Note di Matematica **26,** n. 1, 2006, 79–103.

Cubic order translation planes constructed by multiple hyper-regulus replacement

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Received: 12/4/2005; accepted: 18/4/2005.

Abstract. New classes of mutually disjoint hyper-reguli of order q^3 are determined, which have the property that any subset of at least two such hyper-reguli is non-linear. Each translation plane constructed is not André or generalized André.

Keywords: hyper-regulus, multiple replacement, André hyper-reguli.

MSC 2000 classification: 51E23 (primary), 51A40.

1 Introduction

Recently, the authors have constructed classes of mutually disjoint hyper-reguli of order q^n and degree $(q^n-1)/(q-1)$, for any integer $n > 2$. When $n > 3$, these hyper-reguli are not André hyper-reguli and so the associated translation planes are not André or generalized André planes. In contrast, when $n = 3$, all hyper-reguli are André hyper-reguli. However, if one has a set of mutually disjoint hyper-reguli, it is possible that a subset of cardinality at least two may not be linear and so would produce new translation planes. In this article, new sets of non-André hyper-reguli in $PG(5, q)$ are determined with the property that every subset of cardinality at least two is not linear. Hence, replacement of such a subset will produce a new translation plane.

Of course, it is well known that a regulus R in $PG(3, q)$ is a set of $q + 1$ lines that are covered by another set of $q + 1$ lines, the opposite regulus R^*. Considering this in the associated 4-dimensional $GF(q)$-vector space, we have a set of $q+1$ 2-dimensional $GF(q)$-subspaces that are covered by a set of $q+1$ Baer subplanes that share the zero vector. If we have a regulus R in V_4, considered within a Desarguesian affine plane with spread

$$x = 0, y = xm; m \in K^2 \simeq GF(q^2),$$

we consider the kernel homology group

$$Kern^2 : \left\{\sigma_d : (x, y) \longmapsto (xd, yd); d \in K^2 - \{0\}\right\}.$$

If we select a Baer subplane of R, π_o (line of R^*), then

$$R^* = \pi_o Kern.$$

For example, if we consider the sets

$$A_\delta = \left\{y = xm; m^{q+1} = \delta\right\}, \text{ for } \delta \in F \simeq GF(q),$$

we have a union of $q - 1$ reguli that are disjoint from $x = 0, y = 0$. Any of these may be selected to be derived as above. In this setting the opposite reguli $A^*_\delta = \{y = x^q m; m^{q+1} = \delta\}$.

More generally, if $K^3 \simeq GF(q^3)$, and we consider the Desarguesian affine plane with spread

$$x = 0, y = xm; m \in K^3 \simeq GF(q^3),$$

the sets

$$A_\delta = \left\{y = xm; m^{(q^3-1)/(q-1)} = \delta\right\}, \text{ for } \delta \in F \simeq GF(q),$$

are called the 'André' nets of degree $(q^3-1)/(q-1)$. Here, there are corresponding sets that cover these André nets, defined as follows:

$$A^{q^k}_\delta = \left\{y = x^{q^k} m; m^{(q^3-1)/(q-1)} = \delta\right\}, \text{ for } \delta \in F \simeq GF(q),\ k = 1, 2.$$

The analogous construction process of derivation then may be more generally considered as follows: If $k = 1$ or 2 and π_o is any subspace $y = x^{q^k} m_o$ such that $m_o^{(q^3-1)/(q-1)} = \delta$, then

$$A^{q^k}_\delta = \pi_o Kern^3.$$

1 Definition. A 'hyper-regulus' in a vector-space of dimension $2n$ over $GF(q)$ is a set of $(q^n - 1)/(q - 1)$ n-dimensional $GF(q)$-subspaces that have a replacement set of $(q^n - 1)/(q - 1)$ n-dimensional $GF(q)$-subspaces, so that each subspace of the replacement set 'lies over' each of the original subspaces in a 1-dimensional $GF(q)$-subspace.

Hence, we see that the André nets defined above are hyper-reguli and if the degree is q^n, there are at least $n-1$ replacement sets. These replacement sets are called the 'André' replacements, which are also hyper-reguli. We note that any André replacement hyper-regulus is also an André hyper-regulus using a different Desarguesian affine plane.

Moreover, each André net in a Desarguesian affine plane Σ admits an affine homology group of Σ that is transitive on the components:

$$\left\langle (x,y) \longmapsto (x, ya^{(q-1)}); a \in GF(q^n) \right\rangle$$

acts transitively on A_1. We note the following for André nets of order q^3 and degree $(q^3-1)/(q-1)$.

The following result is in Jha-Johnson [3] where André nets of order q^n, $n > 2$, are considered. Here we restrict attention to $n = 3$.

2 Theorem (Jha-Johnson [3])**.** *Given an André net $\mathcal{A}$ in a Desarguesian plane Σ of order q^3, there is a unique pair of components L and M external to $\mathcal{A}$ such that $\mathcal{A}$ admits affine homologies with axis L and coaxis M, and with axis M and coaxis L.*

The André nets of a Desarguesian plane Σ are in an orbit under $GL(2,q^3)$.

We focus our attention on the following two problems.

3 Problem. Is any hyper-regulus in a 6-dimensional $GF(q)$-vector space always an André hyper-regulus?

4 Problem. Given a set of mutually disjoint hyper-reguli, when is the set 'linear' (see below)?

We note that strictly as an abstract net of degree $(q^3-1)/(q-1)$, and order q^3, an André net may be embedded in a Desarguesian affine plane, and clearly this Desarguesian affine plane is unique. We define the more general term 'André net' in a Desarguesian affine plane of order q^3 as in the following theorem:

5 Theorem (see Jha-Johnson [3])**.** *Let $\mathcal{A}$ be any André net of degree $(q^3-1)/(q-1)$ and order q^3. Then there is a unique Desarguesian affine plane Σ containing $\mathcal{A}$. Now choose any two components L and M of Σ. Then there is a set of $(q-1)$ disjoint André nets of degree $(q^3-1)/(q-1)$ such that the union of these nets with L and M is the spread for Σ. Any such net shall be called an André net. Hence, within any Desarguesian affine plane, there are exactly*

$$(q^3+1)q^3(q-1)/2$$

André nets in Σ. And, there is a corresponding set of

$$(q^3+1)q^3(q-1)$$

replacement sets for André nets of Σ.

There are exactly

$$(|\Gamma L(6,q)| \,/\, |\Gamma L(2,q^3)|)(q^3+1)q^3(q-1)/2$$

André nets in a 6*-dimensional* $GF(q)$*-vector space* V_6.

Certainly each hyper-regulus of order q^n, when $n=2$, produces a derivable net, which is a regulus in some Desarguesian affine plane by the main result of Johnson [6]. Hence, any hyper-regulus is an André hyper-regulus when $n=2$.

When $n=3$, Bruck [1] has shown that every hyper-regulus that sits in a Desarguesian affine plane is an André hyper-regulus. Furthermore, Pomareda [7] proved that each André hyper-regulus of order q^3 can have exactly two possible André replacements. In the process of our analysis, we give alternative proofs of these results of Bruck and Pomareda.

Furthermore, we have given constructions of hyper-reguli that are not André in Jha-Johnson [3], [5] of order q^n, $n>3$, and shown that it is possible to construct sets of mutually disjoint hyper-reguli. The collineation groups of the translation planes obtained are considered in Jha-Johnson [4]. In particular, the authors have obtained a variety of families of translation planes constructed by the replacement of subsets of sets of mutually disjoint hyper-reguli.

6 Theorem (Jha and Johnson [5])**.** *Let* Σ *denote a Desarguesian affine plane of order* q^n*, for* $n>2$*, coordinatized by a field isomorphic to* $GF(q^n)$*. Let* ω *be a primitive element of* $GF(q^n)^*$*, then for* ω^i*, attach an element* $f(i)$ *of the cyclic subgroup of* $GF(q^n)^*$ *of order* $(q^n-1)/(q-1)$*,* $C_{(q^n-1)/(q-1)}$*, and for* $\omega^{-iq^{n-1}}$*, attach an element* $f(i)^{-q^{n-1}}$*. Hence, basically, we have a set of coset representatives* $\{\omega f(1), \omega^2 f(2), \ldots, \omega^{(q-1)} f(q-1)\}$ *for* $C_{(q^n-1)/(q-1)}$*. Let*

$$\mathcal{H}^* = \left\{ \begin{array}{c} y = x^q \omega^{i_j} f(i_j) d^{1-q} + x^{q^{n-1}} (\omega^{i_j})^{-q^{n-1}} f(i_j)^{-q^{n-1}} b d^{1-q^{n-1}}; \\ d \in GF(q^n)^*, \text{for } i_j \in \lambda \subseteq \{1,2,\ldots,q-1\}, \\ \text{assume } b^{(q^n-1)/(q-1)} \notin (\omega^{i_j+i_k})^{(q^n-1)/(q-1)} \end{array} \right\}.$$

