent, and so G is nilpotent by 123. $\boxed{QED}$

3 Groups of Heineken and Mohamed

In the literature two quite different methods for constructing non-nilpotent $\mathcal{N}_1$-groups are known. The first goes back to a celebrated 1968 paper by H. Heineken and I. J. Mohamed, and produces p-groups with trivial centre and no proper subgroup of finite index, while the second one was discovered by H. Smith in 1982, and gives rise to mixed groups that are hypercentral and residually finite. We describe Smith's constructions later in Chapter 6, when we will specifically deal with hypercentral $\mathcal{N}_1$-groups, while to the Haineken-Mohamed groups, which have been much more investigated, we devote the present Chapter.

3.1 General remarks

In their mentioned paper [47], H. Heineken and I. J. Mohamed provided the first examples of $\mathcal{N}_1$-groups with trivial centre. The groups they constructed are (locally finite) p-groups for a prime p, and the extension of an infinite elementary abelian group by a Prüfer group; furthermore, all their proper subgroups are *subnormal and nilpotent.*

Heineken and Mohamed construction was studied and extended by many authors (see e.g. [9], [40], [41], [73], [75]) and it became customary to call a group G of *Heineken-Mohamed type* if G is not nilpotent and all of its proper subgroups are nilpotent and subnormal. In particular, in [48] the same authors show that there exist $2^{\aleph_0}$ non-isomorphic groups sharing these properties, Bruno and Phillips [9] and Möhres [75] studied, respectively, the Schur multiplier and the automorphisms group of certain groups of Heineken-Mohamed type, and Hartley [41] showed that, for every $n \geq 1$, there exist p-groups of Heineken-Mohamed type G such that G' is an abelian group of exponent p^n. For some time all groups thus constructed were metabelian, and the question as to whether a soluble group G of Heineken-Mohamed type may have arbitrary derived length was eventually solved in the affermative by Menegazzo in [72]. In the same paper, Menegazzo gave a very general method for constructing groups of Heineken-Mohamed type, which was in turn inspired by Hartley approach ([40]), and which is the one that we will present here.

Before the actual construction, let us prove the following fact.

125 Proposition. [Heineken and Mohamed [47]] *Let p be a prime and let G be a p-group of Heineken-Mohamed type such that $G \neq G'$. Then*

(i) G is countable;

(ii) $G/G' \simeq C_{p^\infty}$ and $(G')^p \neq G' = \gamma_3(G)$;

(iii) for every $H \leq G$, $G'H = G$ implies $H = G$.

Conversely, if G is a non-nilpotent p-group with a normal nilpotent subgroup N of finite exponent such that $G/N \simeq C_{p^\infty}$ and $NH \neq G$ for every proper subgroup H of G, then G is a group of Heineken-Mohamed type.

Later we shall prove Möhres Theorem that every $\mathcal{N}_1$-group is soluble; thus the extra condition $G \neq G'$ in the statement of Proposition 125 is redundant, and all groups of Heineken-Mohamed type have the properties listed.

For further reference, we isolate part of the proof of 125 in a separate and elementary Lemma.

126 Lemma. [Newman and Wiegold [86]] *Let G be a non-trivial group such that $UV \neq G$ for all pairs of proper normal subgroups U and V. Then there exists* *a prime number p such that G/G' is either a cyclic p-group (possibly trivial) or $G/G' \simeq C_{p^\infty}$ and $G' = \gamma_3(G)$.*

Proof. Let first assume that G is abelian. Let $1 \neq x \in G$; then there exists a prime p such that $\langle x^p \rangle \neq \langle x \rangle$. Let U be a subgroup of G maximal such that $x^p \in U$ but $x \notin U$ (it exists by Zorn's Lemma). Then all subgroups of G/U contain xU. Since G/U is abelian, we have that G/U is either a non-trivial cyclic p-group or isomorphic to C_{p^∞}. If G is not a p-group there exists a $y \in G$ and a prime $q \neq p$ such that $\langle y^q \rangle \neq \langle y \rangle$. Arguing as before, we then get a proper subgroup V of G such that G/V is a q-group. But then, clearly, $G = UV$, contradicting the assumptions on G. Thus, G is a p-group, and from this it easily follows that G is either cyclic or of type C_{p^∞}. Now for the general case we are left to show that $G' = \gamma_3(G)$. But this is immediate, since $H = G/\gamma_3(G)$ is a nilpotent group and $H/H' \simeq G/G'$ is cyclic or a Prüfer group, and so H is abelian. $\boxed{QED}$

Proof of Proposition 125. Since, by definition, G is not nilpotent but all of its proper subgroups are nilpotent, G must be countable by Theorem 33. By assumption, $G' \neq G$ and so G' is nilpotent. It thus follows from Lemma 77 that G/G' is not finitely generated. Also, by Fitting's Theorem, G cannot be the product of two proper normal subgroups and therefore $G/G' \simeq C_{p^\infty}$ by Lemma 126. Finally, suppose that $(G')^p = G'$. Then G' is an abelian divisible group by Lemma 18. Now, every cyclic subgroup X of G is subnormal and so G' is centralized by X by Lemma 34. It follows that $G' \leq \zeta(G)$, which contradicts the non-nilpotence of G. Hence $(G')^p \neq G'$.

For the converse, suppose that the non-nilpotent p-group G satisfies the conditions of the second part of the statement, and let H be a proper subgroup of G. Then $NH \neq G$ and so, since $G/N \simeq C_{p^\infty}$, NH/N is finite. Thus, NH is nilpotent by Corollary 79. Therefore H is nilpotent and subnormal in NH. Since NH is normal in G, it follows that H is subnormal in G. Hence G is a group of Heineken-Mohamed type. $\boxed{QED}$

3.2 Basic construction

As mentioned before, our approach follows closely Menegazzo [72].

For the rest of this section, we fix a prime p and denote by U the Prüfer group C_{p^∞}, which we take with a fixed set of standard generators $u_1, u_2, u_3, \ldots$:

$$U = \langle u_1, u_2, \ldots \mid u_1^p = 1,\ u_{i+1}^p = u_i \ (i \geq 1) \rangle.$$

For each $i \geq 1$, we write $U_i = \langle u_i \rangle$. Also, we denote by $R = \mathbb{F}_p[U]$ the group algebra of U over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and by $\mathfrak{U}$ its augmentation ideal. This means that $\mathfrak{U}$ is the kernel of the (ring) epimorphism $\epsilon : R \to \mathbb{F}_p$ defined by

$$\epsilon\Big(\sum_{u\in U} a_u u\Big) = \sum_{u\in U} a_u.$$

Then, $\mathfrak{U}$ is the ideal of R generated by all the elements of type $u-1$ for $u \in U$. Similarly, for each $i \geq 1$, we put $R_i = \mathbb{F}_p[U_i]$ and let $\mathfrak{U}_i$ denote the augmentation ideal of R_i. Then, clearly, $R = \bigcup_{i\geq 1} R_i$, $\mathfrak{U} = \bigcup_{i\geq 1} \mathfrak{U}_i$, and

$$\mathfrak{U}_i = (u_i - 1)R_i$$

for every $i \geq 1$. Moreover $(u_i-1)^{p^i} = u_i^{p^i} - 1 = 0$; hence all elements of $\mathfrak{U}$ are nilpotent and therefore, by elementary ring theory, all elements of $R \setminus \mathfrak{U}$ are invertible. Our first Lemma is standard and not difficult to prove.

127 Lemma. *The ideals of R_i are exactly the principal ideals*

$$(u_i - 1)^k R_i \quad \textit{for} \quad 0 \leq k \leq p^i.$$

These are all distinct and form a totally ordered set with respect to inclusion.

An immediate consequence is

128 Lemma. *The set of ideals of R is totally ordered.*

Proof. It is enough to show that, for all $u, v \in R$, if u does not belong to vR then v belongs to uR. Now, given $u, v \in R$, there clearly exists $i \geq 1$ such that $u, v \in R_i$. But then, by Lemma 127, either $uR_i \leq vR_i$ or $vR_i \leq uR_i$. Thus, the Lemma is proved. $\boxed{QED}$

We observe a consequence of this, which will be used in the next section.

129 Corollary. *Let M be a (right) R-module and $y \in M$, $r \in R$ with $0 \neq x = yr$. Then* $\mathrm{Ann}_R(y) = r\,\mathrm{Ann}_R(x)$.

Proof. Clearly, $r\,\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(y)$. Since $0 \neq x = y(r1)$, $\mathrm{Ann}_R(y) \not\subseteq rR$, and so (by Lemma 128) $\mathrm{Ann}_R(y) \subseteq rR$. From this the claim easily follows. $\boxed{QED}$

Lemma 127 suggests also a convenient way to parametrize the set of all ideals of R. In fact, let $\mathfrak{I}$ be an ideal of R; then, for each $i \geq 1$, there is a unique $0 \leq k_i \leq p^i$ such that

$$\mathfrak{I} \cap R_i = (u_i - 1)^{k_i} R_i.$$

We thus associate to $\mathfrak{I}$ the sequence $(k_1, k_2, \ldots)$. Since $R = \bigcup_{i\geq 1} R_i$, this sequence uniquely determines $\mathfrak{I}$. Observe also that, since $(\mathfrak{I} \cap R_{i+1}) \cap R_i = \mathfrak{I} \cap R_i$, the sequence is such that, for every $i \geq 1$,

$$p(k_i - 1) < k_{i+1} \leq pk_i. \tag{4}$$

Conversely, a sequence $(k_1, k_2, \ldots)$ of integers $0 \leq k_i \leq p^i$ satisfying (4) is the sequence associated to the ideal $\sum_{i\geq 1}(u_i - 1)^{k_i} R$ of R.

130 Lemma. *Let $\mathfrak{I}$ be a non-zero ideal of R, and let $(k_1, k_2, \ldots)$ be the sequence associated to $\mathfrak{I}$. Then $\mathfrak{I}\mathfrak{U} = \mathfrak{I}$ if and only if for every $i \geq 1$ there exists $j \geq i$ such that $pk_j > k_{j+1}$.*

Proof. Suppose that the sequence for $\mathfrak{I}$ satisfies the condition in the statement and let $i \geq 1$. Choose $j \geq i$ such that $pk_j > k_{j+1}$. Then, for some $t > 0$,

$$(u_j - 1)^{k_j} = (u_{j+1} - 1)^{pk_j} = (u_{j+1} - 1)^{k_{j+1}}(u_{j+1} - 1)^t.$$

This implies $(u_j - 1)^{k_j} \in \mathfrak{I}\mathfrak{U}$ and, consequently, $(u_i - 1)^{k_i} \in \mathfrak{I}\mathfrak{U}$. Therefore $\mathfrak{I}\mathfrak{U}$ has the same sequence as $\mathfrak{I}$ and so $\mathfrak{I}\mathfrak{U} = \mathfrak{I}$.

Conversely, assume $\mathfrak{I}\mathfrak{U} = \mathfrak{I}$, and let $i \geq 1$ with $(u_i - 1)^{k_i} \neq 0$. Then it is easy to see that there exists $t \geq i$ such that $(u_i - 1)^{k_i} \in (\mathfrak{I} \cap R_t)\mathfrak{U}_t$. Since $\mathfrak{I} \cap R_t = (u_t - 1)^{k_t} R_t$ and $\mathfrak{U}_t = (u_t - 1)R_t$, we have that $(u_i - 1)^{k_i} = (u_t - 1)^{k_t+1} v$ for some $v \in R_t$. Hence $(u_i - 1)^{k_i} R < (u_t - 1)^{k_t} R$, and so in the chain

$$(u_i - 1)^{k_i} R \leq (u_{i+1} - 1)^{k_{i+1}} R \leq \ldots \leq (u_t - 1)^{k_t} R$$

at least one of the inclusions is proper, say $(u_j - 1)^{k_j} R < (u_{j+1} - 1)^{k_{j+1}} R$, which means $j_{j+1} < pk_j$. $\boxed{QED}$

We now come to the key definition of an HM-system. Let V be a right R-module which is generated by a sequence of elements $\mathbf{a} = (a_i)_{i \geq \ell}$ (for some positive integer ℓ). For any sequence $\mathbf{v} = (v_i)_{i \geq \ell}$ of elements of V, we set

$$\tau_{i,k}(\mathbf{v}) = -v_i + \sum_{s=0}^{k} a_{i+s}(u_{i+s} - 1)^{p^s - 1} + v_{i+k+1}(u_{i+k+1} - 1)^{p^{k+1} - 1}$$

for all $i \geq \ell$, $k \geq 0$. We then say that $\mathbf{a}$ is a *HM-system* in V if

$$V = \langle \tau_{i,k}(\mathbf{v}) \mid i \geq \ell,\ k \geq 0 \rangle$$

for every sequence $\mathbf{v} = (v_i)_{i \geq \ell}$.

131 Proposition. [Menegazzo [72]] *Let G be a p-group with a normal elementary abelian subgroup $N \neq 1$ such that $[G, N] = N$ and $G/N \simeq C_{p^\infty} = U$. Let $\eta : U \to G/N$ be an isomorphism, and make N into a R-module in the obvious way. For each $i \geq 1$, let $g_i \in R_i$ such that $g_i N = u_i^\eta$, and let $a_i = g_i^{-1} g_{i+1}^p$ (thus $a_i \in N$). Suppose further that $G = \langle g_i \mid i \geq \ell \rangle$ for some $\ell \geq 1$. If the sequence $\boldsymbol{a} = (a_i)_{i \geq \ell}$ is a HM-sequence for N then G is a group of Heineken-Mohamed type.*

Proof. Since $[G, N] = N \neq 1$, G is not nilpotent. Hence, by proposition 125 it suffices to show that $HN = G$ forces $H = G$ for every $H \leq G$. For $n \in \mathbb{N}$ and $u \in U$ we write $n^u = n^{(u^\eta)}$, and for all $i \geq \ell$, $k \geq 0$, we set

$$\sigma_{i,k} = \prod_{s=0}^{k} a_{i+s}^{(u_{i+s} - 1)^{p^s - 1}}.$$

We show, by induction on $k \geq 0$, that $g_{i+k+1}^{p^{k+1}} = g_i\sigma_{i,k}$ for all $i \geq \ell$. For $k = 0$ this is trivial since $\sigma_{i,0} = a_i$. Thus, let $k \geq 1$ and assume $g_{i+k}^{p^k} = g_i\sigma_{i,k-1}$. Then

$$\begin{aligned} g_{i+k+1}^{p^{k+1}} = (g_{i+k+1}^p)^{p^k} = (g_{i+k}a_{i+k})^{p^k} &= g_{i+k}^{p^k} a_{i+k}^{u_{i+k}^{p^k-1}+\ldots+u_{i+k}+1} = \\ &= g_i\sigma_{i,k-1}a_{i+k}^{(u_{i+k}-1)^{p^k-1}} = g_i\sigma_{i,k}. \end{aligned}$$

Now, let $H \leq G$ with $NH = G$. Then, for every $i \geq \ell$, H contains an element of the form g_iv_i with $v_i \in N$. Let $\mathbf{v}$ be the sequence $(v_i)_{i\geq\ell}$. For every $i \geq \ell$, $k \geq 0$, writing $\tau_{i,k} = \tau_{i,k}(\mathbf{v})$, and using the identities established above, we have

$$\begin{aligned} (g_{i+k+1}v_{i+k+1})^{p^{k+1}} &= g_{i+k+1}^{p^{k+1}} v_{i+k+1}^{(u_{i+k+1}-1)^{p^{k+1}-1}} = g_i\sigma_{i,k}v_{i+k+1}^{(u_{i+k+1}-1)^{p^{k+1}-1}} = \\ &= g_iv_i\Big(v_i^{-1}\sigma_{i,k}v_{i+k+1}^{(u_{i+k+1}-1)^{p^{k+1}-1}}\Big) = g_iv_i\tau_{i,k}. \end{aligned}$$

Hence, $\tau_{i,k} \in H$ for every $i \geq \ell$ and $k \geq 0$, and thus H contains the subgroup generated by the elements $\tau_{i,k}$, which is N, since $\mathbf{a}$ is a HM-system. Therefore $H \geq NH = G$, and so $H = G$ as wanted. $\boxed{QED}$

Our next task is then to find R-modules admitting HN-systems. We do that with the aid of Lemmas 128 and 130.

132 Proposition. *Let $\mathfrak{I}$ be a non-zero ideal of R such that $\mathfrak{I} = \mathfrak{I}\mathfrak{U} < \mathfrak{U}$, and let $(k_1, k_2, \ldots)$ be the sequence associated to $\mathfrak{I}$. Fix $\ell \geq 1$ with $0 < k_\ell < p^\ell$, and for each $i \geq \ell$ set*

$$c_i = \begin{cases} (u_i-1)^{k_i} & \text{if } k_{i+1} = pk_i \\ (u_{i+1}-1)^{pk_i-1} & \text{if } k_{i+1} < pk_i. \end{cases}$$

Then $\mathbf{c} = (c_i)_{i\geq\ell}$ is a HM-system for $\mathfrak{I}$ as a R-module.

Proof. We first make sure that $\mathbf{c}$ is a generating set for $\mathfrak{I}$. Thus, let $\mathfrak{J}$ be the ideal (i.e. R-submodule) generated by $\mathbf{c}$. Then $\mathfrak{J} \leq \mathfrak{I}$: in fact $c_i \in \mathfrak{I}$ by definition if $k_{i+1} = pk_i$, and, if $k_{i+1} < pk_i$, $c_i = (u_{i+1}-1)^{pk_i-1} \in (u_{i+1}-1)^{k_{i+1}}R \leq \mathfrak{I}$. For the reverse inclusion, consider first $i \geq \ell$. If $k_{i+1} = pk_i$ then $R_i \cap \mathfrak{I} = c_iR_i \leq \mathfrak{J}$; if $k_{i+1} < pk_i$,

$$R_i \cap \mathfrak{I} = (u_i-1)^{k_i}R_i = (u_{i+1}-1)^{pk_i}R_i = c_i(u_{i+1}-1)R_i \leq \mathfrak{J}.$$

If $1 \leq i < \ell$, then $(u_i-1)^{k_i} \in (u_\ell-1)^{k_\ell}R \leq \mathfrak{J}$. Hence $\mathfrak{J} = \mathfrak{I}$.

We now prove that $\mathbf{c}$ satisfies the requirements of a HM-system for $\mathfrak{I}$ as a R-module. Let $\mathbf{v} = (v_i)_{i\geq\ell}$ be a sequence of elements of $\mathfrak{I}$, and for every $i \geq \ell$, $k \geq 0$, write $\tau_{i,k} = \tau_{i,k}(\mathbf{v})$. We prove that for every $i > \ell$ there exists $k \geq 0$ such that

$$(u_{i+1}-1)^{k_{i-1}} \in \tau_{i,k}R. \tag{5}$$

This of course will imply that $\mathfrak{I}$ is generated by the set $\{\tau_{i,k} \mid i \geq \ell,\ k \geq 0\}$. therefore assuring that $\mathbf{c}$ is a HM-system for $\mathfrak{I}$.
Thus, let $i \geq \ell$. If $k_i = pk_{i-1}$ then, by Lemma 130, there is a $j \geq i$ such that$(u_{i-1}-1)^{k_{i-1}} = (u_{j-1}-1)^{k_{j-1}}$ and $k_j < pk_{j-1}$. Hence we may assume $k_i < pk_{i-1}$. Now, there exists $h > 0$ such that $v_i \in \mathfrak{I} \cap R_{i+h}$, and there exists $k \geq h$ such that $k_{i+k+1} < pk_{i+k}$. Then $c_{i+k} = (u_{i+k+1}-1)^{pk_{i+k}-1}$, and

$$\tau_{i,k} = -v_i + c_i + \ldots + c_{i+k+1}(u_{i+k+1}-1)^{p^{k-1}-1} + w \tag{6}$$

where $w = c_{i+k}(u_{i+k}-1)^{p^k-1} + v_{i+k+1}(u_{i+k+1}-1)^{p^{k-1}-1}$. We then have

$$\begin{aligned} w &= (u_{i+k+1}-1)^{pk_{i+k}-1}(u_{i+k}-1)^{p^k-1} + v_{i+k+1}(u_{i+k+1}-1)^{p^{k+1}-1} = \\ &= (u_{i+k+1}-1)^{pk_{i+k}-1+p^{k+1}-p} + v_{i+k+1}(u_{i+k+1}-1)^{p^{k+1}-1} = \\ &= (u_{i+k+1}-1)^{p^{k+1}-1}\big((u_{i+k+1}-1)^{p(k_{i+k}-1)} + v_{i+k+1}\big) = \\ &= (u_{i+k+1}-1)^{p^{k+1}-1}\big((u_{i+k}-1)^{k_{i+k}-1} + v_{i+k+1}\big). \end{aligned}$$

Now, $v_{i+k+1} \in \mathfrak{I}$ and $(u_{i+k}-1)^{k_{i+k}-1} \notin \mathfrak{I}$, and so it follows from Lemma 128 that $(u_{i+k}-1)^{k_{i+k}-1}$ and $(u_{i+k}-1)^{k_{i+k}-1} + v_{i+k+1}$ generate the same ideal of R. Therefore, there exists an invertible element $\epsilon \in R$ such that

$$(u_{i+k}-1)^{k_{i+k}-1} + v_{i+k+1} = (u_{i+k}-1)^{k_{i+k}-1}\epsilon.$$

Thus, $w = (u_{i+k+1}-1)^{p^{k+1}-1+p(k_{i+k}-1)}\epsilon$. All other summands in the right term of (6) belong to $\mathfrak{I} \cap R_{i+k}$; hence, denoting by w' their sum, we have $w' = (u_{i+k}-1)^m\eta = (U_{i+k+1}-1)^{pm}\eta$ for some $m \geq n_{i+k}$ and some invertible element η of R_{i+k}. By observing that the exponents of $u_{i+k+1}-1$ in w and in w' are not congruent modulo p, we deduce that the ideals $w'R$ and wR are distinct. Therefore, $\tau_{i,k} = w' + w$ generates the largest of the two ideals $w'R$ and wR. In particular,

$$(u_{i+k+1}-1)^{p^{k+1}-1+p(k_{i+k}-1)} = w\epsilon^{-1} \in \tau_{i,k}R. \tag{7}$$

Now, taking into account that $pk_{i-1} \geq k_i + 1$, we have

$$p^{k+2}k_{i-1} \geq p^{k+1}(k_i+1) \geq pk_{i+k} + p^{k+1} > p^{k+1} - 1 + p(k_{i+k}-1),$$

and therefore, by (7), $(u_{i-1}-1)^{k_{i-1}} = (u_{i+k+1}-1)^{p^{k+2}k_{i-1}}$ belongs to $\tau_{i,k}R$. This proves (5) and the Proposition. $\boxed{QED}$

We can now proceed to the construction of Heineken-Mohamed groups.

133 Theorem. [Menegazzo [72]] *To every non-zero ideal $\mathfrak{I}$ of R such that $\mathfrak{I} = \mathfrak{I}\mathfrak{U} < \mathfrak{U}$ there corresponds a group of Heineken-Mohamed type $G = G(\mathfrak{I})$ such that $G/G' \simeq U$ and $G' \simeq \mathfrak{I}$ (as R-modules). Moreover, if $\mathfrak{J}$ is another ideal of R with $\mathfrak{J} = \mathfrak{J}\mathfrak{U} < \mathfrak{U}$ and $\mathfrak{I} \neq \mathfrak{J}$, then $G(\mathfrak{I})$ and $G(\mathfrak{J})$ are not isomorphic.*

Proof. Let $\mathfrak{I}$ be as in the statement and let $(k_1, k_2, \ldots)$ be the associated sequence. Choose $\ell \geq 1$ such that $1 < k_\ell < p^\ell$ and for every $i \geq \ell$ define the element c_i as in Proposition 132. We will inductively define a sequence $(a_i)_{i \geq \ell}$ of elements of R satisfying the following conditions:

$$a_i \in (u_i - 1)R \quad \text{and} \quad a_{i+1}(u_{i+1} - 1)^{p-1} = a_i + c_i \tag{8}$$

for every $i \geq \ell$. Set $a_\ell = 0$, and assume that, for $i \geq \ell$, we have found $a_\ell, \ldots, a_i$ with the desired properties. Now, $k_i \geq k_\ell > 1$ and c_i is either $(u_i - 1)^{k_i}$ or $(u_{i+1} - 1)^{pk_I - 1}$; in any case $c_i \in (u_i - 1)R$ and so there exists $b \in R$ such that $c_i + a_i = (u_i - 1)b = (u_{i+1} - 1)^p b$. By setting $a_{i+1} = (u_{i+1} - 1)b$ we get a new element in the sequence that satisfies (8).

Consider now the semidirect product $W = R \rtimes U$, where R is meant to be the additive group of the ring (thus the multiplication in W is given by $(r, u)(r', u') = (ru' + r', uu')$), and for every $i \geq \ell$, let $g_i = (a_i, u_i)$. Let $G = G(\mathfrak{I})$ be the subgroup of W generated by all the g_i's:

$$G = \langle\ (a_i, u_i) \in W \mid i \geq \ell\ \rangle.$$

Then, for every $i \geq \ell$,

$$g_{i+1}^p = (a_{i+1}(u_{i+1} - 1)^{p-1}, u_{i+1}^p) = (a_i + c_i, u_i) = g_i(c_i, 1),$$

and therefore $G \cap (R \times 1)$ contains the U-invariant subgroup N generated by the set $\{(c_i, 1) \mid i \geq \ell\}$, which, as a U-module, is isomorphic to $\mathfrak{I}$. Clearly $G/N = \langle g_i N \mid i \geq \ell \rangle \simeq U$; moreover, since $\mathfrak{I}\mathfrak{U} = \mathfrak{I}$, we have $N = [N, U] = [N, G]$. Finally, the sequence $(g_i^{-1} g_{i+1}^p)_{i \geq \ell} = ((c_i, 1))_{i \geq \ell}$ is a HM-system for $N \simeq_U \mathfrak{I}$, and so we may apply Proposition 131 to conclude that G is a group of Heineken-Mohamed type.

Now, for the second part of the statement, let $\mathfrak{J}$ be another ideal of R with $\mathfrak{J} = \mathfrak{J}\mathfrak{U} < \mathfrak{U}$, write $G_1 = G(\mathfrak{I})$, $G_2 = G(\mathfrak{J})$, and assume that there is a group isomorphism $\alpha : G_1 \to G_2$. By construction, there are canonical isomorphism $G_1' \simeq_R \mathfrak{I}$ and $G_2' \simeq_R \mathfrak{J}$ (as R-modules). Now, α induces an isomorphism $G_1/G_1' \to G_2/G_2'$, which, combined with the natural isomorphisms with U, gives an isomorphism of U, which we extend by linearity to an automorphism θ of R. Then, for every $x \in \mathfrak{I} = G_1'$ and $u \in R$:

$$(xu)^\alpha = x^\alpha u^\theta.$$

It follows that $\mathrm{Ann}_R(x^\alpha) = \mathrm{Ann}_R(x)$, for every $x \in \mathfrak{I}$. Now, if $x = (u_i - 1)^{mp^{i-k}}$, with $1 \leq k \leq i$ and $(m, p) = 1$, it is easy to see that

$$\mathrm{Ann}_R(x) = (u_k - 1)^{p^k - m} R.$$

Therefore, for all $i \geq \ell$, $\mathrm{Ann}_R(c_i^\alpha) = \mathrm{Ann}_R(c_i)$ implies $c_i^\alpha R = c_i R$. Thus we conclude that $\mathfrak{I} = \mathfrak{I}^\alpha = \mathfrak{J}$. $\boxed{QED}$

Comments. (1) The groups G constructed in Theorem 133 are certainly not nilpotent as $G' = [G, G']$. A similar behaviour has the upper central series of any $G = G(\mathfrak{I})$. In fact, if $0 \neq r \in R$, there exists $u \in U$ such that $ru \neq r$. This implies (with the notation used in the proof of 133) that $\zeta(G) \cap N = 1$, and therefore $[\zeta_2(G), G] \leq \zeta(G) \cap G' \leq \zeta(G) \cap N = 1$, forcing $\zeta_2(G) = \zeta(G)$. Factoring G by $\zeta(G)$ we thus obtain groups of Heineken-Mohamed type with trivial centre. Observe also that $\zeta(G(\mathfrak{I}))$ is contained in U; hence $\zeta(G(\mathfrak{I}))$ is not trivial if and only if $\mathfrak{I}(u_1 - 1) = 0$.

(2) There are $2^{\aleph_0}$ distinct ideal-sequences $(k_1, k_2, \ldots)$ that satisfy the conditions of Lemma 130, each of those is associated to a different ideal of R,. Therefore, by the second part of Theorem 133, we have

134 Corollary. *For every prime p there are $2^{\aleph_0}$ non-isomorphic groups G of Heineken-Mohamed type such that $G/G' \simeq C_{p^\infty}$ and G' elementary abelian.*

A result which was also proved by Heineken and Mohamed [48], Hartley [40] and Meldrum [73].

3.3 Developements

In [72] Menegazzo is able to exploit the tecniques reported above to establish the existence, for every prime p, of a p-group of Heineken-Mohamed type G whose derived subgroup is abelian of infinite exponent (as we are dealing with p-groups, this means that G' contains elements of order p^n for every $n \geq 0$). Since Hartley had previously proved in [41] that there exist Heineken-Mohamed groups with derived subgroup of arbitrary finite exponent p^n, we have the following result, whose proof we do not include here.

135 Theorem. *For every prime p and any $e \in \{p^n \mid n \in \mathbb{N}\} \cup \{\infty\}$ there exists a p-group G of Heineken-Mohamed type such that G' is abelian of exponent e.*

Another important result from [72] is the following one.

136 Theorem. [Menegazzo] *For every prime p and every $n \geq 1$ there exist p-groups of Heineken-Mohamed type whose derived length is exactly n.*

We try at least to indicate the ideas used in the proof of this. We start by describing a method of lifting an action on an abelian group to an action on a nilpotent one, which we will soon specialize to extend the action of U on $R = \mathbb{F}_p[U]$.

Let A be a commutative ring (with identity) of prime characteristic p, and let $1 \leq n \in \mathbb{N}$. To each ordered n-tuple $(a_1, \ldots, a_n) \in A^n$ we associate a unitriangular $(n+1)\times(n+1)$-matrix

$$\Sigma(a_1,\ldots,a_n)=\begin{pmatrix}1 & a_1 & a_2 & \ldots & & a_n\\ 0 & 1 & a_1^p & \ldots & & a_{n-1}^p\\ 0 & 0 & 1 & \ldots & & a_{n-2}^{p^2}\\ & & & & & \\ & & & \ldots & 1 & a_1^{p^{n-1}}\\ & & & & & 1\end{pmatrix}$$

We then set

$$\Sigma_n(A)=\{\Sigma(a_1,\ldots,a_n)\mid a_1,\ldots,a_n\in R\}.$$

It is easily checked that $\Sigma(A)$ is a subgroup of the group of all upper unitriangular A-matrices of order $n+1$. In particular, $\Sigma_n(A)$ is a nilpotent p-group of finite exponent. Also, $Q=Q_n(A)=\{\Sigma(0,a_2,\ldots,a_n)\mid a_2,\ldots,a_n\in A\}$ is a normal subgroup of $\Sigma=\Sigma_n(A)$, Σ/Q is isomorphic to the additive group of A, and the set of matrices $\{\Sigma(a_1,a_2,\ldots,a_n)\mid a_2=\cdots=a_n=0\}$ is a set of coset representatives of Σ modulo Q (all these facts are not hard to check by direct computations). For every $1\le i\le n$, we define $\pi_i:\Sigma_n(A)\to A$ as the natural projection $\Sigma(a_1,\ldots,a_n)\mapsto a_i$.

The following observation may be easily proved by matrix computations, and we omit the details.

137 Lemma. *Let $\alpha=\Sigma(a_1,\ldots,a_n)$ and $\beta=\Sigma(b_1,\ldots,b_n)$ be elements of $\Sigma_n(A)$, and suppose that $1\le t,s\le n$ are such that $a_i=0$ for all $i<t$ and $b_i=0$ for all $i<s$. Let $[\alpha,\beta]=\Sigma(q_1,\ldots,q_n)$. Then $q_i=0$ for all $i<t+s$, and $q_{t+s}=a_tb_s^{p^t}-b_sa_t^{p^s}$.*

Let now X be a group of multiplications of A. Then X acts on $\Sigma_n(A)$ in the following way

$$\Sigma(a_1,a_2,\ldots,a_n)^x=\Sigma(a_1x,a_2x^{p+1},\ldots,a_nx^{p^{n-1}+\ldots+p+1}). \tag{9}$$

That this defines a group action may be seen immediately by observing that (9) coincides with conjugating $\Sigma(a_1,\ldots,a_n)$ (in the group of all invertible A-matrices of order $n+1$) by the diagonal matrix

$$D(x)=\begin{pmatrix}1 & & & & \\ & x & & & \\ & & x^{p+1} & & \\ & & & \ddots & \\ & & & & x^{p^{n-1}+\ldots+p+1}\end{pmatrix}$$

and that $x \mapsto D(x)$ clearly defines a group isomorphism. Under this action the normal subgroup $Q = Q_n(A)$ defined before is X-invariant, and the action of X on the factor group $\Sigma_n(A)/Q$ is equivalent to the natural action by multiplication of X on A.

Now, using the notations of section 3.2, we specialize to the case $X = U = \langle u_1, u_2, \ldots\rangle \simeq C_{p^\infty}$ and $A = \mathbb{F}_p[U] = R$. Recall, in particular, that $\mathfrak{U}$ denotes the augmentation ideal of R.

138 Lemma. *Let $\mathfrak{I}$ be an ideal of R with $\mathfrak{I} = \mathfrak{I}\mathfrak{U} < \mathfrak{U}$.*

(i) Let H be a subgroup of $\Sigma_n(R)$ such that for $1 \le s \le n$, $H\pi_i = 0$ for all $i < s$, and $\mathfrak{I}^{p^{s-1}+\ldots+1} \subseteq H\pi_s$; then $[H,H]\pi_j = 0$ for $j < 2s$, and

$$\mathfrak{I}^{p^{2s-1}+\ldots+1} \subseteq [H,H]\pi_{2s}.$$

(ii) Let $d = [\log_2(n)]$, $m = p^{2^d-1}+\ldots+p+1$, and suppose further that $\mathfrak{I}^m \neq 0$. Let $H \le \Sigma_n(R)$ with $\mathfrak{I} \subseteq H\pi_1$; then H has derived length d.

Proof. Let $(k_1, k_2, \ldots)$ be the sequence associated to $\mathfrak{I}$. Then $k_{i+1} \le pk_i$ for all $i \ge 0$, and, by Lemma 130, $\mathfrak{I}$ is generated by the set $\{(u_i-1)^{k_i} \mid k_{i+1} < pk_i\}$. For $j \ge 1$, we write $\mathfrak{I}_j = \mathfrak{I}^{p^{j-1}+\ldots+1}$.

(i) By Lemma 137, $[H,H]\pi_i = 0$ for every $j < 2s$, and $[H,K]\pi_{s+t}$ contains all elements of the form

$$x(a,b) = ab^{p^t} - ba^{p^s}$$

with $a, b \in \mathfrak{I}_s$. Observe that $x(a,b) \in \mathfrak{I}_{2s}$. For $i \ge 1$ such that $k_{i+1} < pk_i$, we take

$$\begin{array}{l} a = (u_i-1)^{k_i(p^{s-1}+\ldots+1)} = (u_{i+1}-1)^{k_i(p^s+\ldots+p)} \\ b = (u_{i+1}-1)^{k_{i+1}(p^{s-1}+\ldots+1)} \end{array}.$$

Then, we have

$k_{i+1}(p^{2s-1}+\ldots+p^s) - k_{i+1}(p^{s-1}+\ldots+1) < k_i(p^{2s}+\ldots+p^{s+1}) - k_i(p^s+\ldots+p).$

which implies that $ba^{p^s} \in ab^{p^s}R$, and $ba^{p^s}R < ab^{p^s}R$. By the total ordering of R–ideals, it follows that $x(a,b)R = ab^{p^s}R$ contains

$$ba^{p^s}(u_{i+1}-1)^{(pk_i-k_{i+1})(p^{s-1}+\ldots+1)} = (u_i-1)^{k_i(p^{s+t-1}+\ldots+1)}.$$

Since the set of all elements $(u_i-1)^{k_i(p^{s+t-1}+\ldots+1)}$, generates $\mathfrak{I}_{2s}$ as an ideal, the proof of point (i) is completed by observing the $[H.H]\pi_{2s}$ contains the ideal generated by all the elements $x(a,b)$ for $a, b \in \mathfrak{I}_s$. Now, under our assumptions on H, $(xy)\pi_{2s} = x\pi_{2s} + y\pi_{2s}$ for all $x, y \in [H,H]$; moreover, if $a, b \in \mathfrak{I}_s$, and $u \in U$, then

$$x(a,b)u^{p^s+1} = ab^s u^{1+p^s} - a^s b u^{1+p^s} = x(au, bu) \in [H,H]\pi_{2s};$$

since the power p^s+1 is an automorphism of the group U, we conclude that $[H,H]\pi_{2s}$ contains the ideal $\mathfrak{I}_{2s}$.

