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Some partitions in Figueroa planes

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Abstract. Using Grundhöfer's construction of the Figueroa planes from Pappian planes which have an order 3 planar collineation $\widehat{\alpha}$, we show that any Figueroa plane (finite or infinite) has a partition of the complement of any proper $\langle\widehat{\alpha}\rangle$-invariant triangle mostly into subplanes together with a few collinear point sets (from the point set view) and a few concurrent line sets (from the line set view). The partition shows that each Figueroa line (regarded as a set of points) is either the same as a Pappian line or consists mostly of a disjoint union of subplanes of the Pappian plane (most lines are of this latter type) and dually. This last sentence is true with "Figueroa" and "Pappian" interchanged. There are many collinear subsets of Figueroa points which are a subset of the set of points of a Pappian conic and dually.

Keywords: projective plane, Figueroa plane, Pappian plane, subplane, partition, conic, oval, hyperoval

MSC 2000 classification: 51E15, 05B25, 51E30

1 Introduction

A class of non-Desarguesian, proper, finite projective planes of orders q^3 for prime powers $q \not\equiv 1 \pmod 3$ and $q > 2$ were defined by Figueroa [6] in 1982. This construction was generalized to all prime powers $q > 2$ by Hering and Schaeffer [8] later in the same year. We [1] gave a group-coset description of these finite Figueroa planes in 1983. The construction was extended to include infinite planes in 1984 by Dempwolff [5]. These constructions were all algebraic in the sense that they made essential use of collineation groups and coordinates. In 1986 Grundhöfer [7] gave a beautiful synthetic construction which included all these Figueroa planes.

We remind readers of Theo Grundhöfer's elegant synthetic definition of the Figueroa planes. Consider a Pappian plane which has an order 3 planar collinear $\widehat{\alpha}$. The point set (line set) of the Figueroa plane is the same as the point set (line set) of the Pappian plane, but incidence is changed. Letting $\mathrm{I}^{\mathcal{P}}$ and $\mathrm{I}^{\mathcal{F}}$ denote Pappian and Figueroa incidence, respectively, $\mathrm{I}^{\mathcal{F}}$ is defined in terms of $\mathrm{I}^{\mathcal{P}}$ as follows: if either $P^{<\widehat{\alpha}>}$ or $\ell^{<\widehat{\alpha}>}$ is not a proper triangle, then $P \; \mathrm{I}^{\mathcal{F}} \; \ell \Leftrightarrow P \; \mathrm{I}^{\mathcal{P}} \; \ell$; if both $P^{<\widehat{\alpha}>}$ and $\ell^{<\widehat{\alpha}>}$ are proper triangles, then $P \; \mathrm{I}^{\mathcal{F}} \; \ell \Leftrightarrow$ the "vertex" opposite ℓ in $\ell^{<\widehat{\alpha}>}$ $\mathrm{I}^{\mathcal{P}}$ the "side" opposite P in $P^{<\widehat{\alpha}>}$. In what follows we shall denote the "vertex" opposite ℓ in $\ell >^{<\hat{>}\alpha>}$ by $opp(\ell)$ and we shall denote the "side" opposite P in $P^{<\widehat{\alpha}>}$ by *opp*(P).

The map $\widehat{\alpha}$ which is a planar collineation of the Pappian plane remains a planar collineation of the Figueroa plane. Any collineation or polarity of the Pappian plane which commutes with $\widehat{\alpha}$ remains a collineation or polarity, respectively, of the Figueroa plane. Letting α denote the field automorphism associated with the planar automorphism $\widehat{\alpha}$, the Figueroa plane inherits a collineation group isomorphic to $\mathrm{PGL}(3, \mathrm{Fix}(\alpha))$. We describe the orbits of this group.

The set of points P for which $P^{\langle\widehat{\alpha}\rangle}$ is a single point is an orbit. We call points in this orbit *type* I. The set of points P for which $P^{\langle\widehat{\alpha}\rangle}$ is three collinear points is an orbit. We call points in this orbit *type* II. The set of points P for which $P^{\langle\widehat{\alpha}\rangle}$ is a proper triangle is an orbit. We call points in this orbit *type* III. Line orbits have dual descriptions and types of lines have dual definitions. (In general, collinearity of three points (concurrency of three lines) may be different with respect to $\mathrm{I}^{\mathcal{F}}$ and to $\mathrm{I}^{\mathcal{P}}$. But for $P^{\langle\widehat{\alpha}\rangle} = \{P, P^{\widehat{\alpha}}, P^{\widehat{\alpha}^2}\}$ (for $\ell^{\langle\widehat{\alpha}\rangle} = \{\ell, \ell^{\widehat{\alpha}}, \ell^{\widehat{\alpha}^2}\}$) it is the same. So the above definition of types is unambiguous.) Figueroa incidence induces on the orbits of type I points and lines the structure of a subplane isomorphic to $\mathrm{PG}(2, \mathrm{Fix}(\alpha))$.