Then $\mathcal{H}^*$ *is a set of* $|\lambda|$ *mutually disjoint hyper-reguli.*

Now restricting this to $n=3$, we have

7 Theorem. *Let* Σ *be a Desarguesian affine plane of order* q^3*. Let* ω *be a primitive element of* $GF(q^3)$*, then for* ω^i*, attach an element* $f(i)$ *of the cyclic subgroup of* $GF(q^3)^*$ *of order* $(q^3-1)/(q-1)$*,* $C_{(q^3-1)/(q-1)}$*. Let*

$$\mathcal{H}^* = \left\{ \begin{array}{c} y = x^q \omega^{i_j} f(i_j) d^{1-q} + x^{q^2} (\omega^{i_j})^{-q^2} g(i_j) b d^{1-q^2}; \\ d \in GF(q^3)^*, \text{for } i_j \in \lambda \subseteq \{1,2,\ldots,q-1\}, \\ \text{assume } b^{(q^3-1)/(q-1)} \notin (\omega^{i_j+i_k})^{(q^3-1)/(q-1)} \end{array} \right\}.$$

Then $\mathcal{H}^*$ *is a set of* $|\lambda|$ *mutually disjoint hyper-reguli.*

In particular,

(1) then $\mathcal{H}^*_{i_j}$*, for* i_j *fixed, defines a replacement set for a hyper-regulus* $\mathcal{H}_{i_j}$ *of* Σ.

(2) Furthermore,

$$\left\{y = x^q \omega^{i_j} f(i_j) d^{1-q} + x^{q^2} (\omega^{i_j} f(i_j))^{-q^2} b d^{1-q^2}\right\} \text{ and}$$
$$\left\{y = x^q (\omega^{i_j} f(i_j))^{-1} b^q d^{1-q} + x^{q^2} (\omega^{i_j} f(i_j))^{q^2} d^{1-q^2}\right\}$$

are replacements for each other.

A set of hyper-reguli in $PG(5,q)$ is said to be 'linear' if and only if the set belongs to a standard set of André hyper-reguli invariant under the same affine homology group of order $(q^3-1)/(q-1)$. The question is if there are any linear subsets of cardinality > 1. In fact, we show that there are no such sets. That is, there are no two hyper-reguli in a linear set. If there is a set of $t > 1$ hyper-reguli of the type represented above, we show that every subset of $j \leq t$, $j > 1$, will produce a non-André plane and hence a non-generalized-André plane. In this article, we also consider the isomorphism classes of the constructed translation planes.

Ebert and Culbert [2] have constructed several classes of sets of hyper-reguli in $PG(5,q)$, and from the construction it turns out that no two hyper-reguli fall into a linear set. This construction uses the analysis of cubic functions by Sherk [8] and constructs the hyper-reguli directly from the associated Desarguesian affine plane (more precisely from the associated line at infinity), where our construction finds first the replacement hyper-reguli and then only indirectly can one construct the hyper-reguli in the associated Desarguesian plane. Although there may be some overlap between these two classes, this is completely undetermined.

Using algebraic methods concerning linear sets, we prove the following general result:

8 Theorem. *Let* Σ *be a Desarguesian affine plane of order* q^3*. If we have a set of mutually disjoint hyper-reguli of the type* $\{y = x^q \alpha d^{1-q} + x^{q^2} \alpha^{-q^2} b d^{1-q^2}\}$*, where* α *is in* $\lambda \subseteq GF(q^3)^*$*, then no two of these hyper-reguli are in a linear set.*

9 Corollary. *For any subset of at least two mutually disjoint hyper-reguli of the type stated in the previous theorem, for each hyper-regulus, choose one of the two possible replacements. Then a translation plane of order* q^3 *and kernel* $GF(q)$ *is constructed that admits a cyclic collineation group of order* q^3-1 *but is not an André or generalized André plane.*

2 Examples

In this section, we show how to construct a variety of examples. We consider other examples in coming sections.

2.1 Group type examples.

10 Example. For example, take q odd and $\lambda = \{2, 4, \dots, (q-1)/2\}$ and $i_j = 2j$. Then $(\omega^{2j+2k})^{(q^3-1)/(q-1)} \in C_{(q-1)/2}$. Choose b so that $b^{(q^3-1)/(q-1)} \notin C_{(q-1)/2}$. Then, we obtain a set of $(q-1)/2$ mutually disjoint hyper-reguli.

11 Example. Similarly, if we take $\lambda = \{2, 4, \dots, (q-1)/2\} + 1$ and choose b so that $b^{(q^3-1)/(q-1)} \notin \omega C_{(q-1)/2}$, we obtain a set of $(q-1)/2$ mutually disjoint hyper-reguli.

12 Example. For q odd, take $\lambda = \{1, 2, 3, 4, \dots, (q-3)/2\}$. Then note that since for i, j in λ we have $i+j \leq q-3$, we may take $b^{(q^3-1)/(q-1)} = 1$ to construct a set of $(q-3)/2$ mutually disjoint hyper-reguli.

13 Example. Actually, take any set of $(q-3)/2$ elements of $\{1, 2, \dots, q-1\}$ such that $i+j$ is not congruent to 0 mod $q-1$. Then for $b^{(q^3-1)/(q-1)} = 1$, we may construct a set of $(q-3)/2$ mutually disjoint hyper-reguli. More generally, choose any i_0 from $\{1, 2, 3, \dots, q-1\}$. Then choose any set of $(q-3)/2$ elements so that $i+j$ is not congruent to i_0 mod $(q-1)$. Letting $b^{(q^3-1)/(q-1)} = \omega^{i_0(q^3-1)/(q-1)}$ produces a set of $(q-3)/2$ mutually disjoint hyper-reguli.

For example take $q = 9$. Taking $\lambda = \{2, 4, 6, 8\}$ produces sets of four disjoint hyper-reguli. Taking $\lambda = \{1, 2, 3\}$ produces a set with three disjoint hyper-reguli.

14 Example. Let q be even, $\lambda = \{1, 2, 3, \dots, q/2-1\}$, and $b^{(q^3-1)/(q-1)} = 1$. Then for i, j in $\{1, 2, \dots, q/2-1\}$, we have $i+j \leq q-2$, so this will produce a set of $q/2-1$ mutually disjoint hyper-reguli.

15 Example. Let k divide $q-1$. Take $\lambda = \{k, 2k, \dots, (q-1)/k\}$. Then $i+j = kz$ and $\omega^{kz(q^3-1)/(q-1)} \in C_{(q-1)/k}$. Take b so that $b^{(q^3-1)/(q-1)} \notin C_{(q-1)/k}$ to produce a set of $(q-1)/k$ mutually disjoint hyper-reguli.

16 Example. In the previous example, let i_0 be any element of $\{1, 2, \dots, q-1\} - \{3, 6, 9, \dots, (q-1)/k\}$ and consider $\lambda = \{i_0 + ktk; t = 1, 2, \dots, (q-1)/k\} \cup \{k, 2k, \dots, (q-1)/k\}$. Then $(\omega^{i+j})^{(q^3-1)/(q-1)} \in C_{(q-1)/k} \cup \omega^{i_0} C_{(q-1)/k} \cup \omega^{2i_0} C_{(q-1)/k}$. So, if $k > 3$, we may add on to obtain $2(q-1)/k$ mutually disjoint hyper-reguli.

17 Remark. Note when we obtain a set of t mutually disjoint hyper-reguli, we obtain $(t)^{(q^3-1)/(q-1)}$ different sets of t mutually disjoint hyper-reguli.

So, if q is odd, we may obtain $((q-1)/2)^{(q^3-1)/(q-1)}$ sets of $(q-1)/2$ mutually disjoint hyper-reguli and when q is even, we may obtain $(q/2-1)^{(q^3-1)/(q-1)}$ sets of $q/2-1$ hyper-reguli. Furthermore, all of these sets correspond to the same choice for b. Other choices for b that have the same avoidance condition also work. For example, if b works, so does b^* so that $b^{*(q^3-1)/(q-1)} = b^{(q^3-1)/(q-1)}$.

In the group type sets, if $t=(q-1)/2$, for example, then there are actually $(q^3-1)/2$ choices for b. Hence, there are then at least $((q-1)/2)^{(q^3-1)/(q-1)}(q^3-1)/2$ possible sets of $(q-1)/2$ mutually disjoint hyper-reguli.

Moreover, for each coset representative defining a hyper-regulus, we may replace the hyper-regulus with the replacement mentioned above and still produce a set of mutually disjoint hyper-reguli.

3 Hyper-reguli of degree $(q^3-1)/(q-1)$

Consider any subspace of V_{2n} over $GF(q)$, of dimension n over $GF(q)$, that is disjoint from a given subspace, which we represent in the form $x=0$. Represent V_{2n} as $GF(q^n) \oplus GF(q^n)$. Then, we may represent the subspace in the form

$$y = \sum_{i=0}^{n-1} x^{q^i} a_i,$$

for a set of constants a_i, $i=0,1,2,\ldots,n-1$, of $GF(q^n)$, where x is in $GF(q^n)$, as an indeterminate.

