

A Class of Transitive BLT-Sets

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Abstract. We present a class of transitive BLT-sets that contains several known examples and at least one new example over the field with 41 elements with automorphism group $\mathbb{Z}_{21} : (\mathbb{Z}_2 \times \mathbb{Z}_2)$. This example is sporadic in the sense that it does not fall into any of the known infinite families of BLT-sets.

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1 Introduction

Let V be a five-dimensional vector space over \mathbb{F}_q , q odd. Let $Q(x)$ be a nondegenerate quadratic form with corresponding bilinear polar form $f(x, y) = Q(x + y) - Q(x) - Q(y)$. Let \perp be the polarity determined by Q . Let $E = \{e_1, \dots, e_5\}$ be a basis for V . Let Δ be the determinant of the 5×5 matrix whose (i, j) -th entry is $f(e_i, e_j)$. The class of Δ modulo nonzero squares in \mathbb{F}_q is known as the discriminant of Q .

A set of $q + 1$ pairwise non-collinear points in the projective space $\mathcal{P}(V)$ is called BLT-set (after [1]) if any three of the points form a BLT-triple. Three noncollinear points $\langle x \rangle, \langle y \rangle, \langle z \rangle$ form a BLT-triple if

$$-\frac{2f(x, y)f(y, z)f(z, x)}{\Delta} \quad (1)$$

is a non-square in \mathbb{F}_q (see [2, Lemma 4.1]).

For the remainder of this article, the trace and the norm functions from \mathbb{F}_{q^2} to \mathbb{F}_q are denoted as T and N , respectively. Also, we will denote elements in \mathbb{F}_{q^2} using greek letters, whereas we will use ordinary letters for elements in \mathbb{F}_q .

2 The Construction

1 Lemma. *Let q be an odd prime power. Let S be equal to the set of non-squares in \mathbb{F}_q if $q \equiv 1 \pmod{4}$ and let S be equal to the set of nonzero squares in \mathbb{F}_q if $q \equiv 3 \pmod{4}$. Let m be an involution in $\mathbb{Z}_{\frac{q+1}{2}}^\times$, $m \not\equiv -1 \pmod{\frac{q+1}{2}}$. Let s be in $\mathbb{Z}_{\frac{q+1}{2}}^\times$. Let ζ be a primitive root of unity of order $q + 1$ in \mathbb{F}_{q^2} . Let $g \in \mathbb{F}_q^\times \setminus \{-1\}$*

with

$$g\left(T(\zeta^{2k}) - T(\zeta^{2mk})\right) + 2 - T(\zeta^{2mk}) \in S \quad \text{for all } 1 \leq k < \frac{q+1}{2}. \quad (2)$$

Let γ and δ be elements in \mathbb{F}_{q^2} with $N(\gamma) = g$ and $N(\delta) = -1 - g$ and

$$2 + T(\gamma\delta^q\zeta^{2k}) + T(\delta\gamma^q\zeta^{2mk-2m-s}) \in S \quad \text{for all } 1 \leq k \leq \frac{q+1}{2}. \quad (3)$$

Define

$$x_i = (\gamma\zeta^{2i}, \delta\zeta^{2mi}, 1), \quad y_i = (\delta\zeta^{2i}, \gamma\zeta^{2m(i+1)+s}, 1) \quad \text{for } i = 1, \dots, (q+1)/2,$$

and let $P_i = \langle x_i \rangle$ and $Q_i = \langle y_i \rangle$ for $i = 1, \dots, \frac{q+1}{2}$. The set

$$\mathcal{B}(q, m, s, g, \gamma, \delta) := \left\{ P_i \mid i = 1, \dots, \frac{q+1}{2} \right\} \cup \left\{ Q_i \mid i = 1, \dots, \frac{q+1}{2} \right\}$$

is a BLT-set in the parabolic space $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2} \times \mathbb{F}_q$ (viewed as \mathbb{F}_q vector space of dimension 5) with non-degenerate quadratic form