(ii) Let $H \le \Sigma_n(R)$ be such that $H\pi_1 \supseteq \mathfrak{I}$. Then, point (i) and an obvious induction shows that $H^{(r)}\pi_{2^r} \supseteq \mathfrak{I}_{2^r}$, for every $0 \le r \le d$. Therefore

$$H^{(d)}\pi_{2^d} \supseteq \mathfrak{I}_{2^d} = \mathfrak{I}^m \neq 0,$$

and thus H has derived length d (it cannot be more). $\boxed{QED}$

Observe that if the sequence $(k_1, k_2, \ldots)$ associated the ideal $\mathfrak{I} = \mathfrak{I}\mathfrak{U} < \mathfrak{U}$, satisfies $k_j = 1$ for $j \le m$, then $\mathfrak{I}^m \neq 0$; thus, there exist $2^{\aleph_0}$ ideals $\mathfrak{I}$ that satisfy the condition in point (ii) of the Lemma.

From now on, we suppose $n \ge 2$ to be fixed, and simply write $\Sigma = \Sigma_n(R)$. Let W be the semidirect product $W = \Sigma \rtimes U$, and let $\mathfrak{I}$ be a fixed ideal of R such that $\mathfrak{I} = \mathfrak{I}\mathfrak{U} < \mathfrak{U}$. We then refer to the notations used in section 3.2; in particular ℓ is an integer choosen as in Proposition 132, and $(c_i)_{i\ge\ell}$ is the HM-system for the R-module $\mathfrak{I}$ defined in the same Proposition.

Let $(k_i)_{i\ge1}$ be the sequence associate to $\mathfrak{I}$, we define integers r_i (for $i \ge \ell$), as follows:

$$r_\ell = \begin{cases} p^{\ell+1} - pk_\ell & \text{if} \quad k_{\ell+1} = pk_\ell \\ p^{\ell+1} - pk_\ell + 1 & \text{if} \quad k_{\ell+1} < pk_\ell \end{cases} \tag{10}$$

and, for $i > \ell$,

$$r_i = \begin{cases} 0 & \text{if} \quad k_{i+1} = pk_i \quad \text{and} \quad k_i = pk_{i-1} \\ p(pk_{i-1} - k_i - 1) & \text{if} \quad k_{i+1} = pk_i \quad \text{and} \quad k_i < pk_{i-1} \\ 1 & \text{if} \quad k_{i+1} < pk_i \quad \text{and} \quad k_i = pk_{i-1} \\ p(pk_{i-1} - k_i - 1) + 1 & \text{if} \quad k_{i+1} < pk_i \quad \text{and} \quad k_i < pk_{i-1} \end{cases} \tag{11}$$

These numbers are singled out because of the following fact.

139 Lemma. *With the notations of Proposition 132, and definitions (10) and (11), set, for every $i \ge \ell$, $w_i = (u_{i+1} - 1)^{r_i}$. Then the following hold.*

(i) $c_\ell w_\ell = 0$*; and* $c_i w_i = c_{i-1}$ *for all* $i > \ell$.

(ii) Fore every $i \ge \ell$, $\mathrm{Ann}_R(c_i) = \left(\prod_{s=\ell}^{i} w_s\right)R$.

PROOF. Point (i) follows easily from the definitions of the elements c_i and of the numbers r_i. Now, using point (i), Corollary 129 and an obvious induction, we see that, in order to prove (ii), it is enough to observe that $\mathrm{Ann}_R(c_\ell) = w_\ell R$, which is again clear by the definition. $\boxed{QED}$

The relevance of this is in turn motivated by the following Lemma.

140 Lemma. *Let M be a R-module, which is generated by the sequence $(d_i)_{i\geq\ell}$, such that $d_\ell w_\ell = 0$ a, and $d_i w_i = d_{i-1}$ for all $i > \ell$. Then there exists a R-homomorphism $\sigma : \mathfrak{I} \longrightarrow M$, with $\sigma(c_i) = d_i$, for every $i \geq \ell$. In particular, $(d_i)_{i\geq\ell}$ is a HM-system for M.*

Proof. Since $w_\ell \in \operatorname{Ann}_R(d_\ell)$, by Lemma 129, Lemma 139 and an obvious inductive argument, we have $\operatorname{Ann}_R(c_i) \subseteq \operatorname{Ann}_R(d_i)$, for every $i \geq \ell$. Thus, for every $i \geq \ell$, there is the natural projection $R/\operatorname{Ann}_R(c_i) \to R/\operatorname{Ann}_R(d_i)$, which in turn yields a homomorphism of R-modules $\sigma_i : c_iR \to d_dR$ (since $c_iR \simeq_R R/\operatorname{Ann}_R(c_i)$ and $d_iR \simeq_R R/\operatorname{Ann}_R(d_i)$). Now, by Lemma 139

$$\sigma_j(c_i) = \sigma_j\big((\prod_{s=i+1}^{j} w_s)c_j\big) = (\prod_{s=i+1}^{j} w_s)d_j = d_i = \sigma_i(c_i)$$

for every $\ell \leq i < j$. Hence the maps σ_i are compatible, and so the position $c_i \mapsto d_i$ (for $i \geq \ell$), may be extended to a R-homomorphism $\sigma : \mathfrak{I} \longrightarrow M$. The last assertion follows easily from the definition of HM-system. $\boxed{QED}$

Next step is to prove the existence of elements of Σ that will allow to apply Lemma 140 (in suitable abelian factors). Thus, Menegazzo establishes the following crucial fact, whose proof (by induction on n, being the case $n = 1$ part of the proof of Theorem 133) is rather long; and we refer to the original paper [72] for it.

141 Lemma. *There exist elements x_i, y_i in Σ, for all $i \geq \ell$, such that:*

(i) $x_i\pi_j,\ y_i\pi_j \in (u_i - 1)R$ *for every* $i \geq \ell$ *and every* $j = 1, \dots, n$;

(ii) $y_i\pi_1 = c_i$ *for every* $i \geq \ell$; *and* $x_\ell = 1$;

(iii) $[y_{\ell,r_\ell}\, u_{\ell+1}x_{\ell+1}] = 1$, *and* $[y_{i,r_i}\, u_{i+1}x_{i+1}] = y_{i-1}$ *for every* $i > \ell$;

(iv) $x_{i+1}^{u_{i+1}^{p-1}} \dots x_{i+1}^{u_{i+1}} x_{i+1} = x_i y_i$ *for every* $i \geq \ell$.

Now, for the proof of Theorem 136, we set $g_i = u_i x_i$ for every $i \geq \ell$, and consider the subgroup G of W given by

$$G = \langle g_i \mid i \geq \ell \rangle.$$

By property (iv) in Lemma 141, we have, for every $i \geq \ell$.

$$g_{i+1}^p = (u_{i+1}x_{i+1})^p = u_{i+1}^p x_{i+1}^{u_{i+1}^{p-1}} \dots x_{i+1}^{u_{i+1}} x_{i+1} = u_i x_i y_i = g_i y_i. \qquad (12)$$

Write $N = \langle y_i \mid i \geq \ell \rangle^G$. Then (12) shows that $G' \leq N \leq \Sigma \cap G$, and $G/N \simeq U$. In fact, as $G \ni g_\ell = u_\ell$, and $G/N \simeq U$, we have $\Sigma \cap G = \Sigma \cap N\langle u_\ell \rangle = N$.

Also, for $i \geq \ell$, let $j > i$ minimal such that $r_j > 0$; then, by point (iii) of 141, $y_i = y_{j-1} = [y_{i,r_i}\, g_{i+1}] \in [N, G]$. Hence $N = [N, G] = G'$.

Now, let $D = N'N^p$ and write $\overline{N} = N/D$; let also η denote the isomorphism $U \to G/N$ which maps $u_i \mapsto g_iN$ for all $i \geq \ell$. Then, $\overline{N}$ becomes a R-module by letting, for all $u \in U$ and $yD \in \overline{N}$,

$$(yD)^u = y^{u\eta}D.$$

As an R-module, $\overline{N}$ is generated by the sequence $(y_iD)_{i\geq\ell}$. Now, point (iii) in Lemma 141, yields

$$(y_\ell D)^{w_\ell} = [y_{\ell,r_\ell}\, g_{\ell+1}]D = 1$$

and, for every $i > \ell$,

$$(y_iD)^{w_i} = [y_{i,r_i}\, g_{i+1}]D = y_{i-1}D.$$

Thus, by Lemma 140, $(y_iD)_{i\geq\ell}$ is a HM-system for the R-module $\overline{N}$. It then follows from Proposition 131 that G/D is a group of Heineken–Mohamed type.

To deduce that G is also a group of Heineken–Mohamed type it is now easy, and requires only the following observation.

142 Lemma. *Let G be a p-group of finite exponent, and let N be a normal nilpotent subgroup G. If G/N^pN' is a Heineken–Mohamed group, then G is a Heineken–Mohamed group.*

Proof. Let G and N be as in the assumptions, write $K = N^pN'$, and let S be a proper subgroup of G. If $SK < G$, then SK/K is nilpotent and subnormal, whence in particular NS/K is also nilpotent by Lemma 61. Since N/N' has finite exponent, it is easy to deduce that SN/N' is nilpotent. Thus, by P. Hall's nilpotency criterion 56, NS is nilpotent. In particular, S is nilpotent, and $S \lhd\lhd NS \lhd\lhd G$.

Thus, let $KS = G$. In such a case, $KS = G$ by Proposition 125, and so $K(N \cap S) = N \cap KS = N$. Since N is nilpotent, it follows $N \cap S = N$. Thus $N \leq S$, and consequently $S = G$. $\boxed{QED}$

The proof of Theorem 136 will be completed once we prove that the group G constructed above may have arbitrary derived length. As $G' = [G, N] = N$, we have to show that $n \geq 2$ and ideal $\mathfrak{I}$ may be chosen such that N has arbitrary derived length. This is easily achieved by first observing that, by point (ii) of Lemma 141, $\mathfrak{I} = N\pi_1$: in fact $(ab)\pi_1 = a\pi_1 + b\pi_1$ for every $a, b \in N$, and if $u_i \in U$ $(i \geq \ell)$, $N\pi_1 \ni (a^{g_i})\pi_1 = (a\pi_1)u_i$, for every $a \in N$. Now, given $d \geq 1$, we take $n \geq 2^d$, and $\mathfrak{I}$ an ideal with $\mathfrak{I} = \mathfrak{I}\mathfrak{U} < \mathfrak{U}$ and $\mathfrak{I}^m \neq 0$, where $m = p^{2^d-1} + \ldots + p + 1$. Then Lemma 138 yields the desired conclusion. Observe also that, using the remark following the proof of Lemma 138, it is not difficult to show that there exists $2^{\aleph_0}$ pairwise non-isomorphic Heineken–Mohamed p-groups of a given derived length (we recall also that we will see in Chapter 6 that every Heineken-Mohamed group is in fact soluble).

Another construction which somehow extends that of Heineken and Mohamed, and we like to mention, appears in W. Möhres doctoral thesis [75].

143 Proposition. *For every prime number p and every integer $n \geq 1$, there exists a group $G \in \mathcal{N}_1$ such that*

(1) $Z(G) = 1$;
(2) G' is an elementary abelian p-group;
(3) G/G' is isomorphic to the direct product of n copies of the C_{p^∞}.

Clearly (see Proposition 125), if $n \geq 2$, the groups obtained by this Proposition are not of Heineken-Mohamed type. Nevertheless the existence of $\mathcal{N}_1$-groups with the properties described in 143 becomes relevant in view of the content of our final result on periodic $\mathcal{N}_1$-groups (Theorem 225).

3.4 Minimal non-$\mathfrak{N}$ groups

Despite of its simplicity, Möhres' Lemma 116 (and its variations, see e.g. [82]) is often useful in reducing certain problems to the periodic (or to the finitely generated) case. We now leave for a while our main theme to treat just a particular case, somehow related to HM-groups, in which this occurs.

Let $\mathcal{P}$ be a class of groups. A group G is called *minimal non-$\mathcal{P}$* if G does not belong to $\mathcal{P}$, but all its proper subgroups are $\mathcal{P}$-groups. We are interested in minimal non-nilpotent groups (minimal non-$\mathfrak{N}$). Finite minimal non-$\mathfrak{N}$ groups are very well understood by a result of O. J. Schmidt (see [97] 9.19). Infinite examples are the Heineken-Mohamed groups and the infinite dihedral 2-group. We show

144 Proposition. *Let G be a minimal non-nilpotent group. Then, either G is finitely generated or it is a countable locally finite p-group (for some prime p) of one of the following types:*

(i) a perfect group;

(ii) a Černikov p-group;

(iii) a (soluble) group of Heineken-Mohamed type.

Proof. Let G be a minimal non-nilpotent group, and assume that G is not finitely generated. Then G is locally nilpotent and it is countable by Theorem 33. Let T be the torsion subgroup of G.

Suppose $T \neq G$. Then G/T is a countable locally nilpotent torsion-free group, so, by Lemma 116, it admits a proper subgroup H/T with $I_{G/T}(H/T) =$

G/T. Now H (and H/T) is nilpotent by minimality of G, whence G/T is nilpotent by Corollary 114. Let N/T be the derived subgroup of G/T. Since G/T is not trivial, G/N cannot be a p-group (for any prime p), so by Lemma 126 there exist two proper subgroups U/N and V/N of it such that $UV = G$. Now, U and V are then normal nilpotent subgroups of G, and it follows from Fitting's Theorem that G is nilpotent, contradicting our assumption.

Thus $T = G$, and so, being locally nilpotent, G is the direct product of its primary components. If there are two of more such components, then G is the direct product of two proper subgroups and so it is nilpotent. Therefore only one primary component may exist, and so G is a locally finite p-group for some prime p.

Suppose that G is not perfect (which is case i)), and let $N = G'$. Then by Lemma 126 G/N is either cyclic or C_{p^∞}.

Assume firts that G/N is cyclic, and let $x \in G$ such that $G = N\langle x\rangle$. Observe that G/N^p is nilpotent by Corollary 79; in particular $X = \langle x\rangle N^p$ is subnormal in G. Now, if $N^p \neq N = G'$, X is a proper subgroup, and so X^G is also a proper subgroup of G. But then $G = NX^G$ is nilpotent by Fitting's Theorem. Thus, $N^p = N$ or, in other words, N is semi-radical, and it follows from Lemma 18 that N is an abelian divisible p-group, a direct product of groups of type C_{p^∞}. Let $A \leq N$ be such a subgroup; then A has a finite number of conjugates in G, so A^G is the product of finitely many copies of A. If $A^G \neq N$ then $A^G\langle x\rangle$ is nilpotent, forcing $[A^G, x] = 1$, which is a contradiction. Thus, $A^G = N$ has finite rank, and G is a Černikov p-group.

Assume finally that $G/N \simeq C_{p^\infty}$. Let H be a proper subgroup of G, If $NH \neq G$ then NH is nilpotent and normal in G and so H is subnormal in G. Thus G is a group of Heineken-Mohamed type if we show that no proper subgroup H of G exists such that $NH = G$. Suppose, by contradiction, that H is such a subgroup. Then $H \cap N$ is a proper subnormal subgroup of N, and it is normal in H; hence, being N nilpotent, $M = (H \cap N)^N N'$ is a proper subgroup of N which is normalized by $NH = G$. It follows that MH is a proper subgroup of G. We may than assume $M = 1$. Hence N is abelian, and $N \cap H = 1$ (this last condition imply $H \simeq C_{p^\infty}$). Since N is not centralized by H (otherwise $H \trianglelefteq G$ and G is nilpotent), there exists an element $x \in H$ such that $C_N(x) \neq N$. Now, as H is abelian, $C_N(x)$ is normalized by H, so $C_N(x)H$ is a proper, and hence nilpotent, subgroup of G. But also $[N, x] \neq N$, as $N\langle x\rangle$ is nilpotent and $N \cap \langle x\rangle = 1$; whence $[N, x]H$ is nilpotent. It follows that there exists $n \in \mathbb{N}$ such that $[[N, X], {}_nH] = 1$, where $X = \langle x\rangle$. Then, by 13,

$$1 = [N, X, H, \ldots, H] = [N, H, \ldots, H, X]$$

which means that $[N, {}_nH] \leq C_N(X)$. Since we observed above that $C_N(X)H$ is nilpotent, we conclude that H is subnormal, and this implies that G is nilpotent, a contradiction. $\boxed{QED}$

Clearly, not every Černikov p-group is minimal non-$\mathfrak{N}$, and we leave to the reader to work out a more precise description for this case. More relevant is to report that Asar [1] has proved that case (i) cannot occur. It is also important to note that finitely generated minimal non-$\mathfrak{N}$ groups appear to be very difficult to understand: the finitely generated groups with all proper subgroups cyclic (the so-called *Tarski monsters*), constructed by Ol'shanskii [87] and Rips, are, obviously, of this kind (and they can be torsion-free). In view of these examples, it is common in the literature on the argument to restrict investigations to classes of groups that are large enough to comprise important cases but exclude Tarski monsters and objects alike. The usual restriction is to locally graded groups.

Now, let G be a locally graded finitely generated group with all proper subgroups nilpotent, and assume that G is not finite. Then G is a finite extension of a nilpotent group N; since finite minimal non-$\mathfrak{N}$ groups are soluble, we may take $N \geq G'$. We know that (being finitely generated) G/N is a cyclic p-group for some prime p. Also, as a subgroup of finite index of a finitely generated group, N is finitely generated nilpotent infinite group; hence the torsion subgroup $T(N)$ is finite and, by 43, $N/T(N)$ admits a characteristic subgroup X/N with N/X a finite non-trivial p-group. But then $X \trianglelefteq G$ and G/X is a finite p-group, contradicting $N = \gamma_3(G)$ (which in turn follows from Lemma 126).

Thus, together with the aforementioned result of Asar, we have:

145 Theorem. *Let G be a locally graded minimal non-nilpotent group. Then, G is either finite, or a Černikov p-group, or a p-group of Heineken-Mohamed type (in particular - as we will see later - G is soluble).*

In fact, Heineken-Mohamed groups are nilpotent-by-Černikov, and with similar (but more elaborated) methods it is possible to prove the following Theorem.

146 Theorem. *Let G be a locally graded group in which every proper subgroup is nilpotent-by-Černikov. Then G is nilpotent-by-Černikov.*

A result that, as well as Theorem 145, is due to the combined efforts of a number of people; see Newman and Wiegold [86], Bruno [8], Otal and Peña [90], Bruno and Phillips [10], H. Smith [105], Napolitani and Pegoraro [82] and Asar [1].

4 Bounded defects

The main result to be proved in this chapter (at least in view of its subsequent applications in these notes) is a fundamental theorem of Roseblade, stating that a group in which every subgroup is subnormal of defect at most n (for $n \geq 1$) is nilpotent of nilpotency class not exceeding a value depending only on n. We also include some related material (mostly without proofs).

4.1 n-Baer groups

For every $n, r \geq 1$, we denote by $\mathfrak{U}_{n,r}$ the class of all groups in which every subgroup that can be generated by r elements is n-subnormal (i.e. subnormal of defect at most n). By definition, $\mathfrak{U}_{n,r+1} \subseteq \mathfrak{U}_{n,r}$ for every $n, r \geq 1$; we set

$$\mathfrak{U}_n = \bigcap_{r \geq 1} \mathfrak{U}_{n,r}.$$

Then, from Lemma 24 it immediately follows,

147 Proposition. *for every $n \geq 1$, $\mathfrak{U}_n$ is the class of groups in which every subgroup is n-subnormal.*

Given $n \geq 1$, $\mathfrak{U}_{n,1}$ is the class of groups in which every cyclic subgroup is n-subnormal; such groups are usually called *n–Baer* groups. Occurencies of groups of this kind we have already encountered. For instance, Proposition 78 states that a soluble p-group of finite exponent is an n-Baer group, where n depends on the exponent and on the derived length of the group. However, not many general results are known about n-Baer groups, and we have precise informations only for small values of n, which we will briefly report.

Before, let us notice the obvious fact that every n-Baer group G is $(n+1)$-Engel, that is it satisfies the identity $[x_{n+1}y] = 1$. Thus, as a first step in treating n-Baer groups we recall some known facts about n-Engel groups (for n small). Clearly, 1-Engel groups are just the abelian groups. 2-Engel groups are also well understood; their description is essentially due to Levi [65], who also proved that every group of exponent 3 is 2-Engel.

148 Theorem. [Levi [65]]. *Let G be a 2-Engel group. Then $\gamma_4(G) = 1$, and $\gamma_3(G)$ has exponent dividing 3. Thus, a torsion–free 2-Engel group is nilpotent of class at not 2.*

3-Engel groups are much more complicated. They need not be nilpotent: the standard wreath product $G = C \wr A$ of a cyclic group C of order 2 by an infinite elementary abelian 2-group A is not nilpotent (for example $Z(G) = 1$) but it is 3-Engel, as it is easily checked. The fact that 3-Engel groups are locally nilpotent is not at all immediate and was established in [43] by Heineken, who also proved that if G is a 3-Engel group with no elements of order 2 or 5, then $\gamma_5(G) = 1$. On the other hand, Bachmuth and Mochizuki showed in [2] that there exists a 3-Engel group of exponent 5 that is not even soluble (while 3-

Engel 2-groups are soluble, see [33]). The following statements collect the most relevant known facts about 3-Engel groups.

149 Theorem. [N. Gupta, M. Newman [35]] *Let G be a 3-Engel group. Then*

1. *if G is n-generated, with $n > 2$, then it is nilpotent of class at most $2n-1$, if, further, G does not have elements of order 5, then G has class at most $n+2$;*

2. *$\gamma_5(G)$ has exponent dividing 20, and this is best possible;*

3. *the subgroup G^5 generated by the fifth powers of elements of G satisfies the law $[[a,b,c],[d,e]] = 1$*

To complete the statement of point 1. we mention that if G is a 2-generated 3-Engel group, then $|\gamma_4(G)| \leq 2$ (Heineken [43]), and this is best possible (C. K. Gupta, see [34]).

150 Proposition. [L. C. Kappe and W. P. Kappe [50]]. *Let G be a group. The following are equivalent:*

1. *G is a 3-Engel group;*

2. *$\langle x\rangle^G$ is a 2-Engel group for every $x \in G$;*

3. *$\gamma_3(\langle x\rangle^G) = 1$ for every $x \in G$.*

Recently Havas and Vaughan-Lee [42] succeeded in proving that 4-Engel groups are locally nilpotent (see also Traustason [117] for a mostly computer–free approach).

Of course, for n-Baer groups local nilpotency is not in question. We already observed that a n-Baer group is $(n+1)$-Engel group (but not necessarily n-Engel, se e.g. [67]). However, if G is not periodic then G is in fact n-Engel.

151 Lemma. *Let G be a non-periodic group in which the set $\mathrm{Tor}(G)$ of torsion elements is a finite subgroup. Suppose that, for some $n \geq 1$, all elements of infinite order in G are left n-Engel. Then G is n-Engel.*

Proof. Write $T = \mathrm{Tor}(G)$. Since $T \trianglelefteq G$ is finite and G is not periodic, the centralizer $C_G(T)$ contains an element x of infinite order.

Let $g \in G$. If $|g| = \infty$ then g is left n-Engel by assumption. Thus, let $g \in T$. Then, for any $y \in G$, $[y, xg] = [y, g][y, x]^g = [y, g][y, x]$, as $\langle x\rangle^G \leq C_G(T)$. Continuing by induction on $i \geq 1$, we have

$$[y,_{i+1} xg] = [[y,_i g][y,_i x], xg] = [y,_i g, xg]^{[y_i\, x]}[y,_i x, xg]$$

and, since $[y,_i g, xg] \in T$,

$$[y,_{i+1} xg] = [y,_i g, xg][y,_i x, xg] = [y,_{i+1} g][y,_{i+1} x].$$

Thus, for every $k \geq 1$, $[y,_k xg] = [y,_k g][y,_k x]$. Now, both x and xg have infinite order and so are left n-Engel. Hence, in particular,

$$1 = [y,_n xg] = [y,_n g][y,_n x] = [y,_n g].$$

This proves that g is left n-Engel. Therefore, G is n-Engel. $\boxed{QED}$

152 Proposition (see [51])**.** *Let $n \geq 1$. Every non-periodic n-Baer group is n-Engel.*

Proof. Let G be a non-periodic n–Baer group. To show that G is n-Engel, we may clearly assume that G is finitely generated. Then G is nilpotent, and so, in particular, $T = \mathrm{Tor}(G)$ is a finite normal subgroup of G. By Lemma 151, it is then sufficient to show that all elements of infinite order of G are left n-Engel. Thus, let $x \in G$ have infinite order, and let $g \in G$. Then $[g,_n x] \in \langle x \rangle$, and so there exists $m = m(g) \geq 0$ such that $[g,_n x] = x^m$. Hence $x^m \in \gamma_{n+1}(G)$, and

$$x^{m^2} = [g,_{n-1} x, x]^m = [g,_{n-1} x, x^m] \in \gamma_{2n+1}(G).$$

Proceeding in this way, we have, for any $r \geq 1$, $x^{m^r} \in \gamma_{rn+1}(G)$. But G is nilpotent, so there exists $r \geq 1$ such that $x^{m^r} = 1$. Since $|x| = \infty$, the only possibility is then $m = 0$. Thus, $[g,_n x] = 1$, and we are done. $\boxed{QED}$

Clearly, 1-Baer groups are just those groups in which every subgroup is normal. These are the well-known *Dedekind groups* (see [97] 3.5.7)

153 Proposition. [Dedekind]. *$\mathfrak{U}_1 = \mathfrak{U}_{1,1}$, and $G \in \mathfrak{U}_1$ if and only if G is either abelian or the direct product $G = Q \times D$ of a quaternion group of order 8 and a periodic abelian group D that does not have elements of order 4. In particular, if $G \in \mathfrak{U}_1$, then $|\gamma_2(G)| \leq 2$. Torsion–free $\mathfrak{U}_1$ are abelian.*

The class of 2-Baer groups was first studied by Heineken, who proved that if G is a 2-Baer group then $G/\zeta(G)$ is 2-Engel;. from Theorem 148 nilpotency of G follows, together with informations on the lower central factors. These were later completed by Mahdavianary. The combined result is

154 Theorem. [Heineken [44], Mahdavianary [66]] *Let G be a 2-Baer group, then $\gamma_4(G) = 1$.*

Special classes of 2-Baer p-groups ($p = 2, 3$) are classified in further papers of Mahdavianary ([67], [68]), while in [89], E. Ormerod describes all 2-Baer p-groups for $p \geq 5$.

It follows immediately from Proposition 150 that every 3-Engel is a 3-Baer group; thus, 3-Baer groups need not be even be soluble (in fact the Bachmuth-Mochizuki group shows that 3-Baer groups of finite exponent need not be soluble (cfr. Proposition 78)). Also, by 150 and 152, a non-periodic group is 3-Baer if and only if it is 3-Engel. Some positive results on arbitrary 3-Baer groups are to be found in Traustason [118]; in particular, Traustason proves that every 3-Baer group G admits a normal subgroup N, which is nilpotent of class at most 3 (and in fact abelian if G does not have 2-elements) such that G/N is a 3-Engel group. Finally, metabelian n-Baer groups are the subject of a paper by L. C. Kappe and Garrison [29].

4.2 Roseblade's Theorem

As mentioned before, Roseblade's Theorem says (in particular) that, for every $n \geq 1$, there exists a positive integer $\rho(n)$ such that a group in which every subgroup is n-subnormal (thus, a $\mathfrak{U}_n$-group) is nilpotent of nilpotency class bounded by $\rho(n)$. Thus (recalling that $\mathfrak{N}$ denotes the class of all nilpotent groups,

$$\mathfrak{N} = \bigcup_{n \in \mathbb{N}} \mathfrak{U}_n.$$

155 Theorem. [Roseblade [98]] *There exist functions f, $\rho : \mathbb{N} \to \mathbb{N}$ such that, for every $n \geq 1$, a group in which every $f(n)$-generated subgroup is subnormal of defect at most n is nilpotent of nilpotency class not exceeding $\rho(n)$. Thus*

$$\mathfrak{U}_{n,f(n)} \subseteq \mathfrak{N}_{\rho(n)}.$$

Before coming to it, let we mention that the value of $\rho(n)$ that one obtains from the proof of Roseblade's Theorem is quite likely far larger than the real bound. The actual bound has been determined only for $n = 1$ and $n = 2$. Clearly $\rho(1) = 1$, while $\rho(2) = 3$ follows from Mahdavianary Theorem 154 (although it is not hard to see that the class $\mathfrak{U}_2$ is strictly smaller than the class of 2-Baer groups). For $n = 3$, we have the following proposition (to be considered in connection to the mentioned results on 3-Baer groups in [118])

156 Proposition. [Traustason [117]] *Let G be a 3-Engel $\mathfrak{U}_3$-group with no elements of order 2. Then $\gamma_5(G) = 1$.*

(Thus, a $\mathfrak{U}_3$-group with no elements of order 2 has derived length at most 4.)

We already mentioned in Chapter 2, that if $n \leq 4$, and the group $G \in \mathfrak{U}_n$ is torsion-free, then $\gamma_{n+1}(G) = 1$; this is due Stadelmann [115] for $n = 2$ (but this follows at once from 148 and 152), Traustason [117] (the previous Proposition plus 152) for $n = 3$, and Smith and Traustason [114] for $n = 4$.

For the proof of 155 we follow [64], and start with a preliminary result dealing with Engel groups.

157 Lemma. *Let A be a normal abelian subgroup of the group G. Suppose that $G/C_G(A)$ is abelian and there exists $n \geq 1$ with $[a,_n x] = 1$ for all $a \in A$. Then there exists $0 < \beta(n) \in \mathbb{N}$ such that*

$$[A,_{2^{n-1}} G]^{\beta(n)} = 1$$

or, equivalently, $A^{\beta(n)} \leq \zeta_{2^{n-1}}(G)$.

Proof. See e.g. [64], Lemma 6.1.6. $\boxed{QED}$

This Lemma is in fact the key ingredient (together with an inductive argument using Hall"s nilpotency criterion) in the proof of Proposition 69. As we mentioned in Chapter 1, nowadays (thanks to Zelmanov's solution of the restricted Burnside Problem plus some tools from the theory of profinite groups) it is possible to say much more (at least for locally graded groups) as seen in Theorem 68. However, for the proof of Roseblade Theorem, we need only those facts (like. 157) that can be proved without invoking such deep results, and so we proceed along this line.

158 Lemma. *There exists a function $c : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that if $G \in \mathfrak{U}_{n,n}$ is soluble of derived length d, then $G \in \mathfrak{N}_{c(n,d)}$.*

Proof. Clearly $c(n, 1) = 1$ for every $n \geq 1$ and $c(1, d) = 12$. Now, assume $d = 2$, $n \geq 2$, set $t = 2^{n-1}$. Let $A = G'$; then A is a normal abelian subgroup of G and $G/C_G(A)$ is abelian; since G is $(n + 1)$-Engel it follows from Lemma 157 that $A^{\beta(n+1)} \leq \zeta_{2^n}(G)$. Thus, we may assume that A has exponent dividing $b = \beta(n)$. By Lemma 16 we then have that $G/C_G(A)$ is abelian of exponent dividing b^n. Let $x_0, x_1, \ldots, x_n \in G$ and set $H = \langle x_0, x_1, \ldots, x_n\rangle$. Then H' is generated by the H-conjugates of the commutators $[x_i, x_j]$, $0 \leq i < j \leq n$. Since $H' \leq A$ and $H/C_H(A)$ is a $(n + 1)$-generated abelian group of exponent dividing b^n, it follows that the number of generators of H' does not exceed $c = c(n) = \binom{n+1}{2} b^{n(n+1)}$, whence $|H'| \leq b^c$ since H' has exponent dividing b. Now, by Lemma 12,ian; since G is $(n+1)$-Engel it follows from Lemma 157 that $A^{\beta(n+1)} \leq \zeta_{2^n}(G)$. Thus, we may assume that A has exponent dividing $b = \beta(n)$. By Lemma 16 we then have that $G/C_G(A)$ is abelian of exponent dividing b^n. Let $x_0, x_1, \ldots, x_n \in G$ and set $H = \langle x_0, x_1, \ldots, x_n\rangle$. Then H' is generated by the H-conjugates of the commutators $[x_i, x_j]$, $0 \leq i < j \leq n$. Since $H' \leq A$ and $H/C_H(A)$ is a $(n+1)$-generated abelian group of exponent dividing b^n, it follows that the number of generators of H' does not exceed $c = c(n) = \binom{n+1}{2} b^{n(n+1)}$, whence $|H'| \leq b^c$ since H' has exponent dividing b. Now, by Lemma 12,

$$[A, x_0, x_1, \ldots, x_n, g] = [a, g, x_0, x_1, \ldots x_n]$$

for every $g \in G$, and $a \in A$, showing that $K = [A, x_0, x_1, \dots, x_n]$ is a normal subgroup of G. Since $G \in \mathfrak{U}_{n,n}$, $[A, x_0, x_1, \dots, x_{n-1}] \leq H$ and so $K \leq H'$ has order bounded by b^c. As G is locally nilpotent, this implies that K is contained in the b^c-th term of the upper central series G. Thus

$$\gamma_{n+3}(G) = [A,_{n+1} G] \leq \zeta_{b^c}(G).$$

Recalling that we worked modulo A^b, we conclude that G is nilpotent of class at most

$$c(n, 2) = 2^n + b^c + n + 2.$$

We now fix $n \geq 1$ and proceed by induction on the derived length d of $G \in \mathfrak{U}_{n,n}$. Then, by inductive assumption, $N = G' \in \mathfrak{N}_{c(n,d-1)}$, while G/N' is metabelian and so, by what proved above, it is nilpotent of class at most $c(n, 2)$. By Theorem 56, we conclude that G is nilpotent of class at most $c(n, d) = \binom{c(n,d-1)+1}{2} c(n, 2) - \binom{c(n,d-1)}{2}$. $\boxed{QED}$

The following property of the automorphism group of an abelian p-group was first proved by P. Hall for the finite case, and later extended by Baer and Heineken [5].

159 Lemma. *Let A be a abelian p-group of rank r. Then any p-subgroup of $Aut(A)$ can be generated by at most $r(5r - 1)/2$ elements.*

Proof. See [5], or [64] page 178. $\boxed{QED}$

We are now ready to prove Roseblade's Theorem.

Proof. of Theorem 155 By Dedekind's Theorem 153, $f(1) = 1$ and $\rho(1) = 2$. We then let $n \geq 2$ and proceed by induction on n.

We set $d = (n - 1)([\log_2(\rho(n - 1))] + 1)$, and define

$$f(n) = c(n, d) + f(n - 1) + 1$$

where $c(n, d)$ is the value obtained in Lemma 158 (observe that $f(n) \geq n$). Let $G \in \mathfrak{U}_{n,f(n)}$; we have to show that G is nilpotent of bounded class.

Let X be a s-generated subgroup of G, with $s \leq c(n, d) + 1$, and denote as usual by $X^{G,i}$ the i-th term of the normal closure series of X in G. Since $s \leq f(n)$, $X^{G,n} \leq X$. For $i = 1, \dots, n - 1$, and let Y be a $f(n - 1)$-generated subgroup of $H^{G,i}$, then $V = \langle X, Y \rangle$ is generated by $s + f(n-1) \leq f(n)$ elements and so it is subnormal of defect at most n in G, whence it is subnormal of defect at most $n - i$ in $V^{G,i} = X^{G,i}$. Thus (since $f(n - 1) \geq f(n - i)$), we have the following:

$$\frac{X^{G,i}}{X^{G,i+1}} \in \mathfrak{U}_{n-i,f(n-i)}. \tag{13}$$

for all $1 \leq i \leq n-1$. Now, by inductive assumption, $X^{G,i}/X^{G,i+1}$ is nilpotent of class at most $\rho(n-i) \leq \rho(n-1)$, and so its derived length is at most $[\log_2(\rho(n-1))]+1$ by 8. Thus, by the definition of d given above,

$$(X^G)^{(d)} \leq X^{G,n} \leq X. \tag{14}$$

Let $c = c(n,d)+1$ (as we will write from now on). By applying Lemma 158 to (14), we have that *for any c-generated subgroup X of G*

$$\gamma_c(X^G) \leq X \tag{15}$$

In particular, it follows from 14 that for every $x \in G$, $\langle x\rangle^G$ is soluble of derived length at most $d+1$, and therefore it is nilpotent of class not exceeding $\ell = c(n,d+1)$. By Fitting's Theorem it follows that, for every $s \geq 1$, and any $x_1,\dots,x_s \in G$

$$\gamma_{s\ell+1}(\langle x_1,\dots,x_s\rangle^G) = 1. \tag{16}$$

By Proposition 48, we conclude that there exists $r = r(n)$ such that

$$\textit{every } c\textit{-generated subgroup of } G \textit{ has rank at most } r. \tag{17}$$

We now observe that we may assume that G is a p-group for some prime p. In fact, in order to prove that G has bounded nilpotency class it is enough to prove this for a finitely generated (and thus nilpotent) G; but then G is residually finite; thus we may suppose that G is finite and, consequently, a p-group for some prime p.

