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Parallelism

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Abstract. Problems involving the idea of parallelism occur in finite geometry and in graph theory. This article addresses the question of constructing parallelisms with some degree of "symmetry". In particular, can we say anything on parallelisms admitting an automorphism group acting doubly transitively on "parallel classes"?

Keywords: line–parallelism in $PG(3, q)$, circle geometries, one–factorization, two–factorization

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Introduction

The title of the paper [19] by N.L. Johnson is *Two–transitive parallelisms.* The parallelisms in question are the line–parallelisms of 3–dimensional finite projective spaces. "Two–transitive" refers to the existence of an automorphism group with a doubly transitive action. What is not immediately clear is the set of "geometric" objects on which the doubly transitive action is assumed: in the quoted paper this set is that of parallel classes, each of which is in turn a set of lines.

In this expository paper we shall consider other contexts in which an analogous situation can be reproduced. Double transitivity is an assumption involving a lot of symmetry. Since a given automorphism group generally acts on different sets, the double transitivity condition can be imposed separately on each such action. In many cases meaningful examples exist and a full classification can be obtained: the situation of the quoted paper [19] and of the more general result of [18] is one such succesful instance.

We shall deal in particular with problems having to do with parallelism in circle geometries and complete graphs: the formulations in each context will be straightforward once the roles of the various objects are specified.

We have tried to keep notation and terminology as standard as possible.They should be clear from the context: if in doubt we refer to the monographs [10], [13], [26] in the References.

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1 Parallelism and resolvable designs

The classical notion of parallelism in affine planes finds an extremely natural generalization in the context of finite geometry [4, I§5], [1]. In a block design a "parallel class" or "resolution class" can be simply defined as a set of blocks forming a partition of the underlying point–set. If, in turn, the set of all blocks can be partitioned into parallel classes, then we can say that the block design admits a "parallelism" or a "resolution". The former terminology is justified by the circumstance that the parallel classes are then the equivalence classes of an equivalence relation on blocks satisfying the Euclidean postulate: given any point and any block there is a unique block through the given point which is parallel to the given block. A block design admitting a parallelism is often called a "resolvable" design.

Perhaps the most famous instance of a transitivity issue in resolvable designs is Wagner's Theorem [25] for affine planes: if a finite affine plane admits a collineation group acting transitively on points then the plane is a translation plane.

A result which can be derived from the previous one is the celebrated Ostrom–Wagner Theorem [22]: if a finite projective plane admits a collineation group acting doubly transitively on points then the plane is desarguesian and the group contains the projective special linear group.

The previous statement focuses on the action on points, but we can certainly formulate a similar assumption for the action on the parallel classes of a resolvable design. What can be said in general on a 2–(v, k, λ) resolvable design admitting an automorphism group G acting doubly transitively on resolution classes? When reference to the group G is unessential, we shall simply speak of a resolvable design which is doubly transitive on parallel classes.

The parallelisms mentioned in the Introduction fall under this description: a line–parallelism of $PG(3, q)$ exists precisely when the block design $PG_1(3, q)$ formed by the points and lines of $PG(3, q)$ is resolvable. Very few such parallelisms are doubly transitive on parallel classes: the classification result in [19] shows that only two non–isomorphic examples for $q = 2$ have this property.

An infinite class of examples exists if the design is assumed to be an affine plane: the following statement shows that the Desarguesian affine plane $AG(2, q)$ and, more generally, the design $AG_1(d, q)$ is doubly transitive on the classes of parallel lines.

1 Proposition. *The affine linear group $AGL(d, q)$ acts doubly transitively on the classes of parallel lines in $AG(d, q)$.*

Proof. The group $AGL(d, q)$ induces the projective general group on the hyperplane at infinity in its natural doubly transitive permutation representation. Since the points on the hyperplane at infinity are in natural one–to–one correspondence with the classes of parallel lines in $AG(d,q)$, the assertion follows. $\boxed{QED}$

If the design is assumed to be an affine plane, then the same argument used in the previous proof shows that the double transitivity of the collineation group G on the parallel classes is equivalent to the double transitivity of G on the points of the line at infinity.