$$Q(\alpha, \beta, c) = N(\alpha) + N(\beta) + c^2.$$

PROOF. Note that $N(\zeta) = \zeta^{1+q} = 1$ and hence $\zeta^q = \zeta^{-1}$. Also, recall that $T(\omega^q) = T(\omega)$ for all $\omega \in \mathbb{F}_{q^2}$. From

$$Q(x_i) = N(\gamma\zeta^{2i}) + N(\delta\zeta^{2mi}) + 1 = gN(\zeta)^{2i} + (-1 - g)N(\zeta)^{2i} + 1 = 0$$

and

$$Q(y_i) = N(\delta\zeta^{2i}) + N(\gamma\zeta^{2mi+2m+s}) + 1 = 0$$

it follows that the points P_i and Q_i are on the quadric. For $i \neq j$, the points P_i and P_j are not collinear since

$$\begin{aligned} f(x_i, x_j) &= Q(x_i + x_j) - Q(x_i) - Q(x_j) \\ &= Q(\gamma(\zeta^{2i} + \zeta^{2j}), \delta(\zeta^{2mi} + \zeta^{2mj}), 2) \\ &= N(\gamma(\zeta^{2i} + \zeta^{2j})) + N(\delta(\zeta^{2mi} + \zeta^{2mj})) + 4 \\ &= gN(\zeta^{2i} + \zeta^{2j}) + (-1 - g)N(\zeta^{2mi} + \zeta^{2mj}) + 4 \\ &= g(\zeta^{2i} + \zeta^{2j})(\zeta^{-2i} + \zeta^{-2j}) + (-1 - g)(\zeta^{2mi} + \zeta^{2mj})(\zeta^{-2mi} + \zeta^{-2mj}) + 4 \\ &= g(2 + T(\zeta^{2(i-j)})) + (-1 - g)(2 + T(\zeta^{2m(i-j)})) + 4 \\ &= g(T(\zeta^{2(i-j)}) - T(\zeta^{2m(i-j)})) - T(\zeta^{2m(i-j)}) + 2 \end{aligned}$$

which is a nonzero square (if $q \equiv 3 \pmod{4}$) or a nonsquare (if $q \equiv 1 \pmod{4}$) by (2), and hence nonzero.

For $u \neq v$, the points Q_u and Q_v are not collinear since

$$\begin{aligned}
f(y_u, y_v) &= Q(y_u + y_v) - Q(y_u) - Q(y_v) \\
&= Q(\delta(\zeta^{2u} + \zeta^{2v}), \gamma(\zeta^{2m(u+1)+s} + \zeta^{2m(v+1)+s}), 2) \\
&= N(\delta(\zeta^{2u} + \zeta^{2v})) + N(\gamma\zeta^{2m+s}(\zeta^{2mu} + \zeta^{2mv})) + 4 \\
&= (-1 - g)N(\zeta^{2u} + \zeta^{2v}) + gN(\zeta^{2mu} + \zeta^{2mv}) + 4 \\
&= gN(\zeta^{2i} + \zeta^{2j}) + (-1 - g)N(\zeta^{2mi} + \zeta^{2mj}) + 4 \\
&= f(x_i, x_j),
\end{aligned}$$

with i and j determined uniquely by the congruences

$$i \equiv mu \pmod{\frac{q+1}{2}} \quad \text{and} \quad j \equiv mv \pmod{\frac{q+1}{2}}$$

(note that $mi \equiv u$ and $mj \equiv v \pmod{(q+1)/2}$).