Now assume that $n=3$. We first ask the conditions on $y = x^q a_1 + x^{q^2} a_2$, for $a_1 a_2$ so that this is a subspace of a putative hyper-regulus replacement for a hyper-regulus of Σ, a Desarguesian affine plane of order q^3.

We consider the image set of $y=x^q$, under $GL(2,q^3)$, where $GL(2,q^3)$ is written as

$$\left\langle \begin{bmatrix} a & f \\ c & d \end{bmatrix}; ad-cf \neq 0, a,f,c,d \in GF(q^3) \right\rangle.$$

Since $(x,x^q)\begin{bmatrix} a & f \\ c & d \end{bmatrix} = (xa+x^q c, xf+x^q d)$, we see that this is $y = xe_0 + x^q e_1 + x^{q^2} e_2$, if and only if $xa + x^q c \neq 0$ for all x. Note that this condition implies that $-c/a$ is not a $(q-1)^{\mathrm{st}}$ power. Then, we obtain:

$$(xa+x^q c)e_0 + (xa+x^q c)^q e_1 + (xa+x^q c)^{q^2} e_2 = xf + x^q d, \ \forall x \in GF(q^3).$$

This produces the following matrix equation:

$$\begin{bmatrix} a & 0 & c^{q^2} \\ c & a^q & 0 \\ 0 & c^q & a^{q^2} \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} f \\ d \\ 0 \end{bmatrix}.$$

We see that the coefficient matrix has determinant $a^{1+q+q^2}+c^{1+q+q^2}\neq 0$, since otherwise $-(c/a)^{(q^3-1)/(q-1)}=1$, a contradiction to our assumptions.

Hence, we obtain:

$$\begin{aligned} e_0 &= \det\begin{bmatrix} f & 0 & c^{q^2} \\ d & a^q & 0 \\ 0 & c^q & a^{q^2}\end{bmatrix} /(a^{1+q+q^2}+c^{1+q+q^2}) \\ &= (fa^{q+q^2}+dc^{q+q^2})/(a^{1+q+q^2}+c^{1+q+q^2}), \end{aligned}$$

$$\begin{aligned} e_1 &= \det\begin{bmatrix} a & f & c^{q^2} \\ c & d & 0 \\ 0 & 0 & a^{q^2}\end{bmatrix} /(a^{1+q+q^2}+c^{1+q+q^2}) \\ &= (da^{1+q^2}-fca^{q^2})/(a^{1+q+q^2}+c^{1+q+q^2}), \end{aligned}$$

$$\begin{aligned} e_2 &= \det\begin{bmatrix} a & 0 & f \\ c & a^q & d \\ 0 & c^q & 0\end{bmatrix} /(a^{1+q+q^2}-c^{1+q+q^2}) \\ &= (fc^{1+q}-adc^q)/(a^{1+q+q^2}+c^{1+q+q^2}). \end{aligned}$$

Now consider

$$\begin{aligned} &(fc^{1+q}-adc^q)/(a^{1+q+q^2}+c^{1+q+q^2})/((da^{1+q^2}-fca^{q^2})/(a^{1+q+q^2}+c^{1+q+q^2})) \\ &\quad = (fc^{1+q}-adc^q)/((da^{1+q^2}-fca^{q^2})) = -\frac{c^q}{a^{q^2}}(ad-fc)/(da-fc) \\ &\quad = -\frac{c^q}{a^{q^2}} = -\left(\frac{c}{a^q}\right)^q. \end{aligned}$$

Note that $-\left(\frac{c}{a^q}\right)^q$ is a non-$(q-1)^{\text{st}}$ power, and by choosing various a and c (e.g. $a=1$), we may obtain any non-$(q-1)^{\text{st}}$ power. Moreover, we see that it is possible to vary a,b,c,f to obtain, when $ac\neq 0$, all elements $y=xe_0+x^qe_1+x^{q^2}e_1\rho$, where ρ is not a $(q-1)^{\text{st}}$ power. The total number of lying-over subspaces of standard André replacements is

$$((q^3+1)q^3/2)(q-1)2(q^3-1)/(q-1) = q^3(q^3+1)(q^3-1) = q^3(q^6-1).$$

Furthermore, since the stabilizer of $y=x^q$ in $GL(2,q^3)$ clearly has order (q^3-1), it follows that each lying-over subspace of standard André type is an image of $y=x^q$ under an element of $GL(2,q^3)$. Note that since we have shown that every lying-over subspace of the type $y=xa+x^qb+x^{q^2}c$ is an image of $y=x^q$ under some collineation of $GL(2,q^3)$, we have shown that every hyper-regulus is an André hyper-regulus and every replacement of an André hyper-regulus is one of two standard types. The first of these statements was proved by Bruck [1] and the second by Pomareda [7], both by different methods.

18 Theorem. *Let Σ be an affine Desarguesian plane of order q^3 and let $y = xa + x^q b + x^{q^2} c$.*

(1) Then this subspace represents a subspace defining an André hyper-regulus if and only if either $bc = 0$ or $bc \neq 0$ and (b/c) is a non-$(q-1)^{st}$ power.

(2) Every subspace defining a hyper-regulus of Σ defines an André hyper-regulus and is in an orbit under $GL(2,q^3)$ (see Bruck [1]). Furthermore, there are exactly two replacement hyper-reguli for each (see [7]).

19 Corollary. *Every hyper-regulus and every replacement belongs to a unique Desarguesian affine plane.*

Proof. Just note that every replacement is Desarguesian and by cardinality, two Desarguesian planes of order q^3 sharing at least $(q^3-1)/(q-1)$ components must be identical. QED

4 Additional constructions

We list a few additional construction procedures of Jha and Johnson [5] that may be obtained from the general construction procedure.

20 Theorem.

$$\left\{ x^q \alpha\delta + x^{q^2} \beta b; \left(\frac{\alpha\delta}{\beta}\right)^3 \neq b^{(q^3-1)/(q-1)} \right\},$$

$$\left\{ x^q \beta\delta + x^{q^2} \alpha b; \left(\frac{\beta\delta}{\alpha}\right)^3 \neq b^{(q^3-1)/(q-1)} \right\},$$

$$\left\{ x^q \beta + x^{q^2} \alpha\delta b; \left(\frac{\beta}{\alpha\delta}\right)^3 \neq b^{(q^3-1)/(q-1)} \right\},$$

$$\left\{ x^q \alpha + x^{q^2} \beta\delta b; \left(\frac{\alpha}{\beta\delta}\right)^3 \neq b^{(q^3-1)/(q-1)} \right\}$$

define mutually disjoint hyper-reguli, for all δ in $GF(q)$, such that $b^{(q^3-1)/(q-1)} \neq \delta^{\pm 3}$, and $\beta \neq \alpha\delta$, $\alpha \neq \beta\delta$, $\delta \neq 1$.

We ask if the plane obtained could be an André plane (or generalized André plane). If so, there is an affine homology group fixing each André replacement set which is induced from a homology group of the associated Desarguesian affine plane. So, if this is a set of André nets with carrying lines L and M of Σ, assume first that $x = 0$ and $y = xm_0$ are the axis and coaxis. Then mapping by $(x, y) \longmapsto (x, -xm_0 + y)$ moves the axis and coaxis to $x = 0, y = 0$ and maps

$$\left\{x^q\alpha\delta + x^{q^2}\beta b; \left(\frac{\alpha\delta}{\beta}\right)^n \neq b^{(q^n-1)/(q-1)}\right\},$$

$$\left\{x^q\beta\delta + x^{q^2}\alpha b; \left(\frac{\beta\delta}{\alpha}\right)^n \neq b^{(q^n-1)/(q-1)}\right\},$$

$$\left\{x^q\beta + x^{q^2}\alpha\delta b; \left(\frac{\beta}{\alpha\delta}\right)^n \neq b^{(q^n-1)/(q-1)}\right\},$$

$$\left\{x^q\alpha + x^{q^2}\beta\delta b; \left(\frac{\alpha}{\beta\delta}\right)^n \neq b^{(q^n-1)/(q-1)}\right\}$$

to

$$\left\{-xm_0 + x^q\alpha\delta + x^{q^2}\beta b; \left(\frac{\alpha\delta}{\beta}\right)^n \neq b^{(q^n-1)/(q-1)}\right\},$$

$$\left\{-xm_0 + x^q\beta\delta + x^{q^2}\alpha b; \left(\frac{\beta\delta}{\alpha}\right)^n \neq b^{(q^n-1)/(q-1)}\right\},$$

$$\left\{-xm_0 + x^q\beta + x^{q^2}\alpha\delta b; \left(\frac{\beta}{\alpha\delta}\right)^n \neq b^{(q^n-1)/(q-1)}\right\},$$

$$\left\{-xm_0 + x^q\alpha + x^{q^2}\beta\delta b; \left(\frac{\alpha}{\beta\delta}\right)^n \neq b^{(q^n-1)/(q-1)}\right\}.$$