Let$A$ be a normal abelian subgroup of G. Let $x_1,\dots,x_c \in G$ and write $H = \langle x_1,\dots,x_c\rangle$. Then, by (15),

$$[A,_c H^G] \leq \gamma_c(H^G) \leq A \cap H. \tag{18}$$

Thus, by (17) we have that $B = [A,_c H^G]$ is a normal abelian subgroup of G of rank at most r. We may then apply Lemma 159 to conclude that $G/C_G(B)$ is generated by at most $r(5r-1)/2$ elements. Then (16) tells us that $G/C_G(B)$ has nilpotency class at most $r_1 = (r(5r-1)/2)\ell$, and so $[B,\gamma_{r_1+1}(G)] = 1$. In particular, setting $C = \gamma_{r_1+1}(G)$,

$$[A,x_1,x_2,\dots,x_c] \leq C_G(C). \tag{19}$$

Since A is normal and abelian, this yields $[A,_c G, C] = 1$. Since $c \leq r_1+1$ and $[A,\gamma_c(G)] \leq [A,_c G]$, we obtain

$$[A,C'] \leq [A,C,C] = 1 \tag{20}$$

for any abelian normal subgroup A of G. Now, let $x \in G$. If $K = \langle x \rangle^G$, then K/K' is an abelian normal subgroup of G/K', and so $[K, C'] \leq K'$, by (20). Since, by the remark following (15), K has class at most ℓ, a simple inductive argument using the Three Subgroups Lemma, shows that

$$[K,_\ell C'] = 1.$$

This holds for every $x \in G$; in particular we have $\gamma_{\ell+1}(C') = 1$. Therefore, G is soluble of derived length bounded by $[\log_2 r_1] + [\log_2 \ell] + 3$. By Lemma 158, we conclude that G is nilpotent of class at most

$$\rho(n) = c(n, [\log_2 r_1] + [\log_2 \ell] + 3).$$

This completes the proof of the Theorem. $\boxed{QED}$

For torsion-free groups, it follows form Zel'manov deep result on bounded Engel groups (Theorem 67) that groups in $\mathfrak{U}_{n,1}$ are nilpotent of bounded class. Obviously, this is not in general the case: for instance, let $G = C \wr A$ be the wreath product of a group of order 2 by an infinite elementary abelian 2-group, and let B denote its base group; then for every $x \in G$, $\langle x \rangle^G \leq B\langle x \rangle$ is nilpotent of class at most 2 and from Fitting theorem it follows that, for every $n \geq 1$ and $x_1, \ldots x_n \in G$,$\langle x_1, \ldots x_n \rangle^G$ has class at most $2n$; hence, every n-generated subgroup of G has defect at most $2n$ in its normal closure, and so $G \in \mathfrak{U}_{2n+1,n}$ (for every $n \geq 1$), but G is not nilpotent . Indeed groups in $U_{2n+1,n}$ need not even be soluble: using the same argument (via Proposition 150) one shows that the mentioned Bachmuth–Mochizuki group ([2]), which is not soluble, belongs to $\mathfrak{U}_{2n+1,n}$ for every $n \geq 1$. In his original paper, Roseblade asks the following:

2 Question. Is $\mathfrak{U}_{n,n} \subseteq \mathfrak{N}_{\rho_1(n)}$, for some positive integer $\rho_1(n)$?

(in view of the examples given above, the feeling is that $\mathfrak{U}_{n,r} \subseteq \mathfrak{N}$ for some $n/2 \leq r \leq n$). Also, it follows from Roseblade's Theorem that, for every $n \geq 1$, there exists $r(n) \geq 1$, such that $\mathfrak{U}_{n,r(n)} = \mathfrak{U}_n$ (it is plain that $r(1) = 1$ and not difficult to see that $r(2) = 2$); thus, a related question is

3 Question. Find a reasonable bound for $r(n)$.

An Engel-type version of these questions could be the following.

4 Question. Let G be a group, $n \geq 1$, and suppose that

$$[g, x_1, \ldots, x_n] \in \langle x_1, \ldots, x_n \rangle$$

for every $g, x_1, \ldots, x_n \in G$. Is it true that G is nilpotent of class bounded by a function of n?

For locally nilpotent torsion groups, Roseblade's Theorem has been generalized by E. Detomi ([24]) along a direction which is clearly suggested by Brookes' trick (Theorem 98).

160 Theorem. *Let G be a periodic locally nilpotent group. Assume that there exist a finite subgroup F of G, and $n \in \mathbb{N}$, such that every subgroup of G containing F is subnormal of defect at most n in G. Then $\gamma_{\beta(n)+1}(G)$ is finite for a positive integer $\beta(n)$ depending only on n. In particular, G is nilpotent.*

We begin the proof with a rather simple observation.

161 Lemma. *Let A be a normal subgroup of the group G, such that $G/C_G(A)$ is abelian. Suppose that there exist $1 \leq n, m \in \mathbb{N}$ such that $|[A, x_1, \ldots, x_n]| \leq m$ for all $x_1, \ldots, x_n \in G$. Then $|[A,_{2n} G]| \leq g(n, m)$, where $g(1, m) = (m!)^2$, and $g(n + 1, m) = (g(n, m)!)^2$.*

Proof. Observe first that, since $G/C_G(A)$ is abelian, $[A, xy] = [A, yx]$ for every $x, y \in G$. Hence, for all $x, y \in G$ $[A, x]$ is normal in G, and $[A, x, y] = [A, y, x]$.

Assume first $n = 1$ and proceed by induction on m. If $m = 1$, we have nothing to prove. Thus, let $m \geq 2$. If $[A, x, y] = 1$ for all $x, y \in G$ then we are done. Otherwise, there exist $x, y \in G$ such that $[A, x, y] \neq 1$. Let $N = [A, x][A, y]$, and $\overline{G} = G/N$.
Now, if $\overline{z} = zN \in \overline{G}$, then $[\overline{A}, \overline{z}] = [A, z]N/N \cong [A, z]/([A, z] \cap N)$. Suppose that there exists an element $\overline{z} \in \overline{G}$ such that $|[\overline{A}, \overline{z}]| \geq m$. Then $[A, z] \cap N = 1$. In particular, $[N, z] \leq N \cap [A, z] = 1$, which in turn implies

$$[A, x, z] = [A, y, z] = 1.$$

Also, as $[\overline{A}, \overline{x}] = 1$, we get $[\overline{A}, \overline{xz}] = [\overline{A}, \overline{x}][\overline{A}, \overline{z}][\overline{A}, \overline{x}, \overline{z}] = [\overline{A}, \overline{z}]$, and so, by the same argument used above, $N \cap [A, xz] = 1$ and $[a, y, xz] = 1$. Hence, for all $a \in A$, we have

$$[a, y, xz] = [a, xz, y] = [[a, x][a, x, z][a, z], y] = [a, x, y][a, z, y] = [a, x, y].$$

Thus, we reach the contradiction $1 = [A, y, xz] = [A, x, y] \neq 1$. Therefore, $|[\overline{A}, \overline{z}]| \leq m - 1$ for all $\overline{z} \in \overline{G}$. By inductive hypothesis, we then have

$$|[\overline{A},_2 \overline{G}]| \leq ((m - 1)!)^2,$$

and consequently,

$$|[A,_2 G]| \leq |[\overline{A},_2 \overline{G}]||N| \leq ((m - 1)!)^2 m^2 = (m!)^2.$$

Thus, the Lemma is proved for $n = 1$, and we now continue the proof by induction on n. If we fix $x \in G$, then $[A, x] \trianglelefteq G$ and $G/C_G([A, x])$ is abelian, as

$C_G([A,x]) \geq C_G(A)$. Moreover, for all $x_2,\ldots,x_n \in G$, $|[[A,x],x_2,\ldots,x_n]| \leq m$. Hence, by inductive assumption,

$$|[[A,x],_{2(n-1)}G]| \leq g(n-1,m).$$

This holds for every $x \in G$. But $[[A,x],_{2(n-1)}G] = [[A,_{2(n-1)}G],x]$. It then follows by the case $n=1$ that

$$|[A,_{2n}G]| = |[[A,_{2(n-1)}G],_2 G]| \leq g(1,g(n-1,m)) = (g(n-1,m)!)^2 = g(n,m),$$

thus completing the proof. $\boxed{QED}$

To shorten the notation, let us denote by $\mathfrak{U}_n^+$ the class of all *locally nilpotent* groups which admit a finite subgroup F such that all subgroups $F \leq H \leq G$ are subnormal of defect at most n in G.

162 Lemma. *Let $G \in \mathfrak{U}_n^+$, and suppose that G has a nilpotent subgroup N, with finite index in G and nilpotency class c. Then $\gamma_{cn+1}(G)$ is finite.*

Proof. Since N has finite index, we may possibly replace it by its normal core N_G. Thus, we assume $N \trianglelefteq G$. Let T be the torsion subgroup of G. Then G/T is a locally nilpotent torsion-free group with a subgroup of finite index NT/T, which is nilpotent of class at most c; by Corollary 114, G/T is nilpotent of class at most c; thus $\gamma_{c+1}(G) \leq T$.

Let F be a finite subgroup of G such that all subgroups of G containing F are subnormal of defect at most n. Let T be a transversal of G modulo N, and set $H = \langle F,T\rangle$. Then H is finitely generated (hence nilpotent) and subnormal of defect $d \leq n$ in G. If $d=1$, then $G/H = NH/H \simeq N/N\cap H$ is nilpotent of class at most c; hence $\gamma_{c+1}(G) \leq H \cap T$ is finite (because $H \cap T = \mathrm{Tor}(H)$), and we are done. Continuing by induction on d, let $d \geq 2$. Now, $H^G \in \mathfrak{U}_n^+$ and $N \cap H^G$ is a finite-index subgroup of H^G; so $\gamma_{c(d-1)+1}(H^G)$ is finite by inductive assumption. Therefore, by Fitting's Theorem applied to $G = NH^G$, we conclude

$$|\gamma_{cd+1}(G)| \leq |\gamma_{(d-1)c+1}(H^G)| \cdot |\gamma_{c+1}(N)| < \infty$$

which is what we wanted. $\boxed{QED}$

This allows to prove the specific Hall–type reduction needed.

163 Lemma. *There exists a function $f(d,c;n)$ with the following property. Let $G \in \mathfrak{U}_n^+$ and N a normal subgroup of G; if $\gamma_{c+1}(N)$ and $\gamma_{d+1}(G/N')$ are finite, then $\gamma_{f(d,c;n)}(G)$ is finite.*

Proof. Since $\gamma_{c+1}(N)$ is a finite normal subgroup of G, we may well assume that $\gamma_{c+1}(N) = 1$. Now, by a result of P. Hall (Proposition 53), we have that $A/N' = \zeta_{2c}(G/N')$ has finite index in G/N'. Since A/N' has nilpotency class at most $2d$, Fitting Theorem yields that AN/N' has nilpotency class at most

$2d+1$. Then, by Hall criterion (Theorem 56), AN is nilpotent of class at most $m = \binom{c+1}{2}(2d+1) - \binom{c}{2}$. As AN has finite index in G, we finally apply Lemma 162 to get the desired conclusion. $\boxed{QED}$

PROOF. OF THEOREM 160. We proceed by induction on $n \geq 1$. If $n = 1$, $F \trianglelefteq G$ and G/F is a Dedekind group; thus $\beta(1) = 1$.

Assume $n \geq 2$, and let $N = F^G$. Then $N \in \mathfrak{U}^+_{n-1}$, and so, by inductive assumption, $\gamma_{\beta(n-1)+1}(N)$ is finite. By Lemma 163 we are done if we show that $\gamma_k(G/N')$ is finite for some k depending only on n. Thus, we may assume that $N = F^G$ is abelian.

By Roseblade Theorem, G/N is nilpotent of class bounded by $\rho(n)$; in particular the derived length ℓ of $G/C_G(N))$ is bounded by $\log_2(\rho(n))$. Fixed n, we argue by induction on ℓ.

Thus, assume first that $\ell = 1$, i.e. G' centralizes N. Let $\pi = \pi(F)$ be the set of all prime divisors of $|F|$. Then N is an abelian π-group. Given $p \in \pi$, let X_p be the product of all q-components of N with $q \neq p$; then $X_p \trianglelefteq G$ and N/X_p is a p-group. If we prove that $\gamma_\mu(G/X_p)$ is finite for a uniform $\mu = \mu(n)$, then we are done because π is a finite set of primes. Thus, we may suppose that $N = F^G$ is an abelian p-group for some prime p. Let $|F| = p^k$, let p^r be the exponent of F, and observe that p^r is also the exponent of N. Write $\overline{G} = G/N^p$, $\overline{N} = N/N^p$, and so on. Let $x \in G$, and $\overline{x} = xN^p$. By assumption, $\langle \overline{F}, \overline{x} \rangle$ is nilpotent and subnormal; hence, by 61, $\langle \overline{N}, \overline{x} \rangle$ is nilpotent. Also, every subgroup of $\langle \overline{N}, \overline{x} \rangle$ containing $\overline{F}$ has defect at most n, so by Lemma 121 we have

$$[\overline{N},_{f_p(k,n)-1} \overline{x}] = 1.$$

Let s be the smallest power of p grater than $f_p(k,n)-1$. As $\overline{N}$ is an elementary abelian p-group, it follows from 14 that $[\overline{N}, \overline{x}^s] = 1$; i. e.

$$[N, x^s] \leq N^p \quad \text{for all } x \in G. \tag{21}$$

Write now $t = sp^{r(\log_p r+1)}$. Since N has exponent p^r, (21) yields

$$[N, x^t] = 1 \quad \text{for all } x \in G. \tag{22}$$

Thus, the exponent of $G/C_G(N)$ is at most t. Now, take $x_1, x_2, \ldots, x_{\rho(n)} \in G$, and let $H = \langle F, x_1, \ldots, x_{\rho(n)} \rangle = F^H \langle x_1, \ldots, x_{\rho(n)} \rangle$. Since $H/CH(N)$ is abelian (as such, by assumption, is $G/C_G(N)$), its order is at most $t^{\rho(n)}$, and consequently

$$|F^H| \leq |F|^{t^{\rho(n)}}.$$

Now, $F \leq F^H \trianglelefteq NH$, and all subgroups of NH/F^H are subnormal of defect at most n. By Roseblade's Theorem, NH/F^H is nilpotent of class at most $\rho(n)$;

hence $[N,_{\rho(n)} H] \leq F^H$, and, in particular,

$$\left|[N, x_1, \dots, x_{\rho(n)}]\right| \leq \left|F^H\right| \leq |F|^{t^{\rho(n)}}. \tag{23}$$

We may then apply Lemma 161 and have that $[N,_{2\rho(n)} G]$ is finite. Now, G/N is nilpotent of class at most $\rho(n)$, and so we conclude that

$$\gamma_{3\rho(n)+1}(G) = [\gamma_{\rho(n)+1}(G),_{2\rho(n)} G] \leq [N,_{2\rho(n)} G]$$

is finite. Thus, the case in which $G/C_G(N)$ is abelian is done.

Suppose now $\ell \geq 2$. Let $K = C_G(N)G'$. Then, by inductive assumption, $\gamma_\nu(K)$ is finite for some ν which depends only on n and $\ell-1$. Also, $N \leq K$, and $C_G(NK'/K') \geq K \geq G'$. It is now easy to see that we may apply the previous case to the group G/K', concluding that $\gamma_{3\rho(n)+1}(G/K')$ is finite. Applying Lemma 163 we thus conclude that $\gamma_k(G)$ is finite, for some k that ultimately depends only on n. By the remarks made at the beginning, this completes the proof. $\boxed{QED}$

In her paper, Detomi also shows that Theorem 160 does not hold when dropping the assumption of G being locally nilpotent (an example in which $|F| = 2$ and $\gamma_2(G) = \gamma_3(G)$ is infinite is given), nor it is true for locally nilpotent non-periodic groups; although she proves that, in this case, G is hypercentral.

164 Proposition. *Let F be a finitely generated subgroup of the locally nilpotent group G, and suppose that there exists $n \geq 1$ such that every subgroup of G containing F has defect at most n. Then G is hypercentral (and soluble).*

Proof. By assumption F is nilpotent and subnormal in G, and, by Roseblades Theorem, each section of the normal closure series of F in G is nilpotent; thus, G is soluble.

Now, it is clearly enough to prove that G has non-trivial centre. If $n = 1$, then $F \trianglelefteq G$, G/F is nilpotent of class at most two, and F, as a finitely generated normal subgroup of a locally nilpotent group, is contained in some term of the upper central series of G, which is then nilpotent. Thus, letting $n \geq 2$, and assuming that the claim is true for smaller values, we may suppose $Z(F^G) \neq 1$; in particular, F is contained in a normal subgroup N of G which has non-trivial centre. We now proceed by induction on the derived length d of G/N. If $d = 0$, then $G = N$ has non-trivial centre. Let $d \geq 1$, set $G_1 = G'N$ and $A = Z(G_1)$. Then $A \neq 1$, by inductive hypothesis. Also, A is a normal subgroup of G. Let U be a finitely generated subgroup of G containing F. By assumption, U is subnormal of defect at most n in G, hence $[A,_n U] \leq U$ and $[A,_n U]$ is finitely generated,. As $G' \leq C_G(A)$, for every $g \in G$ we get

$$[A,_n U]^g = [A,_n U^g] \leq [A,_n U[U,g]]] = [A,_n U],$$

and so $[A,_n U]$ is normal in G. As $[A,_n U]$ is finitely generated, $[A,_n U] \leq \zeta_k(G)$ for some $k \geq 1$. Thus, if $[A,_n U] \neq 1$, then $Z(G) \neq 1$. Otherwise, if $[A,_n U] = 1$ for any finitely generated subgroup U of G (containing F), then $[A,_n G] = 1$, i.e. $A \leq \zeta_n(G)$, and again $Z(G) \neq 1$. $\boxed{QED}$

4.3 First applications

A first immediate application of Roseblade's Theorem allows to reduce the study of periodic $\mathcal{N}_1$-groups to the case of p-groups.

165 Lemma. *Let G be periodic $\mathcal{N}_1$-group. Then there exists $1 \leq m \in \mathbb{N}$, such that all but finitely many primary components of G are nilpotent of nilpotency class at most m. in particular, G is nilpotent if and only if all of its primary components are nilpotent.*

Proof. Since G is locally nilpotent, it is isomorphic to the direct product of its primary components. In one sense, the implication is trivial. Conversely, suppose that all primary components of G are nilpotent. If the nilpotency class of the components is not bounded, then by Roseblade's Theorem, for each positive integer n there is a primary component P_n of G and a subgroup H_n of P_n of defect n (and $P_n \neq P_k$ if $n \neq k$). But then, the subgroup $H = \langle H_n | n \in \mathbb{N}\rangle$ cannot be subnormal in G. Thus, the nilpotency class of the primary components of G is bounded, and therefore G is nilpotent. $\boxed{QED}$

Then, in conjunction with Brookes' trick, a first step towards the proof of solubility of $\mathcal{N}_1$-groups.

166 Lemma. *Let G be a $\mathcal{N}_1$-group. Then there exists a $1 \leq n \in \mathbb{N}$ such that $G^{(n)} = G^{(n+1)}$.*

Proof. Let G be a $\mathcal{N}_1$-group which we may clearly assume not to be soluble. Then, by Theorem 98 applied to the family on non-soluble subgroups of G, there exists a non-soluble subgroup H of G, a finitely generated subgroup F of H, and a positive integer d, such that for every $F \leq K \leq H$, if K is not soluble, then K has defect at most d in H. Let $\rho = \rho(d)$ be the bound in Roseblade's Theorem 155, and let $d = [\log_2(\rho)] + 1$. Now, if K is a non-soluble subgroup of H containing F, then K^H is not soluble, and so all subgroups of H/K^H have defect at most n. By Roseblade's Theorem, H/K^H is nilpotent of class at most ρ, and so it is soluble of derived length at most d. $\boxed{QED}$

167 Corollary. [[13]]. *A residually soluble $\mathcal{N}_1$-group is soluble.*

Now, an application of Detomi's Theorem.

168 Proposition. [H.Smith [107]]. *A periodic residually finite $\mathcal{N}_1$-group is nilpotent.*

Proof. Let G be a periodic residually finite $\mathcal{N}_1$-group. Since every subgroup of G is residually finite, we may assume that G is countable. By Theorem 98 there exists a subgroup H of finite index, a finitely generated subgroup F of H, and a positive integer d, such that every $F \leq K \leq H$, such that $|H : K|$ is finite, has defect at most d in H. Now, let K be a finitely generated subgroup of H containing F. Since G is periodic, K is finite, whence, by Lemma 30, K is an intersection of subgroups of finite index of H. It follows that K has defect at most d in H. This implies that every subgroup of H containing F has defect at most d in H. By Theorem 160, H is nilpotent. Since $|G : H|$ is finite, we conclude that G is nilpotent. $\boxed{QED}$

Both Corollary 167 and Proposition 168 will be later superseded (respectively, by Theorem 206 and Theorem 198).

5 Periodic $\mathcal{N}_1$-groups

5.1 $\mathcal{N}_1$-groups of finite exponent

In this section we prove that a soluble $\mathcal{N}_1$-group of finite exponent is nilpotent; a most important result, due to W. Möhres, which lies at the core of the whole theory of $\mathcal{N}_1$-groups. Möhres proof is based on a delicate analysis ([76] and [77]) of p-groups which are the extension of two (infinite) elementary abelian groups, and we rather closely follow his approach.

For the next results, up to Proposition 177, we fix the following notation: p is a given prime, A an elementary abelian p-group, and B an elementary abelian p-group acting on A.

We recall a couple of elementary facts. From Lemma 12, we have that if $n \geq 1$, $x_1, \dots, x_n \in B$ and σ is a permutation of $\{1, \dots, n\}$ then

$$[a, x_1, \dots, x_n] = [a, x_{\sigma(1)}, \dots, x_{\sigma(n)}]$$

for any $a \in A$, while from Lemma 14 it follows that $[A,_p x] = 1$ for all $x \in B$.

We set $Z_0 = \{1\}$ and, for every $n \in \mathbb{N}$, $Z_{n+1}/Z_n = C_{A/Z_n}(B)$. Then, for every $a \in A$ and $n \geq 1$, $a \in Z_n$ if and only if $[a, x_1, \dots, x_n] = 1$ for every $x_1, \dots, x_n \in B$. Observe also that if U is a finite B-invariant subgroup of A, then $U \leq Z_{\log_p |U|}$. Finally, if B is finite then, by Corollary 79, the natural semidirect product AB is nilpotent, so there exists $n \in \mathbb{N}$ such that $[A,_n B] = 1$.

The first Lemma we prove is a standard tool in the theory of (soluble) p-groups of finite exponent and Lie algebras in characteristic p.

169 Lemma. *Let $0 \leq n \leq p-1$, $a \in A$ and $x_1, \dots, x_n \in B$. Suppose that $[a, x_1, \dots, x_n] \neq 1$. Then there exists $x \in \langle x_1, \dots, x_n \rangle$, such that $[a,_n x] \neq 1$.*

Proof. We argue by induction on n. If $n = 1$ the claim is trivial. Thus, let $n \geq 2$, and let $X = \langle x_1, \dots, x_n \rangle$. Since X is finite, there exists $k \in \mathbb{N}$ such that $[A,_k X] = 1$. So, in order to prove the Lemma, we may well assume $[A,_{n+1} X] = 1$. By inductive assumption there exists $y \in \langle x_2, \dots, x_n \rangle$ such that $[[a, x_1],_{n-1} y] \neq 1$. For every $i \in \{0, 1, \dots, n\}$ let $b_i = [a,_{n-i} x_1,_i y]^{\binom{n}{i}}$. Now, since $[A,_{n+1} X] = 1$ the substitution of elements from X in commutators of type $[a, t_1, \dots, t_n]$ is linear in every component (see Lemma 49). From this it easily follows that, for every $0 \leq k \leq n$,

$$[a,_n x_1 y^k] = \prod_{i=o}^{n} [a,_{n-i} x_1,_i y^k]^{\binom{n}{i}} = \prod_{i=o}^{n} b_i^{k^i}.$$

Now, the reduction modulo p of the $(n+1) \times (n+1)$ matrix $(k^i)_{k,i=0,\dots,n}$ is a Vandermonde matrix on $\mathbb{Z}/p\mathbb{Z}$, and so its determinant is not zero. Since $b_{n-1} = [a, x_1,_{n-1} y] \neq 1$, it thus follows that there exists $0 \leq k \leq n$ such that $[a,_n x_1 y^k] \neq 1$. As $x_1 y^k \in X$, this is what we wanted to show. $\boxed{QED}$

The case of this Lemma that we will use frequently is when $n = p-1$. Observe that if $a \in A$ and $x \in B$ are such that $[a,_{p-1} x] \neq 1$ then $M = \langle a \rangle^{\langle x \rangle}$ has order p^p (in fact, if $|M| \leq p^{p-1}$ then, as M is $\langle x \rangle$-invariant, $[M,_{p-1} x] = 1$). Thus, both $\{a, a^x, \dots, a^{x^{p-1}}\}$ and $\{a, [a, x], \dots, [a,_{p-1} x]\}$ are independent generating sets for M (in general, if, for some $1 \leq n \leq p-1$, $[a,_n x] \neq 1$, then $a, a^x, \dots, a^{x^n}$ are independent). In other words, M is the regular $\mathbb{F}_p[\langle x \rangle]$–module. Observe also that, for every $a \in A$ and $x \in B$, $[a,_{p-1} x] = a a^x \cdots a^{x^{p-1}}$.

These remarks are further extended in the next Lemma.

170 Lemma. *Let $n \geq 1$, and $a \in A$.*

(i) If $a \notin Z_{n(p-1)}$, there exist $x_1, \dots, x_n \in B$ with $[a,_{p-1} x_1, \dots,_{p-1} x_n] \neq 1$.

(ii) If $x_1, \dots, x_n \in B$ are such that $[a,_{p-1} x_1, \dots,_{p-1} x_n] \neq 1$, then $x_1, \dots, x_n$ are independent in B (whence $\langle x_1, \dots, x_n \rangle = p^n$).

(iii) If $x_1, \dots, x_n \in B$ are such that $[a,_{p-1} x_1, \dots,_{p-1} x_n] \neq 1$, then the set of all elements $[a,_{t_1} x_1, \dots,_{t_n} x_n]$, for every $(t_1, \dots, t_n) \in \{0, 1, \dots, p-1\}^n$ is linearly independent.

Proof. (i) For $n = 1$ the claim follows from Lemma 169. Let $n \geq 2$ and assume the property holds for $n-1$. If $a \in A \setminus Z_{n(p-1)}$ then, by inductive assumption, there exists $x_1, \dots, x_{n-1}$ such that $[a,_{p-1} x_1, \dots,_{p-1} x_{n-1}] \notin Z_{p-1}$, whence by case $n = 1$, we find $x_n \in G$, with $[a,_{p-1} x_1, \dots,_{p-1} x_{n-1},_{p-1} x_n] \neq 1$.

(ii) The fact is trivial for $n = 1$. Thus, arguing by induction on n, we suppose that $x_1, \dots, x_{n-1}$ are linearly independent. Now, $[a,_{p-1} x_1, \dots,_{p-1} x_{n-1}, x_i] = 1$

for every $i = 1, \ldots, n-1$. Hence $b = [a,_{p-1} x_1, \ldots,_{p-1} x_{n-1}]$ is centralized by $Y = \langle x_1, \ldots, x_{n-1} \rangle$. If $x_n \in Y$ we have a contradiction. Therefore $x_n \notin Y$ and $x_1, \ldots, x_{n-1}, x_n$ are linearly independent.

(iii) By induction on n. For $n = 1$ this fact has already been observed. Thus, let $n \geq 2$, Δ a non-empty subset of $\{0, \ldots, p-1\}^n$, and for each let be given an integer k_t with $t \in \Delta$ let $1 \leq k_t \leq p-1$. We have to show that $b = \prod_{t \in \Delta} [a,_{t_1} x_1, \ldots,_{t_n} x_n]^{k_t} \neq 1$. Let $m = \min\{t_n \mid t \in \Delta\}$, $s = p - 1 - m$, and $\Delta_0 = \{t \in \Delta \mid t_n = m\}$. If $c = [a,_{p-1} x_n]$, then

$$[b,_s x_n] = \prod_{t \in \Delta_0} [a,_{t_1} x_1, \ldots,_{t_{n-1}} x_{n-1},_{p-1} x_n]^{k_t} = \prod_{t \in \Delta_0} [c,_{t_1} x_1, \ldots,_{t_{n-1}} x_{n-1}]^{k_t}.$$

By inductive assumption $[b,_s x_n] \neq 1$, whence $b \neq 1$. $\boxed{QED}$

In the hypothesis of point (iii) of the previous Lemma, let $X = \langle x_1, \ldots, x_n \rangle$. It then follows from (ii) and (iii) that $|X| = p^n$ and $|\langle a \rangle^X| = p^{p^n}$. Hence $C_X(a) = 1$ and $\{a^x \mid x \in X\}$ is a set of independent generators of $\langle a \rangle^X$. After these remarks one easily deduce the following Lemma.

171 Lemma. *Let $n \in \mathbb{N}$, $a \in A \setminus Z_{n(p-1)}$, and let $X = \langle x_1, \ldots, x_n \rangle \leq B$, with $[a,_{p-1} x_1, \ldots,_{p-1} x_n] \neq 1$; then*

(i) if $y_1, \ldots, y_m \in X$ are independent, then $[a,_{p-1} y_1, \ldots,_{p-1} y_m] \neq 1$;

(ii) $X \cap C_B(a) = 1$, and so $C_B(a)$ has index at least p^n in B.

We now move to some more specific facts.

172 Lemma. *Let $n, s \in \mathbb{N}$ with $n \geq 1$ and $p^s > n$. If $a_1, \ldots, a_n \in A \setminus Z_{s(p-1)}$ then there exists $x \in B$ such that $[a_a, x] \neq 1$ for every $i = 1, \ldots, n$.*

Proof. By point (i) of Lemma 170 and point (ii) of Lemma 171, we have that $|B : C_B(a_i)| \geq p^s$ for every $i = 1, \ldots, n$. Since $p^s > n$, a result of B. H. Neumann [84] implies $B \neq \bigcup_{i=1}^n C_B(a_i)$, and the claim follows. $\boxed{QED}$

173 Lemma. *Let $n \geq 1$, $t = t_n = (p-1)^{2n-1}$, $a_1, \ldots, a_n \in A$, $x_1, \ldots x_t \in B$, and suppose that $[a_i, x_1, \ldots, x_t] \neq 1$, for every $i = 1, \ldots n$. Then there exists $y \in \langle x_1, \ldots, x_t \rangle$ such that $[a_i,_{p-1} y] \neq 1$ for every $i = 1, \ldots, n$.*

Proof. By induction on n. Case $n = 1$ follows from Lemma 169. Thus, let $n \geq 2$, $t = (p-1)^{2n-1}$, and assume the claim true for $n-1$. Let $s = (p-1)^{2n-2}$; then $t = (p-1)s = (p-1)^2 t_{n-1}$. We show that for each $j = 1, \ldots, p-1$, there exists $y_j \in X_j = \langle x_{(j-1)s+1}, \ldots, x_{js} \rangle$, such that

$$\begin{cases} [a_n, y_1, \ldots, y_j, x_{js+1}, \ldots, x_t] \neq 1 \\ [a_i,_{p-1} y_1, \ldots,_{p-1} y_j x_{js+1}, \ldots, x_t] \neq 1 \quad \text{for } i = 1, \ldots, n-1. \end{cases} \tag{24}$$

We start by finding y_1. For each $i = 1, \ldots, n$, we set $b_i = [a_i, x_{s+1}, \ldots, x_t]$. Then, by assumption, $[b_i, x_1, \ldots, x_s] \neq 1$ for all $i = 1, \ldots, n$. Now, by Lemma 171, $C_{X_1}(b_n)$ has index at least $p^{s/(p-1)} = p^{t_{n-1}}$ in X_1. Thus, there is a linearly independent subset $\{z_1, \ldots, z_{t_{n-1}}\}$ of $\{x_1, \ldots, x_s\}$, such that $Y = \langle z_1, \ldots, z_{t_{n-1}} \rangle$ intersects trivially $C_{X_1}(b_n)$. By the inductive assumption on n, we then find $y_1 \in Y$ such that $[b_{i,p-1}\, y_1] \neq 1$ for for all $j = 1, \ldots, n-1$. Then

$$[a_{i,p-1} y_1, x_{s+1}, \ldots, x_t] = [b_{i,p-1}\, y_1] \neq 1$$

for $i = 1, \ldots, n-1$; and $[a_n, y_1, x_{s+1}, \ldots, x_t] = [b_n, y_1] \neq 1$. So conditions 24 are satisfied for $j = 1$.

Suppose that, for $1 \leq k < p-1$, we are given $y_1, \ldots, y_k$ with the required properties. Then, by setting $b_i = [a_{i,p-1}\, y_1, \ldots,_{p-1}\, y_k, x_{(k+1)s+1}, \ldots, x_t]$ for $i = 1, \ldots, n-1$, $b_n = [a_n, y_1, \ldots, y_k, x_{(k+1)s+1}, \ldots, x_t]$, and repeating the same argument used for $j = 1$ we find $y_{k+1} \in \langle x_{ks+1}, \ldots, x_{(k+1)s} \rangle$ that together with $y_1, \ldots, y_k$ satisfies 24.

Thus, we eventually get elements $y_1, \ldots, y_{p-1} \in \langle x_1, \ldots, x_t \rangle$ such that

$$\begin{cases} [a_n, y_1, \ldots, y_{p-1}] \neq 1 & \\ [a_{i,p-1} y_1, \ldots,_{p-1} y_{p-1}] \neq 1 & \text{for } i = 1, \ldots, n-1. \end{cases}$$

By the first of these inequalities and Lemma 169 it follows that there exists $y \in \langle y_1, \ldots, y_{p-1} \rangle$ such that $[a_{n,p-1}\, y] \neq 1$. But from the remaining inequalities and Lemma 171 we also have $[a_{i,p-1}\, y] \neq 1$ for all $i = 1, \ldots, n-1$, thus finishing the proof. $\boxed{QED}$

174 Proposition. *There exists a function $\alpha : \mathbb{N}\backslash\{0\} \to \mathbb{N}$, with the property that if U is a subgroup of A of order at most p^n and $U \cap Z_{\alpha(n)} = 1$, then there exists $y \in B$ such that $[a_{,p-1}\, y] \neq 1$ for all $1 \neq a \in U$.*

Proof. For $1 \leq n \in \mathbb{N}$, we set $\alpha(n) = n(p-1)^{2p^n-2}$. Let U be a subgroup of A with $|U| \leq p^n$ and $U \cap Z_{\alpha(n)} = 1$. Let $s = (p-1)^{2p^n-3}$; thus $\alpha(n) = n \cdot s \cdot (p-1)$. Since $p^n > |U| \setminus \{1\}$, it follows from Lemma 172 that there exist elements $x_1, \ldots, x_s$ in B such that $[a, x_1, \ldots, x_s] \neq 1$ for all $a \in U \setminus \{1\}$. But $s \geq (p-1)^{2|U\setminus\{1\}|-1}$, and so, by Lemma 173, there exists $y \in B$ such that $[a_{,p-1}\, y] \neq 1$ for all $1 \neq a \in U$. $\boxed{QED}$

Observe that, if U and y are as in the statement of 174, then $|U^{\langle y \rangle}| = |U|^p$. In fact, by the Jordan canonical form, $U^{\langle y \rangle} = \langle u_1 \rangle^{\langle y \rangle} \times \ldots \times \langle u_s \rangle^{\langle y \rangle}$ for suitable $u_1, \ldots, u_s \in U$ with $U = \langle u_1, \ldots, u_s \rangle$. By the remark following Lemma 169, $|\langle u_i \rangle^{\langle y \rangle}| = p^p$ for every $i = 1, \ldots, s$. Hence $|U^{\langle y \rangle}| = p^{ps} = |U|^p$.