It remains to verify that no P_i is collinear to any Q_j (for all i and j modulo $(q+1)/2$). To this end, we compute

$$\begin{aligned}
f(x_i, y_j) &= Q(x_i + y_j) - Q(x_i) - Q(y_j) \\
&= Q(\gamma\zeta^{2i} + \delta\zeta^{2j}, \delta\zeta^{2mi} + \gamma\zeta^{2m(j+1)+s}, 2) \\
&= N(\gamma\zeta^{2i} + \delta\zeta^{2j}) + N(\delta\zeta^{2mi} + \gamma\zeta^{2m(j+1)+s}) + 4 \\
&= (\gamma\zeta^{2i} + \delta\zeta^{2j})(\gamma^q\zeta^{-2i} + \delta^q\zeta^{-2j}) \\
&\quad + (\delta\zeta^{2mi} + \gamma\zeta^{2m(j+1)+s})(\delta^q\zeta^{-2mi} + \gamma^q\zeta^{-2m(j+1)-s}) + 4 \\
&= 2\gamma^{1+q} + 2\delta^{1+q} + \gamma\delta^q\zeta^{2(i-j)} + \delta\gamma^q\zeta^{2(j-i)} + \delta\gamma^q\zeta^{2m(i-j)-2m-s} \\
&\quad + \gamma\delta^q\zeta^{2m(j-i)+2m+s} + 4 \\
&= 2 + T(\gamma\delta^q\zeta^{2(i-j)}) + T(\delta\gamma^q\zeta^{2m(i-j)-2m-s})
\end{aligned}$$

which is nonzero by (3). This shows that the points of $\mathcal{B}(q, m, s, g, \gamma, \delta)$ are pairwise noncollinear.

It remains to verify that any three points form a BLT-triple. We compute the discriminant of f . Note that V has the orthogonal decomposition as $U \perp \langle (0, 0, 1) \rangle$ and so the discriminant is the product of the discriminants of f restricted to each of the direct summands. The space U is hyperbolic, and hence the discriminant on U is 1. The discriminant on $\langle (0, 0, 1) \rangle$ is $f((0, 0, 1), (0, 0, 1)) = 2$. Thus we may take $\Delta = 2 \pmod{\text{squares}}$ in (1). Notice that -1 is a square if and only if $q \equiv 1 \pmod{4}$. Thus we can rephrase (1) as

$$f(x, y)f(y, z)f(z, x) \in \begin{cases} \square & q \equiv 1 \pmod{4} \\ \square & q \equiv 3 \pmod{4} \end{cases}$$

Choose a and b from the set $\{x_i \mid i = 1, \dots, (q+1)/2\} \cup \{y_i \mid i = 1, \dots, (q+1)/2\}$. The above computation shows that $f(a, b)$ is a nonzero square if $q \equiv 3 \pmod{4}$ and $f(a, b)$ is a nonsquare if $q \equiv 1 \pmod{4}$. That is, the triple product $f(x, y)f(y, z)f(z, x)$ is a nonzero square if $q \equiv 3 \pmod{4}$ and it is a nonsquare if $q \equiv 1 \pmod{4}$. Thus, the three points $\langle x \rangle, \langle y \rangle, \langle z \rangle$ form a BLT-triple. Therefore, $\mathcal{B}(q, m, s, g, \gamma, \delta)$ is a BLT-set. \square

Let ζ be a primitive $(q+1)$ th root of unity in \mathbb{F}_{q^2} . We define the integer k (modulo $q+1$) by means of the equation

$$\zeta^k = (\gamma/\delta)^{q-1}.$$

This integer k exists since $\epsilon^{q^2-1} = 1$ for all $\epsilon \in \mathbb{F}_{q^2}^\times$.

2 Lemma. *Consider the BLT-set $\mathcal{B}(q, m, 2, g, \gamma, \delta)$ constructed in Lemma 1. Assume that $q \equiv 1 \pmod{4}$ and that the integer k in $\zeta^k = (\gamma/\delta)^{q-1}$ is even and let $t = -k/2$. Assume further that $(m+1)(t+2)$ is divisible by $q+1$. Then the automorphism group contains a group isomorphic to $\mathbb{Z}_{\frac{q+1}{2}} : (\mathbb{Z}_2 \times \mathbb{Z}_2)$ acting transitively.*