Then, $(x, y) \longmapsto (x, yt^{q-1})$ is a collineation group of the spread, clearly a contradiction. Hence, the axis and coaxis are $y = xm_0$ and $y = xm_1$. The conditions for an affine homology $\begin{bmatrix} a & f \\ c & d \end{bmatrix}$ with axis $y = xm_0$ and $y = xm_1$ are as follows:

$$\begin{bmatrix} a & f \\ c & d \end{bmatrix};\ a = 1 - m_0c,\ f = m_0(d-1),$$
$$(1 + (m_1 - m_0)c)m_1 = (m_1 + m_0)d - m_0.$$

Consider the image of $y = x^q$ under $\begin{bmatrix} a & f \\ c & d \end{bmatrix}$, such that $-(c/a)$ is not a $(q-1)^{\text{st}}$ power as

$$\begin{aligned} y &= x(fa^{q+q^2} + dc^{q+q^2})/(a^{1+q+q^2} + c^{1+q+q^2}) \\ &\quad + x^q(a^{q^2}(da - fc))/(a^{1+q+q^2} + c^{1+q+q^2}) \\ &\quad + x^{q^2}(-c^q)(da - fc)/(a^{1+q+q^2} + c^{1+q+q^2}). \end{aligned}$$

Just to get an example, take q odd and $a = c = 1$, to get $y = x(f + d)/2 + x^q(d - f)/2 + x^{q^2}(-(d - f))/2$. Take $f = -d = -1$ to get

$$y = x^q - x^{q^2}.$$

Thus, $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ will map $y = x^q$ onto $y = x^q - x^{q^2}$ and will map $y = x^q m$ onto

$$\begin{aligned} y &= x(m^{1+q+q^2} - 1)/(m^{1+q+q^2} + 1) + x^q(2m)/(m^{1+q+q^2} + 1) \\ &\quad + x^{q^2}(-2m^{1+q})/(m^{1+q+q^2} + 1). \end{aligned}$$

So, in order to get a term with x-coefficient 0, we must have $m^{1+q+q^2} = 1$. However, then the image is in $\{y = x^q d^{1-q} - x^{q^2} d^{1-q^2}\}$ so can't map to one of our other André hyper-reguli. The same collineation will map $y = x^q m$ onto

$$\begin{aligned} y &= x(m^{1+q+q^2} - 1)/(m^{1+q+q^2} + 1) + x^q(-2m^{1+q^2})/(m^{1+q+q^2} + 1) \\ &\quad + x^{q^2}(2m/(m^{1+q+q^2} + 1)). \end{aligned}$$

Hence, again in order to obtain a 0 x-coefficient, we must map to the same hyper-regulus overlying $y = x^q - x^{q^2}$, a contradiction. Now choose $\left\{x^q\alpha\delta + x^{q^2}\beta b; \left(\frac{\alpha\delta}{\beta}\right)^n \neq b^{(q^n-1)/(q-1)}\right\}$, as $\{x^q - x^{q^2}\}$ such that $\alpha\delta = 1$, $\beta b = -1$. Hence, we require that $\left(\frac{1}{b}\right)^3 \neq b^{(q^n-1)/(q-1)} = b^3$, if and only if b^6 is not 1. $\left\{x^q\beta\delta + x^{q^2}\alpha b; \left(\frac{\beta\delta}{\alpha}\right)^n \neq b^{(q^n-1)/(q-1)}\right\}$ is then $\{x^q(-1/b\alpha) + x^q\alpha b\}$ if and only if $b^3 \neq -\alpha^{-3}$. $\left\{x^q\beta + x^{q^2}\alpha\delta b; \left(\frac{\beta}{\alpha\delta}\right)^n \neq b^{(q^n-1)/(q-1)}\right\}$ is then $\{x^q(-1/b) + x^{q^2}b\}$ if and only if $(-1/b)^3 \neq b^3$, so $b^6 \neq -1$. $\left\{x^q\alpha + x^{q^2}\beta\delta b; \left(\frac{\alpha}{\beta\delta}\right)^n \neq b^{(q^n-1)/(q-1)}\right\}$ is then $\{x^q\alpha + x^{q^2}(-1/\alpha b)\}$ if and only $-\alpha^3 \neq b^3$.

21 Lemma. *If α, β in $GF(q) - \{0\}$ and $\alpha/\beta = d^{1-q} = 1/d^{1-q^{n-1}}$, then $\left(\frac{\alpha}{\beta}\right)^{(n,q-1)} = 1$. Hence, if $\left(\frac{\alpha}{\beta}\right)^{(n,q-1)} \neq 1$ then*

$$\left(\frac{\alpha - \beta d^{1-q}}{\alpha d^{1-q^{n-1}} - \beta}\right)^{(q^n-1)/(q-1)} = 1.$$

PROOF. $\alpha/\beta = d^{1-q} = 1/d^{1-q^{n-1}}$ implies that $d^{(1-q)(1-q)} = 1$. $((1-q)^2, q^n - 1) = (q-1)(q-1, (q^n-1)/(q-1)) = (q-1)(q-1, n)$. We consider

$$\left(\frac{\alpha - \beta d^{1-q}}{\alpha d^{1-q^{n-1}} - \beta}\right)^{(q^n-1)/(q-1)}.$$

Note that

$$\begin{aligned}
&\left(\alpha d^{1-q^{n-1}} - \beta\right)^{1+q+q^2+\cdots+q^{n-1}} \\
&\quad = \left(d^{1-q^{n-1}}\right)^{(q^n-1)/(q-1)} \left(\alpha - \beta d^{q^{n-1}-1}\right)^{1+q+q^2+\cdots+q^{n-1}} \\
&\quad = \prod_{i=0}^{n-1} \left(\alpha - \beta d^{\,(q^{n-1}-1)q^i}\right) \\
&\quad = \prod_{i=0}^{n-1} \left(\alpha - \beta d^{\,(q^{i-1}-q^i)}\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\left(\alpha - \beta d^{1-q}\right)^{1+q+q^2+\cdots+q^{n-1}} \\
&\quad = \prod_{j=0}^{n-1} \left(\alpha - \beta d^{\,(1-q)q^j}\right) \\
&\quad = \prod_{j=0}^{n-1} \left(\alpha - \beta d^{\,q^j - q^{j+1}}\right) \\
&\quad = \prod_{i=0}^{n-1} \left(\alpha - \beta d^{\,q^{i-1} - q^i}\right).
\end{aligned}$$

This completes the proof. QED

5 Multiplicative sets of hyper-reguli

In this section, we generalize the sets of the previous section.

22 Theorem. *Suppose that* $\{\alpha_i; \alpha_i \in GF(q) - \{0\},\ i = 1, 2, \ldots, t\}$ *is a set of elements of* $GF(q)$ *such that*

$$\left(\frac{\alpha_i}{\alpha_j}\right)^{(n,q-1)} \neq 1,\ \alpha_j \neq \alpha_i,$$

and there exists an element b in $GF(q^n)-\{0\}$, where the following conditions hold:

$$b^{(q^n-1)/(q-1)} \neq \left(\frac{\alpha_i}{\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_t}\right)^n,\ b^{(q^n-1)/(q-1)} \neq \left(\frac{1}{\alpha_1\cdots\widehat{\alpha}_i\cdots\widehat{\alpha}_j\cdots\alpha_t}\right)^n,$$

for $\alpha_i \neq \alpha_j$, where $\widehat{\alpha}_i$ indicates that the element α_i is not in the product.

Then

$$\mathcal{R}_t = \Big\{y = x^{q^k}\alpha_i d^{1-q} + x^{q^{n-k}}\alpha_1\alpha_2\cdots\widehat{\alpha}_i\cdots\alpha_t b d^{1-q^{n-1}};\ i=1,2,\ldots,t;\ d\in GF(q^n)-\{0\}\Big\}$$

is a partial spread of degree $t(q^n-1)/(q-1)$ that lies over a set of $t(q^n-1)/(q-1)$ components in a Desarguesian affine plane Σ which is defined by a set of t hyper-reguli. If $\mathcal{M}$ denotes the set of components of $\Sigma-\mathcal{R}_t$, then

$$\mathcal{R}_t \cup \mathcal{M}$$

is a spread with kernel $GF(q)$.

23 Definition. Any set of mutually disjoint hyper-reguli obtained from a set λ as above shall be called a 'multiplicative' set of hyper-reguli. More precisely, we call this a 'multiplicative set of degree 1'.