175 Lemma. *For every $n \geq 1$ there exists a function $f_n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that the following holds:*

if $|B| \geq p^{f_n(r,s)}$, $U \leq Z_n$, $z \in Z_1 \setminus U$, and $|U| \leq p^r$, then there exists $H \leq B$, with $|H| = p^s$ and $z \notin U^H$.

Proof. We argue by induction on n. Clearly, f_1 is given by $f_1(r,s) = s$, for all $(r,s) \in \mathbb{N}\times\mathbb{N}$. We then assume that, for $n \geq 2$, functions f_i have been found for $1 \leq i \leq n-1$, and proceed to define the values $f_n(r,s)$.

Trivially, for any $r,s \in \mathbb{N}$, $f_n(0,s) = s$ and $f_n(r,0) = 0$.

To provide $f_n(1,1)$ let us introduce an auxiliary function $h : \mathbb{N}\setminus\{0\} \to \mathbb{N}$, by setting $h(1) = f_{n-1}(1,1)$ and, for $t \geq 2$, $h(t) = f_{n-1}(n-2, h(t-1)) + 1$. Then let $f_n(1,1) = h((p-1)^2 + 1)$.

Suppose that, for the given A and B, the conclusion of the statement fails for $r = s = 1$ (and holds for $n-1$). Then, there exist

$$1 \neq \mathrm{a} \in \mathrm{Z_n} \setminus \mathrm{Z_{n-1}},\ \mathrm{z} \in \mathrm{Z_1} \setminus \langle \mathrm{a}\rangle \text{ such that } \mathrm{z} \in \langle \mathrm{a}\rangle^{\langle \mathrm{y}\rangle} \text{ for all } 1 \neq y \in B. \tag{25}$$

If $n \geq p+1$ then $a \notin Z_p$ and so, by 169 there exists $y \in B$ with $[a,_{p-1} y] \notin Z_1$, whence $Z_1 \cap \langle a\rangle^{\langle y\rangle} = Z_1 \cap \langle a, [a,x], \dots [a,_{p-1} x]\rangle = 1$. Thus, $n \leq p$.

For every $1 \neq y \in B$ we denote by $d(y)$ the smallest positive integer such that $[a,_{d(y)} y] \neq 1$. Thus $\langle [a,_{d(y)} y]\rangle = C_A(y) \cap \langle a\rangle^{\langle y\rangle}$. By our assumptions, for every $1 \neq y \in B$, we have $1 \leq d(y) \leq n-1$ and $\langle z\rangle = \langle [a,_{d(y)} y]\rangle$. So there is a uniquely determined $1 \leq m(y) \leq p-1$ with $[a,_{d(y)} y] = z^{m(y)}$.

We say that a subset $\{y_1, \dots, y_t\}$ of B is *stable* (with respect to a and z) if

– $y_1, \dots, y_t$ are independent;

– for every $\emptyset \neq \{i_1, \dots, i_s\} \subseteq \{1, \dots, t\}$, $d(x_{i_1} \cdots x_{i_s}) = n-1$ and

$$m(x_{i_1} \cdots x_{i_s}) \equiv \sum_{j=1}^{s} m(x_{i_j}) \pmod p.$$

Let $K \leq B$; we show, by induction on $t \geq 1$, that

$$\text{if } |K| \geq h(t) \text{ then } K \text{ admits a stable subset of cardinality } t. \tag{26}$$

For $t = 1$, $|K| \geq p^{f_{n-1}(1,1)}$. Then, by the inductive assumption on n and the assumption (25), $a \notin Z_{n-1}(K)$, and so, by Lemma 169, there exists $x \in K$ such that $d(x) = n-1$.

Thus, let $t \geq 2$, and suppose $|K| \geq p^{h(t)}$. As above, there exists $x_1 \in K$ such that $d(x_1) = n-1$. Let $V = \langle [a,x_1], \dots, [a,_{n-2} x_1]\rangle$. Since $[a,x_1], \dots, [a,_{n-1} x_1]$ are linearly independent, we have $|V| = p^{n-2}$ and $z = [a,_{n-1} x]^{-m(x_1)} \notin V$. Let $K = K_1 \times \langle x_1\rangle$; then $|K_1| \geq p^{f_{n-1}(n-2,h(t-1))}$. Since $V \leq Z_{n-1}$, the inductive assumption on n implies that there exists $H \leq K_1$ with $|H| = p^{h(t-1)}$ and $z \notin V^H$. Then $z \notin V^H(\langle a\rangle \cap Z_{n-1}) = V^H\langle a\rangle \cap Z_{n-1}$, whence $z \notin V^H\langle a\rangle$. We work with A/V^H acted on by H. If there exists $y \in H$ such that $z \notin V^H\langle a\rangle^{\langle y\rangle}$, then obviously $z \notin \langle a\rangle^{\langle y\rangle}$, which is in contrast with (25). Thus, H satisfies (25) on A/V^H with respect to aV^H and zV^H. By induction on t it follows that $H \leq K_1$ admits a subset $\{x_2, \dots, x_t\}$ of cardinality $t-1$, which is stable with respect to aV^H and zV^H (observe that, since $z \in Z_1$, this means, in particular, that $\{x_2, \ldots, x_t\}$ is stable with respect to a and z).

Now, $\{x_1, x_2, \ldots, x_t\}$ is an independent subset of K, and $d(x_i) = n-1$ for every $i = 1, \ldots, t$. Let $1 \neq y \in \langle x_2, \ldots, x_t \rangle$. Then, as $a \in Z_n$, for $1 \leq k \leq n-1$,

$$[a,_k x_1 y] = [a,_k x_1][a,_k y] \prod_{i=1}^{k-1} [a,_i x_1,_{k-i} y]^{\binom{n-1}{i}} \in [a,_k x_1][a,_k y] V^H. \tag{27}$$

Let $1 \leq s \leq t$, $\{i_1, \ldots, i_s\}$ a subset of $\{2 \ldots, t\}$, $y = x_{i_1} \cdots x_{i_s}$, and $d = d(x_1 y)$. By (27),

$$1 \neq z^{m(x_1 y)} = [a,_d x_1][a,_d y] v,$$

with $v \in V^H$, and so $[a,_{d+1} y] \in V^H$. Since the set $\{x_2, \ldots, x_t\}$ is stable with respect to aV^H and zV^H, necessarily we have $d = n-1$. Moreover, by applying again (27) with $k-n-1$, we have $[a,_{n-1} x_1 y] = [a,_{n-1} x_1][a,_{n-1} y] w$ with $w \in V^H$; but then $w \in V^H \cap \langle z \rangle$, i.e. $w = 1$. Hence

$$z^{m(x_1 y)} = [a,_{n-1} x_1 y] = [a,_{n-1} x_1][a,_{n-1} y] = z^{m(x_1)} z^{m(y)},$$

and, since $\{x_2, \ldots, x_t\}$ is stable with respect to a and z,

$$m(x_1 x_{i_1} \cdots x_{i_s}) = m(x_1 y) \equiv m(x_1) + m(y) \equiv m(x_1) + \sum_{j=1}^{s} m(x_{i_j}) \pmod p.$$

This completes the proof of claim (26).

Now, letting $t = (p-1)^2+1$, if we suppose (by contradiction) that $|B| \geq p^{h(t)}$, then by (26), B admits a stable subset $\{x_1, \ldots, x_t\}$ with respect to a and z, of cardinality t. Since, for each $1 \leq i \leq t$, $m(x_i) \in \{1, 2, \ldots, p-1\}$, there exists a subset $\{i_1, \ldots, i_p\}$ of $\{1, \ldots, t\}$ such that $m = m(x_{i_1}) = m(x_{i_j})$ for all $j = 1, \ldots, p$. But then stability of $\{x_1, \ldots, x_t\}$ implies the contradiction.

$$0 \neq m(x_{i_1} \cdots x_{i_s}) \equiv \sum_{j=1}^{p} m(x_{i_j}) = pm \equiv 0 \pmod p.$$

Therefore, if $|B| \geq p^{h((p-1)^2+1)}$, then B, in its action on A, cannot verify (25). Thus we may define $f_n(1,1) = h((p-1)^2+1)$.

Now, for $s \geq 1$, let $f_n(1,s) = \max\{f_{n-1}(n-1, f_n(1, s-1)) + 1, f_n(1,1)\}$; we prove by induction on s that this setting satisfies the desired property.

For $s = 1$ this has already been established. Thus, let $s \geq 2$, $|B| \geq p^{f_n(1,s)}$, $1 \neq a \in Z_n$ and $z \in Z_1 \setminus \langle a \rangle$. By the inductive assumption on n we may well suppose $a \in Z_n \setminus Z_{n-1}$. Let $x \in B$ such that $z \notin \langle a \rangle^{\langle x \rangle}$ (it exists by case

$s = 1$) and write $D = [\langle a\rangle, \langle x\rangle] = \langle [a, x], \dots, [a,_{n-1} x]\rangle$. Then, $D \le Z_{n-1}$ and $|D| \le p^{n-1}$. Let $B = B_1 \times \langle x\rangle$; then $|B_1| \ge p^{f_{n-1}(n-1, f_n(1,s-1))}$, and so, by the inductive assumption on n, there exists $V \le B_1$, with $|V| = p^{f_n(1,s-1)}$ and $z \notin D^V$. Thus $z \notin D^V(\langle a\rangle \cap Z_{n-1}) = D^V\langle a\rangle \cap Z_{n-1}$, and, in particular, $z \notin D^V\langle a\rangle$. Therefore, by the inductive assumption on s, there exists $W \le V$ with $|W| = p^{s-1}$ and $sD^V \notin \langle a\rangle^W D^V/D^V$; thus $z \notin \langle a\rangle^W D^V$. Let then $H = \langle W, x\rangle$. Since $W \le V \le B_1$, $H = W \times \langle x\rangle$. Thus $|H| = p^s$, and

$$\langle a\rangle^H = (\langle a\rangle^{\langle x\rangle})^W = (D\langle a\rangle)^W = D^W\langle a\rangle^W \not\ni z.$$

This completes the discussion of the case $r = 1$.

To conclude the proof we put, for every $r, s \ge 1$,

$$f_n(r, s) = \max\{f_{n-1}(r, s), f_n(r-1, f_n(1, s))\},$$

and show by induction on r that this satisfies the property in the statement. For $r = 1$ this has been proved above. Thus, let $r \ge 2$. $|B| \ge p^{f_n(r,s)}$, $U \le Z_n$ with $|U| \le p^r$, and let $z \in Z_1 \setminus U$. By induction on n we may also assume $U \notin Z_{n-1}$. Then, let $a \in U \setminus Z_{n-1}$, and let $U = \langle a\rangle \times U_1$, with $U \cap Z_{n-1} \le U_1$. Now, $|U_1| \le p^{r-1}$ and so, by the inductive assumption on r and the definition of $f_n(r, s)$, there exists $V \le B$, with $|V| = p^{f_n(1,s)}$ and $z \notin U_1^V$. Then

$$z \notin U_1[U_1, V] \ge [U_1, V](U \cap Z_{n-1}) = [U_1, V]U \cap Z_{n-1}$$

and so $z \notin [U_1, V]U = U_1^V\langle a\rangle$. Considering the action of V on A/U_1V, we have, by case $r = 1$, that there exists $H \le V$, with $|H| = p^s$ and $zU_1^V \notin \langle a\rangle^H U_1^V/U_1^V$. Then $z \notin \langle a\rangle^H U_1^V \ge \langle a\rangle^H U_1^H = U^H$. This completes the proof of the inductive step on r, and thus the proof of the Lemma. $\boxed{QED}$

We go on by eliminating the role of the parameter n in Lemma 176.

176 Lemma. *There exists a function $\alpha_1 : \mathbb{N} \to \mathbb{N}$, such that, for every $r \in \mathbb{N}$, the following holds:*

if $|B| \ge p^{\alpha_1(r)}$, $U \le A$ with $|U| \le p^r$, and $z \in Z_1 \setminus U$, then there exists $1 \ne x \in B$, with $z \notin U^{\langle x\rangle}$.

Proof. We set $\alpha_1(0) = 1$ and, inductively, $\alpha_1(n) = f_{\alpha(n)+1}(n, \alpha_1(n-1))$, where α and f_k are the functions of Proposition 174 and Lemma 175. We prove by induction on n that α_1 has the desired properties.

Thus, let $n \ge 1$, and $|B| \ge \alpha_1(n)$. Let $U \le A$ with $|U| \le p^n$, and $z \in Z_1 \setminus U$. Write $U = U_1 \times U_2$, where $U_1 = U \cap Z_{\alpha(n)+1}$.

If $U_1 = 1$, then $UZ_1/Z_1 \cap Z_{\alpha(n)}(A/Z_1) = 1$ and so, by Proposition 174 (since $\alpha_1(n) \ge \alpha(n)$), there exists $x \in B$ with $|U^{\langle x\rangle}Z_1/Z_1| = |U|^p$. But then $U^{\langle x\rangle} \cap Z_1 = 1$, and we are done.

Assume now $U_1 \neq 1$; then $|U_2| \leq p^{n-1}$. By the definition of $\alpha_1(n)$ and Lemma 175, there exists $H \leq B$ with $|H| = p^{\alpha_1(n-1)}$ and $z \notin K = U_1^H$. Now, $K \leq Z_{\alpha(n)+1}$, and so $KU_2 \cap Z_{\alpha(n)+1} = K(U_2 \cap Z_{\alpha(n)+1}) = K$; hence $z \notin KU_2$. By considering the action of H on A/K, we know, by the inductive assumption, that there exists $1 \neq x \in H$ such that $zK \notin (KU_2/K)^{\langle x \rangle} = KU_2^{\langle x \rangle}/K$.. Therefore $z \notin KU_2^{\langle x \rangle} \geq U^{\langle x \rangle}$, and we are done. $\boxed{QED}$

We are ready to prove the main result of this part.

177 Proposition. [Möhres [76], Satz 3.5] *There is a function $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every $r, s \in N$ the following holds:*

if $|B| \geq p^{\beta(r,s)}$, $U \leq A$ with $|U| \leq p^r$ and $a \in A \setminus U$, then there exists $H \leq B$ with $|H| = p^s$ and $a \notin U^H$.

Proof. Trivially, $\beta(r, 0) = 0$ for every $r \geq 0$. For $s \geq 1$ and all $r \geq 0$, we set

$$\beta(r, s) = \alpha_1(rp^{s-1}) + s - 1,$$

where α_1 is the function of Lemma 176, and proceed by induction on s to prove that such function β satisfies the desired property.

Thus, let $s \geq 1$ and $|B| \geq p^{\beta(r,s)}$. Let $U \leq A$ with $|U| \leq p^r$, and $a \in A \setminus U$. We may clearly assume that B is finite. Hence $A = Z_m$ for some $m \geq 1$. Let $d \geq 0$ be minimal such that $a \in Z_{d+1}U$. Then $a = zu$ for some $u \in U$ and $z \in Z_{d+1} \setminus U$. Since $a \notin Z_dU$, also $z \notin Z_dU$. Now $zZ_d/Z_d \in Z_1(A/Z_d)$ and so, by Lemma 176, since (as $s \geq 1$) $\beta(r, s) \geq a_1(n)$, there exists $x \in B$ such that $zZ_d \notin (UZ_d/Z_d)^{\langle x \rangle}$. Thus $z \notin U^{\langle x \rangle}$, and consequently $a \notin U^{\langle x \rangle}$.

If $s = 1$ we are done. Otherwise, let Y be a complement of $\langle x \rangle$ in B. Then $|Y| \geq p^{\beta(r,s)-1}$. Now, $\beta(r, s) - 1 = \alpha_1(rpp^{s-2}) + (s - 1) - 1 = \beta(rp, s - 1)$. Since $|U^{\langle x \rangle}| \leq |U|^p \leq p^{rp}$, there exists, by the inductive assumption, $W \leq Y$ with $|W| = p^{s-1}$ and $a \notin (U^{\langle x \rangle})^W$. Then $H = W\langle x \rangle = W \times \langle x \rangle$ has order p^s and $a \notin (U^{\langle x \rangle})^W = U^H$. This completes the proof. $\boxed{QED}$

We now move to actual group extensions. Given a prime number p, we denote by Φ the set of all pairs (G, A) where G is a p-group, A a normal elementary abelian subgroup of G, and G/A is elementary abelian. In this case, by letting $B = G/A$ we may apply the results proved so far.

We begin with a couple of elementary observations.

178 Lemma. *Let $(G, A) \in \Phi$, $n \geq 1$, $x_1, \ldots, x_n \in G$, and $X = \langle x_1, \ldots, x_n \rangle$. Then*

(i) AX is nilpotent of class at most $n(p - 1) + 2$;

(ii) $|X| \leq p^{\gamma(n)}$, where $\gamma(1) = 2$, and $\gamma(n) = 2n + p^n \binom{n}{2}$ for $n \geq 2$.

PROOF. (i) This follows easily from the fact that, for every $x \in G$, $[A,_p x] = 1$, and elementary commutator calculus.

(ii) Let $S = \langle [x_i, x_j] \mid i,j = 1,\dots,n\rangle$. Then, by Lemma 4, $X' = S^X \leq A$. Now, X has exponent dividing p^2, and so $|X/X'| \leq p^{2n}$. Also, $|S| \leq p^{\binom{n}{2}}$ and $[X : N_X(S)] \leq [X : X \cap A] \leq p^n$. Thus

$$|X| \leq p^{2n}|X'| \leq p^{2n} \cdot |S|^{p^n} \leq p^{2n+p^n\binom{n}{2}},$$

which is what we wanted. $\boxed{QED}$

The next fundamental result (Proposition 180) is somehow more general than we actually need in the present contest, but in this form it will be useful in later applications. For its proof, we need a special variation of the Chevalley–Warning Theorem (see e.g. [100]).

179 Lemma. [[77]] *For every $m,d,n \in \mathbb{N} \setminus \{0\}$, there exists a value $\alpha(m,d,n)$, such that if $s \geq \alpha(m,d,n)$, and $f_1,\dots,f_m \in \mathbb{Z}(x_1,\dots x_s)$ are homogeneous polynomials of degree at least 1, with $\sum_{i=1}^m \deg f_i \leq d$, and p is a prime, then there exists $(a_1,\dots a_s) \in \mathbb{Z}^s$ with at least one entry a_j not a multiple of p, and*

$$f_i(a_1,\dots,a_s) \equiv 0 \pmod{p^n}$$

for all $i = 1,\dots,m$.

PROOF. See Möhres [77], Lemma 1.5 $\boxed{QED}$

180 Proposition. *Let G be a nilpotent p-group of class $c \geq 2$, and suppose that $\gamma_c(G)$ has rank 1. Let F be subgroup of G with $|F| \leq p^n$ and $\gamma_c(F) = 1$. Let H be a normal subgroup of G such that G/H is elementary abelian of order at least $\alpha((n+1)^c,(n+1)^c c,n)$. Then there exists $y \in G\setminus H$ such that $\gamma_c(\langle F,y\rangle) = 1$.*

PROOF. Let $s = \alpha((n+1)^c,(n+1)^c c,n)$, and let $\{Hy_1\dots,Hy_s\}$ be a set of s independent elements of G/H. Let also $\{x_1,\dots,x_n\}$ be a set of generators of F (which certainly exists since $|F| \leq p^n$).

Denote by $\mathcal{S}$ be the set of all functions $\sigma : \{1,\dots,c\} \to \{0,1,\dots,n\}$, such that $0 \in Im(\sigma) \neq \{0\}$. Observe that $|\mathcal{S}| < (n+1)^c$.

For $\sigma \in \mathcal{S}$, let $q = q_\sigma = |\sigma^{-1}(0)|$ (then $1 \leq q \leq c-1$), and write $\sigma^{-1}(0) = \{\overline{\sigma}(1),\dots\overline{\sigma}(q)\}$ where $\overline{\sigma}(1) < \dots < \overline{\sigma}(q)$. We define a map $\phi_\sigma : G^q \to \gamma_c(G)$ by setting, for all $g_1,\dots,g_q \in G$, $\phi_\sigma(g_1,\dots,g_q) = [z_1,\dots,z_c]$, where $z_i = x_{\sigma(i)}$ if $\sigma(i) \neq 0$, and $z_i = g_\ell$ if $i \in \sigma^{-1}(0)$ and $i = \overline{\sigma}(\ell)$. Finally, for all $g \in G$, we set $\omega_\sigma(g) = \phi_\sigma(g,\dots,g)$.

Since G has class $c \geq 2$, $\gamma_c(G)$ is locally cyclic and $\gamma_c(F) = 1$, it follows from Corollary 50 that, for every $g \in G$,

$$\gamma_c(\langle F,g\rangle) = \{\omega_\sigma(g) \mid \sigma \in \mathcal{S}\}. \tag{28}$$

Let $z \in \gamma_c(G)$ be a generator of the unique subgroup of order p^n of $\gamma_c(G)$. Now, in any of the commutators $\phi_\sigma(g_1, \dots, g_q)$ (with $\sigma \in \mathcal{S}$ and $g_1, \dots, g_q \in G$) there appears at least one element of F, and therefore (Lemma 46)

$$\phi_\sigma(g_1, \dots, g_q) \in \langle z \rangle. \tag{29}$$

Given $\sigma \in \mathcal{S}$, let us write $q = q_\sigma$, and denote by $\mathcal{J}_\sigma$ the set of all q-tuples $j = (j(1), \dots j(q))$ of elements in $\{1, \dots, s\}$. By (29), for every $\sigma \in \mathcal{S}$ and every $j \in \mathcal{J}_\sigma$, there exists a unique element $a_{\sigma,j} \in I = \{0, 1, \dots, p^n - 1\}$ such that

$$\phi_\sigma(y_{j(1)}, \dots, y_{j(q)}) = z^{a_{\sigma,j}}. \tag{30}$$

Now, let $t_1, \dots t_s$ be independent indeterminates over $\mathbb{Z}$, and for every $\sigma \in \mathcal{S}$ let

$$f_\sigma = \sum_{j \in \mathcal{J}_\sigma} a_{\sigma,j} t_{j(1)} \cdots t_{j(q)} \in \mathbb{Z}[t_1, \dots, t_s]. \tag{31}$$

Then each such f_σ is homogeneous of degree $q = q_\sigma \leq c - 1$.

By Lemma 49, the commutators of weight c in G are homomorphisms in each component; moreover, since $|\langle z \rangle| = p^n$, commutators that involve elements from F (like the ϕ_σ), behave linearly modulo $p^n\mathbb{Z}$ in each component. Thus, if $(m_1, \dots, m_s) \in \mathbb{Z}/p^n\mathbb{Z}$, we have, for every $\sigma \in \mathcal{S}$,

$$\begin{aligned} \omega_\sigma(y_1^{m_1} \cdots y_s^{m_s}) &= \phi_\sigma(y_1^{m_1} \cdots y_s^{m_s}, \dots, y_1^{m_1} \cdots y_s^{m_s}) = \\ &= \prod_{j \in \mathcal{J}_\sigma} \phi_\sigma(y_{j(1)}^{m_{j(1)}}, \dots, y_{j(q)}^{m_{j(q)}}) = \\ &= \prod_{j \in \mathcal{J}_\sigma} \phi_\sigma(y_{j(1)}, \dots, y_{j(q)})^{m_{j(1)} \cdots m_{j(q)}} = \\ &= \prod_{j \in \mathcal{J}_\sigma} z^{a_{\sigma,j} m_{j(1)} \cdots m_{j(q)}} = z^{f_\sigma(m_1, \dots m_s)}. \end{aligned} \tag{32}$$

Now, since $\sum_{\sigma \in \mathcal{S}} \deg f_\sigma \leq |\mathcal{S}|(c-1) < (n+1)^c c$, by Lemma 179, there exists a s-tuple $(k_1, \dots, k_s) \in \mathbb{Z}$ such that not all the entries k_i are multiples of p, and $f_\sigma(k_1, \dots, k_s) \equiv 0 \pmod{p^n}$ for every $\sigma \in \mathcal{S}$. Thus, if $y = y_1^{k_1} \cdots y_s^{k_s}$, then $y \notin H$, as at least one of the k_i's is not zero (mod p), and $\gamma_c(\langle F, y \rangle) = 1$ by (28) and (32). $\boxed{QED}$

181 Remark. We will use Proposition 180 in its full force in the next section. At the moment, for groups in the class Φ, one may well suppose $|\gamma_c(G)| = p$. In this case, the polynomials in (31) induce $\mathbb{F}_p$-multilinear maps, and the standard Chevalley–Warning Theorem (see e.g. [100] p. 5, or [83] p. 50) may be applied instead of Proposition 180, with the smaller bound $s = (n+1)^c c$ to get the desired conclusion (we leave the details to the reader). Thus

182 Lemma. *Let G be a nilpotent p-group of class $c \geq 2$, with $|\gamma_c(G)| = p$, and let F be a subgroup of G with $|F| \leq p^n$ and $\gamma_c(F) = 1$. Let H be a normal subgroup of G such that G/H is elementary abelian and $|G/H \geq (n+1)^c c$. Then there exists $y \in G \setminus H$ such that $\gamma_c(\langle F, y\rangle) = 1$.*

Repeated applications of this Lemma easily yield the following.

183 Corollary. *Let $(G, A) \in \Phi$, with G nilpotent of class $c \geq 2$, let $n \geq 0$ and suppose that $|G/A| \geq p^{n^c c+n-1}$. Then there exists $Y \leq G$ such that $\gamma_c(Y) = 1$ and $|AY : A| = p^n$.*

We immediately apply this.

184 Lemma. *Let $n \geq 1$. There exists a function $g_n : \mathbb{N} \to \mathbb{N}$, such that if $(G, A) \in \Phi$ and $|G/A| \geq p^{g_n(c)}$, then*

$$\bigcap\{X \leq G \mid |AX/A| = p^n\} \leq \gamma_{c+1}(G).$$

Proof. We may clearly put $g_n(0) = n$, and $g_n(1) = n + 1$. Let $c \geq 2$ and suppose we have already found $g_n(c-1)$ with the desired property. We set $g_n(c) = g_n(c-1)^c c + n$, and show that it satisfies our requirement.

Let $(G, A) \in \Phi$, with $|G/A| \geq p^{g_n(c)}$; let $\mathcal{W} = \bigcap\{X \leq G \mid |AX : A| = p^n\}$, and $K = \gamma_c(G)$. Since $g_n(c) > g_n(c-1)$, we have, by inductive assumption, $W \leq K$. If $\gamma_{c+1}(G) = [K, G] = K$, there is nothing more to prove. Thus, let $\gamma_{c+1}(G) < K$. Take $\gamma_{c+1}(G) \leq T < K$ with $|K : T| = p$. Then $T \trianglelefteq G$ and $\gamma_c(G/T) = K/T$ is cyclic of order p. By Corollary 183 and the choice of $g_n(c+1)$, there exists a subgroup H/T of G/T with $\gamma_c(H) \leq T$ and $|AT/A| = p^{g_n(c-1)}$. By inductive assumption we have

$$\bigcap\{X \leq H \mid |(A \cap H)X/(A \cap H)| = p^n\} \leq \gamma_c(H) \leq T.$$

But, for $X \leq H$, $|(A \cap H)X/(A \cap H)| = |X/A \cap X| = |AX/A|$, and so $\mathcal{W} \leq T$. Now, this holds for every maximal subgroup $T/\gamma_{c+1}(G)$ of $K/\gamma_{c+1}(G)$. Since $K/\gamma_{c+1}(G)$ is elementary abelian, we conclude that $\mathcal{W} \leq \gamma_{c+1}(G)$. $\boxed{QED}$

We now look to a kind of opposite situation, that is when G admits 'long' non–trivial commutators.

185 Lemma. *Let $(G, A) \in \Phi$, $n \geq 1$. Let $x_1, \ldots, x_n, y \in G$ and $X = \langle x_1, .., x_n\rangle$. Suppose that $A \cap X = 1$ and $|[A,_{p-1} x_1, \ldots,_{p-1} x_n,_{p-1} y]| \geq p^{p^{n+1}}$. Then for every $1 \neq a \in A$ there exists $c \in A$ such that $a \notin \langle X, yc\rangle$.*

Proof. Let Δ be the set of all n-tuples $t = (t_1, \ldots, t_n)$ of integers $0 \leq t_i \leq p-1$, with $t_i \neq 0$ for at least one $i \in \{1, \ldots, n\}$.

For every $t = (t_1, \ldots, t_n) \in \Delta$, $0 \leq j \leq p-2$, and $g \in X$, we define $\omega_{t,j}(g) = [g,_{t_1} x_1, \ldots,_{t_n} x_n,_j g]$ and $\tau_t(g) = [g^p,_{t_1} x_1, \ldots,_{t_n} x_n]$. We show that, for every $g \in X$,

$$A \cap \langle X, g\rangle = \langle g^p,\, \tau_t(g),\, \omega_{t,j}(g) \mid t \in \Delta,\, 0 \le j \le p-2\rangle\,. \tag{33}$$

Observe that $A \cap X = 1$ implies X elementary abelian. It is thus clear that $A \cap \langle X, g\rangle = \langle g^p\rangle\langle X, g\rangle'$. So it will suffice to show that

$$\langle X, g\rangle' = \langle \tau_t(g),\, \omega_{t,j}(g) \mid t \in \Delta,\, 0 \le j \le p-2\rangle\,. \tag{34}$$

By Lemma 169, $\langle X, g\rangle'$ is generated by all the conjugates of the elements $[g, x_i]$ $(i = 1, \ldots, n)$. Since $\langle X, g\rangle' \le A$ is abelian, we deduce that $\langle X, g\rangle'$ is generated by the set of all the elements $[g, x_s]^{x_1^{k_1}\cdots x_n^{k_n} g^j}$, and so it is generated by the set of all commutators

$$[g, x_{s}, {}_{k_1} x_1, \ldots, {}_{k_n} x_n, {}_j\, g] \tag{35}$$

with $s \in \{1, \ldots n\}$, $0 \le k_i \le p-1$ for $i = 1, \ldots, n$, and $0 \le j \le p-1$. Now, since $x_i x_j = x_j x_i$ and the commutators $[g, x_i]$, $[g, x_j]$ also commute, we see that $[g, x_i, x_j] = [g, x_j, x_i]$ for sll $i, j = 1, \ldots, n,$; moreover, as $x_i^p = 1$, we have $[g, {}_p\, x_i] = [g, x_i][g, x_i]^{x_i} \cdots [g, x_i]^{x_i^{p-1}} = 1$, and similarly $[g, x_i, {}_{p-1}\, g] = [g^p, x_i]$, for all $i = 1, \ldots, n$. These observations allow to freely rearrange the elements $x_1, \ldots, x_n$ in (35), to deduce that $k_s \le p-2$, and eventually to rewrite (35) as a commutator of type $\omega_{t,j}(g)$ (if $j \ne p-1$), or of type $\tau_t(g)$ (if $j = p-1$); thus proving identity (34), and consequently establishing (33).

Let $x_1, \ldots, x_n, y$ be as in the statement of the Lemma, and $I = \{0, \ldots, p-2\}$. Let $\mathcal{S}$ be the set of all functions from $\Delta \times I \cup \Delta \cup \{0\}$ in $\{0.1, \ldots, p-1\}$. Then $\log_p |\mathcal{S}| = |\Delta \times I \cup \Delta \cup \{0\}| = p^{n+1} - p + 1$.

For every $\sigma \in \mathcal{S}$ and every $c \in A$, let

$$b_\sigma(c) = (yc)^{p\sigma(0)} \cdot \prod_{t\in\Delta} \tau_t(yc)^{\sigma(t)} \cdot \prod_{(t,j)\in\Delta\times I} \omega_{t,j}(yc)^{\sigma(t,j)}. \tag{36}$$

Then, by (35), for every $c \in A$, we have

$$A \cap \langle X, yc\rangle = \{b_\sigma(c) \mid \sigma \in \mathcal{S}\}. \tag{37}$$

Let K be the kernel of the linear map on A,

$$a \mapsto [a_{n-1} x_1, \ldots, {}_{n-1}\, x_n, {}_{n-1}\, y].$$

Then, by hypothesis, $|A/K| \ge p^{p^{n+1}} > |\mathcal{S}|$.

Let now $a \in A$ with $a \in \langle X, yc\rangle$, for every $c \in A$. Then, by (37) there exist $c, c' \in A$ with $Kc \ne Kc'$, and $\sigma \in \mathcal{S}$, such that

$$b_\sigma(c) = a = b_\sigma(c'). \tag{38}$$

Now, for every $t \in \Delta$, $j \in I$, and every $c \in A$, we have

$$\begin{aligned}
&\omega_{t,j}(yc) = \omega_{t,j}(y)[c,_{t_1} x_1, \dots ,_{t_n} x_n,_j y]\\
&\tau_t(yc) = \tau_t(y)[c,_{t_1} x_1, \dots ,_{t_n} x_n,_{p-1} y]\\
&(yc)^p = y^p[c,_{p-1} y].
\end{aligned}$$

Thus, setting $b = c^{-1}c'$, (38) and (36) entail

$$1 = [b,_{p-1} y] \prod_{t \in \Delta} [b,_{t_1} x_1, \dots ,_{t_n} x_n,_{p-1} y]^{\sigma(t)} \prod_{(t,j)} [b,_{t_1} x_1, \dots ,_{t_n} x_n,_j y]^{\sigma(t,j)}$$

But, as $Kc \neq Kc'$, $b = c^{-1}c' \notin K$, whence, by Lemma 170, all the commutators that appear in the above product are linearly independent. It then follows that σ is the zero–constant, and so $a = 1$. This proves the Lemma. $\boxed{QED}$

We may now complete Lemma 184.

186 Lemma. *There exists a function* $\alpha_3 : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ *such that, for every* $n \geq 1$, *if* $(G, A) \in \Phi$ *and* $|G/A| \geq p^{\alpha_3(n)}$, *then*

$$\bigcap \{X \leq G \mid |AX/A| = p^n\} = 1.$$

Proof. For a given $n \geq 1$, let $s = \max\{2\gamma(n-1),\ \beta(\gamma(n-1), 1)\} + n$, where γ and β are, respectively, the functions defined in 178 and 177; then take $c = s(p-1) + p^n + 1$, and finally define $\alpha_3(n) = g_n(c)$, where g_n is the function determined in Lemma 184.

Then, let $(G, A) \in \Phi$, with $|G/A| \geq p^{\alpha_3(n)}$, and $1 \neq a \in A$. We prove that there exists $X \leq G$, with $|AX/A| = p^n$ and $a \notin X$.

If $\gamma_{c+1}(G) = 1$, the claim follows at once by Lemma 184. Thus, assume $\gamma_{c+1}(G) \neq 1$. Then $A \not\leq \zeta_{c-1}(G) = \zeta_{s(p-1)+p^n}(G)$, hence, if $W = \zeta_{p^n}(G) \cap A$, $A/W \not\leq \zeta_{s(p-1)}(G/W)$. Then, by Lemma 170, there exist $y_1, \dots, y_s \in G$ such that $[A,_{p-1} y_1, \dots ,_{p-1} y_s] \not\leq W$. By 170, $Ay_1, \dots, Ay_s$ are independent in G/A, and we may well suppose $G = A\langle y_1, \dots y_s\rangle$. Now, for $\ell \leq s$, let $\{Az_1, \dots, Az_\ell\}$ be a set of independent elements in G/A; we may complete it to a base $Az_1, \dots, Az_\ell, Az_{\ell+1}, \dots, Az_s$ of G/A. Then, by Lemma 171, we have $[A,_{p-1} z_1, \dots ,_{p-1} z_s] \not\leq W$, and so $[A,_{p-1} z_1, \dots ,_{p-1} z_\ell] \not\leq \zeta_{p^n+(s-\ell)(p-1)}(G) \cap A$, which in turn yields (as $[A,_{p-1} z_1, \dots ,_{p-1} z_\ell]$ is normal in G),

$$|[A,_{p-1} z_1, \dots ,_{p-1} z_\ell]| \geq p^{p^n+(s-\ell)(p-1)}. \tag{39}$$

Let $x_0 = 1$; we show that, for every $0 \leq i \leq n$, there exist $x_0, x_1, \dots, x_i \in G$ such that $Ax_1, \dots, Ax_i$ are independent in G/A and $a \notin \langle x_0, \dots, x_i\rangle$. Suppose that, for some $0 \leq i \leq n-1$, we have already found $x_0, \dots, x_i$ with these properties, and let $U = \langle x_0, \dots, x_i\rangle$. Then, by 178, $|A \cap U| \leq p^{\gamma(i)} \leq p^{\gamma(n-1)}$.