PROOF. We define the following three mappings ψ, μ, ν :

$$\begin{aligned} (\alpha, \beta, c)\psi &= (\alpha\zeta^2, \beta\zeta^{2m}, c), \\ (\alpha, \beta, c)\mu &= (\beta\zeta^t, \alpha\zeta^{-t}, c), \\ (\alpha, \beta, c)\nu &= (\alpha^q\gamma^{1-q}, \beta^q\delta^{1-q}, c). \end{aligned}$$

In the remainder of this proof, we will show that all three mappings are automorphisms of the BLT-set, satisfying the relations

$$\psi^{(q+1)/2} = \mu^2 = \nu^2 = 1, \quad \psi^\mu = \psi^m, \quad \psi^\nu = \psi^{-1}, \quad \mu^\nu = \mu.$$

Thus, they generate a group isomorphic to $\mathbb{Z}_{\frac{q+1}{2}} : (\mathbb{Z}_2 \times \mathbb{Z}_2)$. It is clear that this group is transitive on the points of the BLT-set.

Notice that the given elements lie in the orthogonal linear group. At first, we verify that ψ, μ and ν are automorphisms of the BLT-set. In the following calculations, the integer $t = -k/2$ sometimes appears as exponent of ζ^2 . In this case, t is determined modulo $(q+1)/2$, and the quantity $t/2$ is defined since $q \equiv 1 \pmod{4}$ and hence $(q+1)/2$ is odd.

The assumption that $q+1$ divides $(m+1)(t+2)$ leads to

$$mt + 2m + 2 \equiv -t \pmod{q+1} \tag{4}$$

and hence also to

$$2mt + 2m + 2 \equiv -2t - 2m - 2 \pmod{q+1}. \tag{5}$$

The action of ψ on the BLT-set is

$$\begin{aligned} x_i\psi &= (\gamma\zeta^{2i+2}, \delta\zeta^{2mi+2m}, 1) = x_{i+1}, \\ y_i\psi &= (\delta\zeta^{2i+2}, \gamma\zeta^{2mi+2m+2+2m}, 1) = y_{i+1} \end{aligned}$$

with i modulo $(q+1)/2$. Therefore, $\psi^{(q+1)/2} = 1$. For μ , we get

$$\begin{aligned} x_i\mu &= (\gamma\zeta^{2i}, \delta\zeta^{2mi}, 1)\mu \\ &= (\delta\zeta^{2mi+t}, \gamma\zeta^{2i-t}, 1) \\ &= (\delta\zeta^{2(mi+t/2)}, \gamma\zeta^{2m(mi+t/2)+2m+2}, 1) \\ &= y_{mi+t/2}. \end{aligned}$$

Using (4), we compute

$$\begin{aligned} y_i\mu &= (\delta\zeta^{2i}, \gamma\zeta^{2m(i+1)+2}, 1)\mu \\ &= (\gamma\zeta^{2m(i+1)+2+t}, \delta\zeta^{2i-t}, 1) \\ &= (\gamma\zeta^{2mi+2m+2+t}, \delta\zeta^{2i+2+2m+mt}, 1) \\ &= (\gamma\zeta^{2(mi+m+1+t/2)}, \delta\zeta^{2m(mi+m+1+t/2)}, 1) \\ &= x_{mi+m+1+t/2}. \end{aligned}$$

The element ν acts in the following way on the x_i :

$$\begin{aligned} x_i\nu &= (\gamma\zeta^{2i}, \delta\zeta^{2mi}, 1)\nu \\ &= (\gamma^q\zeta^{-2i}\gamma^{1-q}, \delta^q\zeta^{-2mi}\delta^{1-q}, 1) \\ &= (\gamma\zeta^{-2i}, \delta\zeta^{-2mi}, 1) \\ &= x_{-i}. \end{aligned}$$