24 Corollary. $\{x^q\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}d^{1-q}+x^{q^{n-1}}\alpha_1\cdots\widehat{\alpha}_{i_1}\cdots\widehat{\alpha}_{i_k}\cdots\alpha_t b d^{1-q^{n-1}}\}$, *for k fixed, is a partial spread and consists of t hyper-reguli if*

$$\left(\frac{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}}{\alpha_{j_1}\cdots\alpha_{j_k}}\right)^{(n,q-1)} \neq 1,$$

$$\left(\frac{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}}{\alpha_1\cdots\widehat{\alpha}_{i_1}\cdots\widehat{\alpha}_{i_k}\cdots\alpha_t}\right)^n \neq b^{(q^n-1)/(q-1)},$$

and

$$\left(\frac{1}{\alpha_1\cdots\widehat{\alpha}_{i_1}\cdots\widehat{\alpha}_{i_k}\cdot\widehat{\alpha}_{j_1}\cdots\widehat{\alpha}_{j_k}\cdots\alpha_t}\right)^n \neq b^{(q^n-1)/(q-1)},$$

where the term indicated is removed exactly once.

25 Definition. The sets of hyper-reguli obtained as above are called 'multiplicative sets of degree k'.

6 When a subset is linear

Suppose we have two hyper-reguli $\mathcal{H}^*_{\alpha,\rho,b} = \{y = x^q\alpha d^{1-q} + x^{q^2}\rho b d^{1-q^2}\}$, where $\alpha, \rho \in GF(q^3)^*$, and $\mathcal{H}^*_{\alpha^*,\rho^*,b}$.

Now consider the mapping $\begin{bmatrix} 1 & b^{q+1} \\ b^q & 1 \end{bmatrix}$, and note that it maps $\mathcal{H}^*_{1,1,b}$ onto $y = x^q$, since

$$\left(x, x^q\alpha d^{1-q} + x^{q^2}\rho b d^{1-q^2}\right)\begin{bmatrix} 1 & b^{q+1} \\ b^{q^2} & 1 \end{bmatrix}$$
$$= \left(x + \left(x^q\alpha d^{1-q} + x^{q^2}\rho b d^{1-q^2}\right)b^{q^2}, x b^{q+1} + x^q\alpha d^{1-q} + x^{q^2}\rho b d^{1-q^2}\right).$$

If $\alpha = \rho$ and $d = 1$ then note that

$$\left(x + \left(x^q + x^{q^2} b\right) b^{q^2}\right)^q = x b^{q+1} + x^q + x^{q^2} b$$

Then the question is where this mapping takes $\mathcal{H}^*_{\alpha^*,\rho^*,b}$.

A generator $y = \left(x^q\alpha^* + x^{q^2}\rho^* b\right)$ for this hyper-regulus then maps to $\{x + \left(x^q\alpha^* + x^{q^2}\rho^* b\right)b^{q^2}, x b^{q+1} + \left(x^q\alpha^* + x^{q^2}\rho^* b\right)\}$. Assuming these two distinct hyper-reguli $\mathcal{H}^*_{\alpha^*,\rho^*,b}$ and $\mathcal{H}^*_{1,1,b}$ form a linear set, this is true if and only if the image is $y = x^{q^k} d$, where k is either 1 or 2 and $d \in GF(q^3)^*$, since there is a unique linear set in the Desarguesian affine plane Σ lying across $y = x^q$. If $k = 1$, this implies the following:

$$d = \alpha^*,\ (\alpha^*)^q\, bd = \rho^* b,\ (\rho^*)^q\, b^{q+1} d = b^{q+1}.$$

This in turn implies that

$$(\alpha^*)^{1+q+q^2} = 1,\ \text{and}\ \rho^* = (\alpha^*)^{1+q}.$$

Since α^* is a $(q-1)^{\text{st}}$ power, let $d^{1-q} = \alpha^*$, then $\rho^* = d^{1-q^2}$. However, this means that we have exactly the same hyper-regulus.

We recall that

$$\left\{y = x^q \omega^{i_j} f(i_j) d^{1-q} + x^{q^2}(\omega^{i_j} f(i_j))^{-q^2} b d^{1-q^2}\right\}\ \text{and}$$
$$\left\{y = x^q (\omega^{i_j} f(i_j))^{-1} b^q d^{1-q} + x^{q^2}(\omega^{i_j} f(i_j))^{q^2} d^{1-q^2}\right\}$$

are replacements for one another. Then

$$\left\{x^q\alpha d^{1-q} + x^{q^2}\alpha^{-q^2} b d^{1-q^2}\right\}\ \text{and}$$
$$\left\{x^q\alpha^{-1} b^q d^{1-q} + x^{q^2}\alpha^{q^2} d^{1-q^2}\right\}$$

are replacements for each other.

Thus, if $k=2$, we obtain:

$$d=\rho^{*}b,\ (\alpha^{*})^{q^{2}}b^{q}d=b^{q+1},\ (\rho^{*})^{q^{2}}b^{q^{2}+q}d=\alpha^{*}.$$

This implies that

$$\rho^{*}=(\alpha^{*})^{-q^{2}}\ \text{ and } b^{1+q+q^{2}}=(\alpha^{*})^{1+q+q^{2}}.$$

If we now work out the condition for the union of $\mathcal{H}^{*}_{1,1,b}$ and $\mathcal{H}^{*}_{\alpha^{*},\rho^{*},b}$ to be a partial spread, we require that $b^{1+q+q^{2}}\neq(\alpha^{*})^{1+q+q^{2}}$. To see this, we note that $y=x^{q}\alpha^{*}+x^{q^{2}}(\alpha^{*})^{-q^{2}}b$ and $y=x^{q}+x^{q^{2}}b$ do not have a non-trivial solution if and only if

$$\left((\alpha^{*}-1)/\left(1-(\alpha^{*})^{-q^{2}}\right)\right)^{(q^{3}-1)/(q-1)}\neq b^{(q^{3}-1)/(q-1)}.$$

However, $\left(1-(\alpha^{*})^{-q^{2}}\right)=(\alpha^{*})^{-q^{2}}\left((\alpha^{*})^{q^{2}}-1\right)$ and $(\alpha^{*}-1)=\left((\alpha^{*})^{q^{2}}-1\right)^{q}$. But then

$$\left((\alpha^{*}-1)/\left(1-(\alpha^{*})^{-q^{2}}\right)\right)^{(q^{3}-1)/(q-1)}=(\alpha^{*})^{q^{2}(q^{3}-1)/(q-1)}=(\alpha^{*})^{(1+q+q^{2})}.$$

Hence, if $k=2$, we have a replacement for the same hyper-regulus.

We now obtain our main theorem.

26 Theorem. *Let Σ be a Desarguesian affine plane of order q^{3}. If we have a set of mutually disjoint hyper-reguli of the type $\{y=x^{q}\alpha d^{1-q}+x^{q^{2}}\alpha^{-q^{2}}bd^{1-q^{2}}\}$, where α is in $\lambda\subseteq GF(q^{3})^{*}$, then no two of these hyper-reguli are in a linear set.*

Proof. From the discussion above, it is evident that no two hyper-reguli of the form $\mathcal{H}^{*}_{1,1,b}$ and $\mathcal{H}^{*}_{\alpha^{*},\rho^{*},b}$ can form a linear set. Similarly, it is easily checked that $y=x^{q}\alpha+x^{q^{2}}\alpha^{-q^{2}}b$ maps to $y=x^{q}$ under the mapping

$$\begin{bmatrix}\alpha^{1+q+q^{2}} & \alpha^{-1}b^{q+1}\\ b^{q^{2}} & 1\end{bmatrix}.$$

The question then is where this mapping will take $y=x^{q}\beta+x^{q^{2}}\beta^{-q^{2}}b$. To be linear, again the image must be $y=x^{q^{k}}d$ for $k=1$ or 2.

The image is

$$\left(x\alpha^{1+q+q^{2}}+\left(x^{q}\beta+x^{q^{2}}\beta^{-q^{2}}b\right)b^{q^{2}},x\alpha^{-1}b^{q+1}+x^{q}\beta+x^{q^{2}}\beta^{-q^{2}}b\right).$$

This is $y=x^{q}d$ if and only if

$$\left(x\alpha^{1+q+q^{2}}+\left(x^{q}\beta+x^{q^{2}}\beta^{-q^{2}}b\right)b^{q^{2}}\right)^{q}d=x\alpha^{-1}b^{q+1}+x^{q}\beta+x^{q^{2}}\beta^{-q^{2}}b.$$

The requirements then are

$$\alpha^{1+q+q^2}d=\beta,\ \beta^q bd=\beta^{-q^2}b,\ \beta^{-1}b^{q+1}d=\alpha^{-1}b^{q+1}.$$

This implies

$$\alpha^{1+q+q^2}=\beta^{1+q+q^2}=\alpha.$$

This implies that α is in $GF(q)$ and that $\alpha^3=\alpha$, or $\alpha^2=1$, so $\alpha=\pm 1$. If $\alpha=1$, we may apply the previous argument to arrive at a contradiction. If $\alpha=-1$, map $y=-x^q-x^{q^2}b$ and $y=x^q\beta+x^{q^2}\beta^{-q^2}b$ to $y=x^q+x^{q^2}b$ and $y=x^q(-\beta)+x^{q^2}(-\beta)^{-q^2}b$ by the mapping $(x,y)\longmapsto(x,-y)$. Since we retain the same form as previously considered, we have a contradiction if $k=1$.