This situation has been studied extensively by many authors. Perhaps the most recent contribution in this direction is contained in [21] and it states that one of the following occurs:

(a) the plane is desarguesian of order q and $SL(2,q) \trianglelefteq G$;

(b) the plane has order q^2 with q an odd 2–power and $Sz(q) \trianglelefteq G$;

(c) the plane has order q^3 with q an even 2–power and $PSU(3,q) \trianglelefteq G$, where G fixes an affine point.

Earlier results in [23], [11] imply that if the plane is a translation plane then it is either Desarguesian or Lüneburg–Tits. The affine Lüneburg–Tits planes are indeed doubly transitive on parallel classes, hence non–desarguesian examples exist even within the class of translation planes, where the translation group always lies in the kernel of the action on parallel classes.

2 Parallelism in finite circle planes

Circle planes are the incidence structures generalizing the geometries of the plane sections of a non–degenerate quadric in projective 3–space, and 'circle' in this context is just another word for "block". There are three families of circle planes, called Möbius, Laguerre and Minkowski planes in the german terminology [3], [17], [5]. Möbius planes are also called inversive planes and correspond to the plane sections of an elliptic quadric in the classical case, while Laguerre and Minkowski planes correspond to the plane sections of quadratic cones and hyperbolic quadrics, respectively. Circle planes arising from quadrics are generally referred to as being miquelian, because they can be characterized through a purely configurational condition called Miquel's axiom [12], [3].

In the finite case circle planes can be described as 3–designs with special parameters. That is true without restrictions for finite inversive planes, which are 3–$(n^2+1, n+1, 1)$ designs, the parameter n being the so called "order", which is just the order of the derived affine plane.

For Laguerre and Minkowski planes, the condition that three distinct points determine a unique block must be understood in the more restricted sense that any three *pairwise independent* points determine a unique block.

As a matter of fact, a little care is required when speaking of parallelism in Laguerre and Minkowski planes, since in these geometric structures the word "parallelism" denotes an equivalence relation on points. In a Minkowski plane we even have two distinct point–parallelisms. Parallel points cannot lie together on a block and the terminology 'independent' points is the same as "non–parallel" points. The equivalence classes of parallel points in miquelian Laguerre and Minkowski planes are precisely the families of ruling lines on the corresponding quadrics, one family for the cone and two for the hyperbolic quadric, respectively.

It is immediately clear that a finite inversive plane of order n cannot admit a resolution class of blocks according to the usual definition given for block designs. Since each block has size $n+1$ and since n^2+1 is the total number of points, we have that a set of pairwise disjoint blocks has size at most $n-1$: if such a set of size $n-1$ exists, then all but two points of the inversive plane are covered by the blocks in the set, and so it is quite reasonable to call such a set a"flock," in analogy with the corresponding definition for an elliptic quadric in projective 3–space, see [12], [16].

A finite Laguerre plane of order n has $n(n+1)$ points; a finite Minkowski plane of order n has $(n+1)^2$ points; in either case a block consists of $n+1$ points and so there is no arithmetical obstruction to the existence of a resolution class, which will consist of n and $n+1$ blocks, respectively. We can thus define a "flock" in a finite Laguerre or Minkowski plane to be a resolution class of blocks: the analogy with the corresponding definitions for quadratic cones and hyperbolic quadrics in projective 3–space persists, see [16].

The total number of blocks in a finite Möbius, Laguerre and Minkowski plane of order n is $n(n^2+1)$, n^3 and $(n+1)n(n-1)$, respectively.

The number $n-1$ divides $n(n^2+1)$ if and only if $n=3$. Therefore it is not possible to partition the set of blocks of a finite Möbius plane of order n into flocks, except, possibly, in the case $n=3$.

On the other hand, no numerical constraint forbids the existence of a partition of the set of blocks of a finite Laguerre or Minkowski plane of order n into flocks, which should then consist of n^2 and $n(n-1)$ flocks respectively. If such a partition exists we shall say that the Laguerre or Minkowski plane is "resolvable" and we shall refer to the partition as to a "resolution."