For the action on the y_i , we obtain using (5)

$$\begin{aligned} y_i\nu &= (\delta\zeta^{2i}, \gamma\zeta^{2mi+2m+2}, 1)\nu \\ &= (\delta^q\zeta^{-2i}\gamma^{1-q}, \gamma^q\zeta^{-2mi-2m-2}\delta^{1-q}, 1) \\ &= \left(\delta\left(\frac{\delta}{\gamma}\right)^{q-1}\zeta^{-2i}, \gamma\left(\frac{\gamma}{\delta}\right)^{q-1}\zeta^{-2mi-2m-2}, 1\right) \\ &= (\delta\zeta^{2t-2i}, \gamma\zeta^{-2mi-2m-2-2t}, 1) \\ &= (\delta\zeta^{2(t-i)}, \gamma\zeta^{-2mi+2mt+2m+2}, 1) \\ &= (\delta\zeta^{2(t-i)}, \gamma\zeta^{2m(t-i)+2m+2}, 1) \\ &= y_{t-i} \end{aligned}$$

Also,

$$(\alpha, \beta, c)\mu^2 = (\beta\zeta^t, \alpha\zeta^{-t}, c)\mu = (\alpha, \beta, c),$$

and

$$\begin{aligned} (\alpha, \beta, c)\nu^2 &= (\alpha^q\gamma^{1-q}, \beta^q\delta^{1-q}, c)\nu \\ &= (\alpha^{q^2}\gamma^{q-1}\gamma^{1-q}, \beta^{q^2}\delta^{q-1}\delta^{1-q}, c) \\ &= (\alpha, \beta, c), \end{aligned}$$

i.e., $\mu^2 = \nu^2 = 1$.

Furthermore,

$$\begin{aligned} (\alpha, \beta, c)\mu^{-1}\psi\mu &= (\beta\zeta^t, \alpha\zeta^{-t}, c)\psi\mu \\ &= (\beta\zeta^{t+2}, \alpha\zeta^{-t+2m}, c)\mu \\ &= (\alpha\zeta^{2m}, \beta\zeta^2, c) \\ &= (\alpha, \beta, c)\psi^m \end{aligned}$$

which shows that $\psi\mu = \psi^m$.

Also,

$$\begin{aligned} (\alpha, \beta, c)\nu^{-1}\psi\nu &= (\alpha^q\gamma^{1-q}, \beta^q\delta^{1-q}, c)\psi\nu \\ &= (\alpha^q\gamma^{1-q}\zeta^2, \beta^q\delta^{1-q}\zeta^{2m}, c)\nu \\ &= (\alpha^{q^2}\gamma^{q-1}\zeta^{-2}\gamma^{1-q}, \beta^{q^2}\delta^{q-1}\zeta^{-2m}\delta^{1-q}, c) \\ &= (\alpha\zeta^{-2}, \beta\zeta^{-2m}, c) \\ &= (\alpha, \beta, c)\psi^{-1}, \end{aligned}$$

which shows that $\psi\nu = \psi^{-1}$.

It remains to show that ν and μ commute:

$$\begin{aligned} x_i\nu^{-1}\mu^{-1}\nu\mu &= (\gamma\zeta^{2i}, \delta\zeta^{2mi}, 1)\nu\mu\nu\mu \\ &= (\gamma^q\zeta^{-2i}\gamma^{1-q}, \delta^q\zeta^{-2mi}\delta^{1-q}, 1)\mu\nu\mu \\ &= (\delta^q\zeta^{-2mi}\delta^{1-q}\zeta^{-k/2}, \gamma^q\zeta^{-2i}\gamma^{1-q}\zeta^{k/2}, 1)\nu\mu \\ &= (\delta\zeta^{-2mi-k/2}, \gamma\zeta^{-2i+k/2}, 1)\nu\mu \\ &= (\delta^q\zeta^{2mi+k/2}\gamma^{1-q}, \gamma^q\zeta^{2i-k/2}\delta^{1-q}, 1)\mu \\ &= (\gamma^q\zeta^{2i-k/2}\delta^{1-q}\zeta^{-k/2}, \delta^q\zeta^{2mi+k/2}\gamma^{1-q}\zeta^{k/2}, 1) \\ &= (\gamma(\frac{\gamma}{\delta})^{q-1}\zeta^{2i}\zeta^{-k}, \delta(\frac{\delta}{\gamma})^{q-1}\zeta^{2mi}\zeta^k, 1) \\ &= (\gamma\zeta^{2i}, \delta\zeta^{2mi}, 1) \\ &= x_i \end{aligned}$$