Let $A \leq H \leq G$ such that $G/A = H/A \times AU/A$. Then H/A has rank $s-i$ and, by choice of s, $s-i \geq s-(n-1) \geq \beta(\gamma(n-1),1)$. By Theorem 177 there exists $y \in H \setminus A$ such that $a \notin (A \cap U)^{\langle y \rangle} = D$. Now, $DU \cap A = D(U \cap A) = D$, and so $A/D \cap DU/D = 1$, and $aD \neq 1$. Also, $Ax_1, \ldots, Ax_i, Ay$ are independent in G/A. Let $K = [A,_{p-1} x_1, \ldots,_{p-1} x_i,_{p-1} y]$; then, by (39) and the choice of s,

$$|K| \geq p^{p^n+(s-(i+1))(p-1)} \geq p^{p^n+(s-n)(p-1)} \geq p^{p^n+\gamma(n-1)p} \geq p^{p^n}|D|. \qquad (40)$$

Now, D is $\langle U, y \rangle$-invariant; passing modulo D, (40) yields

$$[A/D,_{p-1} Dx_1, \ldots,_{p-1} Dx_i,_{p-1} Dy] \geq p^{p^n} \geq p^{p^{i+1}}.$$

Then, by Lemma 185, there exists $Db \in A/D$ such that $Da \notin \langle XD/D, byD \rangle$. By letting $x_{i+1} = by$, we have that $Ax_1, \ldots, Ax_i, Ax_{i+1}$ are independent, and $a \notin \langle x_1, \ldots, x_i, x_{i+1} \rangle$. The inductive proof is now complete, hence we eventually find $x_1, \ldots, x_n \in G$ independent modulo A, such that $a \notin X = \langle x_1, \ldots, x_n \rangle$. $\boxed{QED}$

We are now in a position to deduce a major step in the proof.

187 Theorem. [Möhres [77], Satz 2.2] *There exists a function $\mu : \mathbb{N} \to \mathbb{N}$ such that, for all $n \geq 1$, the following holds:*

if $(G, A) \in \Phi$ is such that $|G/A| \geq p^{\mu(n)}$, and $U \leq G$ has order at most p^n, then

$$\bigcap_{x \in G \setminus AU} \langle U, x \rangle = U.$$

Proof. Let $n \geq 1$ be fixed, and let β and α_3 the functions defined, respectively, in 177 and 184. We put $\tau_n(0) = \beta(n, \alpha_3(1))$, and inductively, for $d \geq 1$, $\tau_n(d) = \beta(n, \alpha_3(\tau_n(d-1)))$.

1) We first consider the case in which $U \cap A = 1$.

Then U is elementary abelian. Let $|U| = p^n$, and let $(x_1, \ldots, x_n)$ be an ordered set of independent generators of U. Let m_1 be the smallest positive integer such that $[A,_{m_1} x_1] \neq 1$, and for every $1 < i \leq n$, let m_i; be the smallest positive integer such that $[A,_{m_1} x_1, \ldots,_{m_{i-1}} x_{i-1},_{m_i} x_i] \neq 1$. Finally, let $d = d_U(A) = \sum_{i=1}^n m_i$ (observe that $0 \leq d \leq n(p-1)$).

Arguing by induction on d, we show that if $|G/A| \geq p^{\tau_n(d)+n}$, then

$$W = \bigcap_{x \in G \setminus AU} \langle U, x \rangle = U. \qquad (41)$$

Let first $d = 0$. Then U centralizes A and $(G, AU) \in \Phi$. Let $1 \neq a \in A$. By definition of $\tau_n(0)$, we have $|G/AU| \geq p^{\beta(n,\alpha_3(1))}$, so, by Proposition 177, there exists a subgroup $AU \leq H \leq G$, $|H/AU| = p^{\alpha_3(1)}$ and $a \notin U^H$, Thus, by Lemnma 186, the intersection of all subgroups $U^H\langle x \rangle$ with $x \in H \setminus AU$ is U^H.

In particular, there exists $x \in H \setminus AU$ such that $a \notin U^H\langle x\rangle$, and, a fortiori, $a \notin \langle U, x\rangle$. This shows that $A \cap W = 1$. But then

$$W = W \cap U^H \leq W \cap AU = (W \cap A)U = U,$$

Assume now $d \geq 1$. Then there exists a largest index $1 \leq t \leq n$ such that $m_t \neq 0$. Let

$$N = [A,_{m_1} x_1, \ldots ,_{m_t} x_t].$$

Then $N \trianglelefteq G$, $A/N \cap NU/N = 1$ and $d_U(A/N) \leq d-1$. Since $\tau_n(d) \geq \tau_n(d-1)$, by inductive assumption we have $W \leq NU$.

Let K be the kernel of the surjective homomorphism $\phi : A \to N$ given by $\phi(v) = [v,_{m_1} x_1, \ldots ,_{m_t} x_t]$. Let $1 \neq a \in N$ and take $b \in A$ such that $\phi(b) = a$.

Then, $K \trianglelefteq G$, and for every $1 \leq i \leq n$ and $v \in A$, $\phi([v, x_i]) = 1$. Hence $[A, U] \leq K$. Also $A \cap KU = K(A \cap U) = K$, so AU/K is elementary abelian and $(G/K, AU/K) \in \Phi$. By definition of $\tau_n(d)$, $|G/AU| \geq p^{\beta(n,\alpha_3(\tau_n(d-1)))}$. Thus, by Proposition 177, there exists a subgroup $AU \leq H \leq G$, with $|H/AU| = p^{\alpha_3(\tau_n(d-1))}$ and $bK \notin KU^H/K$. In turn, by Lemma 186 (working in H/KU^H), there exists a subgroup $KU^H \leq Y \leq H$, with $|AY : AU| = p^{\tau_n(d-1)}$, and $b \notin Y$.

Now, $|Y : A \cap Y| = |AY : A| = |AY : AU||AU : A| = p^{\tau_n(d-1)+n}$, whence, by inductive assumption, $W \leq [Y \cap A,_{m_1} x_1, \ldots ,_{m_t} x_t]U = \phi(Y \cap A)U$. If $a = \phi(c)$ for some $c \in A \cap Y$, then $bc^{-1} \in K$, and so $b \in K\langle c\rangle \leq Y$, a contradiction. Thus, $a \notin \phi(Y \cap A)$ and, consequently, $a \notin W$. This holds for every $1 \neq a \in N$. Hence, $W \cap A = W \cap NU \cap A = W \cap N(U \cap A) = W \cap N = 1$. This, as above, yields the desired conclusion (41).

Now, as observed before, $d_U(A) \leq n(p-1)$; so, by letting, for every $n \geq 1$, $\overline{\mu}(n) = \tau_n(n(p-1)) + n$, we have the following :

if $|G/A| \geq p^{\overline{\mu}(n)}$ and $U \leq G$, with $|U| = p^n$ and $A \cap U = 1$, then (41) holds.

2) Now, for the general case, let, for each $n \geq 1$,

$$\mu(n) = \beta(n, \overline{\mu}(n)).$$

Let $|G/A| \geq p^{\mu(n)}$, and $U \leq G$ with $|U| \leq p^n$. Let also $a \in A \setminus U$.

Since $|A \cap U| \leq p^n$, Proposition 177 guarantees the existence of a subgroup $A \leq H \leq G$, such that $|H/A| = p^{\overline{\mu}(n)}$ and $a \notin (A \cap U)^H = D$. Clearly, we may let $H \geq AU$. Then $(U/D, A/D) \in \Phi$, $aD \neq 1$, and $A/D \cap UD/D = 1$ (as $A \cap DU = D(A \cap U) = D$). Therefore, by the case discussed in point 1), there exists $x \in H \setminus AU$ such that $aD \notin \langle UD/D, xD\rangle$, and so, a fortiori, $a \notin \langle U, x\rangle$. This proves that $1 = A \cap W$, where $W = \bigcap_{x \in G\setminus AU}\langle U, x\rangle$. But then, as usual $W = W \cap AU = U(W \cap A) = U$. The proof is thus complete. $\boxed{QED}$

Let us extend this theorem in a rather obvious way, and in a form that we will be able to apply more directly.

188 Proposition. *For every $n, m, k \geq 1$, there exists $\psi(n, m, k) \in \mathbb{N}$, such that the following holds:*

let $(G, A) \in \Phi$ with $|G/A| \geq \psi(n, m, k)$; if U is a n-generated subgroup of G and X a subset of A of order k, with $X \cap U = \emptyset$, then there exist $y_1, \dots, y_m \in G$, such that $\emptyset = A \cap V = \langle U, y_1, \dots, y_m\rangle$ and $|AV : AU| = p^m$.

Proof. We begin with defining ψ for $k = 1$ and all n, m. Thus, for $n, m \geq 1$, we set $\psi(n, m, 1) = \mu(\gamma(n+m-1))$ (where $\gamma(i)$ is as in Lemma 178, and μ is the function determined in Theorem 187), and show that it satisfies the required property, arguing by induction on m.

Let $x_1, \dots, x_n \in G$, $U = \langle x_1, \dots, x_n\rangle$, and let $a \in A \setminus U$. For $m = 1$, we have $\psi(n, 1, 1) = \mu(\gamma(n))$ and the claim follows from 187. Let $m \geq 2$. Then, as $\psi(n, m, 1) \geq \psi(n, m-1, 1)$, by inductive assumption there exist $y_1, \dots, y_{m-1}$ such that $a \notin T = \langle U, y_1, \dots, y_{m-1}\rangle$ and $|AT : AU| = p^{m-1}$. By Lemma 178, $|T| \leq p^{\gamma(n+m-1)}$, and so Theorem 187 again implies the existence of $y_m \in G$ with $a \notin V = \langle T, y_m\rangle = \langle U, y_1, \dots, y_{m-1}, y_m\rangle$ and $|AV : AT| = p$. Thus $|AV : AU| = |AV : AT||AT : AU| = p^m$, and we are done.

Thus, we have ψ for all cases in which $k = 1$. Its extension to all $k \geq 1$ is by induction: for $n, m, \geq 1$, $k \geq 2$, we set $\psi(n, m, k) = \psi(n, \psi(n, m, 1), k-1)$. To show that this satisfies the desired property is now an easy induction. $\boxed{QED}$

The analisys of the case Φ now comes to an end.

189 Proposition. *Let $(G, A) \in \Phi$. If $G \in \mathcal{N}_1$ then G is nilpotent.*

Proof. Let (G, A) in Φ. We prove that if G is not nilpotent then it has a subgroup which is not subnormal.

Thus, let G be not nilpotent. For every $1 \leq n \in \mathbb{N}$ write $\sigma(n) = n(n+1)/n$. We prove, inductively on $n \geq 1$, the existence of sequence of subgroups U_n of G and of elements a_n of A, such that, for every $n \geq 1$, U_n is $\sigma(n)$–generated, and for every $1 \leq i \leq j$, $U_i \leq U_j$ and $a_i \in [A,_{i(p-1)} U_i] \setminus U_j$.

Since G is not nilpotent, there exists, by 169, an element $y \in G$ such that $[A,_{p-1} y] \neq 1$. Let $1 \neq a_1 \in [b,_{p-1} y]$ (for some $b \in A$), by possibly replacing y with $b^k y$ (for a suitable $0 \leq k \leq p-1$) we have $a_1 \notin \langle y\rangle$; so let $U_1 = \langle y\rangle$.

Now, for $n \geq 2$, suppose we have already established the existence of a chain of subgroups $U_1 \leq \ldots \leq U_{n-1}$ of G, and of elements $a_1, \dots, a_{n-1}$ of A with the prescribed properties. Write $U = U_{n-1}$ and $X = \{a_1, \dots, a_n\}$. Let $G/A = AU/A \times K/A$. Since U is finite and G is not nilpotent, K/A is not nilpotent and, in particular, it is infinite.. Let $s = \psi(\gamma(\sigma(n-1)), n, n-1)$, where ψ is the function defined in 188 and γ that defined in 178. Since K is not nilpotent, by Lemma 170 there exist elements $y_1, \dots, y_s \in K \setminus A$ such that

$$[A,_{p-1} y_1, \dots,_{p-1} y_s] \geq p^{\gamma(\sigma(n))+1} \tag{42}$$

(see also the proof of 185). Let $H = \langle A, y_1, \ldots, y_s \rangle$; then $|H/A| = p^s$, as $Ay_1, \ldots, Ay_s$ are independent by 170. In fact, $|HU/AU| = p^s$, and by inductive assumption, $|U| \leq p^{\gamma(\sigma(n-1))}$. Then, bt Lemma 188, there exists a subgroup $V \leq HU$ such that $U \leq V$, $X \cap V = \emptyset$, and $|AV : AU| = p^n$. Now, as $V = V \cap HU = (V \cap H)U$, we may take elements $x_1, \ldots, x_n \in H$ such that, setting $U_n = \langle x_1, \ldots, x_n \rangle$, we have $U_n \leq V$ (hence $X \cap U_n = \emptyset$) and $AU_n = AV$. This defines U_n; observe, in fact, that, as $U = U_{n-1}$ is $\sigma(n-1)$–generated, U_n is generated by $\sigma(n-1) + n = \sigma(n)$ elements.

As $Ax_1, \ldots, Ax_n$ are independent in H/A, by Lemma 171 and condition 42 we have $[A,_{p-1} x_1, \ldots,_{p-1} x_n] > p^{\gamma(\sigma(n))} \geq p^{|U_n|}$. Therefore, there exists $b \in A$ such that $a_n = [b,_{p-1} x_1, \ldots,_{p-1} x_n] \notin U_n$. This completes the inductive step.

We then find in this way the desired infinite sequence $U_1 \leq U_2 \ldots$ of finitely generated subgroups of G and elements $a_n \in A$ such that $a_i \in [A,_{i(p-1)} U_i] \setminus U_j$, for all $1 \leq i \leq j$. Now, let $S = \bigcup_{n \geq 1} U_n$. Then $S \leq G$ and $S \cap \{a_1, a_2, \ldots\} = \emptyset$. It follows that S is not subnormal in G; for, if it were, there existed a positive integer d such that $[A,_d S] \leq S$, whence, as $a_d \in [A,_{d(p-1)} U_d] \leq [A,_d S]$, the contradiction $a_d \in S$. This completes the proof. $\boxed{QED}$

The main result of this section follows by standard arguments. We need the following variation on P. Hall nilpotency criterion (56); the easy proof (using 56 and the elementary observations at the end of section 1.1) we leave to the reader.

190 Lemma. *Let G be group and N a normal p-subgroup of finite exponent. If N and $G/N'N^p$ are nilpotent then G is nilpotent*

191 Theorem. [Möhres [77]] *A soluble $\mathcal{N}_1$-group of finite exponent is nilpotent.*

Proof. Let G be a soluble $\mathcal{N}_1$-group of finite exponent. Then G is the direct product of p-groups for a finite set of primes p. Thus, we may well suppose that G is a p-group for some prime p. Since G is soluble and has finite exponent, it admits a finite normal series with p-elementary abelian factors. We let d be the shortest length of such a series, and argue by induction on d.

If $d = 1$, G is abelian. Thus, let $d \geq 2$ and write $N = G'G^p$. Then G/N is the largest elementary abelian quotient of G, whence by inductive assumption N is nilpotent. Let $K = N'N^p$; then G/K is an extension of the elementary abelian p-group N/K by the elementary abelian p-group G/N; hence, by Proposition 189, G/K is nilpotent. By Lemma 190, G is nilpotent. $\boxed{QED}$

5.2 Extensions by groups of finite exponent

In this section we prove another important Theorem of Möhres, saying that a periodic $\mathcal{N}_1$-group which is the extension of a nilpotent group by a (soluble) group of finite exponent, is nilpotent.

We start with a fundamental result, which finds applications also in other contexts.

192 Theorem. [Möhres [79]] *Let G be a nilpotent p-group, and N a normal subgroup such that G/N is an infinite elementary abelian group. Then, for every finite subgroup U of G and any $a \in G \setminus U$, there exists a subgroup V of G with $U \leq V$, $a \notin V$ and NV/N infinite.*

Proof. We proceed by induction on the class c of G. If $c = 1$ the claim follows easily from the basic theory of abelian groups. Thus, suppose $c \geq 2$, and assume the statement true for all p-groups of class less or equal to $c - 1$.

Let U be a finite subgroup of G, and $a \in G \setminus U$. Let $K = \gamma_c(G)$. If $a \notin KU$, then we are done by inductive assumption (observe that, since $c \geq 2$, $N \geq G' \geq K$). Hence, we may assume $a \in KU$, that is $a = bu$ for some $b \in K$ and $u \in U$: clearly, we may now replace a by b if necessary, and so suppose $a \in K$. Let M be a subgroup of K maximal subject to $K \cap U \leq M$ and $a \notin M$. Then, since K is central in G (in particular, it is abelian). $M \trianglelefteq G$ and K/M has rank 1. Then $a \notin MU$ (for, otherwise, $a \in MU \cap K = M(U \cap K) = M$), and so we may assume $M = 1$, i.e. $K = \gamma_c(G)$ is abelian of rank 1.

In this setting, we have $U \cap K = 1$, and so $\gamma_c(\langle U, a\rangle) = 1$. Then, by repeated applications of Proposition 180, we conclude that there exists a subgroup H of G, containing $\langle U, a\rangle$, with HN/N infinite and $\gamma_c(H) = 1$. Now, $H/(H \cap N) = NH/N$ is an infinite elementary abelian group, and so we may apply the inductive assumption and conclude that there exists $V \leq H$ such that $U \leq V$, $a \notin V$ and $V(H \cap N)/(H \cap N)$ infinite. Clearly then VN/N is infinite and we are done. $\boxed{QED}$

Let us state an immediate consequence, specialized to our purposes.

193 Lemma. *Let G be a nilpotent p-group, $N \trianglelefteq G$ with G/N is an infinite elementary abelian group. Let F be a finitely generated subgroup and $c \geq 1$ an integer such that every $H \leq G$ with $F \leq H$ and NH/N infinite is subnormal of defect at most c in G. Then, every subgroup of G containing F has defect at most c (whence $\gamma_{\beta(c)+1}(G)$ is finite by 160)..*

Proof. Let $F \leq U \leq G$. In order to show that it has defect at most c, we may assume that U is finitely generated. Suppose that there exists $a \in [G,_c U] \setminus U$. Then. by Theorem 192, here exists $V \leq G$ with NV/N infinite, $U \leq V$, and $a \notin V$. By hypothesis, $[G_c U] \leq [G,_c V] \leq V$;hence the contradiction $a \in V$. $\boxed{QED}$

Now we consider nilpotent–by–(finite exponent) $\mathcal{N}_1$-groups. As in the previous section, the basic case is that of a metabelian p-group. We need a few preparatory lemmas (see [79]).

194 Lemma. *Let $G \in \mathcal{N}_1$ be the extension of an abelian group A by a soluble group of finite exponent. If G satisfies an Engel condition, then G is nilpotent.*

PROOF. Let G and A be as in the statement, and suppose that there exists $n \geq 1$ such that $[x,_n y] = 1$ for every $y, x \in G$. Hence $[A,_n x] = 1$ for every $x \in G$. Let e be the exponent of G/A. Then, since $A \leq C_G(A)$, by applying point (i) of 16 we get $\left[A^{e^{n-1}}, x\right]$ for every $x \in G$, that is $B = A^{e^{n-1}} \leq Z(G)$. Now, G/B is a soluble $\mathcal{N}_1$-group of finite exponent, so it is nilpotent by Theorem 191. Thus, G is nilpotent. $\boxed{QED}$

195 Lemma. *Let $G \in \mathcal{N}_1$ be the extension of an abelian group A by an elementary abelian p-group (p a prime). If G is not nilpotent then there is a non-nilpotent subgroup K of G, with $A \leq K$ and such that a subgroup H of K is nilpotent if and only if HA/A is finite.*

PROOF. Since G is a Baer group, every element of G is a bounded left Engel element. In particular, for each $x \in G$, there is a largest positive integer $n(x)$ such that $[A,_n x] \neq 1$. Since G is not nilpotent, and $[G, x] \leq A$ for every $x \in G$, by Lemma 194 we have $\sup\{n(x) \mid x \in G\} = \infty$. Thus, there is an infinite sequence $(x_i)_{i \geq 1}$ of elements of G, such that $n(x_1) \geq 1$ and

$$n(x_n) \geq n + \sum_{i=1}^{n-1} n(x_i) \tag{43}$$

for all $n \geq 2$. Let $K = A\langle x_i \mid i \geq 1 \rangle$.

Let $x, y \in G$ and $m = n(x) + n(y) + 1$. Then,

$$[A,_m xy] \leq \prod_{m \leq i+j \leq 2m} [A,_i x,_j y] = 1.$$

Thus, $n(xy) \leq m - 1 = n(x) + n(y)$. From this it follows that, for every $x, y \in G$,

$$n(xy) \geq n(x) - n(y^{-1}) = n(x) - n(y).$$

Now, for some $n \geq 1$, take $x \in K \setminus A\langle x_1, \ldots, x_n \rangle$. Then, there exist $t > n$, and $0 \leq m_j \leq p - 1$ ($j = 1, 2, \ldots, t$), with $m_t \neq 0$, such that $xA = x_1^{m_1} \ldots x_t^{m_t} A$. Hence, recalling (43),

$$n(x) \geq n(x_t) - \sum_{j=1}^{t-1} n(x_j^{m_j}) \geq n(x_t) - \sum_{j=1}^{t-1} n(x_j) \geq t > n. \tag{44}$$

Let $H \leq K$ and suppose that H is nilpotent. Then, since H is subnormal, AH is nilpotent, say of class c. But then $n(x) \leq c$ for every $x \in AH$, and thus it follows from (44) that $AH \leq A\langle x_1, \ldots, x_c \rangle$. In particular, $|AH/A| \leq p^c$.

Conversely, if $H \leq K$ is such that AH/A is finite, then AH is nilpotent by Lemma 77. $\boxed{QED}$

196 Lemma. *Let G be a p-group in $\mathcal{N}_1$, and let A be a normal abelian subgroup of G, such that G/A is elementary abelian. Then G is nilpotent.*

PROOF. By Proposition 99, we may suppose that

$$A^\omega = \bigcap_{m \geq 1} A^{p^m} = 1.$$

For every $m \geq 1$, write $K_m = A^{p^m}$. By Theorem 207, G/K_m is nilpotent for every $m \geq 1$.

Suppose, by contradiction, that G is not nilpotent. Then, by 77, G/A is infinite. By Theorem 98 and by Lemma 195, we may also assume that there is a finite subgroup F of G and a $n \geq 1$ such that all subgroups H of G with $F \leq H$ and AH/A infinite, have defect at most n in G.

Now, let $m \geq 1$; then, $\overline{G} = G/K_m$ is nilpotent, and $(G/K_m)/(A/K_m)$ is an infinite elementary abelian p-group. Also, every subgroup $U/K^m = \overline{U}$ of $\overline{G}$ containing $\overline{F} = K^mF/K^m$ and such that UK/K is infinite has defect at most n in $\overline{G}$. Thus, by Lemma 193, every subgroup of $\overline{G}$ containing $\overline{F}$ has defct at most n in $\overline{G}$. This holds for every $m \geq 1$. Now, let H be a finitely generated subgroup of G with $F \leq H$. Then, by what we have just observed, $G,_c H] \leq HK^m$ for every $m \geq 1$. But H is finite, hence, by Lemma 29,

$$H = \bigcap_{m \geq 1} K_m H.$$

This shows that H has defect at most c in G. Then, every subgroup of G contag F has bounded defect, and so G is nilpotent by 160. $\boxed{QED}$

197 Theorem. [Möhres [79]] *A $\mathcal{N}_1$-group which is the extension of a periodic nilpotent group by a soluble group of finite exponent is nilpotent.*

PROOF. Let G be a periodic $\mathcal{N}_1$ group, with a nilpotent normal subgroup N such that G/N is soluble of finite exponent. By Lemma 165 we may assume that G is a p-group for some prime p. As G/N is soluble of finite exponent, it admits a finite normal series all of whose factors are elementary abelian. Proceeding by induction on the shortest length d of such a series, we reduce to the case in which G/N is elementary abelian. Now, by P. Hall criterion 56, we may also assume that N is abelian. So we are in a position to apply Lemma 195, and conclude that G is nilpotent. $\boxed{QED}$

As a first application of Theorem 197, we prove a result of H. Smith [111] (see also [17]).

Let H be a subgroup of the group G. We write $H \leq_b G$ if there exists an integer $m \geq 1$ such that $g^m \in H$ for all $g \in G$. This is equivalent to say that G/H_G is a group of finite exponent. Observe that if $K \leq_b H \leq_b G$ then $K \leq_b G$.

198 Theorem. *A residually nilpotent periodic group in $\mathcal{N}_1$ is nilpotent.*

PROOF. Let $G \in \mathcal{N}_1$ be a periodic residually nilpotent group. By Lemma 165, we may assume that G is a p-group for some prime p. Let

$$G^\omega = \bigcap_{n \in \mathbb{N}} G^{p^n}.$$

By Lemma 98, there exist a subgroup $H \leq_b G$, a finitely generated subgroup F of H, and a positive integer d, such that every $F \leq K \leq_b H$ has defect at most d in H. If H is nilpotent, then $G \in \mathcal{N}_1$ is the extension of the normal nilpotent subgroup H_G by a group of finite exponent. By Theorem 197, G is nilpotent. Thus, we may assume that $H = G$.

For $n \geq 1$, let $G_n = G^{p^n}$. Then $K \leq_b G$, for all subgroups $G_n \leq K \leq G$. It follows that all subgroups of G/G_n that contain the finite subgroup FG_n/G_n have defect at most d in G/G_n. By Theorem 160,

$$\gamma_{\beta(d)+1}(G/G_n) = \frac{\gamma_{\beta(d)+1}(G)G_n}{G_n}$$

is finite. By Proposition 53,

$$Z_n/G_n = \zeta_{2\beta(d)}(G/G_n)$$

has finite index in G/G_n. Let $Y = \bigcap_{n \in \mathbb{N}} Z_n$; then G/Y is a periodic residually finite $\mathcal{N}_1$-group. By Proposition 168, G/Y is nilpotent, of nilpotency class c, say. It follows that, for all $n \geq 1$, G/G_n is nilpotent of class at most $m \leq 2\beta(d) + c$. Hence, G/G^ω is nilpotent of class m. Now,

$$\frac{\gamma_{m+1}(G)}{\gamma_{m+3}(G)} \leq \frac{G^\omega}{\gamma_{m+3}(G)} = \left(\frac{G}{\gamma_{m+3}(G)}\right)^\omega$$

is contained in the centre of $G/\gamma_{m+3}(G)$ by Lemma 18. Then $\gamma_{m+3}(G) = \gamma_{m+2}(G)$, whence, since G is residually nilpotent, $\gamma_{m+2}(G) = 1$, thus proving that G is nilpotent. $\boxed{QED}$

This Theorem does not hold in the non-periodic case, as the groups of H. Smith (section 6.3) show (which indeed are non-nilpotent residually finite $\mathcal{N}_1$-groups). However we shall later prove (Theorem 221) that a residually nilpotent $\mathcal{N}_1$-group is hypercentral.

5.3 Periodic hypercentral $\mathcal{N}_1$-groups

Heineken–Mohamed groups have trivial centre. We show in this section that this is not an accident; in fact (Theorem 202) every non-nilpotent periodic $\mathcal{N}_1$-group must have a centreless non-nilpotent quotient. Needless to say, this also is due to W. Möhres.

199 Lemma. *Let $G \in \mathcal{N}_1$ be p-group, and G' be elementary abelian. Then*

(1) $C_G(G')/Z(G)$ is an elementary abelian p-group;

(2) a subgroup H of G is nilpotent if and only if $HZ(G)/Z(G)$ has finite exponent.

PROOF. (1) Let $a \in C = C_G(G')$, and $x \in G$; then $[a, x, a] = 1$, whence $[a^p, x] = [a, x]^p$, Thus $C^p \leq Z(G)$. Let now $a, b \in C$ and $x \in G$; then $[a, b, x] = [b, xa]^{-1}[x, a, b]^{-1} = 1$, showing that $C' \leq Z(G)$.

(2) Let $Z = Z(G)$ and let $H \leq G$. If HZ/Z has finite exponent, then it is nilpotent by Theorem 191. Thus H is nilpotent. Conversely, let H be nilpotent. Then $G'H$ is nilpotent, whence, by Lemma 14, there exists $n \geq 1$ such that $[G', H^{p^n}] = 1$. Thus, for $x \in G$ and $y \in H^{p^n}$, $[x, y^p] = [x, y]^p = 1$, showing that $H^{p^{n+1}} \leq Z$. $\boxed{QED}$

200 Lemma. *Let G be a hypercentral p-group in $\mathcal{N}_1$, such that G' is elementary abelian. Let $C = C_G(G')$ and, for every $i \geq 1$, let $K_i = \langle x \in G \mid x^{p^i} \in C \rangle$. Suppose that G is not nilpotent; then, for every $i \geq 1$, K_{i+1}/K_i is an infinite elementary abelian p-group.*

PROOF. Observe that $K_1 \geq G'$, hence all factors K_{i+1}/K_i are elementary abelian p-groups. Observe also that, by Lemma 199 (1) and Theorem 191, K_i is nilpotent for every $i \geq 1$. Assume that, for some $i \geq 1$, K_{i+1}/K_i is finite. Then the abelian group G/K_i has finite rank, and so it is the direct product of a finite group T/K_i by a divisible group (of finite rank) R/K_i. Now, T is a finite extension of the nilpotent group K_i, and so T is nilpotent. Since R/C is abelian and K_i/C has finite exponent, a standard fact of abelian groups implies that there exists a divisible subgroup D/C of R/C such that $R = DK_i$. Write $Z = Z(G)$. Now, D is hypercentral; let $W/Z(G) = \zeta_2(D/Z) \cap C/Z$. Then $[W, D, D] \leq z$. Since, by 199, C/Z is elementary abelian, we have

$$Z \geq [W, D]^p = [W, D^p] = [W, D].$$

This shows that $C/Z \leq \zeta(D/Z)$. Hence, D is a normal nilpotent subgroup of G. Therefore, $G = TD$ is nilpotent. This contradiction shows that K_{i+1}/K_i is infinite. $\boxed{QED}$

201 Lemma. *Let G be a hypercentral p-group in $\mathcal{N}_1$, such that G' is elementary abelian. Then G is nilpotent.*

Proof. Suppose that G is not nilpotent. Then by 98 we may assume that there is a finite subgroup F of G, and a $n \geq 1$, such that all non-nilpotent subgroups of G containing F have defect at most n in G. We show that every subgroup V with $F \leq V$ has defect at most n. As in Lemma 200, let $Z = Z(G)$, $C = C_G(G')$, and, for every $i \geq 1$, $K_i = \langle x \in G \mid x^{p^i} \in C \rangle$.

Let V be a finitely generated subgroup of G containing F, and suppose by contradiction that V has defect larger than n. Then there exists $a \in [G,_n V] \setminus V$. Clearly $a \in G'$ and $F \leq V \leq K_m$ for some $m \geq 1$. We construct a series of subgroups $V \leq V_m \leq V_{m+1} \leq V_{m+2} \leq \dots$ such that, for every $j \geq m$, $V_j \leq K_j$, $V_j \not\leq K_{j-1}$, and $a \notin V_j$. Now, for every $j \geq m$, as observed in the proof of 200, K_j is nilpotent, and K_j/K_{j-1} is an infinite elementary abelian p-group by Lemma 200. Thus, the existence of the subgroups V_j with the desired properties is guaranteed by repeated applications of Theorem 192. Let

$$H = \bigcup_{j \geq m} V_j;$$

then $a \notin H \geq V$. On the other hand, H is not contained in any of the K_j's, thus the exponent of HC/C is infinite, and so, by Lemma 199 (2), H is not nilpotent. Since $F \leq H$ it follows that H has defect at most n in G, and this yields the contradiction $a \in [G,_n V] \leq [G,_n H] \leq H$.

This shows that all subgroups V of G, with $F \leq VB$ are subnormal of defect at most n in G; since G is locally nilpotent and F finite, Theorem 160 implies that G is nilpotent. $\boxed{QED}$

202 Theorem. [Möhres [81]] *A periodic hypercentral $\mathcal{N}_1$-group is nilpotent.*

Proof. Let G be a periodic hypercentral $\mathcal{N}_1$-group. By 165, we may assume that G is a p-group for some prime p.

By Lemma 11, non-trivial hypercentral groups cannot be perfect, and so, by Lemma 166, G is soluble. By Theorem 56 and the remark which follows, we may then assume that G is metabelian. Let $N = G'$, and

$$K = N^{\omega} = \bigcap_{n \geq 1} N^{p^n}.$$

Now, G/N^p is nilpotent by Lemma 201. It then follows from Lemma 190 that G/N^{p^n} is nilpotent for every $n \geq 1$. Thus, G/K is residually nilpotent and therefore it is nilpotent by Theorem 198. Since $K \leq Z(G)$ by Lemma 17, we conclude that G is nilpotent. $\boxed{QED}$

In the next chapter we will describe examples of H. Smith which show that this result too does not extend to arbitrary $\mathcal{N}_1$-groups.

6 The structure of $\mathcal{N}_1$-groups

6.1 Solubility of $\mathcal{N}_1$-groups

In this section we prove what is perhaps the most relevant result on $\mathcal{N}_1$-groups; i.e. that they are soluble; a fact that has been established by W. Möhres and appears in print in [80]. We follow his approach, that my well have applications to other problems.

Let $x_1, x_2, \ldots$ be an alphabet. We define the set of all *outer commutator words* inductively as follows:

(i) every x_i is an outer commutator word;

(ii) let $m, n \in \mathbb{N}$; if $\phi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_m)$ are outer commutator words, then $[\phi(x_1, \ldots, x_n), \psi(x_1, \ldots, x_m)]$ is an outer commutator word.

203 Lemma. [[80]] *Let G be a perfect locally finite p-group, such that for every proper subgroup T of G, T is soluble and $T^G \neq G$. Then there exist a finite subgroup U and a proper normal subgroup N of G, such that $Z(G/N) = 1$ and*

$$\bigcap_{x \in G \setminus N} \langle U, x \rangle \neq U.$$

Proof. We assume the Lemma to be false. Then, let T be a proper subgroup of G, and let $Z/T^G = Z(G/T^G)$. Since $T^G < G$, and G is perfect, we have $G \neq Z$, and $Z(G/Z) = 1$ (by Grün's Lemma 11). If U is a finite subgroup of G, and $a \in G \setminus U$, then by our assumption there exists $y \in G \setminus Z$ with $a \notin \langle U, y \rangle$.

Arguing by induction on $n \geq 1$, we show that given any finite subgroup U of G, any $a \in G \setminus U$, any proper subgroup T of G, and any outer commutator word $\phi(x_1, \ldots, x_n)$, there exist elements $y_1, \ldots, y_n \in G$, such that

$$\phi(y_1, \ldots, y_n) \notin T \quad \text{and} \quad a \notin \langle U, y_1, \ldots, y_n \rangle. \tag{45}$$

For $n = 1$, (45) means that there is an element $y \in G \setminus T$, such that $a \notin \langle U, y \rangle$, and this is what we had above.

Thus, let $n \geq 2$, and assume that the claim holds for smaller integers. Let U, T, and a as above, and $\phi(x_1, \ldots, x_n)$ an outer commutator word. Since $n \geq 2$, we may suppose that there is a $1 \leq k \leq n-1$, and there are outer commutator words $\phi_1(x_1, \ldots, x_k)$ and $\phi_2(x_{k+1}, \ldots, x_n)$ such that

$$\phi(x_1, \ldots, x_n) = [\phi_1(x_1, \ldots, x_k), \phi_2(x_{k+1}, \ldots, x_n)].$$

Let $Z/T^G = Z(G/T^G)$; then, as before, $Z \neq G$ and $Z(G/Z) = 1$. By inductive assumption, there exist elements $y_1, \ldots, y_k \in G$ with

$$\phi_1(y_1, \ldots, y_k) \notin Z \quad \text{and} \quad a \notin \langle U, y_1, \ldots, y_k \rangle,$$

and there exist elements $y_{k+1}, \ldots, y_n \in G$ such that

$$\phi_2(y_{k+1}, \ldots, y_n) \notin C_G(\phi_1(y_1, \ldots, y_k)Z)$$

and

$$a \notin \langle U, y_1, \ldots, y_k, y_{k+1} \ldots, y_n \rangle, \tag{46}$$

Therefore

$$\phi(y_1, \ldots, y_n) = [\phi_1(y_1, \ldots, y_k), \phi_2(y_{k+1}, \ldots, y_n)] \notin Z,$$

which, since $Z \geq T$, together with (46) is what we wanted. Thus, the claim leading to (45) is proved.