Resolvable finite Minkowski planes are considered in [6], where it is pointed out first of all that miquelian Minkowski planes are resolvable. A purely geometric proof can be obtained by embedding one of the two reguli lying on a hyperbolic quadric of $PG(3,q)$ into a regular line–spread. For each line of the regular spread which is not on the quadric, we form its linear flock (consisting of all circles obtained by intersecting the quadric with the planes through the given line). No two such flocks share a circle, because the corresponding lines are in the spread and are therefore skew; direct counting shows then that the union of all such flocks covers the set of circles of the quadric.

An explicit determination of all resolutions of a miquelian finite Minkowski plane follows from the Bader–Lunardon classification [2] of the flocks of hyperbolic quadrics in $PG(3,q)$. To that purpose it is useful to consider the description of a finite Minkowski plane of order n as arising from a sharply triply transitive set G of permutations on a set of cardinality $n+1$, see [3]. In the miquelian case G is actually a group, namely $G = PGL(2,q)$ in its natural sharply triply transitive permutation representation on the points of the projective line $PG(1,q)$. Flocks in this description are precisely the sharply transitive subsets of $G = PGL(2,q)$: the corresponding version of the classification theorem [2] can be formulated by stating that every sharply transitive sub*set* of $PGL(2,q)$ is actually a coset of a sharply transitive sub*group*. Using this description, Proposition 2 in [6] can be formulated as follows.

2 Proposition. *Every resolution of the miquelian Minkowski plane of order q arises from the partition of $PGL(2,q)$ consisiting of the cosets of some sharply transitive subgroup.*

Non–miquelian finite Minkowski planes of odd order exist. The sharply triply transitive sets of permutations from which they arise are subsets of the projective semilinear group $P\Gamma L(2,p^m)$ for some odd prime p. Once a non–trivial field automorphism σ is chosen (which forces $m > 1$) the set can be described as

$$G(p^m,\sigma) = PSL(2,p^m) \cup \sigma\big(PGL(2,p^m) - PSL(2,p^m)\big)$$

The sharply triply transitive permutation set $G(p^m,\sigma)$ is a group if and only if the automorphism σ is involutory. In any case $G(p^m,\sigma)$ is the union of two cosets of $PSL(2,p^m)$.

It is pointed out in [6] that if $PSL(2,p^m)$ contains a sharply transitive subgroup, then the cosets of this subgroup which are contained in $G(p^m,\sigma)$ do form a resolution of the Minkowski plane. The only possibility for such a subgroup is a dihedral subgroup and that occurs precisely for $p^m \equiv -1 \pmod 4$. This relation is never satisfied when $G(p^m,\sigma)$ is a group. As a matter of fact, if $G(p^m,\sigma)$ is a group then it contains no sharply transitive subgroup and it may even happen that it contains no sharply transitive subset whatsoever: that was proved in [15] for $p^m = 9$ by an exhaustive computer search. The corresponding Minkowski plane does not admit any resolution class and so 'a fortiori' cannot be resolvable.

Proposition 4 in [6] shows that the non–miquelian Minkowksi plane arising from $G(p^m, \sigma)$ admits a resolution which does not consist of the cosets of a sharply transitive subgroup in the following cases:

- $p^m \equiv -1 \mod 4$;
- setting $x^\sigma = x^{p^t}$ then $p^t \equiv 1 \mod 4$.

The automorphisms of the Minkowski planes arising from the sets $G(p^m, \sigma)$ have been explicitely described in [14]. Hence, at least in principle, the determination of the automorphism groups of the previously described resolutions could be done, although it would certainly lead to rather involved calculations. Whether the miquelian Minkowski planes can be characterized by some assumption on the symmetry of their resolutions might be of some interest.

For a quadratic cone in $PG(3, q)$ the spread construction that was outlined above for a hyperbolic quadric will also produce a partition of the set of circles into q^2 classes, each of size q. In fact, consider the q^2 lines of the spread not through the vertex of the cone. There are precisely q planes not containing the vertex through each such line. Each such plane is not a tangent plane and so the intersection with the cone will produce a circle of the Laguerre plane. Any two lines of the spread are skew and so they yield no common circle. A class of q circles in the previous partition is a resolution class of the Laguerre plane if and only if the corresponding line of the spread is disjoint from the cone. At least one of the lines of the spread avoiding the vertex has to meet the cone at a point which is not the vertex. We conclude that the previous partition of the circle set is not a resolution of the Laguerre plane. Although many flocks of quadratic cones have been found over the years, we are not aware of any attempt of arranging a suitable number of such flocks so as to form a resolution of the corresponding Laguerre plane.