We define the *characteristic* of a Figueroa plane to be the the *characteristic* of the field which coordinatises the Pappian plane from which the Figueroa plane is constructed.

2 The partitions

We use Grundhöfer's construction together with homogeneous coordinates to show the following:

1 Theorem. *For every (finite or infinite) Figueroa plane and every $\langle\widehat{\alpha}\rangle$-invariant proper triangle (and for the Pappian plane from which it is constructed by Grundhöfer):*

1. *There is a partition of the point set complement of the vertices of the triangle into point subsets each of which is either a collinear set of points or is the point set of a subplane isormorphic to* $\mathrm{PG}(2, \mathrm{Fix}(\alpha))$. *(The partition is the same in both planes, but collinearity or planarity of the parts of the partition differs in some cases.) In each plane the collection of planar parts of the partition is a partition of the point set complement of the set of points incident with the "sides" of triangle.*

2. *There is a partition of the line set complement of the sides of the triangle into line subsets each of which is either a concurrent set of lines or is the line set of a subplane isomorphic to* $\mathrm{PG}(2, \mathrm{Fix}(\alpha))$. *(The partition is the same in both planes, but concurrency or coplanarity of the parts of the*

partition differs in some cases.) In each plane the collection of coplanar parts of the partition is a partition of the line set complement of the set of lines incident with the "vertices" of triangle.

3. *For each of our big planes there is a one to one correspondence between the planar point set parts of the point partition and the coplanar line set parts of the line partition such that corresponding parts are the point set and line set of the same subplane. (This correspondence is not the same in the two planes. It is not the same in the sense that the domains and images of the correspondence differ. It is also not the same in the sense that the the elementwise correspondence is mostly not the same.) There is a collineation group S which acts as a Singer group on every one of these subplane parts (the same group S in both big planes).*

Proof. Without loss of generality we may choose as our planar automorphism the map $\widehat{\alpha}$ which acts on points as $\langle(x, y, z)\rangle \mapsto \langle(z^\alpha, x^\alpha, y^\alpha)\rangle$ and on lines as

$$\left\langle \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right\rangle \mapsto \left\langle \begin{pmatrix} f^\alpha \\ d^\alpha \\ e^\alpha \end{pmatrix} \right\rangle.$$

Define

$$m(x, y, z) = \begin{bmatrix} x & y & z \\ z^\alpha & x^\alpha & y^\alpha \\ y^{\alpha^2} & z^{\alpha^2} & x^{\alpha^2} \end{bmatrix}$$

and

$$m \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{bmatrix} d & f^\alpha & e^{\alpha^2} \\ e & d^\alpha & f^{\alpha^2} \\ f & e^\alpha & d^{\alpha^2} \end{bmatrix}.$$

A point $P = \langle(x, y, z)\rangle$ is of type III $\Leftrightarrow$ $\det m(x, y, z) \neq 0$ and a line

$$\ell = \left\langle \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right\rangle$$

is of type III $\Leftrightarrow$ $\det m \begin{pmatrix} d \\ e \\ f \end{pmatrix} \neq 0$. By transitivity of our PGL group on points (and lines) of type III we may assume that our special proper $\langle\widehat{\alpha}\rangle$-triangle has vertex set $\langle(1, 0, 0)\rangle^{\langle\widehat{\alpha}\rangle}$ and line set $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle^{\langle\widehat{\alpha}\rangle}$.

The point action of the PGL group acting on this representation of the Figueroa plane becomes the action of the group of all invertible matrices

$$\begin{bmatrix} a & b & c \\ c^\alpha & a^\alpha & b^\alpha \\ b^{\alpha^2} & c^{\alpha^2} & a^{\alpha^2} \end{bmatrix},$$

acting by right matrix multiplication on the point coordinates, modulo the invertible $\mathrm{Fix}(\langle\alpha\rangle)$-multiples of the identity matrix. The group induced by all matrices of this form with $b = c = 0$ will be used very often and will be denoted by S. The plane is self dual under the polarity

$$\langle (x, y, z) \rangle \leftrightarrow \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle.$$

The parts of our partitions will be the nonsingleton orbits of the group S. Note that S acts semiregularly on the union of these orbits and therefore regularly on each orbit.