Now, write $\phi_1(x_1) = x_1$, and, for each $j \geq 1$

$$\phi_{j+1}(x_1, \ldots, x_{2^j}) = [\phi_j(x_1, \ldots, x_{2^{j-1}}), \phi_j(x_{2^{j-1}+1}, \ldots, x_{2^j})]. \tag{47}$$

Take $1 \neq a \in G$, and set $U_0 = 1$. Suppose that, for $i \geq 0$, we have found finite subgroups $U_0 \leq U_1 \leq \ldots \leq U_i$, with $a \notin U_i$. Then, by what we had before, there exist elements $y_{i,1} \ldots, y_{i,2^i} \in G$ such that $a \notin \langle U_{i-1}, y_{1,i} \ldots, y_{i,2^i} \rangle = U_i$ and $\phi_{i+1}(y_{i,1}, \ldots, y_{i,2^i}) \neq 1$.

Let $U = \bigcup_{i \in \mathbb{N}} U_i$. Then $a \notin U$, and so U is a proper subgroup of G. Hence, by hypothesis, U is soluble, of derived length, say, $d \geq 1$. But this contradicts $1 \neq \phi_{d+1}(y_{d,1}, \ldots, y_{d,2^d}) \in U^{(d)}$. $\boxed{QED}$

204 Lemma. *Let G be a locally finite p-group, such that for every proper subgroup T of G, T is soluble and $T^G \neq G$. If G is a Fitting group, then G is soluble.*

Proof. Let G be as in the assumptions, and suppose by contradiction that G is not soluble. Then G is perfect and, by Lemma 203, there exist a finite subgroup U and a proper normal subgroup N of G, such that $Z(G/N) = 1$ and there is an element $a \in \bigcap_{x \in G \setminus N} \langle U, x \rangle \setminus U$. Now, G is a Fitting group, and N is a proper normal subgroup; thus there exists an element $g \in G$, with $\langle g \rangle^G N/N$ a non-trivial elementary abelian p-group. Since, moreover, $Z(G/N) = 1$, we have that $\langle g \rangle^G N/N \simeq \langle g \rangle^G / \langle g \rangle^G \cap N$ is infinite. Now, $\langle g \rangle^G$ is nilpotent, and so we may apply Theorem 192 to conclude that there exists $z \in \langle g \rangle^G \setminus N$, such that $a \notin \langle U, z \rangle$. As $z \notin N$, this is a contradiction. $\boxed{QED}$

205 Lemma. *Let G be p-group in $\mathcal{N}_1$, and assume that all proper subgroups of G are soluble. Then G is soluble.*

Proof. By Lemma 204 it is enough to show that G is a Fitting group. Thus, let $x \in G$; then $K = \langle x \rangle^G$ is soluble because it is a proper subgroup of G. We prove that K is nilpotent arguing by induction on the derived length d

of K. If $d = 1$, K is abelian. Thus, let $d \geq 2$, and $A = K^{(d-1)}$. Then $A \trianglelefteq G$, and $K/A = \langle xA \rangle^{G/A}$; so, by inductive assumption, K/A is nilpotent. Since it is generated by conjugates of x (hence by elements of bounded order), K/A has finite exponent. Thus, K is a periodic $\mathcal{N}_1$-group which is an extension of an abelian group by a soluble group of finite exponent, and so, by Theorem 197, $K = \langle x \rangle^G$ is nilpotent. Therefore, G is is a Fitting group, and we are done. $\boxed{QED}$

We are now in a position to prove the main Theorem.

206 Theorem. [Möhres [80]] *Every $\mathcal{N}_1$-group is soluble.*

Proof. Let G be $\mathcal{N}_1$-group. By 120 and 165, we may assume that G is a p-group, for some prime p. Suppose that G is not soluble; then, by 98, there exists a non-soluble subgroup H of G, a finitely generated subgroup F of H, and a positive integer d, such that every non-soluble subgroup K of H with $F \leq K$ has defect at most d in H. Let $H = H_0 \geq H_1 \geq \ldots \geq H_d = F$ be the normal closure series of F in H, and let $B = H_i$ be the smallest non -soluble term of it. Then F^B is soluble, and so $K = B/F^B$ is not soluble. Furthermore, all non-soluble subgroups of K have defect at most d. It then follows from Roseblade's Theorem that all non-soluble subgroups of K contain the limit D of the derived series of K. Then, all proper subgroups of D are soluble and so, by Lemma 205, D is soluble. But then, Lemma 166, D is soluble, which is a contradiction. $\boxed{QED}$

Having proved that every $\mathcal{N}_1$-group is soluble makes of course redundant this assumption in Theorems like 191, 197 or in proposition 125. Specifically, for further reference, we restate as a Proposition, an argument used in the proof of Lemma 205.

207 Proposition. *Let $G \in \mathcal{N}_1$, and suppose that G is generated by elements of finite bounded order. Then G is nilpotent of finite exponent.*

6.2 Fitting Groups

Proposition 207 implies that in a $\mathcal{N}_1$ group every element of finite order belongs to the Fitting radical. In this section, we generalize this by showing that every $\mathcal{N}_1$-group is a Fitting group. This answers a question of D. Robinson, and completes the information about the inclusion relations among some relevant classes of locally nilpotent groups, as mentioned in the second volume of [96].

In fact, we shall prove something more, i.e. that in a $\mathcal{N}_1$-group every nilpotent subgroup is contained in a normal nilpotent subgroup.

We start with an observation that is certainly well known.

208 Lemma. *Let G be a nilpotent group such that its torsion subgroup T has finite exponent. Then there exists a $1 \leq k \in \mathbb{N}$ such that $G^k \cap T = 1$.*

Proof. Let q be the exponent of T.

We first assume that G/T is abelian, and proceed by induction on the minimal integer m such that $T \leq \zeta_m(G)$. If $m = 1$, then $G' \leq T \leq \zeta(G)$. Now, for all $x, y \in G$, Lemma 2 yields

$$(xy)^{2q} = x^{2q} y^{2q} [y, x]^{q(2q-1)} = x^{2q} y^{2q}.$$

Thus $G^{2q} = \{x^{2q} \mid x \in G\}$. Also, if $a = x^{2q} \in G^{2q} \cap T$, then $1 = a^q = x^{2q^2}$. So $x \in T$ and, consequently, $a = x^{2q} = 1$. Hence $G^{2q} \cap T = 1$.

Let now $m \geq 2$, and set $X = \zeta(G) \cap T$. Then T/X is the torsion subgroup of G/X, and is contained in $\zeta_{m-1}(G/X)$. By inductive hypothesis, there is a $s \geq 1$ such that $G^s \cap T \leq X$. Now, $G^s \cap T$ is the torsion subgroup of G^s and is contained in its centre. By the case $m = 1$, we have that $G^{2sq} \cap X = 1$ and so $G^{2sq} \cap T = G^{2sq} \cap T \cap G^s = G^{2sq} \cap X = 1$.

We now prove the general case by proceeding by induction on the nilpotency class c of G/T. Let Z/T be the centre of G/T. As Z/T is abelian, there exists, by the case $c = 1$ discussed above, an $s \geq 1$ such that $Y = Z^s$ has trivial intersection with T. Now, G/Z is torsion-free (see Proposition 103). Thus, Z/Y is the torsion subgroup of G/Y, and has finite exponent. Since the nilpotency class of G/Z is $c - 1$, by inductive assumption there exists $k \geq 1$ such that $(G/Y)^k = G^k Y/Y$ has trivial intersection with Z/Y. In other words, $G^k \cap Z \leq G^k Y \cap Z = Y$, which in turn gives $G^k \cap T \leq Y \cap T = 1$. $\boxed{QED}$

Now, we prove a technical but useful Lemma. Recall (see 198) that, for $H \leq G$, $H \leq_b G$ means that there exists an integer $m \geq 1$ such that $g^m \in H$ for all $g \in G$.

209 Lemma. *Let $G \in \mathcal{N}_1$ be such that the torsion subgroup A of G is nilpotent, and G/A^n is nilpotent for every $n \geq 1$. Assume that there exists a finitely generated subgroup F of G, and an integer $d \geq 1$, such that every subgroup H, with $F \leq H \leq_b G$, has defect at most d in G. Then there exists $c \geq 1$, which depends only on d and the nilpotency class of G/A, such that every subgroup of G containing F has defect at most c.*

Proof. By Proposition 99 we may assume $A^\omega = \cap_{n \geq 1} A^n = 1$. As A is the torsion subgroup of the locally nilpotent group G, G/A is torsion–free and so it is nilpotent by Theorem 124. Let r be the nilpotency class of G/A, let $\beta(d)$ as defined by Theorem 160, and set $m = \max\{r, \beta(d)\} + 1$.

Let $n \geq 1$. Then G/A^n is nilpotent by assumption, whence, by Lemma 208, there exists a normal subgroup M_n of G such that $M_n \cap A = A^n$, and G/M_n has finite exponent; in particular, $H \leq_b G$ for any $H/M_n \leq G/M_n$. Thus, by assumption, all subgroups of G/M_n containing the finite subgroup FM_n/M_n have defect at most d. By Theorem 160, $\gamma_m(G/M_n) = \gamma_m(G)M_n/M_n$ is finite.

Also, by choice of m, $\gamma_m(G) \leq A$, so that $\gamma_m(G) \cap M_n = \gamma_m(G) \cap A \cap M_n = \gamma_m(G) \cap A^n$. It follows that

$$\frac{\gamma_m(G)A^n}{A^n} \cong \frac{\gamma_m(G)}{A^n \cap \gamma_m(G)} = \frac{\gamma_m(G)}{M_n \cap \gamma_m(G)} \cong \frac{\gamma_m(G)M_n}{M_n}$$

is finite. By Proposition 53, $\zeta_{2m}(G/A^n)$ has finite index in G/A^n.

Let H be a subgroup of G be such that $A^nF \leq H$ for some $n \geq 1$. Then, setting $Z/A^n = \zeta_{2m}(G/A^n)$, we have, by what just proved, that HZ has finite index in G, and so, by assumption, that its defect is at most d in G. Now, clearly, H/A^n has defect at most $2m$ in ZH/A^n. Hence H has defect at most $2m$ in ZH, and so H has defect at most $c = d + 2m$ in G. (this holds for all $n \geq 1$).

Now, to show that every subgroup $H \geq F$ has defect at most c in G, we may well assume that H is finitely generated.

By what proved before, for every $n \geq 1$, A^nH has defect at most $c = 2m + d$ in G. Also, by the definition of c, we have that $[G,_c H] \leq A$. Thus,

$$[G,_c H] \leq \bigcap_{n \geq 1}(A^nH \cap A) = \bigcap_{n \geq 1} A^n(H \cap A).$$

But H is finitely generated nilpotent group,, and so $A \cap H \leq \mathrm{Tor}(H)$ is finite. Since we are assuming $\cap_{n\geq 1}A^n = 1$, we conclude by Lemma 29 that

$$[G_c H] \leq H \cap A \leq H.$$

This proves that H has defect at most c in G. $\boxed{QED}$

We now generalize Theorem 197.

210 Theorem. [H. Smith [107]] *Let G be a $\mathcal{N}_1$-group. If G is the extension of a nilpotent group by a group of finite exponent, then G is nilpotent.*

For the proof, we need the following observation.

211 Lemma. *Let A be an abelian p-group, and X an elementary abelian p-group of automorphisms of A. Then, for every $n \geq 1$,*

$$[A,_n X]^{p^n} \leq [A,_{2n} X].$$

Proof. By induction on n. When $n = 1$, set $\overline{A} = A/[A, X, X]$. Then, for every $\overline{a} \in \overline{A}$ and every $x \in X$, $[\overline{a}, x, x] = 1$, whence, by 2, $[\overline{a}, x]^p = [\overline{a}, x^p] = 1$, showing that $[A, X]^p \leq [A, X, X]$. Let now $n \geq 2$, then

$$[A,_n X]^{p^n} = [[A,_{n-1} X]^{p^{n-1}}, X]^p$$

and so, by the inductive assumption and case $n = 1$,

$$[A,_n X]^{p^n} \leq [[A,_{2(n-1)} X], X]^p \leq [A,_{2n-2} X,_2 X] = [A,_{2n} X],$$

thus proving the Lemma. $\boxed{QED}$

Proof. of Theorem 210. Suppose that G is a counterexample to the theorem. Then, by an obvious inductive argument (using the fact that a $\mathcal{N}_1$-group of finite exponent is nilpotent) we may assume that G admits a normal nilpotent subgroup N such that G/N is an elementary abelian p-group for some prime p. Also, by P. Hall's nilpotency criterion, we may reduce to the case in which N is abelian. Let A be the torsion subgroup of N. Since G is locally nilpotent and $G/C_G(A)$ is a p-group, it follows that the p'-component of A is central in G; thus we may assume that A is a p-group. If T is the torsion subgroup of G, then $T \cap N = A$ and T is nilpotent by Theorem 197; if R is a subgroup of G such that $G/N = TN/N \times R/N$, then R is not nilpotent and $T \cap R = A$. We may therefore replace G by R, and assume that A is the torsion subgroup of G; in particular $C_G(A) \geq N$.

By Lemma 195 we may furthermore suppose that a subgroup H of G is nilpotent if and only if HN/N is finite. Thus, by Brookes' trick 98, we may finally assume that there are a finitely generated subgroup F of G and a positive integer d such that every subgroup H of G which contains F and such that HN/N is infinite has defect at most d in G. Since $FN/N \simeq F/(F \cap N)$ is finite, FN is nilpotent and normal in G; by invoking again P. Hall's nilpotency criterion, we may reduce to the case $(FN)' = 1$; in particular, $F^G \leq FN$ is abelian (and it is easy to see that all other assumptions on A may be mantained).

For $n \geq 1$, let $A_n = A^{p^n}$. by Proposition 99, we may suppose

$$\bigcap_{n \geq 1} A_n = 1. \tag{48}$$

Now, since $A^p[A,_p x] = A^p[A, x^p] = A^p$, for every $x \in G$ (Lemma 14), we deduce that, for every $n \geq 1$, G/A_n is a bounded Engel group, and so it is nilpotent by Lemma 194. By Lemma 208 there exists a normal subgroup M_n of G, with $A \cap M_n = A_n$, $M_n \leq N$, and G/M_n a p-group of finite exponent. By Lemma 193, applied to the group G/M_n, its normal subgroup N/M_n and FM_n/M_n, we deduce that every subgroup of G containing FM_n has defect at most d. This holds for any $n \geq 1$, and so we may apply Lemma 209 and conclude that there exists $c \geq 1$ such that every subgroup of G containing F has defect at most c.

Let $x_1, \ldots, x_c \in G$, and $X = \langle x_1, \ldots, x_c \rangle$. Then

$$[A, x_1, \ldots, x_c] \leq [A,_c X] \leq \langle F, X \rangle. \tag{49}$$

Also, $C_X(F) \geq X \cap N \trianglelefteq X$, whence $|X/C_X(F)| \leq p^c$. Since F^G is abelian, the rank of F^X is bounded by $rk(F)p^c$. Moreover, all subgroups of $\langle F, X \rangle / F^X$ have defect at most c, and so $\langle F, X \rangle / F^X$ is nilpotent of class at most $\rho(c)$ by

Roseblade's Theorem. Since $\langle F, X\rangle/F^X$ is generated by c elements, it follows from Proposition 48 that its rank is bounded by a function of c. Therefore, for all choices of $x_1, \ldots, x_c \in G$, the rank of $\langle F, X\rangle$ is bounded uniformely by a value ℓ (depending on c and $rk(F)$). In particular, from (49) we get

$$rk[A, x_1, \ldots, x_c] \leq \ell. \tag{50}$$

Let now $D = [A,_{2c} G]^p = \Phi([A,_{2c} G])$, and write $\overline{A} = A/D$. By Lemma 211, for any $x_1, \ldots, x_c \in G$ we have

$$[\overline{A}, x_1, \ldots, x_c]^{p^c} \leq [\overline{A},_{2c} G] = [A,_{2c} G]/D;$$

hence $[\overline{A}, x_1, \ldots, x_c]$ has exponent dividing p^{c+1}. From (50), we therefore deduce that

$$|[\overline{A}, x_1, \ldots, x_c]| \leq p^{(c+1)\ell}$$

for every $x_1, \ldots x_c \in G$. Since $G/C_G(A)$ is (elementary) abelian, we may apply Lemma 161 obtaining that $[\overline{A},_{2c} G] = [A,_{2c} G]/D$ is finite. Since $D = [A,_{2c} G]^p$ and $[A,_{2c} G]$ is reduced (by (48)), we conclude that $[A,_{2c} G]$ is a normal finite subgroup of G. Thus, $[A,_{2c} G] \leq \zeta_t(G)$ for some $t \geq 1$. As G/A is nilpotent, we finally obtain that G is nilpotent. $\boxed{QED}$

212 Corollary. *Let $G \in \mathcal{N}_1$. If G admits a nilpotent subgroup H with $H \leq_b G$, then G is nilpotent.*

PROOF. Let G, H be as in the hypotheses. Then G/H_G has bounded exponent (argue by induction on the defect of H in G). Thus, by Theorem 210, G is nilpotent. $\boxed{QED}$

We are now ready to prove the main result of this section.

213 Theorem. *Let G be a group with all subgroups subnormal, and let S be a nilpotent subgroup of G. Then S^G is nilpotent.*

PROOF. Let $G \in \mathcal{N}_1$, S a nilpotent subgroup of G, and $W = S^G$. We fix the notation A for the torsion subgroup of W. Then W/A is a torsion-free $\mathcal{N}_1$-group, and so it is nilpotent by Theorem 124. We want to prove that W is nilpotent. Arguing by induction on the defect of S in G, we may assume that S is normal in W. We begin by proving

(1) The torsion subgroup A of W is nilpotent.

By Lemma 165, it is enough to show that every primary component of A is nilpotent. By factoring modulo the product of all p'-components, we may assume that A is a p-group for some prime p. By induction on the derived length of A, we may also assume that A has a characteristic abelian subgroup X such that A/X is nilpotent. Now, let c be the nilpotency class of S, and let $x \in S$. Then

$$[X,_{c+1} x] = [[X, x],_c x] \leq [S,_c x] = 1 \ .$$

Let q be a power of p greater than $c+1$. Then, by Lemma 14, every G-invariant elementary abelian section of X is centralized by x^q. This holds for every $x \in S$. It follows that $K = \langle (x^g)^q \mid x \in S, g \in G\rangle$ centralizes every G-invariant elementary abelian section of X. Now, X is normal in G and has an ascending characteristic series with elementary abelian factor groups, and so it follows that $X \cap K$ is hypercentral in K. Since $A \cap K/X \cap K \cong X(A \cap K)/X \leq A/X$ is nilpotent, $A \cap K$ is a hypercentral periodic $\mathcal{N}_1$-group. By Theorem 202, $A \cap K$ is nilpotent. Finally, as W is generated by the conjugates of S, W/K is nilpotent of finite exponent by Proposition 207. In particular, $A/A \cap K$ is nilpotent of finite exponent. Hence, by Theorem 197, A is nilpotent.

(2) A is abelian.

If W/A' is nilpotent, then, since A is nilpotent, W is nilpotent by Theorem 56. Hence we may assume A to be abelian.

Let $N = [G, S]$. Then $W = NS$ and, by Fitting's Theorem, W is nilpotent if and only if N is such. We then prove that N is nilpotent. Suppose that N is not nilpotent. By Corollary 212 and Theorem 98, there exist a subgroup $H \leq_b N$, a finitely generated subgroup F of H, and a positive integer d such that all subgroups L of H with $F \leq L \leq_b H$ have defect at most d in H.

Now, $[S, G]$ is generated by all the commutators $[x, g]$, with $x \in S$, $g \in G$, and so there exist a finitely generated subgroup S_1 of S, and a finitely generated subgroup G_1 of G, such that, writing $V = \langle S_1, G_1\rangle$

$$F \leq [S_1, G_1] \leq V' \, .$$

(3) H satisfies the hypotheses of Lemma 209.

Let $n \geq 1$. Let $B = A \cap N$ be the torsion subgroup of N, and let $n \geq 1$. Then B/B^n has finite exponent and is invariant for W. Let $x \in S$; then, as in point (1), $[B,_{c+1} x] = 1$ so, by Lemma 16, there exists a $q \geq 1$ such that x^q centralizes B/B^n. Arguing as in point (1), we have that, if $K = C_W(B/B^n)$, then W/K has finite exponent, and so $N/N \cap K$ has finite exponent. Since N/B is nilpotent (because it is torsion-free), we have that $(K \cap N)/B^n$ is nilpotent. Thus, by Theorem 210, N/B^n is nilpotent. This holds for every $n \geq 1$, and so N satisfies the hypotheses of Lemma 209. Observe now that, since $H \leq_b N$, H satisfies these same hypotheses and so, by Lemma 209, every subgroup of H containing F has defect at most c in H, for some $c \geq 1$.

Let $U = BV$, $Y = NU = NV$, and $U = U_0 \trianglelefteq U_1 \trianglelefteq \ldots \trianglelefteq U_n = Y$ be the normal closure series of U in Y. For each j, let $R_j = U_j \cap N$. Notice that R_j is normal in U_j and contains BF. Given a j, let $Q_j = F^{R_j \cap H}$. By Roseblade's Theorem, $(R_j \cap H)/Q_j$ is nilpotent. Since $F^{R_j} \cap H \geq Q_j$, $(R_j \cap H)/(F^{R_j} \cap H)$ is nilpotent. As $R_j \cap H$ has finite index in R_j, it follows that R_j/F^{R_j} is nilpotent.

Now, by induction on i, we prove that U_i is nilpotent. This is trivial for $i = 0$, as $U = BV$ is a Baer group, and an extension of an abelian group by a finitely generated group. Thus, assume that U_i is nilpotent. Then R_i is nilpotent and is normalized by U_{i+1}. Now, $F \leq V' \leq U_i' \trianglelefteq U_{i+1}$, whence $F^{U_{i+1}} \leq U_i' \cap R_{i+1}$. Thus, a fortiori, $F^{R_{i+1}} \leq U_i' \cap R_{i+1}$, and, by what we have proved above, we have that

$$\frac{R_{i+1}}{U_i' \cap R_{i+1}} \cong \frac{R_{i+1}U_i'}{U_i'}$$

is nilpotent. Since $U_{i+1}/R_{i+1} = U_{i+1}/(U_{i+1} \cap N)$ is isomorphic to a subgroup of the finitely generated group Y/N, U_{i+1}/R_{i+1} is finitely generated, and so $U_{i+1}/(R_{i+1} \cap U_i')$ is nilpotent. In particular, U_{i+1}/U_i' is nilpotent, and so, by P. Hall's criterion, U_{i+1} is nilpotent. This completes the induction. Thus we conclude that $Y = U_n$ is nilpotent, which forces N to be nilpotent. $\boxed{QED}$

As a particular case of the previous Theorem, we have the following

214 Corollary. *A $\mathcal{N}_1$-group is a Fitting group.*

Another immediate corollary of 213 answers a question of H. Smith [102].

215 Corollary. *Let G be group with all subgroups subnormal. If $G = \langle H, K\rangle$ where H, K are nilpotent subgroups, then G is nilpotent.*

6.3 Hypercentral and Smith's groups

We have already seen that periodic hypercentral $\mathcal{N}_1$-groups are nilpotent (Theorem 202); thus, this section on hypercentral groups will focus on non-periodic (indeed, mixed) groups, beginning with H. Smith's construction of non-nilpotent hypercentral $\mathcal{N}_1$-groups, which we have already mentioned on several occasions. The relevance of the hypercentral case in the study of $\mathcal{N}_1$-groups (in particular, for non–periodic groups) may be for instance gathered from Theorem 229.

Smith's method constructs mixed $\mathcal{N}_1$-groups which have some common features, but may be adapted to produce hypercentral $\mathcal{N}_1$-groups with additional properties (see [101] and [112]). I will restrict to a full presentation of one single case (the first produced by Smith).

216 Theorem. [H. Smith [101]] *There exists a non-nilpotent group G with the following properties:*

(1) *all subgroups of G are subnormal;*

(2) *G is hypercentral of length $\omega + 1$;*

(3) *G is locally metacyclic and residually finite;*

(4) *every subgroup H of G has finite index in the second term $H^{G,2}$ of its normal closure series.*

PROOF. Let $p_1, p_2, p_3, \ldots$ be an infinite sequence of distinct prime numbers. For every $n \geq 1$, let

$$H_n = \langle x_n, y_n \mid x_n^{p_n^n} = 1 = y_n^{p_n^{n-1}},\ x_n^{y_n} = x_n^{p_n+1} \rangle.$$

Thus, each H_n is the semidirect product of normal a cyclic group $X_n = \langle x_n \rangle$ by a cyclic group $\langle y_n \rangle$, where y_n acts by conjugation on X_n as an automorphism of order p_n^{n-1}. Let F be the cartesian product of the groups H_n:

$$F = Car_{n \geq 1} H_n.$$

Then F is metabelian and residually finite. Also, clearly, $X_n \leq \zeta_n(F)$ for each $n \geq 1$, and so F is hypercentral of length $\omega + 1$.

For every pair $n, m \geq 1$ with $n \neq m$ let $u_{n,m} \in \mathbb{N}$ be such that

$$u_{n,m} p_m^{m-1} \equiv 1 \pmod{p_n^n}. \tag{51}$$

Let $\bar{z}$ be the element of F defined by $\bar{z}(i) = x_i^{-1}$ for every $i \geq 1$; and, for each $n \geq 1$, let $\bar{x}_n, \bar{y}_n \in F$ such that

$$\bar{x}_n(i) = \begin{cases} x_n & \text{if} \quad i = n \\ 1 & \text{if} \quad i \neq n \end{cases} \qquad \bar{y}_n(i) = \begin{cases} y_n & \text{if} \quad i = n \\ x_i^{-u_{i,n}} & \text{if} \quad i \neq n \end{cases} \tag{52}$$

Notice the following commutator relations; for every $n, m \geq 1$:

$$\begin{array}{rll} [\bar{x}_n, \bar{y}_m] = & \bar{x}_n^{p_n} & \text{if} \quad m = n \\ [\bar{x}_n, \bar{y}_m] = & 1 & \text{if} \quad m \neq n \\ [\bar{y}_n, \bar{y}_m] = & \bar{x}_n^{p_n u_{n,m}} x_m^{-p_m u_{m,n}} & \text{if} \quad m \neq n \\ [\bar{y}_n, z] = & \bar{x}_n^{p_n} & \end{array} \tag{53}$$

Also, from (51), for every $n \geq 1$ we have

$$\bar{y}_n^{p_n^{n-1}} = \bar{x}_n z. \tag{54}$$

We then consider the subgroup G of F:

$$G = \langle \bar{x}_n,\ \bar{y}_n,\ z \ \mid\ n \geq 1 \rangle.$$

Let $X = Dir_{n \geq 1} X_n = \langle \bar{x}_n \mid n \geq 1 \rangle$. Then X is normal in G, it is periodic, locally cyclic, and contained in $\zeta_\omega(G)$. By the relations (53) we also have that $X \geq G'$, and that G/X is an abelian group of rank 1 (a subgroup of the additive group of the rationals). Thus, X is the torsion subgroup of G, and G is locally metacyclic. Furthermore G is residually finite because such is F. Hence G satisfies property (3) in the statement.

Then observe that the fourth relation in (53) implies that, for every $n \geq 1$, $[z,_{n-1} \bar{y}_n] \neq 1$. Hence $z \notin \zeta_\omega(G)$, and so G is hypercentral of length $\omega + 1$, i.e. property (2) in the statement is satisfied by G.

We now prove that every subgroup of G is subnormal and satisfies (4). Let $A = X\langle x\rangle$; then A is an abelian normal subgroup of G, and G/A is a direct product of cyclic p_n-groups.

Let $S \leq G$. If $S \cap A = S \cap X$, then

$$\frac{S}{S \cap X} = \frac{S}{S \cap A} = \frac{SA}{A}$$

is periodic, hence S is periodic and so $S \leq X$, which implies that S is subnormal of defect at most 2 in G

Suppose $S \cap A > S \cap X$. Then there exist $x \in X$ and $r > 0$ such that $xz^r \in S$. Since x has finite order, we get that there exists $s > 0$ such that $z^s \in S$. Let

$$X^* = \langle \bar{x}_n \mid (p_n, s) = 1\rangle.$$

We prove that X^* normalizes S. Let $g \in S$; then there exist an element $a \in A$ and integers $t \in \mathbb{N}$, $\beta_1, \dots \beta_t \geq 1$, such that

$$g = a\bar{y}_{i_1}^{\beta_1} \cdots \bar{y}_{i_t}^{\beta_t}.$$

Let $n \geq 1$ with $(p_n, s) = 1$. Then $[\bar{x}_n, g] = [\bar{x}_n, \bar{y}_{i_1}^{\beta_1} \cdots \bar{y}_{i_t}^{\beta_t}.]$. Hence $[\bar{x}_n, g] = 1 \in S$ if $n \notin \{i_1, \dots, i_t\}$; otherwise, $n = j_j$ for some $j \in \{1, \dots, t\}$ and, letting $\beta = \beta_j$,

$$[\bar{x}_n, g] = [\bar{x}_n, \bar{y}_n^\beta]. \tag{55}$$

Now, since G/A is abelian, by (54) there is a p_n'-number k such that $g^k = a'\bar{y}_n^\beta$.

Then $S \ni [z^s, g^k] = [z, g^k]^s = [z, \bar{y}_n^\beta]^s$. Now, by (53), $[z, \bar{y}_n^\beta] = [\bar{x}_n^{-1}, \bar{y}_n^\beta]$ belongs to $\langle \bar{x}_n\rangle$, and so has order coprime to s. It then follows that

$$[\bar{x}_n, g] = [\bar{x}_n, \bar{y}_n^\beta] = [z, \bar{y}_n^\beta]^{-1} \in \langle [z^s, g^k]\rangle \leq S.$$

Thus, we have proved that X^* normalizes S. Let the $X_* = \langle \bar{x}_n \mid p_n \ s\rangle$. Then X_* is a finite normal subgroup of G, and $X = X^* X_*$. Hence

$$S^{G,2} = S[G,_2 S] \leq S[X, S] = S[X^* X_*, S] = S[X_*, S] \leq SX_*,$$

so $|S^{G,2} : S|$ is finite, and property (4) is satisfied. Finally, X_* is contained in some term $\zeta_m(G)$ of the upper central series of G; therefore

$$[G,_{m+1} S] \leq [X,_m S] \leq S[X_*,_m S] \leq S.$$

This shows that S is subnormal and completes the proof. $\boxed{QED}$

H. Smith's method, in all of its occurencies in papers, gives groups of hypercentral length $\omega + 1$, and it is not immediate how it could be implemented in order to obtain hypercentral $\mathcal{N}_1$-groups of different type. In particular, we ask

5 Question. For every integer $n \geq 1$ construct a hypercentral $\mathcal{N}_1$-group of length $\omega + n$ (or prove that there are not any).

The above question is also motivated by the fact that there do not exist hypercentral $\mathcal{N}_1$-groups of length exactly ω; this was proved by H. Smith [113] (see also [16]).

217 Theorem. *Let G be a hypercentral group of hypercentral length at most ω. If all subgroups of G are subnormal, then G is nilpotent.*

Proof. Let $G \in \mathcal{N}_1$ be hypercentral of hypercentral length at most ω. This means that $G = \bigcup_{n\in\mathbb{N}} \zeta_n(G)$.

By Theorem 98 and Corollary 212, there exist a subgroup $H \leq_b G$, a finitely generated subgroup F of H, and a positive integer d, such that all subgroups K of H, with $F \leq K \leq_b H$ have defect at most d in H. Since $H \leq_b G$ and $F \leq \zeta_n(G)$ for some $n \in \mathbb{N}$, we may assume $F = 1$ and $H = G$.

Let A be the torsion subgroup of G. By Theorem 202, A is nilpotent, hence, by 56, we may also assume that A is abelian. If G/A^n is nilpotent for all $n \geq 1$, then G is nilpotent by Lemma 209 and Roseblade's Theorem 155. So, we are left with the case in which A is an abelian group of finite exponent. Let $C = C_G(A)$. Then, by Lemma 16, G/C is periodic. Now $A \leq C$ and C/A is torsion-free and thus nilpotent. Hence, C is nilpotent and, by Lemma 208, there exists an integer $k \geq 1$ such that $C^k \cap A = 1$. Now, $C^k \trianglelefteq G$, and G/C^k is periodic and hypercentral. Thus G/C^k is nilpotent. Since G/A is also nilpotent, we conclude that $G = G/(A \cap C^k)$ is nilpotent. $\boxed{QED}$

Recently, Martinelli ([70]) gave a more complete statement, which further motivates Question 5.

218 Theorem. *Let G be a hypercentral non–nilpotent group in $\mathcal{N}_1$. Then G has hypercentral length $\omega + n$ for some $1 \leq n \in \mathbb{N}$.*

In the same work, Martinelli provides an extension of Theorem 198, by showing that a residually nilpotent $\mathcal{N}_1$-group is hypercentral. To approach the proof of this, let us first introduce the class $\mathfrak{X}_0$ of all locally nilpotent groups G such that $A = \mathrm{Tor}(G)$ is nilpotent and G/A^n is also nilpotent for every $n \geq 1$.

219 Lemma. *Let $G \in \mathcal{N}_1$ and $H \leq_b G$. If $H \in \mathfrak{X}_0$ then $G \in \mathfrak{X}_0$.*

Proof. Let $G \in \mathcal{N}_1$, $H \leq_b G$ with $H \in \mathfrak{X}_0$, and write $N = H_G$. Then G/N has finite exponent, and in particular, if $B = \mathrm{Tor}(N)$, $B \geq (\mathrm{Tor}(H))^m$ for some $m \geq 1$. From this it easily follows that $N \in \mathfrak{X}_0$.

Let $A = \mathrm{Tor}(G)$; then $B = A \cap N$ and $A/B \simeq AN/N$ has finite exponent. Since B is nilpotent, we have that A is nilpotent by Theorem 197.

Now, let $n \geq 1$. Then $A^n \geq B^n$, and so, by assumption, A^nN/A^n is nilpotent. Thus, G/A^n is the extension of the nilpotent normal subgroup A^nN/A^n by a group of finite exponent. From Theorem 210 we conclude that G/A^n is nilpotent, thus proving that G belongs to $\mathfrak{X}_0$. $\boxed{QED}$

We also need to strengthen Proposition 99.

220 Lemma. *Let $G \in \mathcal{N}_1$, and let A be a normal nilpotent and periodic subgroup of G. Then there exists an integer $m \geq 0$ such that*

$$(A/N)^\omega \leq \zeta_m(G/N)$$

for any normal subgroup N of G contained in A.

Proof. We may assume that G/A is countable (this is slightly less immediate than in the proof of 99: if the property fails for some G, then for every positive integer n there exist a normal subgroup $N_n \leq A$ and a finitely generated subgroup X_n of G such that $[(A/N_n)^\omega,_n X_n] \neq 1$; then consider the subgroup of G generated by A and X_n for every $n \geq 0$). Thus, let $\{Ax_1, Ax_2, Ax_3, \ldots\}$ be an enumeration of the elements of G/A.

Now, as in the proof of 99, using the chain of finitely generated nilpotent groups $\langle x_1 \rangle \leq \langle x_1, x_2 \rangle \leq \ldots$, one shows that there exists a subgroup U of G, with $A \cap U = 1$, and the property that for each $x \in G$ there exists $1 \leq k \in \mathbb{N}$ such that $x^k \in U$ (this last property follows from the fact it holds modulo A by construction of U, A is normal and periodic, and U is subnormal in AU). Let m be the defect of U in G; and let N be a normal subgroup of G contained in A. Then $A \cap NU = N(A \cap U) = N$, and so, writing $\overline{A} = A/N$, $\overline{U} = UN/N$,

$$[\overline{A},_d \overline{U}] \leq \overline{A} \cap \overline{U} = 1.$$

Let $x_1, \ldots, x_m \in G$, and let $k_1, \ldots, k_m \in \mathbb{N}$ with $x_i^{k_i} \in U$. By Lemma 21

$$[\overline{A}^\omega, Nx_1, \ldots, Nx_m] \leq [\overline{A}, \langle Nx_1^{k_1} \rangle, \ldots, \langle Nx_m^{k_m} \rangle] \leq [\overline{A},_d \overline{U}] = 1,$$

and this proves the Lemma. $\boxed{QED}$

Recall from Chapter 1 (section 1.2), that a group G is hypocentral if $\{1\}$ is a term of the extended lower central series of G. For a group G we also write

$$\gamma_\omega(G) = \bigcap_{1 \leq n \in \mathbb{N}} \gamma_n(G);$$

thus G is residually nilpotent if and only if $\gamma_\omega(G) = 1$.