and

$$\begin{aligned}
y_i \nu^{-1} \mu^{-1} \nu \mu &= (\delta \zeta^{2i}, \gamma \zeta^{2m(i+1)+2}, 1) \nu \mu \nu \mu \\
&= (\delta^q \zeta^{-2i} \gamma^{1-q}, \gamma^q \zeta^{-2m(i+1)-2} \delta^{1-q}, 1) \mu \nu \mu \\
&= (\gamma^q \zeta^{-2m(i+1)-2} \delta^{1-q} \zeta^{-k/2}, \delta^q \zeta^{-2i} \gamma^{1-q} \zeta^{k/2}, 1) \nu \mu \\
&= (\gamma \zeta^{2m(i+1)+2} \delta^{q-1} \zeta^{k/2} \gamma^{1-q}, \delta \zeta^{2i} \gamma^{q-1} \zeta^{-k/2} \delta^{1-q}, 1) \mu \\
&= (\gamma \zeta^{2m(i+1)+2} \left(\frac{\delta}{\gamma}\right)^{q-1} \zeta^{k/2}, \delta \zeta^{2i} \left(\frac{\gamma}{\delta}\right)^{q-1} \zeta^{-k/2}, 1) \mu \\
&= (\delta \zeta^{2i} \left(\frac{\gamma}{\delta}\right)^{q-1} \zeta^{-k}, \gamma \zeta^{2m(i+1)+2} \left(\frac{\delta}{\gamma}\right)^{q-1} \zeta^k, 1) \\
&= (\delta \zeta^{2i}, \gamma \zeta^{2m(i+1)+2}, 1) \\
&= y_i.
\end{aligned}$$

This completes the proof. \square

3 Theorem. *The construction described in Lemma 1 gives rise to a new BLT-set over \mathbb{F}_{41} . The automorphism group coincides with the group described in Lemma 2 (of order 84 in this case). It acts transitively on the points of the set.*

PROOF. Let ω be a primitive element for \mathbb{F}_{41^2} with $\omega^2 + 3\omega + 6 = 0$. We define $\zeta = \omega^{40}$, a primitive $q + 1$ -th root of unity. A computer calculation shows that $(q, m, s, g, \gamma, \delta) = (41, 8, 2, 12, \omega^{827}, \omega^{1411})$ solves (2) and (3) and hence gives rise to a BLT-set. Since $(\gamma/\delta)^{q-1} = \omega^{160} = \zeta^4$, we have $t = -2$ and hence $(m+1)(t+2) = 0$ which is divisible by $q+1$. Therefore, all assumptions of Lemma 2 are satisfied and we conclude that the group described in that lemma is a subgroup of the automorphism group of the BLT-set. A computer calculation shows that this group is the full automorphism group of this BLT-set (of order 84). To see that this BLT-set is new, we consider the known BLT-sets over \mathbb{F}_{41} with transitive automorphism group. The Linear and the Fisher and the Fisher-Thomas-Walker BLT sets all have larger groups (see [3] for a description of these families). The group of the Mondello BLT set [5] is a Dihedral group of order 84, and hence is not isomorphic to the group of the example. Other transitive BLT-sets over \mathbb{F}_{41} are not known. \square

4 Theorem. *The construction described in Lemma 1 gives rise to a BLT-set over \mathbb{F}_{23} described in [4]. The automorphism group of this BLT-set is the Weyl group of type F_4 , isomorphic to $\mathbb{Z}_2 : (\text{Sym}_4 \times \text{Sym}_4)$ (of order 1152).*