For $\det m(x, y, z) = 0$ or $\det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} = 0$ distinct points of $\langle (x, y, z) \rangle^S$ are incident, in both planes, with a line of $\left\langle \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} \right\rangle^S$. (It is an immediate application of the definition of incidence that, in both planes, the line incident with distinct points $\langle (bx, b^\alpha y, b^{\alpha^2} z) \rangle$ and $\langle (cx, c^\alpha y, c^{\alpha^2} z) \rangle$ is

$$\left\langle \begin{pmatrix} (bc^\alpha - b^\alpha c)^\alpha yz \\ (bc^\alpha - b^\alpha c)^{\alpha^2} zx \\ (bc^\alpha - b^\alpha c) xy \end{pmatrix} \right\rangle).$$

For $\det m(x, y, z) \neq 0$ and $\det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} \neq 0$, in the Pappian plane distinct points of $\langle (x, y, z) \rangle^S$ are incident with a line of $\left\langle \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} \right\rangle^S$ by the same immediate application.

For $\det m(x, y, z) \neq 0$ and $\det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} \neq 0$, in the Figueroa plane we shall show that distinct points of $\langle(x,y,z)\rangle^S$ are incident with a line of

$$\left\langle\begin{pmatrix} xYZ^{\alpha^2} \\ yZX^{\alpha^2} \\ zXY^{\alpha^2}\end{pmatrix}\right\rangle^S$$

where $X = x^{1+\alpha} - yz^\alpha,\ Y = y^{1+\alpha} - zx^\alpha,\ Z = z^{1+\alpha} - xy^\alpha$.

We shall use the easy facts

$$XY^\alpha Z^{\alpha^2} \;=\; (xy^\alpha z^{\alpha^2})^{\alpha^2} \det m(x,y,z) \;-\; \det m\begin{pmatrix} yz \\ zx \\ xy\end{pmatrix},$$

$$X^{\alpha^2+1} - Y^{\alpha^2}Z = x \det m(x,y,z),$$

$$Y^{\alpha^2+1} - Z^{\alpha^2}X = y \det m(x,y,z),$$

$$Z^{\alpha^2+1} - X^{\alpha^2}Y = z \det m(x,y,z)$$

and

$$N(X) - N(x)\det m(x,y,z) \;=$$

$$N(Y) - N(y)\det m(x,y,z) \;=$$

$$N(Z) - N(z)\det m(x,y,z) \;=$$

$$-\det m\begin{pmatrix} yz \\ zx \\ xy\end{pmatrix} \in \mathrm{Fix}(\alpha).$$

(Here $N(t)$ denotes $t^{1+\alpha+\alpha^2}$ that is N is the *relative norm* function.) We shall also use several times:

$$\begin{aligned} x^{\alpha+\alpha^2}Y^{\alpha^2}Z - y^\alpha z^{\alpha^2}X^{\alpha^2+1} &= y^{\alpha+\alpha^2}Z^{\alpha^2}X - z^\alpha x^{\alpha^2}Y^{\alpha^2+1} \\ &= z^{\alpha+\alpha^2}X^{\alpha^2}Y - x^\alpha y^{\alpha^2}Z^{\alpha^2+1} = \;-\;\det m\begin{pmatrix} yz \\ zx \\ xy\end{pmatrix}. \end{aligned}$$

Then

$$\det m\begin{pmatrix} xYZ^{\alpha^2} \\ yZX^{\alpha^2} \\ zXY^{\alpha^2} \end{pmatrix} = -\Big(\det m(x,y,z)\Big)\left(\det m\begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}\right)^2$$

which shows that under our hypotheses the lines of

$$\left\langle \begin{pmatrix} xYZ^{\alpha^2} \\ yZX^{\alpha^2} \\ zXY^{\alpha^2} \end{pmatrix} \right\rangle^S$$

are of type III.