221 Theorem. *Let $G \in \mathcal{N}_1$. Then the following conditions are equivalent.*

(1) $G \in \mathfrak{X}_0$;

(2) *G is hypercentral;*

(3) $\gamma_\omega(G) \leq \zeta_m(G)$ *for some* $m \in \mathbb{N}$;

(4) *G is hypocentral.*

Proof. Let G be a $\mathcal{N}_1$-group with $G \in \mathfrak{X}_0$, and let $A = \mathrm{Tor}(A)$. We first prove the following claim:

$$\textit{there exists } m \geq 1 \textit{ such that } \frac{\gamma_m(G)A^n}{A^n} \textit{ is finite for all } n \geq 1. \tag{56}$$

Clearly, we may assume that G is not nilpotent. However, G/A is nilpotent by Theorem 124; let c be the nilpotency class of G/A. We first assume that there is a finitely generated subgroup F of G, and a positive integer d, such that every $H \leq_b G$ containing F has defect at most d. Let $\beta(d)$ as defined in Theorem 160, and let $m = \max\{c, \beta(d)\} + 1$. Fix $n \geq 1$. Then G/A^n is nilpotent by hypothesis, and so, by Lemma 208, there exists a normal subgroup M_n of G such that $M_n \cap A = A^n$, and G/M_n has finite exponent. Now, FM_n/M_n is finite, and for each $H/M_n \leq G/M_n$, $H \leq_b G$. Thus, by our assumption, all subgroups of G/M_n containing the finite subgroup FM_n/M_n have defect at most d. By Theorem 160, $\gamma_m(G/M_n) = \gamma_m(G)M_n/M_n$ is finite. Also, by choice of m, $\gamma_m(G) \leq A$, so that $\gamma_m(G) \cap M_n = \gamma_m(G) \cap A \cap M_n = \gamma_m(G) \cap A^n$. It there follows that

$$\frac{\gamma_m(G)A^n}{A^n} \cong \frac{\gamma_m(G)}{A^n \cap \gamma_m(G)} = \frac{\gamma_m(G)}{M_n \cap \gamma_m(G)}$$

is finite. For the general case, by Theorem 98 we know that there exists $H \leq_b G$ which satisfies the condition we have assumed above. Since G/H_G has finite exponent, $B = A \cap H \geq A^k$ for some $k \geq 1$. This implies that for each $n \geq 1$, $B^n \geq A^{kn}$ and H/B^n, being a section of G/A^{kn}, is nilpotent by hypothesis. Thus, there is a $m \geq 1$ such that $\gamma_m(H)B^n/B^n$ is finite for all $n \geq 1$. So,

$$\frac{\gamma_m(H)A^n}{A^n} \cong \frac{\gamma_m(H)}{A^n \cap \gamma_m(H)}$$

being a factor of $\gamma_m(H)/(\gamma_m(H) \cap B^n)$ it is finite. Let $H \trianglelefteq H_1$. Then $\gamma_m(H)$ is normal in H_1. Since H_1/H has finite exponent, Theorem 210 yields that $H_1/\gamma_m(H)$ is nilpotent, that is $\gamma_s(H_1) \leq \gamma_m(H)$ for some $s \in \mathbb{N}$. Then we have that $\gamma_s(H_1)A^n/A^n$ is a subgroup of $\gamma_m(H)A^n/A^n$ and so it is finite. By repeating this argument along the normal closure series of H in G, we finally get claim (56).

Now we prove implication (1) $\Rightarrow$ (2). We suppose that $G \in \mathfrak{X}_0$ is a counterexample, and thus that it is not hypercentral. By Lemma 90 and Brookes' trick 98, we then have that there exist a (non-hypercentral) subgroup H of finite index in G, a finitely generated subgroup F of H, and a positive integer d, such that every finite index subgroup of H containing F has defect at most d in H. By Lemma 90, we may assume $H = G$. As above, let A be the torsion subgroup of G. By 99, we may assume $A^\omega = 1$. Let $m \geq 1$ as definied in claim (56); then, arguing as in the second half of the proof of Lemma 209 (using claim (56) in place of the first half), one shows that every subgroup of G containing F has defect at most $c = 2m + 1$. But then, by Theorem 164 we have that G is hypercentral.

(2) $\Rightarrow$ (1). Let G be a hypercentral $\mathcal{N}_1$-group. Then its torsion subgroup A is nilpotent by Theorem 202. Let $n \geq 1$ and $C_n = C_G(A/A^n)$; G/C_n is periodic by Corollary 21, and since AC_n/A^n is nilpotent, it follows from Lemma 208 that there exists a normal subgroup M_n of AC_n such that $M_n \cap A = A^n$ and AC_n/M_n has finite exponent. Hence, G/M_n is periodic and therefore nilpotent by Theorem 202. Since G/A is nilpotent (being torsion–free) we get that G/A^n is nilpotent for every $n \geq 1$, and so G is a $\mathfrak{X}_o$-group.

(1) $\Rightarrow$ (3). Let $G \in \mathcal{N}_1$ be a $\mathfrak{X}_o$-group, and let $A = \mathrm{Tor}(G)$. Then, by the definition of $\mathfrak{X}_0$, and the fact that, as a torsion–free $\mathcal{N}_1$-group, G/A is nilpotent, it follows that

$$\gamma_\omega(G) = \bigcap_{n \geq 0} \gamma_n(G) \leq A^\omega.$$

Since A is nilpotent by assumption, 99 implies that $A^\omega \leq \zeta_m(G)$ for some $m \in \mathbb{N}$.

(3) $\Rightarrow$ (4). This is clear by the definition of extended lower central series.

(4) $\Rightarrow$ (1). Let G be a hypocentral $\mathcal{N}_1$-group. Then its torsion subgroup A is nilpotent by Theorem 198, and so there exists a positive integer m which satisfies the conclusion of Lemma 220.

Suppose that G does not belong to $\mathfrak{X}_0$; then, by Lemma 219 and Theorem 98, we may assume that there exists a finitely generated subgroup F of G and a positive integer d such that all subgroups $H \leq_b G$ that contain F have defect at most d in G. G/A is nilpotent of class, say, r.

Let $t \geq r$ (so that $\gamma_t(G) \leq A$), and write $D/\gamma_t(G) = (A/\gamma_t G))^\omega$. Now, $G/\gamma_t(G)$ is trivially a $\mathfrak{X}_0$-group, and application of Lemma 209 to it yields that all subgroups of G/D that contain FD have defect at most c, where c depends only on r and d. As, by Lemma 220, $[D,_m G] \leq \gamma_t(G)$, we conclude that every subgroup of G that contains $F\gamma_t(G)$ has defect at most $c + m$ in G. Now, this holds (with the same c and m) for every $t \geq r$; then, if U is a finitely generated subgroup of G containing F, we have

$$[G,_{c+m} U] \leq \bigcap_{t\geq r}(U\gamma_t(G) \cap A) = \bigcap_{t\geq r}\gamma_t(G)(U \cap A) = \gamma_\omega(G)(U \cap A)$$

where the last equality holds because $U \cap A$ is finite. This implies that every subgroup of G that contains $F\gamma_\omega(G)$ has defect at most $c+m$ in G. Thus $G/\gamma_\omega(G)$ is hypercentral by Theorem 164.

Let $K = \gamma_{\omega+m+1}(G)$; then G/K is also hypercentral and so it is a $\mathfrak{X}_0$-group by the already proved implication (2) $\Rightarrow$ (1). But then, if $Y/K = (A/K)^\omega$, $\gamma_\omega(G) \leq Y$ (by definition of $\mathfrak{X}_0$) and $Y/K \leq \zeta_m(G/K)$ (by Lemma 220). Hence $K \geq \gamma_{\omega+m}(G)$. Since G is hypocentral, it follows that

$$K = \gamma_{\omega+m+1}(G) = \gamma_{\omega+m}(G) = 1.$$

Hence G is hypercentral and a $\mathfrak{X}_0$-group. $\boxed{QED}$

6.4 The structure of periodic $\mathcal{N}_1$-groups

In this final section, we prove that a $\mathcal{N}_1$-group is metanilpotent, and, in particular, that a periodic $\mathcal{N}_1$-group is the extension of a nilpotent group by an abelian divisible group of finite rank.

In a Heineken–Mohamed group G, G' is nilpotent and the factor group G/G' is a Prüfer group C_{p^∞}. As we have seen in Chapter 3, this has to be the case if all proper subgroups of G are nilpotent and subnormal. Here, we prove that a similar condition is satisfied in general by periodic groups with all subgroups subnormal.

Let A be an abelian p-group. For $i \in \mathbb{N}$ we set

$$\Omega_i(A) = \{a \in A \mid a^{p^i} = 1\}.$$

Then $\Omega_i(A) \leq A$ and, for all $i \in \mathbb{N}$, $\Omega_{i+1}(A)/\Omega_i(A)$ is an elementary abelian p-group. We say that an abelian p-group A is *large* if $\Omega_{i+1}(A)/\Omega_i(A)$ is infinite for all $i \in \mathbb{N}$; otherwise we say that A is *small*. It is easy to see that an abelian p-group is small if and only if it is the direct product of a divisible group of finite rank by a group of finite exponent.

222 Lemma. *Let $G \in \mathcal{N}_1$ be a p-group, and A a normal elementary abelian subgroup of G, such that $G' \leq A$. Then $G/C_G(A)$ is small.*

Proof. Let G be a counterexample, and let $C = C_G(A)$. Observe that G/C is abelian. Let Θ be the family of all subgroups X of G such that XC/C is large. By Lemma 98, there exists a Θ-subgroup H of G, a finitely generated subgroup

F of H, and a positive integer d, such that every Θ-subgroup of H containing F has defect at most d in H. For each $i \geq 0$ we set

$$H_i/(H \cap C) = \Omega_i(H/(H \cap C)) = \langle g \in H \mid g^{p^i} \in C \rangle.$$

Then, as $H/H \cap C \cong HC/C$ is large, H_{i+1}/H_i is an infinite elementary abelian p-group for all $i \geq 0$. Also H_i is nilpotent, by Theorem 197, since $H_i \in \mathcal{N}_1$ is the extension of the normal nilpotent subgroup $H \cap C$ by a group of finite exponent.

Now, as G is a locally nilpotent p-group, F is finite. If all subgroups of H containing F have defect at most d in H, then H is nilpotent by Theorem 160. But in that case, by Lemma 199, HZ/Z has finite exponent. Since $Z \leq C$, it follows that HC/C has finite exponent, contradicting the choice of $H \in \Theta$.

Thus, there exists a subgroup $K \geq F$ of H, such that $d(K, H) \geq d + 1$. Then $[H,_d K] \not\leq K$. It follows that there exists a finitely generated subgroup $V = V_0$ of K, such that $[H,_d V] \not\leq K$. Clearly, we may assume that $F \leq V$. Let $a \in [H,_d V] \setminus V$, and let m be the smallest integer such that $V \leq H_m$. By induction on i we construct a series

$$V = V_0 \leq V_1 \leq \ldots \leq V_i \leq \ldots$$

of finite subgroups of H such that, for all $i \in \mathbb{N}$, $a \notin V_i$, $V_i \leq H_{m+i}$, and

$$\left| \frac{V_{i+1}}{V_{i+1} \cap H_{m+i}} \right| = p^{i+1}.$$

Suppose we have already found $V_0, \ldots, V_i$. Then, by Theorem 192, applied to the nilpotent group H_{m+i+1} modulo H_{m+i}, there exists a subgroup X of H_{m+i+1} such that $V_i \leq X$, $a \notin X$, and $X/(X \cap H_{m+i}) \cong XH_{m+i}/H_{m+i}$ is infinite. Hence, we may choose elements $x_0, x_1, \ldots, x_i$ in X such that

$$\frac{\langle x_0, \ldots, x_i \rangle H_{m+i}}{H_{m+i}}$$

has order p^{i+1}. We put $V_{i+1} = \langle V_i, x_0, \ldots x_i \rangle \leq X \leq H_{m+i+1}$. Then $a \notin V_{i+1}$, and $V_{i+1}/(V_{i+1} \cap H_{m+i}) \cong V_{i+1}H_{m+i}/H_{m+i}$ has order p^{i+1}.

We now consider the subgroup

$$Y = \bigcup_{i \in \mathbb{N}} V_i.$$

Then, by construction, $F \leq Y \leq H$, and $a \notin Y$. We show that $Y \in \Theta$. Suppose, by contradiction, that $\overline{Y} = YC/C$ is small. Then there exist positive integers n, k such that $|\Omega_{n+1}(\overline{Y})/\Omega_n(\overline{Y})| \leq p^k$. By elementary facts on abelian p-groups, it follows that $|\Omega_{j+1}(\overline{Y})/\Omega_j(\overline{Y})| \leq p^k$ for all $j \geq n$. For all $i \in \mathbb{N}$, let $Y_i/(Y \cap C) = \Omega_i(Y/(Y \cap C))$. Then $Y_i/(Y \cap C) \cong \Omega_i(\overline{Y})$, and $Y_i = H_i \cap Y$. Let $t \geq \max\{n, k\}$. Then, we have

$$p^k \geq \left| \frac{\Omega_{t+m+1}(\overline{Y})}{\Omega_{t+m}(\overline{Y})} \right| = \left| \frac{Y_{t+m+1}}{Y_{t+m}} \right|.$$

But, by construction of Y,

$$\frac{Y_{t+m+1}}{Y_{t+m}} = \frac{H_{t+m+1} \cap Y}{H_{t+m} \cap Y} \cong \frac{(H_{t+m+1} \cap Y)H_{t+m}}{H_{t+m}} \geq \frac{V_{t+1}H_{t+m}}{H_{t+m}}$$

has order at least p^{k+1}, and this gives a contradiction.

Hence $Y \in \Theta$, and so, by the choice of H, Y has defect at most d in H. But then,

$$a \in [H,_d V] \leq [H,_d Y] \leq Y$$

which is the final contradiction. $\boxed{QED}$

223 Lemma. *Let G be a p-group, and D a divisible subgroup of G of finite rank such that $G' \leq D \leq Z(G)$.*

(1) *If G/D is small, then G is the extension of a group of finite exponent by an abelian divisible group of finite rank.*

(2) *If G/D is large, then there exists a large abelian subgroup X of G such that $D \cap X = 1$.*

Proof. (1) Suppose that G/D is small. Then G/D is the direct product of a divisible group D_1/D of finite rank by a group of finite exponent. Since D_1 is then a divisible subgroup of the nilpotent p-group G, $D_1 \leq Z(G)$ by Lemma 18. Thus, we may assume that G/D has finite exponent p^n. Let $g, x \in G$. Then g^{p^n} and $[x, g]$ belong to $D \leq Z(G)$, whence

$$[x, g]^{p^n} = [x, g^{p^n}] = 1.$$

Hence G' is a subgroup of finite exponent of D. Now, D/G' is a divisible subgroup of the abelian group G/G', so there exists a direct summand H/G' of D/G' in G/G'. Then, $H \trianglelefteq G$ has finite exponent, and $G/H \cong D/G'$ is divisible of finite rank.

(2) Suppose that G/D is large. Since D has finite rank, $A = \Omega_1(D)$ is finite. Now, the same argument used to construct Y in the proof of the previous Lemma, can be employed to find a subgroup H of G such that $H/H \cap D$ is large and $H \cap A = 1$. But, trivially, this forces $H \cap D = 1$, whence H is abelian, large, and has trivial intersection with D. $\boxed{QED}$

224 Lemma. *Let $G \in \mathcal{N}_1$ be a p-group, such that G' is nilpotent. Then there exists a normal nilpotent subgroup N of G such that G/N is an abelian divisible p-group of finite rank.*

Proof. Let $H = G'$, and C be the centralizer in G of $H/H'H^p$. Then $H \leq C$ and $C/H'H^p$ is nilpotent. By Lemma 1, it follows that $C/H'H^{p^n}$ is nilpotent, for all $i \in \mathbb{N}$. Thus, if $K/H' = (H/H')^\omega$, C/K is residually nilpotent. By Theorem 198, C/K is nilpotent. We then have, by Lemma 99, that C/H' is nilpotent. Since H is nilpotent by assumption, Theorem 56 allows to conclude that C is nilpotent. Now, by Lemma 222 applied to $G/H'H^p$, G/C is a small abelian p-group. By what we have observed earlier, G/C is the direct product $(D/C) \times (N/C)$ where D/C is a divisible p-group of finite rank, and N/C is a group of finite exponent. By Theorem 197, N is nilpotent. Since G/N is isomorphic to D/C, the proof is done. $\boxed{QED}$

225 Theorem. *Let G be a periodic group with all subgroups subnormal. Then G has a normal nilpotent subgroup N such that G/N is an abelian divisible group of finite rank.*

Proof. Let G be a periodic $\mathcal{N}_1$-group. By Lemma 165, we may assume that G is a p-group, for some prime p. G is soluble by Möhres Theorem 206. We proceed by induction on the derived length of G. Then, by inductive assumption, $H = G'$ is the extension of a normal nilpotent subgroup by a divisible abelian subgroup of finite rank. Among such normal nilpotent subgroups of H, choose K such that the rank r of the divisible group H/K is as small as possible (possibly $r = 0$). By Theorem 213, K^G is nilpotent. Also, H/K^G is divisible of rank at most r, so we may take K to be normal in G.

Now, G/K is nilpotent by Lemma 99, and H/K is central in G/K by Lemma 18. Thus, we are in a position to apply Lemma 223 to the group G/K. Assume first that G/H is small. Then G/K is the extension of a normal subgroup N/K of finite exponent, by an abelian divisible group of finite rank. By Theorem 197, N is nilpotent, and we are done.

Thus, assume that G/H is large. Let W/K be a normal subgroup of G/K maximal such that $W \cap H = K$. We claim that G/HW is small. Suppose not, then by Lemma 223 there exists a large abelian subgroup X/W of G/W such that $X \cap HW = W$. Then $X \cap H = K$ and $X/K \cong XH/H$ is abelian. By Lemma 224, X admits a normal nilpotent subgroup $U \geq K$ such that X/U is divisible of finite rank. By Theorem 213, U^G is nilpotent, $U^G \leq HU$ and, by the choice of K, as $H/(H \cap U^G)$ is divisible, the rank of $H/(H \cap U^G)$ is r. It follows that $(H \cap U^G)/K \cong U^G/U$ is finite. Now, $XU^G/U^G \cong X/(U^G \cap X)$ is a divisible subgroup of the nilpotent p-group G/U^G. By Lemma 18, XU^G is normal in G, i.e. $XU^G = X^G$. Moreover,

$$X^G/X = XU^G/X \cong U^G/(U^G \cap X) = U^G/U \cong (H \cap U^G)/K$$

is finite. Then, there exists an integer $n \geq 0$ such that, if $M = (X^G)^{p^n}$, then $M \leq X$. Now, $X \geq WM \trianglelefteq G$, and $WM \cap H = K$. By the choice of W, we get $M \leq W$, which implies in particular $X^{p^n} \leq W$, contradicting the fact that X/W is large.

Thus G/HW is small. Again by Lemma 224, W has a normal nilpotent subgroup $U \geq K$ (which we may assume to be normal in G by [13]) such that W/U is divisible of finite rank. Since $HW/W \cong H/K$ is divisible of finite rank and G/HW is small, we have that G/U is the extension of a divisible abelian subgroup of finite rank by an abelian group of finite exponent. By applying the same argument used in the case G/H small, we get the desired conclusion. $\boxed{QED}$

226 Remark. It follows from examples constructed by W. Möhres (Proposition 143), that the rank of G/N in the above statement cannot be bounded further. In fact, let $n \geq 1$, let G be a p-group as in the statement of 143, and suppose that N is a nilpotent subgroup containing G'. It is then a standard argument, since $Z(G) = 1$ and G' is elementary abelian, to show that N/G' does not contain any copy of C_{p^∞}, and so that the rank of G/N cannot be less that n.

Let us also mention a curious corollary of 225, that maybe confirms the feeling that periodic $\mathcal{N}_1$-groups do not differ much from Heineken-Mohamed groups. These latter have no proper non-nilpotent subgroups; for the general case we have:

227 Corollary. *Let G be a periodic group in $\mathcal{N}_1$. Then there exists $d \geq 1$ such that every non-nilpotent subgroup of G has defect at most d.*

(Smith's residually finite $\mathcal{N}_1$-groups show that this is not the case for non-periodic groups).

Theorem 225 comprises all other results on periodic $\mathcal{N}_1$-groups that we have included in these notes; as such, together with the nilpotency of the torsion-free case (Theorem 124), it represents a reaching point in the effort of describing $\mathcal{N}_1$-groups. What is not yet very well understood is the mixed case; by applying together Theorems 225 and 124, we have the following fact.

228 Theorem. *Let G be a group with all subgroups subnormal. Then there exists a normal nilpotent periodic subgroup N of G such that G/N is nilpotent.*

Proof. Let $G \in \mathcal{N}_1$, and let T be the torsion subgroup of G. Then, by Theorem 124, G/T is nilpotent. Also, by Theorem 225, there exists a normal nilpotent subgroup K of T such that T/K is a periodic divisible abelian group.

By Theorem 213, $N = K^G$ is nilpotent. Now, T/N is a normal periodic divisible abelian subgroup of G/N. Since G/T is nilpotent, by Lemma 99 we conclude that G/N is nilpotent, thus proving the Theorem. $\boxed{QED}$

6 Question. Is it true that a $\mathcal{N}_1$-group is the extension of a nilpotent group by a periodic (abelian) group of finite rank?

In this direction, using 225 and some of the techniques developed in sections 6.2, 6.3, the following result can be proved.

229 Theorem. *Every $\mathcal{N}_1$-group is the extension of a hypercentral group by an abelian periodic divisible group of finite rank.*

I will not include here a proof of this: it will (possibly) appear elsewhere.

7 Beyond $\mathcal{N}_1$

7.1 Generalizing subnormality

Having reached a reasonably good knowledge of the class $\mathcal{N}_1$, what is perhaps the most immediate question is to ask for groups in which every subgroup satisfies one of the natural generalizations of subnormality; like seriality, ascendancy or descendancy.

Serial subgroups. Imposing seriality to all subgroups is not a very restrictive conditions. By Corollary 65, all locally nilpotent groups satisfy it, and we mentioned J. Wilson's construction in [121] of infinite finitely generated p-groups in which every subgroup is serial (we notice that, following Wilson's line, one may also construct finitely generated non-nilpotent torsion-free groups in which every subgroup is serial). The groups constructed by Wilson, being of Golod-type, are also residually finite, and therefore belong to the class of locally graded groups. On the other hand it is clear that groups in the class $\mathfrak{W}$ and all subgroups serial are locally nilpotent[1].

Descendant subgroups. A subgroup H of the group G is *descendant* if it is a term of a descending series of G. Like seriality, for finite groups descendancy is equivalent to subnormality. Thus, the class $\mathcal{D}$ of groups all of whose subgroups are descendant is a class of generalized nilpotent groups. The following is an easy observation.

230 Lemma. *A group G belongs to the class $\mathcal{D}$ if and only if $H^K < K$ for all $H < K \leq G$.*

However, it is not even clear if groups in $\mathcal{D}$ are locally nilpotent. Consideration of the infinite dihedral group D_∞ shows that (contrary to ascendancy) to

[1]Recall from Chapter 1 that a group G belongs to the class $\mathfrak{W}$ if every finitely generated subgroup of G either is nilpotent or has a non-nilpotent finite image.

assume that all cyclic subgroups of a group G are descendant is not enough to ensure local nilpotency of G. More generally, we make the following remark.

231 Proposition. *Let G be a countable residually nilpotent group. Then every finite and every nilpotent subgroup of G is descendant.*

PROOF. Let F be a finite subgroup of the residually nilpotent group G. Then all subgroups $\gamma_n(G)F$ $(n \in \mathbb{N})$ are subnormal in G, so their chain can be refined to get a descending series of G. Now, $\bigcap_{n\in\mathbb{N}} \gamma_n(G) = 1$, so by Lemma 29, $F = \bigcap_{n\in\mathbb{N}} \gamma_n(G)F$, showing that F is descendant.

Suppose now that the subgroup H of G is nilpotent; we show by induction on its nilpotency class c that H is descendant. If $c = 1$ H is abelian; by Lemma 31 $C_G(H) = \bigcap_{n\in\mathbb{N}} \gamma_n(G)C_G(H)$, so $C_G(H)$ is descendant and thus H is descendant. Let $c > 1$ and let $Y = C_G(\zeta(H))$. By the same argument used before $Y = \bigcap_{n\in\mathbb{N}} \gamma_n(G)Y$ is descendant. Now, Y is residually nilpotent and $Z = \zeta(Y) = \bigcap_{n\in\mathbb{N}} \gamma_n(Y)Z$, so Y/Z is residually nilpotent. Now $HZ/Z \simeq H/(H \cap Z) = H/\zeta(H)$ is a nilpotent subgroup of Y/Z of class $c - 1$, and by inductive assumption HZ is descendant in Y, but $H \trianglelefteq HZ$ so H is descendant in Y. Since Y is descendant in G, we conclude that H is descendant in G. $\boxed{QED}$

Remembering that a free group is residually nilpotent, we have,

232 Corollary. *In a countable free group every cyclic subgroup is descendant.*

Apparently, it is not known whether there exists a finitely generated infinite p-group which is residually finite and such that every subgroup of it is either finite or has finite index. If such a group exists, then, by what observed above, it will have all subgroups descendant.

7 Question. Does there exists a non-trivial perfect (locally nilpotent) group in which all subgroups are descendant?

Ascendant subgroups. The class of groups in which every subgroup is ascendant is of course the class N of all groups satisfying the normalizer condition. Apart from the basic facts that we recalled in Chapter 1 (it is a class of Gruenberg groups that contains every hypercentral group), little more I know in general about this class. The following old question is still open.

8 Question. Is every N-group hyperabelian?

Now, this seems very difficult, but nevertheless I think that some of the techniques developed for studying $\mathcal{N}_1$-groups, in addition to other conditions (like solubility) may prove fruitful also for the broader class N. For instance, Möhres, using the methods we reported in chapter 5, has proved the following.

233 Proposition. *Let G be an N-group which is the extension of a nilpotent p-group of finite exponent by an elementary abelian p-group. Then G is hypercentral.*

The following question is now natural.

9 Question. Is a soluble N-group of finite exponent hypercentral?

and its corrispective in the torsion-free case.

10 Question. Does there exist a (soluble) torsion-free N-group with trivial centre?

Local subnormality. A class which is intermediate between $\mathcal{N}_1$ and N is the class (which we denote by $\mathcal{N}_2$) of groups in which every subgroup is *locally subnormal*; where a subgroup H of a group G is called locally subnormal if $H \lhd\lhd \langle H, X \rangle$ for all finite $X \subseteq G$.

Trivially, in a locally nilpotent group every finitely generated subgroup is locally subnormal. Thus, the existence of locally nilpotent groups with trivial Gruenberg radical shows that a locally subnormal subgroup need not be ascendant. On the other hand, it is clear that a group in which every subgroup is locally subnormal satisfies the normalizer condition, and so it is locally nilpotent.

234 Example. Let $G = C_{p^\infty} \wr X$, where $X = \langle x \rangle$ is cyclic of order p^2. G is hypercentral by Lemma 207. Let $C \simeq C_{p^\infty}$ be one of the coordinate subgroups in the base group of G, and $H = \langle C, x^p \rangle$. Then $H \simeq C_{p^\infty} \wr C_p$ and, clearly, $\langle H, x \rangle = \langle C, x \rangle = G$. On the other hand, H is ascendant but not subnormal in G, so H is not locally subnormal.

This example shows that the class $\mathcal{N}_2$ does not contain all hypercentral groups, and so it is a proper subclass of N (and clearly contains $\mathcal{N}_1$, in particular the Heineken-Mohamed groups which are not hypercentral). Every direct product of nilpotent groups and, more generally, every hypercentral group of length ω is a $\mathcal{N}_2$-group, while the infinite dihedral 2-group is a $\mathcal{N}_2$-group which is not a Fitting group. I do not know much more about this class of locally nilpotent groups.

11 Question. Is every group in $\mathcal{N}_2$ hyperabelian?

Of course, this will follow from a positive answer to question 8; in general, the questions we suggested for the class N make sense for the smaller class $\mathcal{N}_2$ too.

Other generalizations of subnormality. A subgroup H of a group G is *almost subnormal* if H has finite index in a subnormal subgroup of G, and *virtually subnormal* if H is subnormal in a subgroup that has finite index in

G. Both these definitions are included in that of f-subnormality, introduced by Phillips [91]: a subgroup H of G is f-subnormal if there exists a finite series $H_0 = H \leq H_1 \leq \ldots \leq H_n = G$ such that $|H_i : H_{i-1}| < \infty$ or $H_{i-1} \trianglelefteq H_i$ for every $i \in \{1, \ldots, n\}$.

When applied to a single subgroup, these conditions are all different, but things change if we consider all subgroups.

235 Proposition. [see [19]] *For any group G the following are equivalent:*

(1) *every subgroup of G is almost subnormal;*

(2) *every subgroup of G is virtually subnormal;*

(3) *every subgroup of G is f-subnormal.*

We denote by SF the class of groups in which every subgroup is f-subnormal. For finitely generated groups there is a neat characterization of such groups.

236 Theorem. [[64], Theorem 6.3.3] *A finitely generated group is finite by nilpotent if and only if every subgroup is f-subnormal.*

For the general case, we have the following

237 Theorem. [Casolo, Mainardis [19], [20]] *Let G be an SF-group, and let $D(G)$ be the subgroup generated by the nilpotent residuals of the finitely generated subgroups of G. Then*

(1) *$D(G)$ is finite by nilpotent and contained in the torsion part of the FC-centre of G;*

(2) *$G/D(G) \in \mathcal{N}_1$;*

(3) *G is finite by solvable;*

(4) *if G is torsion-free then G is nilpotent;*

(5) *if G is periodic then G is finite-by-$\mathcal{N}_1$.*

Stronger conditions than those assumed in Theorem 237 have been considered. In these cases, the results should be viewed as generalizations both of Roseblade's Theorem and of a Theorem of B. Neumann saying that: *The derived subgroup of a group in which every subgroup has finite index in its normal closure is finite.* We mention only a couple of these results.

238 Theorem. [Lennox [63]] *Let G be a group and suppose that there exists positive integers m, n such that $|H^{G,n} : H| \leq m$, for all $H \leq G$. Then*

$$|\gamma_{\mu(m+n)}(G)| \leq m!$$

for some integer $\mu(n+m)$.

In the same paper, Lennox obtains similar results for those groups G in which every subgroup is subnormal of bounded defect in a subgroup of finite bounded index in G, and for SF-groups with suitable bounds imposed on the finite–by–subnormal series (see also [64] for a fuller account of this particular topic).

More recently, Detomi [25] was able to partly extend Lennox' result.

239 Theorem. *Let G be a group, and suppose that there exists $n \geq 1$ such that $|H^{G,n} : H| < \infty$ for all $H \leq G$. If G is either periodic or torsion–free, then $\gamma_{\delta(n)}(G)$ is finite for some $\delta(n) \in \mathbb{N}$.*

It should be noted that this last Theorem does not carry over to arbitrary groups: H. Smith's hypercentral $\mathcal{N}_1$-groups that we will describe in Section 6.3, satisfy $|H^{G,2} : H| < \infty$ for every $H \leq G$ but they are not finite by nilpotent; similar examples, in which $\gamma_3(G) = G'$ is infinite, are constructed in [19].

Groups in which every subgroup is approached from below by a subnormal subgroup are much less tractable, even in the special case in which H/H_G is finite for every subgroup of G (Ol'shanski infinite groups in which every proper subgroup has order p are examples of groups of this kind). The many problems connected with this class of groups (even when suitably restricted) have stimulated several people, and a number of articles have appeared on this topic, starting perhaps with a paper by Buckley, Lennox, B. H. Neumann, H. Smith and J. Wiegold [11] (this subject involves also some non-trivial questions about finite p-groups, and we mention paper [22], where more complete references may be found). Regarding the class of groups in which every subgroup contains a subgroup of finite index which is subnormal in the whole group, I only am aware of a paper by H. Heineken [45], from which I quote the following Proposition: *In a locally finite group G in which every subgroup H contains a subgroup S with $|H : S| < \infty$ and $S \lhd\lhd G$, the Hirsch–Plotkin radical has finite index.* There might well be some room left for more research on this subject: for instance

12 Question. Is a locally nilpotent (or soluble by finite) group with all subgroups subnormal by finite, a finite extension of a $\mathcal{N}_1$-group ?

(A nilpotent torsion-free group with all subgroups subnormal by finite is certainly nilpotent, while any non-nilpotent Černikov p-group is an example of a locally nilpotent group with this property which is not in $\mathcal{N}_1$.)

7.2 Groups with many subnormal subgroups

Under this label are denoted in the literature groups in which the set of non-subnormal subgroups satisfies certain (usually of finitary type) restrictions; given a specific restriction to the set of non-subnormal subgroups, the usual target is to describe (if any) those groups that satisfy such a restriction and do not belong to $\mathcal{N}_1$ or to the class of groups in which the set of all subgroups satisfies that restriction.

This kind of investigations goes back to Černikov, who studied groups in which many subgroups have a prescribed property $\mathcal{P}$ (structural or of embedding); in particular, close to what we are going to consider here, the case when $\mathcal{P}$ is the property of being ascendant (see [21] for a survey on Černikov's work). Perhaps even closer in methods is a 1978 paper [92] by Phillips and Wilson (in which the class $\mathfrak{W}$ was introduced), where $\mathfrak{W}$-groups with "many" serial or locally nilpotent subgroups are studied; although not explicitely referring to subnormality, we report part of the main result of [92].

240 Theorem. *Let G be a $\mathfrak{W}$-group. The following are equivalent:*

(1) *the set of all non-serial non-locally nilpotent subgroups of G satisfies the minimal condition;*

(2) *either G is a Černikov group, or every subgroup of G is serial or locally nilpotent;*

and in this case, if G is not a Černikov group, then G is locally nilpotent by finite cyclic.

This is a topic that has recently seen a lot of activity, its only bound being the imagination of the scholars. Therefore, I am probably not completely aware of all the developments, and in my report I will describe only a few cases, and provide a couple of proofs. just in order to try giving a flavour of this line of investigation and an idea of the arguments involved.

As in Phillips–Wilson, we begin with the minimal condition.

241 Theorem. [Franciosi, de Giovanni [27]] *Let the group G satisfy the minimal condition on non–subnormal subgroups.*

(1) *If G is a Baer group, then $G \in \mathcal{N}_1$.*

(2) *If G is not periodic, then $G \in \mathcal{N}_1$.*

(3) *If $G \in \mathfrak{W}$, then G is either a Černikov group or $G \in \mathcal{N}_1$.*

Proof. (1) Let G be a Baer group satisfying the minimal condition on non–subnormal subgroup, and suppose by contradiction that $G \notin \mathcal{N}_1$. Thus, let H be a minimal non-subnormal subgroup of G. Then all proper subgroups of H are subnormal; in particular, by Möhres Theorem, $K = H' < H$. Since H cannot be the product of two proper subgroups, H/K is either cyclic or isomorphic to C_{p^∞} for some prime p. Now, G is a Baer group, so if H/K were cyclic, then $H = K\langle x\rangle$ would be the product of two subnormal subgroups. Hence

$H/K \simeq C_{p^\infty}$. Let $G = K_0 > K_1 > \ldots > k_d = K$ be the normal closure series of K in G (since H normalizes K, all K_j are normalized by H), and let $i \geq 1$ be minimal such that HK_i is not subnormal in HK_{i-1}. Then $K_i \trianglelefteq HK_{i-1}$ and

$$\frac{HK_i}{K_i} \simeq \frac{H}{H \cap K_i}$$

is a proper quotient of $H/K \simeq C_{p^\infty}$. Hence $HK_i/K_i \simeq C_{p^\infty}$, and we may replace G by HK_{i-1}/K_i, and H by HK_i/K_i, and thus assume that $H \simeq C_{p^\infty}$ for some prime number p. Clearly we may then also suppose that G is a p-group.