3 Factorizations of complete graphs

If K_v denotes the complete graph on v vertices, then it can be seen as a 2–$(v, 2, 1)$ design, in other words a 2–design in which each block has size 2. More generally, one can consider the t–$(v, t, 1)$ design consisting of all t–subsets of a point–set of size v. These designs are called "complete" designs in [10].

We restrict our attention to the case $t = 2$ and observe that a resolution class in the complete design is nothing but a 1–factor of K_v, the existence of which forces v to be even; a parallelism of a 2–$(v, 2, 1)$ design is thus precisely a one–factorization of the complete graph.

There is a huge literature on one–factorizations of complete graphs [26]. The relevant point of view for our considerations is that of automorphism groups. If

$\mathcal{F} = \{F_1, F_2, \ldots, F_{v-1}\}$ is a 1–factorization of the complete graph K_v, v even, then an automorphism group G of $\mathcal{F}$ is a permutation group on the vertices of the complete graph leaving $\mathcal{F}$ setwise invariant. In other words G is a subgroup of $Sym(v)$ and so the action of G on $V(K_v)$ is faithful by definition.

The action of G on $\mathcal{F}$ need not be faithful, but we know from [10, Thm. 1.3] that if h is an automorphism of $\mathcal{F}$ inducing the identity on $\mathcal{F}$ (which means $(F_i)^h = F_i$ for $i = 1, 2, \ldots, v-1$), then h is either the identity or a fixed–point–free involution on $V(K_v)$: the kernel N of the action of G on $\mathcal{F}$, is thus an elementary abelian 2–group acting semiregularly on $V(K_v)$. In particular, if G fixes a vertex x, then N reduces to the identity permutation and the action of G on $\mathcal{F}$ is faithful.

3 Proposition. *If G fixes a vertex x, then G acts as a permutation group on $\mathcal{F}$ and this action is equivalent to the action of G on $V(K_v) \setminus \{x\}$.*

Proof. Since the kernel N is trivial, the action of G on $\mathcal{F}$ is faithful. For each one–factor F_i in $\mathcal{F}$ let $[x, y_i]$ denote the uniquely determined edge of F_i through x. For an automorphism $g \in G$ we have $y_i^g = y_j$ if and only if $F_i^g = F_j$. QED

It follows from the previous Proposition that if G fixes x and acts doubly transitively on the remaining vertices, then G will also act doubly transitively on $\mathcal{F}$. In this case, if for each one–factor F_j we denote by $[x, y_j]$ the edge in F_j through x, then $H_j = F_j \setminus \{[x, y_j]\}$ is a near one–factor on $V(K_v) \setminus \{x\}$ and $\mathcal{H} = \{H_1, H_2, \ldots, H_{v-1}\}$ is a near one–factorization of K_{v-1} admitting G as an automorphism group acting doubly transitively on vertices.

The process can be reversed in a rather straightforward manner. Assume $\mathcal{H}$ to be a near one–factorization of K_{v-1} admitting an automorphism group L acting doubly transitively on vertices. Add a new vertex x to $V(K_{v-1})$ and extend the action of each permutation g in L by defining $x^g = x$, thus obtaining a permutation group G on $V(K_{v-1}) \cup \{x\}$. For each near one–factor H_j in $\mathcal{H}$ form a one–factor F_j on $V(K_{v-1}) \cup \{x\}$ by adding the edge $[x, y_j]$ where y_j is the unique vertex of K_{v-1} which is uncovered by H_j. The collection $\mathcal{F} = \{F_1, F_2, \ldots, F_{v-1}\}$ is a one–factorization of K_v and G acts as an automorphism group of $\mathcal{F}$ acting doubly transitively on one–factors.