For distinct points $P_b = \langle (bx, b^\alpha y, b^{\alpha^2} z)\rangle, P_c = \langle (cx, c^\alpha y, c^{\alpha^2} z\rangle$ in $\langle (x,y,z)\rangle^S$,

$$opp(P_b) = \left\langle \begin{pmatrix} b^{\alpha+\alpha^2} X^\alpha \\ b^{\alpha^2+1} Y^\alpha \\ b^{1+\alpha} Z^\alpha \end{pmatrix} \right\rangle, \quad opp(P_c) = \left\langle \begin{pmatrix} c^{\alpha+\alpha^2} X^\alpha \\ c^{\alpha^2+1} Y^\alpha \\ c^{1+\alpha} Z^\alpha \end{pmatrix} \right\rangle$$

and both of these are Pappian incident with

$$opp(\ell_f) = opp\left\langle \begin{pmatrix} fxYZ^{\alpha^2} \\ f^\alpha yZX^{\alpha^2} \\ f^{\alpha^2} zXY^{\alpha^2} \end{pmatrix} \right\rangle = \langle (f^{-1}(YZ)^\alpha, f^{-\alpha}(ZX)^\alpha, f^{\alpha^{-2}}(XY)^\alpha)\rangle$$

for $f = (bc)^{\alpha+\alpha^2}(b^\alpha c - bc^\alpha)^{-\alpha}$.

The $\langle (x,y,z)\rangle^S$ with exactly one of x, y, z equal to zero are collinear sets of points in the Pappian plane. Each of the other $\langle (x,y,z)\rangle^S$ is, in the Pappian plane, the point set of a subplane as can be seen by the isomorphism:

$$\langle (bx, b^\alpha y, b^{\alpha^2} z)\rangle \mapsto \langle (b, b^\alpha, b^{\alpha^2})\rangle, \quad \left\langle \begin{pmatrix} dyz \\ d^\alpha zx \\ d^{\alpha^2} xy \end{pmatrix} \right\rangle^S \mapsto \left\langle \begin{pmatrix} d \\ d^\alpha \\ d^{\alpha^2} \end{pmatrix} \right\rangle^S$$

from $\left(\langle (x,y,z)\rangle^S, \left\langle \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} \right\rangle^S\right)$ to $\Big($ points of type I, lines of type I$\Big)$.

In the Figueroa plane case, the collinear $\langle (x,y,z)\rangle^S$ consist of all $\langle (x,y,z)\rangle^S$ with exactly one of x, y, z equal to zero and either $\det m(x,y,z) = 0$ or

$$\det m\begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} = 0$$

together with all $\langle (x,y,z)\rangle^S$ with none of x,y,z equal to zero, exactly one of X,Y,Z equal to zero, and $\det m(x,y,z)\neq 0$, $\det m\begin{pmatrix} yz\\ zx\\ xy\end{pmatrix}\neq 0$. Each of the other $\langle (x,y,z)\rangle^S$, in the Figueroa case, is the point set of a subplane. An isomophism which shows this has the same formula as the one in the Pappian case if $\det m(x,y,z)=0$ or $\det m\begin{pmatrix} yz\\ zx\\ xy\end{pmatrix}=0$. Otherwise an isomorphism is:

$$\langle (bx, b^{\alpha}y, b^{\alpha^2}z)\rangle \mapsto \langle (b^{-1}, b^{-\alpha}, b^{-\alpha^2}\rangle, \quad \left\langle \begin{pmatrix} dxYZ^{\alpha^2}\\ d^{\alpha}yZX^{\alpha^2}\\ d^{\alpha^2}zXY^{\alpha^2}\end{pmatrix}\right\rangle \mapsto \left\langle \begin{pmatrix} d^{-1}\\ d^{-\alpha}\\ d^{-\alpha^2}\end{pmatrix}\right\rangle$$

from

$$\left(\langle (x,y,z)\rangle^S, \left\langle \begin{pmatrix} xYZ^{\alpha^2}\\ yZX^{\alpha^2}\\ zXY^{\alpha^2}\end{pmatrix}\right\rangle^S \right)$$

to (points of type I, lines of type I).

This proves part 1 of the theorem. Part 2 follows by duality.

The one to one correspondence claimed in part 3 of the theorem is, in the Pappian case:

$$\langle (x,y,z)\rangle^S \leftrightarrow \left\langle \begin{pmatrix} yz\\ zx\\ xy\end{pmatrix}\right\rangle^S$$

with domain $x,y,z\neq 0$. (Note that $x,y,z\neq 0 \Leftrightarrow yz,zx,xy\neq 0$.)