Now if $k=2$,

$$\left(x\alpha^{1+q+q^2}+\left(x^q\beta+x^{q^2}\beta^{-q^2}b\right)b^{q^2}\right)^{q^2}d=x\alpha^{-1}b^{q+1}+x^q\beta+x^{q^2}\beta^{-q^2}b.$$

The conditions now are

$$\alpha^{1+q+q^2}d=\beta^{-q^2}b,\ \beta^{q^2}b^q d=\alpha^{-1}b^{q+1}d,\ \text{and}\ \beta^{-q}b^{q^2+q}d=\beta.$$

This implies

$$\alpha^{q+q^2}=1,\ \text{so that}\ \alpha^{q+1}=1.$$

However, $(q+1,q^3-1)=(2,q-1)$. If q is even then $\alpha=1$ and if q is odd then $\alpha^2=1$ and $\alpha=\pm 1$. In either case, by considering the mapping $(x,y)\to(x,-y)$, we have a contradiction as before. QED

Hence, we have proved the following:

27 Theorem. *Let $\mathcal{H}$ be a set of hyper-reguli of the form $\mathcal{H}^*_{\alpha,\alpha^{-q^2},b}$ where $\alpha\in\lambda\subseteq GF(q^3)^*$. Choose any subset of hyper-reguli of cardinality at least two. Then the corresponding translation plane is not generalized André or André.*

7 The collineation group

We have shown that no two hyper-reguli of our sets of mutually disjoint hyper-reguli are linear. We now consider a different type of question: Can two of the hyper-reguli fall into a Desarguesian spread? If one of these hyper-reguli is mapped to $y=x^q$ then the other hyper-regulus cannot be of the form $y=x^{q^k}n$, for $k=1,2$ and n in $GF(q^3)^*$. But the unique Desarguesian spread containing $y=x^q d^{1-q}$, $d\in GF(q^3)^*$, has spread components

$$x=0,y=0,y=x^q m;m\in GF(q^3).$$

Hence, we have then

28 Proposition. *No two of the hyper-reguli of our sets of mutually disjoint hyper-reguli can be contained in the same Desarguesian spread.*

In the authors' work [4] on the groups of these sorts of sets of hyper-reguli of order q^n, $n > 2$, it is shown that if the number of hyper-reguli in the set is $\leq (q-1)/2$ or $q/2-1$, for q odd or even, respectively, then either the set has cardinality $(q-1)/2$ or the full collineation group is the group inherited from the associated Desarguesian affine plane Σ used in the construction of the set. Furthermore, when the number of hyper-reguli is $(q-1)/2$, it is shown that either the full collineation group is the group inherited from the Desarguesian plane Σ or the set of $(q-1)/2$ hyper-reguli is Desarguesian. Hence, we may apply this result and the above note to conclude that the full collineation group is in the inherited group when $n = 3$, whenever the number of hyper-reguli is $\leq (q-1)/2$ or $q/2-1$, for q odd or even, respectively.

29 Theorem. *If a translation plane π of order q^3 and kernel $GF(q)$ is obtained by the replacement of mutually disjoint hyper-reguli with replacements*

$$\left\{y = x^q \alpha d^{1-q} + x^{q^2} \alpha^{-q^2} b d^{1-q^2}; d \in GF(q^3)^*\right\} \text{ or}$$

$$\left\{y = x^q \alpha^{-1} b^q d^{1-q} + x^{q^2} \alpha^{q^2} d^{1-q^2}; d \in GF(q^3)^*\right\}$$

for each $\alpha \in \lambda \subseteq GF(q^3)^*$, where $|\lambda| \leq (q-1)/2$, or $q/2-1$, respectively, as q is odd or even, then the full collineation group is the group inherited from the associated Desarguesian affine plane.

8 Final comments and a few details on isomorphisms

We have provided a great variety of new and different translation planes by the replacement of subsets of at least two mutually disjoint hyper-reguli of a specific type. There must be more types of hyper-reguli, that is, other than linear André sets and the types that we have determined, so the discussion is far from definitive. Moreover, because of the varied nature of our constructions, we have not made much of an attempt to establish the isomorphism classes. For the more general case, of order q^n, for $n > 3$, the situation is much more manageable since the hyper-reguli that we are considering then are never André. So, it might be advisable to provide here a few details on how one might go about establishing both the full collineation group of a particular translation plane and whether two given translation planes are isomorphic.

In this section, we indicate very general methods to show how two translation planes constructed by the methods of this article could be determined isomorphic.

Now assume that we have two translation planes of order q^3 and kernel $GF(q)$ obtained from a Desarguesian plane by the replacements of the type under consideration, one using a set λ with constant b, and another with a set $\widehat{\lambda}$ and constant $\widehat{b}$. The same sort of argument that establishes that the collineation group is inherited applies to isomorphisms, provided that λ and $\widehat{\lambda}$ both have cardinality $\leq (q-1)/2$.

30 Lemma. *The stabilizer of $y = x^q$ in $GL(2, q^3)$ is*

$$\left\langle \begin{bmatrix} a & 0 \\ 0 & a^q \end{bmatrix}; a \in GF(q^3)^* \right\rangle.$$

Proof. Assume that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ fixes $y = x^q$. Then $(xa + cx^q)^q = xb + x^q d$. This is true if and only if $a^q = d$ and $c = b = 0$. QED

31 Lemma. *The stabilizer of $y = x^q\alpha + x^{q^2}\alpha^{-q^2}b$ in $GL(2, q^3)$ is*

$$\left\langle \begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^q \end{bmatrix} \begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix}^{-1}; a \in GF(q^3)^* \right\rangle.$$

Proof. Just note that $\begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix}$ maps $y = x^q\alpha + x^{q^2}\alpha^{-q^2}b$ onto $y = x^q$, then apply the previous lemma. QED

32 Lemma. *An element σ of $GL(2, q^3)$ maps $x^q\alpha + x^{q^2}\alpha^{-q^2}b$ onto $y = x^q\beta + x^{q^2}\beta^{-q^2}\widehat{b}$ if and only if*

$$\sigma = \begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^q \end{bmatrix} \begin{bmatrix} \beta^{1+q+q^2} & \beta^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}^{-1},$$

for some element a of $GF(q^3)^*$.

Proof.

$$\begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix}$$

maps $y = x^q\alpha + x^{q^2}\alpha^{-q^2}b$ onto $y = x^q$ and

$$\begin{bmatrix} \beta^{1+q+q^2} & \beta^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}^{-1}$$

maps $y = x^q$ onto $y = x^q\beta + x^{q^2}\beta^{-q^2}\widehat{b}$. So,

$$\begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix}^{-1} \sigma \begin{bmatrix} \beta^{1+q+q^2} & \beta^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}$$

fixes $y = x^q$ and hence has the form $\begin{bmatrix} a & 0 \\ 0 & a^q \end{bmatrix}$. This proves the assertion. $\boxed{QED}$

If we assume that we are always using the first type of replacement, with the b-term associated with the x^{q^2}-term, then suppose that an isomorphism is linear in the sense that it is in $GL(2, q^3)$ (it must be in $\Gamma L(2, q^3)$ by the previous arguments). Hence, we see that there are exactly $(q^3 - 1)/(q - 1)$ elements of $GL(2, q^3)$ that map given elements $y = x^q\alpha + x^{q^2}\alpha^{-q^2}b$ and $y = x^q\beta + x^{q^2}\beta^{-q^2}\widehat{b}$ one to the other.