The near one–factorizations with an automorphism group acting doubly transitively on vertices were classified by P. J. Cameron in [9, Theorem 2] and form two infinite families. In the former one the underlying vertex–set is a vector space over a prime field of odd characteristic and the near one–factor avoiding the given vector a consists of all edges in which the sum of the two vertices equals $2a$. In the latter one the underlying vertex–set is a projective space over the binary field and the near one–factor avoiding the given point a consists of all edges in which the two vertices are collinear with a.

Let $\mathcal{F}$ be a one–factorization of K_v admitting an automorphism group H acting triply transitively on vertices. The one–point–stabilizer H_x acts doubly transitively on the remaining vertices and so $G = H_x$ acts doubly transitively on one–factors by Proposition 3. It is proved in Theorem 6.4 in [10] that a one–factorizations with an automorphism group acting triply transitively on vertices is necessarily the line–parallelism of an affine space over the binary field. The affine space $AG(d, 2)$ can be obtained from the projective space $PG(d, 2)$ by selecting a hyperplane at infinity, consequently the one–factorization arising from the line–parallelism of $AG(d, 2)$ can be obtained from the previously described near one–factorization based on $PG(d, 2)$, by considering only the near one–factors whose "avoided points" lie on the hyperplane at infinity. The collineation group of $AG(d, 2)$ can be described as the collineation group of $PG(d, 2)$ fixing the hyperplane at infinity: it acts as $PGL(d, 2)$ on $PG(d-1, 2)$ and so it is doubly transitive on the points at infinity, whence also on the parallel classes of lines of $AG(d, 2)$. We have thus another proof of the fact that the one–factorization based on $AG(d, 2)$ is doubly transitive on one–factors.

We have thus described essentially two infinite families of examples for one–factorizations which are doubly transitive on factors. Whether they account for all such one–factorizations is still an open question: some contribution towards their classification will be the subject of a forthcoming paper [8].

The problem described for one–factorizations can be formulated for r–factorizations in a completely similar manner. A r–factor of a given graph is a r–regular spanning subgraph, that is a regular subgraph of degree r which is incident with each vertex of the graph. A r–factorization is a collection of r–factors forming a partition of the edge–set of the graph. In our case the given graph is the complete graph K_v and so r is a divisor of $v - 1$. The case $r = 2$ yields two–factorizations of K_v, which exist if and only if v is odd. A two–factor is the union of disjoint cycles and, differently from one–factors, non–isomorpphic two–factors on a given set of vertices exist. Non–isomorhic two–factors may well be found within one and the same two–factorization of K_v. On the other hand, if it is assumed that the given two–factorization admits an automorphism group acting doubly transitively — whence also transitively — on two–factors, then these two–factors will be pairwise isomorphic.

The investigation of two–factorizations of K_v which are doubly transitive on two–factors was begun in [20]: infinite families were found and several necessary conditions on the parameters were determined. Their classification is not complete yet, although it is complete if it is assumed that each two–factor consists of a single cycle — a so called Hamiltionian cycle.

A complete classification is also available if the action of the automorphism group is assumed to be doubly transitive on vertices rather than on two–factors, as it follows from the resuls in [24], [7]. These two–factorizations arise from an affine space $AG(d,p)$ over the prime field of odd characteristic p and admit the affine group $AGL(d,p)$ as an automorphism group.

Differently from one–factorizations arising from $AG(d,2)$, these two–factorizations arising from $AG(d,p)$ turn out to be doubly transitive on factors only for $p = 3$. As a matter of fact each class of parallel lines yields $(p-1)/2$ two–factors consisting of p–cycles, each of which covers the points on a line of the class. It has been pointed out in [20] that these two–factors form a non–trivial set of imprimitivity for the action of the affine group on two–factors, unless $(p-1)/2 = 1$, that is $p = 3$.

For every value of $r \geq 3$ there exist r–factorizations of complete graphs which are doubly transitive on r–factors. A construction starting from near one–factorizations which are doubly transitive on vertices is described in [20]: the resulting r–factors have the same "shape," as double transitivity implies, and they consist of a complete subgraph K_r tohgether with a certain number of complete bipartite subgraphs $K_{r,r}$. Here again we are aware of no result indicating the possibility of classifying such objects.

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