The one to one correspondence, in the Figuera case, has a two piece definition:

One piece of the correspondence is:

$$\langle (x,y,z)\rangle^S \leftrightarrow \left\langle \begin{pmatrix} yz\\ zx\\ xy\end{pmatrix}\right\rangle^S$$

for $x,y,z\neq 0$ and either $\det m(x,y,z)=0$ or $\det m\begin{pmatrix} yz\\ zx\\ xy\end{pmatrix}=0$. (Again note:

$x, y, z \neq 0 \Leftrightarrow yz, zx, xy = 0$. Also note that in this case $\det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} = 0$ or

$\det m\Big((yz)(zx), (zx)(xy), (xy)(yz)\Big) = N(xyz) \det m(z, x, y) = 0 \Leftrightarrow \det m(x, y, z) = 0$

or $\det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} = 0$.)

The other piece of the correspondence, in the Figueroa case, is:

$$\langle (x, y, z) \rangle \leftrightarrow \left\langle \begin{pmatrix} xYZ^{\alpha^2} \\ yZX^{\alpha^2} \\ zXY^{\alpha^2} \end{pmatrix} \right\rangle$$

for $X, Y, Z \neq 0, \det m(x, y, z) \neq 0, \det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} \neq 0$. To verify this it helps to let $\widehat{x} = xYZ^{\alpha^2}$, $\widehat{y} = yZX^{\alpha^2}$, $\widehat{z} = zXY^{\alpha^2}$ and $\widehat{X} = \widehat{x}^{1+\alpha} - \widehat{y}\widehat{z}^{\alpha}, \widehat{Y} = \widehat{y}^{1+\alpha} - \widehat{z}\widehat{x}^{\alpha}, \widehat{Z} = \widehat{z}^{1+\alpha} - \widehat{x}\widehat{y}^{\alpha}$. Then

$$\widehat{X} = -YZ \det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}$$

$$\widehat{Y} = -ZX \det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}$$

$$\widehat{Z} = -XY \det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}$$

and

$$\widehat{x}\widehat{Y}\widehat{Z}^{\alpha^2} = x(XYZ)^{\alpha^2+1}\left(\det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}\right)^2$$

$$\widehat{y}\widehat{Z}\widehat{X}^{\alpha^2} = y(XYZ)^{\alpha^2+1}\left(\det m \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}\right)^2$$

$$\widehat{z}\widehat{X}\widehat{Y}^{\alpha^2} = z(XYZ)^{\alpha^2+1}\left(\det m\begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}\right)^2$$

which shows that the second power of the correspondence is the identity. (Note that$\widehat{X}, \widehat{Y}, \widehat{Z} \neq 0 \Leftrightarrow X, Y, Z \neq 0$. Also note that under these conditions

$$\det m\begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{pmatrix} = \det m\begin{pmatrix} xYZ^{\alpha^2} \\ yZX^{\alpha^2} \\ zXY^{\alpha^2} \end{pmatrix} = -\Big(\det m(x,y,z)\Big)\left(\det m\begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}\right)^2 \neq 0$$

and

$$\det m\begin{pmatrix} \widehat{x}\widehat{Y}\widehat{Z}^{\alpha^2} \\ \widehat{y}\widehat{Z}\widehat{X}^{\alpha^2} \\ \widehat{z}\widehat{X}\widehat{Y}^{\alpha^2} \end{pmatrix} = \Big(N(XYZ)\Big)^2\left(\det m\begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}\right)^6 \det m(x,y,z) \neq 0$$

$\Leftrightarrow \quad \det m(x,y,z) \neq 0,\ \det m\begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} \neq 0$.) This proves part 3 of the theorem.

This proves the theorem. $\boxed{QED}$

2 Remark. If the Pappian plane has an $\widehat{\alpha}$-invariant subplane on which the restriction of $\widehat{\alpha}$ is an order 3 planar collineation, the Figueroa plane constructed using the subplane and the restriction of $\widehat{\alpha}$ to the subplane is a Figueroa subplane of the (first) big Figueroa plane. (From this it follows that the Figueroa plane of order q^3 is a subplane of the Figueroa plane of order q^{3r} for any prime power q and any $r \equiv 1$ or $2(\mod 3)$). Let partitions be based, both in the big plane and in the subplane, on the same $\widehat{\alpha}$-invariant proper triangle of the subplane. Then the partition of the subplane embeds in the partition of the big plane in the sense that any part of the the big plane partition either meets the subplane in a part of the subplane partition or has empty intersection with the subplane.