Suppose $y = x^q\alpha + x^{q^2}\alpha^{-q^2}b$ maps to $y = x^q\beta + x^{q^2}\beta^{-q^2}\widehat{b}$ by a mapping in $GL(2, q^3)$: then it can only have the form

$$\begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^q \end{bmatrix} \begin{bmatrix} \beta^{1+q+q^2} & \beta^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}^{-1},$$

for some element a of $GF(q^3)^*$. Then $y = x^q\delta + x^{q^2}\delta^{-q^2}b$ maps to

$$(x, x^q\delta + x^{q^2}\delta^{-q^2}b) \begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^q \end{bmatrix} \begin{bmatrix} \beta^{1+q+q^2} & \beta^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}^{-1}.$$

This subspace is of the form $y = x^q\gamma + x^{q^2}\gamma^{-q^2}\widehat{b}$ if and only if we obtain the following conditions:

$$\left\{ (x, x^q\delta + x^{q^2}\delta^{-q^2}b) \begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^q \end{bmatrix} \begin{bmatrix} \beta^{1+q+q^2} & \beta^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}^{-1} \right\},$$

which is

$$\left\{ (x, x^q\delta + x^{q^2}\delta^{-q^2}b) \begin{bmatrix} \alpha^{1+q+q^2}a & \alpha^{-1}b^{q+1}a^q \\ b^{q^2}a & a^q \end{bmatrix} \begin{bmatrix} \beta^{1+q+q^2} & \beta^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}^{-1} \right\},$$

and, in turn, is

$$\left\{ \begin{array}{c} (X, Y) \begin{bmatrix} \beta^{1+q+q^2} & \beta^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}^{-1} \\ (X, Y) \begin{bmatrix} 1 & -\beta^{-1}\widehat{b}^{q+1} \\ -\widehat{b}^{q^2} & \beta^{1+q+q^2} \end{bmatrix} / (\beta^{1+q+q^2} - \beta^{-1}\widehat{b}^{1+q+q^2}) \\ X = x\alpha^{1+q+q^2}a + (x^q\delta + x^{q^2}\delta^{-q^2}b)b^{q^2}a \\ Y = x\alpha^{-1}b^{q+1}a^q + (x^q\delta + x^{q^2}\delta^{-q^2}b)a^q \end{array} \right\}.$$

Let this set product be $\{(x^*y^*)\}$, where $\Delta = (\beta^{1+q+q^2} - \beta^{-1}\widehat{b}^{1+q+q^2})$ and

$$x^* = \Big[x\alpha^{1+q+q^2}a + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)b^{q^2}a - \Big(x\alpha^{-1}b^{q+1}a^q + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)a^q\Big)\widehat{b}^{q^2}\Big]/\Delta$$

and y^* is

$$y\Big[-\Big(\big(x\alpha^{1+q+q^2}a + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)b^{q^2}a\big)\beta^{-1}\widehat{b}^{q+1} + \big(x\alpha^{-1}b^{q+1}a^q + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)a^q\big)\beta^{1+q+q^2}\Big)\Big]/\Delta.$$

Hence, this is $y = x^q\gamma + x^{q^2}\gamma^{-q^2}\widehat{b}$ if and only if

$$\begin{aligned}
&\Big\{\Big[\big(x\alpha^{1+q+q^2}a + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)b^{q^2}a\big) - \\
&\big(x\alpha^{-1}b^{q+1}a^q + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)a^q\big)\widehat{b}^{q^2}\Big]/\Delta\Big\}^q\gamma \\
&+\Big\{\Big[\big(x\alpha^{1+q+q^2}a + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)b^{q^2}a\big) - \\
&\big(x\alpha^{-1}b^{q+1}a^q + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)a^q\big)\widehat{b}^{q^2}\Big]/\Delta\Big\}^{q^2}\gamma^{-q^2}\widehat{b} \\
= \quad &\Big[-\Big(\big(x\alpha^{1+q+q^2}a + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)b^{q^2}a\big)\beta^{-1}\widehat{b}^{q+1} + \\
&\big(x\alpha^{-1}b^{q+1}a^q + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)a^q\big)\beta^{1+q+q^2}\Big)\Big]/\Delta.
\end{aligned}$$

Note that

$$\begin{aligned}
&\Big[\big(x\alpha^{1+q+q^2}a + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)b^{q^2}a\big) - \big(x\alpha^{-1}b^{q+1}a^q + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)a^q\big)\widehat{b}^{q^2}\Big] \\
&= x\big(\alpha^{1+q+q^2}a - \alpha^{-1}b^{q+1}a^q\widehat{b}^{q^2}\big) + x^q\big(\delta b^{q^2}a - \delta a^q\widehat{b}^{q^2}\big) \\
&\qquad + x^{q^2}\big(\delta^{-q^2}b^{1+q^2}a - \delta^{-q^2}ba^q\widehat{b}^{q^2}\big)
\end{aligned}$$

and

$$\begin{aligned}
&\Big[-\Big(\big(x\alpha^{1+q+q^2}a + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)b^{q^2}a\big)\beta^{-1}\widehat{b}^{q+1} + \big(x\alpha^{-1}b^{q+1}a^q \\
&\qquad + \big(x^q\delta + x^{q^2}\delta^{-q^2}b\big)a^q\big)\beta^{1+q+q^2}\Big)\Big] \\
&= x\big(-\alpha^{1+q+q^2}a\beta^{-1}\widehat{b}^{q+1} + \alpha^{-1}b^{q+1}a^q\beta^{1+q+q^2}\big) \\
&\qquad + x^q\big(-\delta b^{q^2}a\beta^{-1}\widehat{b}^{q+1} + \delta a^q\beta^{1+q+q^2}\big) \\
&\qquad + x^{q^2}\big(-\delta^{-q^2}b^{1+q^2}a\beta^{-1}\widehat{b}^{q+1} + \delta^{-q^2}ba^q\beta^{1+q+q^2}\big).
\end{aligned}$$

Thus, we require

$$\left\{\begin{array}{c}\left[x\left(\alpha^{1+q+q^2}a-\alpha^{-1}b^{q+1}a^q\widehat{b}^{q^2}\right)+x^q\left(\delta b^{q^2}a-\delta a^q\widehat{b}^{q^2}\right)+\right.\\ \left.x^{q^2}\left(\delta^{-q^2}b^{1+q^2}a-\delta^{-q^2}ba^q\widehat{b}^{q^2}\right)\right]/\Delta\end{array}\right\}^q\gamma$$
$$+\left\{\begin{array}{c}\left[x\left(\alpha^{1+q+q^2}a-\alpha^{-1}b^{q+1}a^q\widehat{b}^{q^2}\right)+x^q\left(\delta b^{q^2}a-\delta a^q\widehat{b}^{q^2}\right)+\right.\\ \left.x^{q^2}\left(\delta^{-q^2}b^{1+q^2}a-\delta^{-q^2}ba^q\widehat{b}^{q^2}\right)\right]/\Delta\end{array}\right\}^{q^2}\gamma^{-q^2}\widehat{b}$$
$$=\Big[x\left(-\alpha^{1+q+q^2}a\beta^{-1}\widehat{b}^{q+1}+\alpha^{-1}b^{q+1}a^q\beta^{1+q+q^2}\right)$$
$$+x^q\left(-\delta b^{q^2}a\beta^{-1}\widehat{b}^{q+1}+\delta a^q\beta^{1+q+q^2}\right)$$
$$+x^{q^2}\left(-\delta^{-q^2}b^{1+q^2}a\beta^{-1}\widehat{b}^{q+1}+\delta^{-q^2}ba^q\beta^{1+q+q^2}\right)\Big]/\Delta.$$

Hence, we obtain the three conditions:

$$(*):\quad \left\{\left(\delta^{-q^2}b^{1+q^2}a-\delta^{-q^2}ba^q\widehat{b}^{q^2}\right)/\Delta\right\}^q\gamma+\left\{\left(\delta b^{q^2}a-\delta a^q\widehat{b}^{q^2}\right)/\Delta\right\}^{q^2}\gamma^{-q^2}\widehat{b}$$
$$=\left(-\alpha^{1+q+q^2}a\beta^{-1}\widehat{b}^{q+1}+\alpha^{-1}b^{q+1}a^q\beta^{1+q+q^2}\right)/\Delta,$$

$$(**):\quad \left\{\left(\alpha^{1+q+q^2}a-\alpha^{-1}b^{q+1}a^q\widehat{b}^{q^2}\right)/\Delta\right\}^q\gamma$$
$$+\left\{\left(\delta^{-q^2}b^{1+q^2}a-\delta^{-q^2}ba^q\widehat{b}^{q^2}\right)/\Delta\right\}^{q^2}\gamma^{-q^2}\widehat{b}$$
$$=\left(-\delta b^{q^2}a\beta^{-1}\widehat{b}^{q+1}+\delta a^q\beta^{1+q+q^2}\right)/\Delta,$$

and

$$(***):\quad \left\{\left(\delta b^{q^2}a-\delta a^q\widehat{b}^{q^2}\right)/\Delta\right\}^q\gamma$$
$$+\left\{\left(\alpha^{1+q+q^2}a-\alpha^{-1}b^{q+1}a^q\widehat{b}^{q^2}\right)/\Delta\right\}^{q^2}\gamma^{-q^2}\widehat{b}$$
$$=\left(-\delta^{-q^2}b^{1+q^2}a\beta^{-1}\widehat{b}^{q+1}+\delta^{-q^2}ba^q\beta^{1+q+q^2}\right)/\Delta.$$