PROOF. Let ω be a primitive element for \mathbb{F}_{23^2} satisfying $\omega^2 + 2\omega + 5 = 0$. A computer calculation shows that $(q, m, s, g, \gamma, \delta) = (23, 5, 5, 6, \omega^{172}, \omega^{426})$ solves (2) and (3) and hence gives rise to a BLT-set. Another computer calculation

shows that the full automorphism group of this BLT set is $F_4 \simeq \mathbb{Z}_2 : (Sym_4 \times Sym_4)$ of order 1152. \square

5 Theorem. *The construction described in Lemma 1 gives rise to the BLT-set over \mathbb{F}_{47} with automorphism group $\mathbb{Z}_2 : (Sym_4 \wr \mathbb{Z}_2)$ of order 2304 described in [4].*

PROOF. Let ω be a primitive element for \mathbb{F}_{47^2} satisfying $\omega^2 + 2\omega + 5 = 0$. A computer calculation shows that $(q, m, s, g, \gamma, \delta) = (47, 17, 2, 21, \omega^{788}, \omega^{1336})$ solves (2) and (3) and hence gives rise to a BLT-set. Another computer calculation shows that the full automorphism group of this BLT set is $\mathbb{Z}_2 : (Sym_4 \wr \mathbb{Z}_2)$ of order 2304. \square

We remark without proof that for $q \equiv -1 \pmod{8}$, the construction of Lemma 1 yields the Fisher BLT-sets (with $m = -3/4$, $s = 2$, $g = -1/2$).

3 The Plane Invariant

It is desirable to have at hand easy means of telling whether or not things are isomorphic. This is often difficult. Nevertheless, in many cases it suffices to have a procedure that can tell things apart, for instance by examining certain invariants that are preserved under isomorphism. If the invariants differ, the objects are not isomorphic.

We will now discuss such an invariant that applies to sets of points in projective space (not necessarily BLT-sets). Let $\mathcal{B} = \{P_1, \dots, P_\ell\}$ be a set of points in projective space $PG(k, q)$, $k, \ell \geq 3$. Let m be the largest plane intersection number of \mathcal{B} , that is, the largest integer m such that there is a plane π with $|\pi \cap \mathcal{B}| = m$. Let π_1, \dots, π_s be all the planes that intersect \mathcal{B} in m points. Define $a_{i,j} = |\pi_i \cap \pi_j \cap \mathcal{B}|$. The $s \times s$ matrix

$$A_{\mathcal{B}} = [a_{i,j}]$$

is the *plane intersection invariant* of \mathcal{B} . The matrix $A_{\mathcal{B}}$ is unique up to reordering of rows and columns (simultaneously). It should be pointed out that the set of planes π_1, \dots, π_s as above can be computed relatively fast, namely by considering the planes spanned by each of the $\binom{s}{3}$ triples of points chosen from the set \mathcal{B} (as opposed for instance to looking at *all* planes in $PG(k, q)$ which would be very expensive).

The plane invariant is an effective way to tell BLT-sets apart, as demonstrated by the following examples:

The plane invariant of the linear BLT-set is

$$[q + 1].$$

The plane invariant of the Fisher BLT-set for $q \geq 9$ is

$$\begin{bmatrix} \frac{q+1}{2} & 0 \\ 0 & \frac{q+1}{2} \end{bmatrix}.$$

This corresponds to the fact that the points of the Fisher BLT-set are distributed evenly on two planes, as can be seen from the description of these BLT-sets in [5].

The plane invariant of the Mondello and Fisher-Thas-Walker BLT-sets are rather large.

The plane invariant of the BLT-set of order 41 constructed in Theorem 3 is

$$\begin{bmatrix} 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

The plane invariant of the Mondello BLT-set of order 41 is of size 147×147 . This is clearly not permutation equivalent to the 6×6 matrix above, furnishing a second, independent proof of the fact that the BLT-sets are different.

The plane invariants for the examples in Theorem 4 and 5 are matrices of size 16×16 and 18×18 , respectively.

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