Let $X = N_G(H)$. Then $N_G(X) = X$ (by 34 and 35), and $H \leq Z(X)$. Also, X/H satisfies Min and so X is a Černikov p-group. Now, G is a Baer group, hence all proper subgroups of H are subnormal in G; clearly, there exists a proper (cyclic) subgroup Y of H such that $Y^G \not\leq X$. Let M be the smallest term of the normal closure series of Y in G such that $M \not\leq X$. Since $Y \leq Z(X)$, M is normalized by X. Also, $Y^M \leq X$ and so, since Y has finite exponent, Y^M is a finite p-group. Since M is generated by normal conjugates of Y^M, it follows that M is nilpotent of finite exponent. Let $N = N_M(M \cap X)$; then $N > M \cap X$ and N is normalized by X . Let $A/M \cap X$ be the subgroup of all elements of order p in $Z(N/M \cap X)$. Then $A/M \cap X \neq 1$, because M is nilpotent, and A is normalized by X. If $A/M \cap X$ is finite, then $1 < |AX : X|$ is finite and therefore $X \lhd\lhd AX$, which contradicts $X = N_G(X)$. Thus, $A/M \cap X$ is an infinite elementary abelian p-group normalized by X (and by H). Let $B = H(X \cap M)$; then $B/X \cap M \simeq C_{p^\infty}$ and $N_G(B) = X$, whence $N_A(B) = A \cap B$. This, in particular, says that B is not maximal in any subgroup S with $B < S \leq AB$. So there exists an infinite chain of subgroups $AB > S_1 > S_2 > \ldots$, with $B[A, B] > S_i > B$ for all $i \geq 1$. By our assumption on G there exists $t > 1$ such that S_t is subnormal in G. But $S_t = B(S_t \cap A)$ and so $S_t/S_t \cap A \simeq C_{p^\infty}$. It then follows from 34 and 35 that $S_t/S_t \cap A$ is normal in $BA/S_t \cap A$ and so $S_t \trianglelefteq AB$. Therefore $[A, B] \leq [A, S_j] \leq S_j$, which is a contradiction.

(2) Suppose that G is not periodic, and let $g \in G$ be an element of infinite order. Then there exists integers $m, n \geq 1$ such that $U = \langle g^{2^n} \rangle$ and $V = \langle g^{3^n} \rangle$ are subnormal in G, whence $\langle g \rangle = UV$ is subnormal in G. Thus, the Baer radical B of G contains all elements of infinite order. Our claim will be proved if we show that G is generated by elements of infinite order. This is equivalent to prove that for every pair a, b of elements of finite order of G, the product $y = ab$ has finite order. Suppose, to the contrary that $|y| = \infty$. Then y belongs to the Baer radical of $\langle a, b \rangle$, and so $H = \langle a, b \rangle = \langle a, y \rangle$ is the extension of the finitely generated nilpotent group $Y = \langle y \rangle^{\langle a \rangle}$ by the finite group $\langle a \rangle$. Thus H is policyclic and nilpotent by cyclic. As the torsion subgroup of Y is finite, we may well assume that Y is torsion free. Then, if p is a prime which does not divides the order of a, by Theorem 43 there exists an infinite descending chain

$Y > N_1 > N_2 >$ of normal subgroups N_i with Y/N_i a finite p-group. As a has finite order, we may find a chain of this kind with all N_i are normal in H. Thus, by our assumption on G, there exists $t \geq 1$ such that $\langle N_j, a\rangle$ is subnormal in H for all $j \geq t$. Then, for all $i \geq 1$, $H/N_i/N_i$ is a nilpotent group, and the direct product of its p-component Y/N_i and the cyclic p'-group $\langle N_j, a\rangle/N_i$. Thus $[Y, a] \leq N_i$ for all $i \geq 1$. This yields $\langle a\rangle \trianglelefteq H$, and so $H = \langle a, b\rangle$ is finite. This proves that $B = G$ and so, by point (1), that $G \in \mathcal{N}_1$.

(3) Let $G \in \mathfrak{W}$ be a group in which the set of non-subnormal subgroups satisfies the minimal condition. By (2) we may assume that G is periodic. Now, it is easy to see that a finitely generated periodic group in $\mathfrak{W}$ with the minimal condition on non-subnormal subgroups is finite; therefore G is locally finite.

Suppose that G is not Černikov; then, by the Šunkov, Kegel–Wehrfritz Theorem 39, G admits non-Černikov abelian subgroups, and by our assumption on G there exist subnormal such subgroups. Hence, the Baer radical B of G does not satisfy the minimal condition on subgroups. Bu point (1) we are done if we prove that $B = G$. Clearly, it is enough to prove that any element of prime power order of G belongs to B.

Thus, let $g \in G$ be an element of order a power of a prime p., and let A be the p-component of B. Suppose first that A is not Černikov. By Möhres Theorem 206, A is soluble. Let $M = A^{(m)}$, be the smallest term of the derived series of A which is not a Černikov group (it exists because the class of groups with Min is closed by extensions), and let $K = A^{(m=1)} = M'$. Observe that K is a Černikov Baer group and so it is contained in some finite term of the upper central series of M; therefore M is nilpotent. If we prove that $K\langle g\rangle$ is subnormal in G, then in particular $M\langle g\rangle/K$ is nilpotent by 61 and so $M\langle g\rangle$ is nilpotent bt Hall's criterion 56; consequently $\langle g\rangle \lhd\lhd K\langle g\rangle \lhd\lhd G$. Thus, we assume $K = 1$ and $G = M\langle g\rangle$. Since M is a non-Černikov abelian group it has an infinite characteristic elementary abelian subgroup X. Since g has finite order, there is an infinite descending chain of g-invariant subgroups X_i of X, with $X_i \geq X \cap \langle g\rangle$, and then $X_m, \langle g\rangle \lhd\lhd G$ for some $m \geq 1$. But, $X_m\langle g\rangle$ is a soluble p-group of finite exponent ans so, by Proposition 78, $\langle g\rangle \lhd\lhd X_m\langle g\rangle \lhd\lhd G$ and we are done.

Suppose then that the p-component A of B is Černikov. Then, since B does not satisfy Min, it follows that the p'-component U of B is not Černikov. Again, U is soluble. Arguing exactly as in the previous case, we find a characteristic section M/K of U such that it is enough to show that $K\langle g\rangle$ is contained in some subnormal subgroup of G contained in $M\langle g\rangle$. As before, we may assume $K = 1$. Let X be the subgroup generated by all elements of prime order of M. Since M is not Černikov, X is infinite. Let $D = [X, \langle g\rangle]$. By a standard fact for coprime actions on abelian groups, $[D, g] = D$. Now, if D is infinite, as before we find a proper $\langle g\rangle$-invariant subgroup D_0 of D such that $D_0\langle g\rangle \lhd\lhd G$, which yields the contradiction $[D, g] \leq D_0 < D$. Thus, D is finite. This means that $C_X(g)$ is infinite. But then we find a subgroup R of $C_X(g)$ such that $R\langle g\rangle \lhd\lhd G$. Since $\langle g\rangle \trianglelefteq R\langle g\rangle$ we again conclude that $\langle g\rangle \lhd\lhd G$. This completes the proof that G is a Baer subgroup and therefore (3) is established. $\boxed{QED}$

In the same paper, Franciosi and de Giovanni consider groups with only a finite number of conjugacy classes of non-subnormal subgroups, proving that locally graded such groups are either finite or $\mathcal{N}_1$.

Moving to the maximal condition, the following has been proved.

242 Theorem. [Kurdachenko, Smith [57]] *Let the group G satisfy the maximal condition on non–subnormal subgroups.*

(1) *If G is locally nilpotent, or infinite locally finite, then $G \in \mathcal{N}_1$.*

(2) *G is locally (soluble–by–finite) if and only if G satisfies one of the following conditions:*

(i) *G is polycyclic by finite;*

(ii) *$G \in \mathcal{N}_1$;*

(iii) *$G \neq B(G)$, $B(G)$ is nilpotent, $G/B(G)$ is polycyclic–by–finite torsion–free, and for every $g \in G \setminus B(G)$, and every $N \trianglelefteq G$, with $N \leq B(G)$, the group $\langle N, g\rangle$ is finitely generated.*

We isolate in a Lemma one of the technical arguments involved in the proof.

243 Lemma. *Let A be a normal abelian subgroup of the soluble group G, and let $g \in G\setminus A$, with $gA \in Z(G/A)$. Suppose that G/A is not finitely generated while A is finitely generated as $\mathbb{Z}\langle g\rangle$-module. Then the centralizer of g in G contains a subgroup that is not finitely generated*

Proof. Since A is abelian, $[A, \langle g\rangle] = [A, g] = \{[a, g] \mid a \in A\}$. Also, $[A, g] \trianglelefteq G$ because gA is central in G/A. Now, by assumption, $B = A\langle g\rangle$ is finitely generated, and so $B/[A, g]$ is a finitely generated abelian group. Let $C = C_G(B/[A, g])$; then $C \geq B$ and G/C is finitely generated (indeed, it is polycyclic, see for instance [96], 3.2.7). As G/A is not finitely generated, we get that C/A is not finitely generated. Now, let $x \in C$; then $x \in C$, $[x, g] \in [B, C] \leq [A, g]$, and so there exists $a \in A$ such that $[x, g] = [a, g]$. We have

$$[xa^{-1}, g] = [x, g]^{a^{-1}}[a^{-1}, g] = [x, g][a, g]^{-1} = 1$$

which means that $xa^{-1} \in C_G(g)$. This shows that $C \leq AC_G(g)$. Since C/A is not finitely generated, we conclude that $C_C(g) = C \cap C_G(g)$ is not finitely generated. $\boxed{QED}$

Proof. of Theorem 242. Let us denote by $\overline{\mathcal{S}}$ the class of groups satisfying the maximal condition on non–subnormal subgroups. We begin with a rather immediate observation.

(A) *Let G belong to $\overline{\mathcal{S}}$, and let $F < H \leq G$ with F finitely generated and H not finitely generated; then there exists a finitely generated T with $F \leq T < H$ and $T \lhd\lhd G$.*

From this, one immediately deduces,

(B) *A locally nilpotent group in $\overline{\mathcal{S}}$ is a Baer group.*

Now, for the proof of point (1) of the statement, we may just deal with Baer groups.

(C) *Let G be a Baer group in $\overline{\mathcal{S}}$, and $1 \neq H \leq G$; then $H' \neq H$ and $H' \lhd\lhd G$.*

Proof. Let $U \leq H$ be a maximal non-subnormal subgroup of H, or $U = 1$ if there are not any. If $U = 1$ let $N = 1$; otherwise, there exists a proper and subnormal subgroup V of H containing U, then set $N = V^H$. In any case N is a proper normal subgroup of H, and H/N belongs to $\mathcal{N}_1$. It then it follows $H' < H$ by Theorem 206. Now, if $H \lhd\lhd G$, then H' is also subnormal. Thus, assume H is not subnormal in G. If H/H' is not finitely generated, then it does not satisfies Max, and so there exists $H' \leq L \leq H$ with $L \lhd\lhd G$; as $H' \trianglelefteq L$, $H' \lhd\lhd G$. If H/H' is finitely generated, then $H = H'X$ for some finitely generated subgroup X of H. Then, $H'X^H = H$, and since H is a Baer group, $X = H$. Thus, H is finitely generated and so subnormal in G.

(D) *Let $G = AH$ be a Baer group in $\overline{\mathcal{S}}$, with A, H abelian and $A \trianglelefteq G$. Then H cannot be a maximal non-subnormal subgroup of G.*

Proof. Observe that $A \cap H \trianglelefteq G$, whence we may suppose $A \cap H = 1$. Assume that H is a maximal non-subnormal subgroup of G, and let X be a cyclic subgroup of H such that $[A, X] \neq 1$. Now, $X \lhd\lhd G$, and so $C_A(X) \neq 1$. Since H is abelian $C_A(X)$ is normalized by H, and $H < C_A(X)H$. Thus, $C_A(X)H$ is subnormal in G, and therefore $[A,_m H] \leq C_A(X)H \cap A = C_A(X)$ for some $m \in \mathbb{N}$, which we take the smallest such. Since $[A, X] \neq 1$, we have $m \geq 1$. But then, since A and H are abelian

$$[A,_{m-1} H, X, H] = [A,_{m-1} H, H, X] \leq [C_A(X), X] = 1.$$

Thus, $[A,_{m-1} H, X] \leq C_A(H) = 1$, which means $[A,_{m-1} H] \leq C_A(X)$, against the choice of m.

(E) *Let G be a Baer group in $\overline{\mathcal{S}}$. Then $\langle x\rangle^G$ is soluble for every $x \in G$.*

Proof. Let $x \in G$, and $K = \langle x\rangle^G$. Arguing by induction on the defect of $\langle x\rangle$ in G, we may assume that $\langle x\rangle^K$ is soluble, and so that K is generated by normal soluble subgroups. Another obvious inductive argument reduces us to prove that a $\overline{\mathcal{S}}$-group K which is generated by normal abelian subgroups is soluble. Suppose that K is not in $\mathcal{N}_1$, let H be a maximal non-subnormal subgroup of K, and let

N be a normal abelian subgroup of K such that $N \not\leq H$. Then $H < NH \lhd\lhd G$. Now, $H \cap N \trianglelefteq NH$; let $D = H'(N \cap H)$. Then $D < H$ and $D \lhd\lhd G$ by point (C). In particular, $D \lhd\lhd ND$ (and $D < ND$). Let A be the last but one term of the normal closure series of D in AD; then A is normalized by H, and the group AH/D violates point (D). Thus, $K \in \mathcal{N}_1$, and so K is soluble.

Proof of point (1). We first suppose that $G \in \overline{\mathcal{S}}$ is locally nilpotent, and so, by point (B), a Baer group. Assume that G is not in $\mathcal{N}_1$; then there exists a maximal non-subnormal subgroup H of G. Now, $H' \lhd\lhd G$ by point (C); let K be the smallest term of the normal closure series of H' in G such that $K \not\leq H$ (possibly, $K = G$). Then K is normalized by H and $H < KH$, whence $KH \lhd\lhd G$; so, we may replace G by KH if necessary. Then $H' \leq H^K < H$, $H^K \trianglelefteq HK = G$, and we may also assume $H^K = 1$, in particular, that H is abelian. Let X be a cyclic subgroup of H such that $K = X^G \not\leq H$. Then K is soluble by point (E); since $H \cap K \trianglelefteq KH$. there is a subgroup A of K such that $H \cap K < A \trianglelefteq HK$, and $A/(H \cap K)$ is abelian. But this again contradicts point (D). thus, $G \in \mathcal{N}_1$, and we are done.

Now, assume that G is an infinite locally finite group in $\overline{\mathcal{S}}$. Let $x \in G$, with $|x| = p^n$ for some prime p. If there is an infinite p-subgroup containing x, then $\langle x \rangle$ is subnormal in G by point (A). Thus, let P be a maximal p-subgroup of G containing $\langle x \rangle$ and assume that P is finite. Then P is a Sylow p-subgroup of every finitely generated subgroup that contains it. By point (A) there is a finite subnormal subgroup T of G with $P \leq T$; let $N = P^T$. Then $N \lhd\lhd G$ and therefore $N = P^S$ for every finitely generated subgroup S of G, with $T \leq S$ (remember that in a finite group the smallest subnormal subgroup containing a Sylow subgroup is its normal closure). Thus, $N \trianglelefteq G$. Hence $G/C_G(N)$ is finite. In particular $C_G(x)$ has finite index in G, and so it is not finitely generated. Point (A) then ensures that there is a subnormal subgroup U of G with $x \in U \leq C_G(x)$, and so $\langle x \rangle \lhd\lhd G$. Thus, we have proved that every element of G of prime–power order is contained in the Baer radical of G. It clearly follows that G is a Baer group, and we are done.

Proof of point (2). Let G be a locally (soluble by finite) group in $\overline{\mathcal{S}}$, and let $B = B(G)$ be the Baer radical of G. By point (1), $B \in \mathcal{N}_1$, and in particular B is soluble.

If B is finitely generated, then it is polyciclic and so $G/C_G(B)$ is polycyclic–by–finite (because it is a locally (soluble by finite) subgroup of $Aut(B)$; see, for instance, [99], Ch. 8). If $C_G(B)B/B$ is not finite, it contains (by point (A)) a subnormal finitely generated subgroup, hence a non-trivial subnormal abelian subgroup A/B; and this implies that A is contained in the Baer radical of G, a contradiction. Thus, $C_G(B)B/B$ is finite, and consequently, G/B is polycyclic by finite. We conclude that G itself is polycyclic by finite. Conversely, a polycyclic by finite group certainly belongs to $\overline{\mathcal{S}}$ as it satisfies Max.

We are left with the case in which the Baer radical $B = B(G)$ is not finitely generated, and $B \neq G$.

Let $g \in G \setminus B$, and let N be a normal subgroup of G contained in B. Suppose, by contradiction, that $\langle N, g\rangle = N\langle g\rangle$ is not finitely generated. Then, by (A), there exists a finitely generated X with $\langle g\rangle \leq X \leq N\langle g\rangle$ and $X \lhd\lhd G$; in particular, $N\langle g\rangle$ is subnormal in G, and there exists a smallest $n \in \mathbb{N}$, such that $[N,_n \langle g\rangle]\langle g\rangle$ (the n-th term of the normal closure series of $\langle g\rangle$ in $N\langle g\rangle$) is finitely generated. Since we are assuming that $N\langle g\rangle$ is not finitely generated, we must have $n \geq 1$. We write $U = [N,_n \langle g\rangle]$, and consider, $D = [N,_{n-1} \langle g\rangle]\langle g\rangle$. Then, D/U is soluble and, by choice of n, it is not finitely generated; also, $gU \in Z(D/U)$. Now, U/U' is a normal abelian subgroup of the soluble group D/U', and is finitely generated as a $\mathbb{Z}\langle g\rangle$-module. We may then apply Lemma 243: since D/U is not finitely generated, we obtain that the centralizer of gU' in D/U' contains a non-finitely generated subgroup. By observation (A), this implies that $\langle gU'\rangle \lhd\lhd D/U'$. In particular, $U\langle g\rangle/U'$ is nilpotent; since it is also finitely generated, it follows that U/U' is finitely generated. But U is a Baer group, and so U is a finitely generated nilpotent group. As $U\langle g\rangle/U'$ is also nilpotent, P. Hall's nilpotency criterion (Theorem 56) yield that $U\langle g\rangle$ is nilpotent. This means that $\langle g\rangle \lhd\lhd U\langle g\rangle = [N,_n \langle g\rangle]\langle g\rangle$, and so in $\langle g\rangle$ is subnormal in $N\langle g\rangle$, which in turn is subnormal in G. Therefore $\langle g\rangle \lhd\lhd G$, and the contradiction $g \in B$. The last assertion in the statement of the Theorem is thus established.

Now, let $g \in G$ and suppose that $g^n \in B$ for some $n \geq 1$. If $g \notin B$, then, by what we have just proved $B\langle g\rangle$ is finitely generated, hence B, which has finite index in it, is finitely generated, which is against our assumptions. This proves that G/B is torsion free.

We now prove that B is nilpotent. Fix an element $g \in G \setminus B$, and let T denote the torsion subgroup of B. Then $T\langle g\rangle$ is finitely generated by what we proved; and since T is soluble, it easily follows that T has finite exponent. Therefore, T is nilpotent by Theorem 191, and $B/C_B(T)$ is periodic by Lemma 16. Moreover $W = TC_B(T)$ is nilpotent and so, by Lemma 208, there is a $k \geq 1$ such that, writing $N = W^k$, $N \cap T = 1$. Since N is a characteristic subgroup of B, $N \trianglelefteq G$. Now, B/N is periodic and $B\langle g\rangle/N$ is finitely generated (being a quotient of $B\langle g\rangle$), and so the same argument used for T shows that B/N is nilpotent. Since B/T is nilpotent by Theorem 124 and $T \cap N = 1$, we conclude that B is nilpotent.

Let now C/B be the Baer radical of G/B. If C/B is finitely generated, then by what we observed at the beginning of the proof of point (2), G/B is polycyclic by finite, and we are done. Thus suppose, by contradiction, that C/B is not finitely generated. Then, by Theorem 96, C/B admits an abelian subgroup

A/B which is not finitely generated. Let $g \in A \setminus B$; then application of Lemma 243 to the group A/B' implies that the centralizer of gB' in A/B' contains a non-finitely generated subgroup. It turns out that $\langle gB' \rangle$ is subnormal in A/B', and so that $\langle B, g\rangle/B'$ is nilpotent. Since B is nilpotent, it follows from P. Hall's criterion that $\langle B, g\rangle$ is nilpotent, and in particular that $\langle g\rangle$ is subnormal in $\langle B, g\rangle$. Since $\langle B, g\rangle \lhd\lhd G$, we end up with the contradiction $g \in B$.

It remains to show that groups satisfying the conditions in point (2) of the statement do belong to $\overline{\mathcal{S}}$. This is trivial for polycyclic by finite groups (which satisfy Max) and $\mathcal{N}_1$-groups. We then suppose that the group G satisfies the conditions in (iii). Then, if B is the Baer radical of G, G/B satisfies Max. Suppose, by contradiction, that G does not belong to $\overline{\mathcal{S}}$, and let $Z = \gamma_c(B)$ be the smallest term of the lower central series of the nilpotent group B such that $G/Z \in \overline{\mathcal{S}}$; then, we may clearly assume$\gamma_{c+1}(B) = 1$ (i.e. Z central in B). Let $H = H_1 \leq H_2 \leq H_3 \leq \ldots$ be an ascending chain of non-subnormal subgroups of G; then $H \not\leq B$, and since G/Z belongs to $\overline{\mathcal{S}}$, we may suppose that $ZH_i \lhd\lhd G$ for every $i \geq 1$. Let $x \in H \setminus B$; then, since $\langle B, x\rangle$ is finitely generated, and $x \in ZH \lhd\lhd BH$, we have that B/Z is finitely generated, and therefore G/Z is polycyclic by finite and satisfies Max. Thus there exists $k \geq 1$ such that $ZH_i = ZH_k$ for every $i \geq k$. Now, $\langle Z, x\rangle$ is finitely generated, which means that Z is finitely generated as a $\mathbb{Z}\langle x\rangle$-module. Since $\mathbb{Z}\langle x\rangle$ is noetherian, Z is also noetherian, i.e. it satisfies the maximal condition on $\mathbb{Z}\langle x\rangle$–submodules. This implies that there exists an index ℓ such that $Z \cap H_i = Z \cap H_\ell$ for all $i \geq \ell$. Now let $t = \max\{k, \ell\}$; then for every $i \geq t$,

$$H_i = H_i \cap ZH_t = H_t(H_i \cap Z) = H_t(H_t \cap Z) = H_t,$$

and the proof is now complete. (It should be noted that, in this situation, Kurdachenko and Smith actually show that G/B must be abelian by finite; but the proof of this requires one more page, and we thus omit it). $\boxed{QED}$

Of course, consideration of Tarski monsters shows that the conclusions of Theorems 241 and 242 (as well as that of most of the results we will mention in this section) do not hold without some restrictions on the class of groups considered; on the other hand, the questions as to whether 241 and 242 may be extended to larger classes (locally graded and $\mathfrak{W}$ groups, respectively) remain open, and seem very difficult.

Weak forms of maximal and minimal conditions on non-subnormal subgroups are considered in [58], [59]. In [30], de Giovanni and Russo show that infinite groups with *dense* subnormal subgroups are $\mathcal{N}_1$ (a family $\mathcal{S}$ of subgroups of the group G is dense if for every $H < K \leq G$, and H not maximal in K, there exists a $S \in \mathcal{S}$ such that $H < S < K$; see also Mann [69]).

Groups in which non-subnormal subgroups satisfy certain embedding restrictions have also been considered. For instance, combined results of Franciosi, de Giovanni [26], and Kurdachenko, Smith [60], yield the following.

244 Theorem. *Let G be a group in which every non-subnormal subgroup is self-normalizing.*

(1) *If G is not periodic, then $G \in \mathcal{N}_1$;*

(2) *if G is locally nilpotent, then $G \in \mathcal{N}_1$;*

(3) *if G is locally graded and is not locally nilpotent, then $G = \langle g \rangle \rtimes Q$, where g is an element of order a power of a prime p and Q a nilpotent periodic p'-group.*

The subclass of groups with all subgroups either subnormal or abnormal is described by De Falco, Kurdachenko and Subbotin [23], while in [28], Franciosi, de Giovanni and Kurdachenko characterize those groups in which every (infinite) non-subnormal subgroup has a finite number of conjugates.

Along another line of research (but strictly related to the previous one, as it is already evident in [92]), one imposes inner properties to non-subnormal subgroups. We mention only a couple of relevant results. The proofs are in these cases too long to be included.

245 Theorem. [Smith [109] [110]] *Let G be a $\mathfrak{W}$–group in which every subgroup is either subnormal or nilpotent. Then*

(1) *G is soluble;*

(2) *if G is torsion-free then G is nilpotent;*

(3) *if G is locally finite, then G admits a normal subgroup of finite index which belongs to $\mathcal{N}_1$.*

Together with Theorem 225 a corollary of this is an extension of 144;

246 Corollary. *A locally finite group in which all non-nilpotent subgroups are subnormal is nilpotent by Černikov.*

We observe that locally nilpotent groups with all subgroups subnormal or nilpotent need not belong to $\mathcal{N}_1$; for instance, let p be a prime, and let $G = A \rtimes \langle \alpha \rangle$, where $A \simeq C_{p^\infty}$ and α the automorphism $a \mapsto a^{p+1}$ (for all $a \in A$); then G is locally nilpotent and all non-nilpotent subgroups of it contain A (and so are normal); however, G is not even a Baer group (indeed, Smith proves that Baer groups with all subgroups nilpotent or subnormal are $\mathcal{N}_1$-groups).

We mention one more result, dealing with a class of groups which may be seen as sort of opposite to that of Baer groups.

247 Theorem. [Heineken, Kurdachenko [46]] *Let G be group in which every subgroup is either subnormal or finitely generated.*

(1) *If G is locally finite, then either G is Černikov or $G \in \mathcal{N}_1$;*

(2) *if G is locally nilpotent, then either G has finite rank or $G \in \mathcal{N}_1$;*

(3) *if G is generalized radical not nilpotent, and $B(G)$ is its Baer radical, then $G/B(G)$ is finitely generated and abelian–by–finite.*

We recall that a group is "generalized radical" if it admits a normal ascending series whose factors are either locally nilpotent or locally finite, and that locally nilpotent groups with finite rank have been fully described by Mal'cev. Groups with all subgroups either subnormal or of finite rank are studied in [61].

Needless to say, many of these and similar questions may be varied by imposing conditions on the family of all subgroups that are not subnormal with defect not exceeding a prescribed bound $d \geq 1$; aiming in this case at obtaining results that resemble Roseblade's Theorem. This aspect is often considered in the same articles that treat the unbounded case, and we will not say more about it, leaving the interested reader to check the original papers.

7.3 The subnormal intersection property.

A group G is said to satisfy the *subnormal intersection property* (abbreviated s.i.p.) if the intersection of any family of subnormal subgroups of G is subnormal. The class of all groups satisfying s.i.p. is usually denote by $\mathfrak{S}_\infty$.

Since the s.i.p. condition does not necessarily mean the occurrence of many subnormal subgroups (for instance, every simple group has the s.i.p.), but rather it becomes effective when there are already many subnormal subgroups, presence of the class $\mathfrak{S}_\infty$ in this chapter may be not fully justified; however, I decided to include a few comments on it, in view of the fact that, at least in certain specific cases, some of the methods developed to study $\mathcal{N}_1$-groups apply with some success to $\mathfrak{S}_\infty$. Before coming to this, let me remind one of the few general results on $\mathfrak{S}_\infty$ available, namely a rather old theorem of D. Robinson [93] which states that *a finitely generated soluble group G belongs to $\mathfrak{S}_\infty$ if and only if G is finite-by-nilpotent.*

Here, we are mainly interested in $\mathfrak{S}_\infty$-groups that are also Baer groups (clearly, every $\mathcal{N}_1$-group is of this kind). First, one proves a version of Brookes trick 98 for $\mathfrak{S}_\infty$-groups. The not difficult adaptation is left to the reader.

248 Lemma. *Let G be a group in $\mathfrak{S}_\infty$, and let Θ be a family of subnormal subgroups of G such that $G \in \Theta$. Then there exist a $H \in \Theta$, a finitely generated subgroup F of H, and a positive integer d, such that every $F \leq K \leq H$, with $K \in \Theta$, has defect at most d in H.*

With this and Roseblade's Theorem, we may prove the following extension of 167.

249 Theorem. *A residually soluble Baer group with the subnormal intersection property is soluble.*

PROOF. Let G be a residually soluble Baer group in $\mathfrak{S}_\infty$, and suppose by contradiction that G is not soluble. By Lemma 248 applied to the family Θ of all subnormal non-soluble subgroups of G, there exist $H \in \Theta$, a finitely generated subgroup F of H, and a positive integer d, such that all non-soluble subnormal subgroups of H containing F have defect at most d in H. Clearly, we may replace G by H, and assume that d is minimal for a counterexample.

Since H is not soluble, $H^{(m)}$ is not soluble for every $n \geq 1$; so, if V be a finitely generated subgroup containing F, $VH^{(m)}$ is not soluble. On the other hand, $VH^{(m)}$ is subnormal in H, as H is a Baer group and V is finitely generated. Therefore the defect of $VH^{(m)}$ in H is at most d.

We have $d \neq 1$. In fact, if $d = 1$, then, by what we have just observed, for all $m \geq 1$, all subgroups of $H/H^{(m)}$ containing $FH^{(m)}/H^{(m)}$ are normal. Therefore

$$H^{(2)} \leq \bigcap_{m \in \mathbb{N}} FH^{(m)},$$

Hence, if F has derived length t,

$$H^{(2+t)} \leq \bigcap_{m \in \mathbb{N}} H^{(m)} = 1 \ ,$$

thus contradicting the choice of H.

Let now $d \geq 1$. Then, by minimality of d, the normal closure F^H of F is soluble, and, for any $m \geq 1$, all subgroups of $H/F^H H^{(m)}$ are subnormal of defect at most d. By Roseblade's Theorem, there is an integer k such that $H^{(k)} \leq F^G H^{(m)}$, for all $m \geq 1$. But then, if t is the derived length of F^G,

$$H^{(k+t)} \leq \bigcap_{m \in \mathbb{N}} H^{(m)} = 1 \ ,$$

a contradiction that concludes the proof. $\boxed{QED}$

One cannot remove from this theorem the hypothesis that G is a Baer group. In fact (see [14]) for every prime p, there exist residually soluble, non-soluble, locally finite p-groups in which every subnormal subgroup has defect at most four (whence they belong to $\mathfrak{S}_\infty$). The main result that may then be proved, using methods directly derived from Möhres' arguments, is an extension of Theorem 198 (for a proof, we refer to [18]).

250 Theorem. *A periodic residually nilpotent group with the subnormal intersection property is nilpotent.*

(It is an easy exercise to show that residually nilpotent groups with the s.i.p. are in fact Baer groups). Indeed, as for the $\mathcal{N}_1$ case, the crucial step is to prove the statement for groups of finite exponent. However, even in this case one cannot remove the assumption of residual nilpotence: in fact, contrary to the case of $\mathcal{N}_1$, in [18] examples are given of metabelian p-groups of exponent p^2 that belong to $\mathfrak{S}_\infty$ but are not nilpotent. To get one more example showing that the class of Baer $\mathfrak{S}_\infty$-groups is much larger that $\mathcal{N}_1$, one may consider P. Hall generalized wreath power Wr $C_p^{\mathbb{N}}$ (where C_p is a cyclic group of order p) which is not difficult to check being a non-soluble Baer p-group satisfying s.i.p. (for the details, see [18] or Volume II of [96]).

However, I believe that there is still some room left for research on Baer groups in $\mathfrak{S}_\infty$. For instance, the following question should not be terribly difficult to answer.

13 Question. Is every residually nilpotent group in $\mathfrak{S}_\infty$ a $\mathcal{N}_1$-group?

Some more questions (which I have not really meditated on, and thus might well be either trivial or very difficult).

14 Question. Do there exist non-soluble torsion–free Bear groups in $\mathfrak{S}_\infty$?

15 Question. Do there exist non-soluble Baer p-groups of finite exponent in $\mathfrak{S}_\infty$?

Perhaps, more could be proved for the class $\overline{\mathfrak{S}}_\infty$ of groups in which every subgroup satisfies s.i.p. (this class still contains $\mathcal{N}_1$). Of course, Tarski monsters belong to $\overline{\mathfrak{S}}_\infty$, thus some extra conditions are required also in this case.

16 Question. Are locally graded p-groups in $\overline{\mathfrak{S}}_\infty$ locally finite? (the same question is also open for $\mathfrak{S}_\infty$).

7.4 Other classes of locally nilpotent groups

Strongly Baer groups. We say that G is a *strongly Baer group* if every nilpotent subgroup of G is subnormal. Clearly, strongly Baer groups are Baer groups. For every $n \geq 1$, let D_n be the dihedral group of order 2^n; then the direct product $Dir_{n\geq 1} D_n$ is a hypercentral Fitting group with all subgroups locally subnormal, but it is not a strongly Baer group.

One of the difficulties in studying strongly Baer groups might well be the fact that this class, which is obviously closed by subgroups, it is not closed by quotients, as the following example shows. It also proves that strongly Baer groups need not satisfy the normalizer condition (nor in fact belong to $\mathcal{N}_2$).

251 Example. Let H be one of the p-groups constructed by Heineken and Mohamed. Then $A = H'$ is an infinite elementary abelian p-group, $H/A \simeq C_{p^\infty}$ and no proper subgroup of H supplements A. Let K be the wreath product $C_p \wr C_{p^\infty}$, and write $K = BC_{p^\infty}$, where B is the base group. In the direct product $H \times K$, let $W = A \times B$. Then $(H \times K)/W = HW/W \times KW/W$ is the direct product of two copies of C_{p^∞}; we take $G \leq H \times K$ to be such that G/W is a diagonal subgroup of $(H \times K)/W$. Let S be a nilpotent subgroup of G. Then, since $G/B \simeq H$, SB/B is a proper subgroup of G/B and so $SW < G$. Hence SW/W is finite because $G/W \simeq C_{p^\infty}$. Since W is elementary abelian it follows from Lemma 14 that WS is nilpotent. Also, $SW \trianglelefteq G$, so S^G is nilpotent and thus certainly S is subnormal in G. Therefore, G is a strongly Baer group. But $G/A \simeq K = C_p \wr C_{p^\infty}$ is not a strongly Baer group, and does not satisfy $\mathcal{N}_2$.

17 Question. Does there exist a strongly Baer group which is not hyperabelian? Does there exists a (soluble) strongly Baer group that is not a Fitting group?

It may be worth mentioning that many of the classical non-elementary constructions of Baer groups (like McLain groups, P. Hall's generalized wreath powers or Dark's examples of Bear groups with trivial Fitting radical) do not provide, except in trivial cases, any strongly Baer group.

Strong normalizer condition. Let us conclude with mentioning a class of groups which lies strictly between $\mathcal{N}_1$ and the class $\mathfrak{N}$ of all nilpotent groups.

Given a subgroup H of a group G we define the series of the metanormalizers of H by setting $N_G^1(H) = N_G(H)$ and, for $n \geq 1$, $N_G^{n+1}(H) = N_G(N_G^n(H))$. We say that H is *metanormal* in G if $N_G^n(H) = G$ for some $1 \leq n \in \mathbb{N}$. It is then clear that every metanormal subgroup is subnormal, and that in a nilpotent group every subgroup is metanormal. On the other hand, it is easy to see that subnormality does not in general imply metanormality: in the symmetric group S_4 the subgroup $H = \langle (12)(34) \rangle$ is subnormal but not metanormal (in fact $N_G(H)$ is a Sylow 2-subgroup of S_4 and is selfnormalizing). A group satisfies the Strong Normalizer Condition (SNC) if all of its subgroups are metanormal. A group satisfying SNC is clearly a $\mathcal{N}_1$-group, but need not be nilpotent, as groups constructed by H. Smith in [101] show. On the other hand the groups constructed by Heineken and Mohamed, as observed by J. Lennox, do not satisfy SNC; so SNC is a proper subclass of $\mathcal{N}_1$. Since Smith's group are not periodic it seems reasonable to ask the following:

18 Question. Is every periodic group satisfying SNC nilpotent?

19 Question. Is every SNC-group hypercentral?

20 Question. Is it true that a group G is a SNC-group if and only if for each $H \leq G$ there exists a positive integer n such that $\gamma_n(G) \leq N_G(H)$?

An affirmative answer to any of these three questions will imply affirmative answers of the previous ones.

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