3 Corollary. *Every type I line and every type II line is incident with exactly the same set of points in the Figueroa plane as in the Pappian plane. Every type III line ℓ in the two planes is incident with the same set of type II points and with the two points $(opp(\ell))^{\alpha}$ and $(opp(\ell))^{\alpha^2}$, but all other incident points are different: the Figueroa (Pappian) line is incident with the points of a disjoint union of Pappian (Figueroa) subplanes, with both points and lines of type III, from the partition associated with $\ell^{\langle\alpha\rangle}$.*

Proof. By the definition of incidence and the transitivity of the PGL group on the set of type III lines, it is sufficieint to prove the last sentence for

$$\ell = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

Note that the type III line $\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ is Figueroa incident with all the points in the disjoint union $\cup_{z,N(z)\neq 0,1}\langle (z^{1+\alpha}, 1, z)\rangle^S$ which is the set of all the points in the disjoint union of Pappian subplanes

$$\left(\langle (z^{1+\alpha}, 1, z)\rangle^S, \left\langle \begin{pmatrix} 1 \\ z^{1+\alpha} \\ z^{\alpha} \end{pmatrix} \right\rangle^S \right);\ z, N(z) \neq 0, 1.$$

Also the type III line $\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ is Pappian incident with all the points in the disjoint union $\cup_{x,N(x)\neq 0,-1}\langle (x, 1, 0)\rangle^S$ which is the set of all the points in the disjoint union of Figueroa subplanes $\left(\langle x, 1, 0)\rangle^S, \left\langle \begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right\rangle^S \right);\ x, N(x) \neq 0, -1.$
This proves the corollary. $\boxed{QED}$

4 Corollary. *Let L be the set of points of a line of one of these subplanes, of the partition, which is formed from points of type III and lines of type III. (The majority of the subplanes of the partition are of this type.) In the Figueroa case L is the intersection of the set of points of the subplane with the set of points of a Pappian conic through the vertices of the triangle.*

Proof. Any part of the Figueroa partition of this type is

$$\left(< a, b, c)^S,\ \left\langle \begin{pmatrix} aBC^{\alpha^2} \\ bCA^{\alpha^2} \\ cAB^{\alpha^2} \end{pmatrix} \right\rangle^S \right)$$

where $\det m(a, b, c) \neq 0$, $\det m \begin{pmatrix} bc \\ ca \\ ab \end{pmatrix} \neq 0$ and where $A = a^{1+\alpha} - bc^{\alpha}$, $B = b^{1+\alpha} - ca^{\alpha}$, $C = c^{1+\alpha} - ab^{\alpha}$ and where $a, b, c, A, B, C \neq 0$. Let L be the set of points of point part $\langle(a,b,c)\rangle^S$ which are I^F with line $\left\langle\begin{pmatrix} taBC^{\alpha^2} \\ t^{\alpha}bCA^{\alpha^2} \\ t^{\alpha^2}cAB^{\alpha^2}\end{pmatrix}\right\rangle$ for some fixed $t \neq 0$. Then L is the set of points $\langle(sa, s^{\alpha}b, s^{\alpha^2}c)\rangle$ satisfying $s^{-1}t^{-1} + s^{-\alpha}t^{-\alpha} + s^{-\alpha^2}t^{-\alpha^2} = 0$ which is the same as the set of points of $\langle(a,b,c)\rangle^S$ which are Pappian incident with the conic having equation $t^{-\alpha^2}cxy + t^{-1}ayz + t^{-\alpha}bzx = 0$.

This proves the corollary. $\boxed{QED}$

5 Remark. It is tempting to conjecture that the last sentence of the above corollary might be true, in characteristic not two, with Figueroa and Pappian planes interchanged and with Pappian conics replaced by (Figueroan) Ovali di Roma [2, 3]. But such a conjecture is not the true. A general reason for this is that there are not enough Ovali di Roma. More specifically, if the conjecture were true, then most of the Ovali di Roma would have to be constructed from a conic which was not $\widehat{\alpha}$ invariant (violating a hypothesis of the construction of Ovali di Roma). Also we cannot prove a converse in the finite characteristic two case using the Figueroa hyperovals of de Resmini and Hamilton [4] because these hyperovals are also hyperovals in the Pappian planes.

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