However, this mapping must also be of the form

$$\begin{bmatrix}\delta^{1+q+q^2} & \delta^{-1}b^{q+1}\\ b^{q^2} & 1\end{bmatrix}\begin{bmatrix}e & 0\\ 0 & e^q\end{bmatrix}\begin{bmatrix}\gamma^{1+q+q^2} & \gamma^{-1}\widehat{b}^{q+1}\\ \widehat{b}^{q^2} & 1\end{bmatrix}^{-1}$$

so that

$$\begin{bmatrix} \alpha^{1+q+q^2} & \alpha^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a^q \end{bmatrix}\begin{bmatrix} \beta^{1+q+q^2} & \beta^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \delta^{1+q+q^2} & \delta^{-1}b^{q+1} \\ b^{q^2} & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^q \end{bmatrix}\begin{bmatrix} \gamma^{1+q+q^2} & \gamma^{-1}\widehat{b}^{q+1} \\ \widehat{b}^{q^2} & 1 \end{bmatrix}^{-1}.$$

Solving for $\begin{bmatrix} e & 0 \\ 0 & e^q \end{bmatrix}$ produces the following conditions, after a short computation:

$$(*_4): \quad \big(\delta^{1+q+q^2} - \alpha^{1+q+q^2}\big)\big(\gamma^{1+q+q^2} - \beta^{-1}b^{1+q+q^2}\big) \cdot \big(\delta^{-1} - \alpha^{-1}\big)\big(\beta^{1+q+q^2} - \gamma^{-1}b^{1+q+q^2}\big)$$
$$= \big(\gamma^{1+q+q^2} - \beta^{1+q+q^2}\big)\big(\delta^{1+q+q^2} - \alpha^{-1}b^{1+q+q^2}\big) \cdot \big(\gamma^{-1} - \beta^{-1}\big)\big(\alpha^{1+q+q^2} - \delta^{-1}b^{1+q+q^2}\big),$$

and

$$(*_5): \quad \Big\{a\big(\alpha^{1+q+q^2} - \delta b^{1+q+q^2}\big)\big(\gamma^{1+q+q^2} - \beta^{-1}b^{1+q+q^2}\big)/\Delta_\delta\Delta_\beta = e\Big\}^q$$
$$= \Big\{b^{1+q+q^2}a^q\big(\alpha^{-1} - \delta^{-1}\big)\big(\gamma^{1+q+q^2} - \beta^{1+q+q^2}\big)/\Delta_\delta\Delta_\beta\Big\},$$

where $\Delta_\rho = \rho^{1+q+q^2} - \rho^{-1}b^{1+q+q^2}$.

It is also true that the equality between the forms of the mapping means that the previous three conditions $(*), (**), (***)$ are valid replacing a by e, and interchanging δ with α, and β with γ.

Thus, we have:

$$(*)': \quad \Big\{\big(\alpha^{-q^2}b^{1+q^2}e - \alpha^{-q^2}be^q\widehat{b}^{q^2}\big)/\Delta\Big\}^q\beta + \Big\{\big(\alpha b^{q^2}e - \alpha e^q\widehat{b}^{q^2}\big)/\Delta\Big\}^{q^2}\beta^{-q^2}\widehat{b}$$
$$= \big(-\alpha^{1+q+q^2}e\beta^{-1}\widehat{b}^{q+1} + \alpha^{-1}b^{q+1}e^q\beta^{1+q+q^2}\big)/\Delta,$$

$$(**)': \quad \Big\{\big(\delta^{1+q+q^2}e - \delta^{-1}b^{q+1}e^q\widehat{b}^{q^2}\big)/\Delta\Big\}^q\beta$$
$$+ \Big\{\big(\alpha^{-q^2}b^{1+q^2}e - \alpha^{-q^2}be^q\widehat{b}^{q^2}\big)/\Delta\Big\}^{q^2}\beta^{-q^2}\widehat{b}$$
$$= \big(-\alpha b^{q^2}e\gamma^{-1}\widehat{b}^{q+1} + \alpha e^q\gamma^{1+q+q^2}\big)/\Delta,$$

and

$$(***)': \quad \left\{\left(\alpha b^{q^2}e - \alpha e^q \widehat{b}^{q^2}\right)/\Delta\right\}^q \beta + \left\{\left(\delta^{1+q+q^2}e - \delta^{-1}b^{q+1}e^q\widehat{b}^{q^2}\right)/\Delta\right\}^{q^2}\beta^{-q^2}\widehat{b} = \left(-\alpha^{-q^2}b^{1+q^2}e\gamma^{-1}\widehat{b}^{q+1} + \alpha^{-q^2}be^q\gamma^{1+q+q^2}\right)/\Delta.$$

8.1 When $b = \widehat{b}$

Now assume that $b = \widehat{b}$ to obtain:

$$(*): \ \left\{\left(\delta^{-q^2}b^{1+q^2}(a-a^q)\right)/\Delta\right\}^q\gamma + \left\{\delta^{q^2}b^q(a-a^q)/\Delta\right\}^{q^2}\gamma^{-q^2}b = b^{1+q}\left(-\alpha^{1+q+q^2}a\beta^{-1} + \alpha^{-1}a^q\beta^{1+q+q^2}\right)/\Delta,$$

$$(**): \ \left\{\left(\alpha^{1+q+q^2}a - \alpha^{-1}b^{q+1}a^qb^{q^2}\right)/\Delta\right\}^q\gamma + \left\{\left(\delta^{-q}b^{(q^2+q)}(a-a^q)\right)/\Delta\right\}^{q^2}\gamma^{-q^2}b = \left(-\delta b^{q^2}a\beta^{-1}b^{q+1} + \delta a^q\beta^{1+q+q^2}\right)/\Delta,$$

and

$$(***): \ \left\{\left(\delta b^{q^2}(a-a^q)\right)/\Delta\right\}^q\gamma + \left\{\left(\alpha^{1+q+q^2}a - \alpha^{-1}b^{q+1}b^{q^2}a^q\right)/\Delta\right\}^{q^2}\gamma^{-q^2}b = -\delta^{-q^2}b\left(b^{1+q+q^2}a\beta^{-1} + a^q\beta^{1+q+q^2}\right)/\Delta.$$

Note that if $a = a^q$ then $(*)$ gives $\alpha^{2+q+q^2} = \beta^{2+q+q^2}$, implying that $\alpha = \beta$ since $(2+q+q^2, q^3-1) = 1$. But then the collineation is a kernel homology of order necessarily dividing $q-1$, so all 'components' are fixed.

Hence, we may assume that $a \neq a^q$. We may easily solve for γ using $(*)$ and $(**)$ by taking $\delta^{-q^2}b^{-q}(*) - \delta^q b^{-(q^2+q)}(**)$, which eliminates the $\gamma^{-q^2}b$, term.

Hence, we obtain:

$$\left\{\delta^{-q^2}b^{-q}\left\{\left(\delta^{-q^2}b^{1+q^2}(a-a^q)\right)/\Delta\right\}^q - \delta^q b^{-(q^2+q)}\left\{\left(\alpha^{1+q+q^2}a - \alpha^{-1}b^{q+1}a^q\right)/\Delta\right\}^q\right\}\gamma$$
$$= \delta^{-q^2}b^{-q}\left\{\left(-\alpha^{1+q+q^2}a\beta^{-1}b^{q+1} + \alpha^{-1}b^{q+1}a^q\beta^{1+q+q^2}\right)/\Delta\right\} - \delta^q b^{-(q^2+q)}\left\{\left(-\delta b^{1+q+q^2}a\beta^{-1} + \delta a^q\beta^{1+q+q^2}\right)/\Delta\right\}.$$

This uniquely specifies γ unless the coefficient of γ is zero. We may repeat this using a choice of two of the three conditions involving γ. If one of the three of the corresponding coefficients is not zero, this places a condition independent of γ on α, β, δ. Such a condition must be valid for α, β and any other value δ of λ. By using the conditions $(*)'$, etc., then we would similarly obtain a condition valid for β and λ and any other value α of λ.

These conditions although complex could be used to determine isomorphisms in specific situations since any isomorphism arises from an element of $\Gamma L(2, q^3)$.

Furthermore, these ideas can be utilized when determining collineation groups of particular translation planes.

In general, each plane admits a collineation group of order $q^3 - 1$, which is the kernel homology group of the associated Desarguesian affine plane, that acts transitively on each hyper-regulus.

Finally, we again point out that some of the classes of translation planes are similar to combinatorics as those found by Culbert and Ebert [2], in the following sense. We have found classes of hyper-reguli of size $(q-1)/2$, $(q-3)/2$ and $q/2-1$ such that no subset of at least two hyper-reguli can be embedded as a linear André set of hyper-reguli. In other words, replacement by any one hyper-regulus leaves to an André plane, but replacement of more than one hyper-regulus never leads to a hyper-regulus. Since Culbert and Ebert arrive at their constructions using Sherk spaces and work directly on the line $PG(1, q^3)$, whereas we work from the replacement side, finding first the replacement for the putative hyper-regulus that lies in an associated Desarguesian plane, the two approaches are completely different and it is not clear how the two sets of translation planes obtained are related to each other. We leave this as an open problem.

33 Problem. Determine if our classes of hyper-reguli of sizes $(q-1)/2, (q-3)/2$ and $q/2-1$ and those of Culbert and Ebert are isomorphic?

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