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From long exact sequences to spectral sequences

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Abstract. Two long exact sequences $\ldots \to A^{i-1} \to A^i \to A^{i+1} \to \ldots$ and $\ldots \to B^{i-1} \to B^i \to B^{i+1} \to \ldots$ are called bonded at level a mod 3, if $A^j \cong B^j$ for all $j \equiv a$ mod 3. A tree (or compound) of long exact sequences is a tree, whose vertices carry a spectral sequence and whose edges are labelled by a number a mod 3, such that the two long exact sequences adjacent to an edge are bonded at level a. Furthermore the edges incident to a common vertex carry different labels mod 3. Then I show that there is a spectral sequence associated to every compound of long exact sequences. As an example for this approach I construct the Grothendieck spectral sequence for the derived functors of a composition of two functors.

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A long exact sequence can be viewed as a composition of short exact sequences coinciding in the last and first entries respectively. The exact sequence

$$\ldots \to M^{i-1} \to M^i \to M^{i+1} \to \ldots$$

corresponds to

$$\begin{array}{ccccccccccccccccc}
\ldots & & & & & & & & & & & & & & & & \\
0 & \to & A^{i-1} & \to & M^{i-1} & \to & A^i & \to & 0 & & & & & & & & \\
 & & & & & & \| & & & & & & & & & & \\
 & & & & 0 & \to & A^i & \to & M^i & \to & A^{i+1} & \to & 0 & & & & \\
 & & & & & & & & & & \| & & & & & & \\
 & & & & & & & & 0 & \to & A^{i+1} & \to & M^{i+1} & \to & A^{i+2} & \to & 0 \\
 & & & & & & & & & & & & & & & & \ldots
\end{array}$$

The connecting modules A^i do not appear explicitly in the long exact sequence. In analogy to chemistry a short exact sequence could be called of valence two, because it can bind to at most two other short exact sequences to give a linear structure.

Now long exact sequences can bind to each other as well. In the chemical analogy a long exact sequence will be of valence three. The resulting structure will be a tree.

A compound of long exact sequences gives rise to a spectral sequence. Two long exact sequences are said to form a bond, if they coincide at every third entry:

$$\begin{array}{ccccccccccc} \dots & \to & A^i & \to & A^{i+1} & \to & A^{i+2} & \to & A^{i+3} & \to & \dots \\ & & \| & & & & & & \| & & \\ \dots & \to & B^i & \to & B^{i+1} & \to & B^{i+2} & \to & B^{i+3} & \to & \dots \end{array}$$

We say (A^i) forms a bond with (B^i) at level a mod 3, if there is a distinguished a mod 3 such that $A^j = B^j$ for all $j \equiv a$ mod 3.

1 Definition. A **tree** (or **compound**) of bonded long exact sequences is a tree, whose vertices are long exact sequences and whose edges are bonds between them, such that the bonds corresponding to the edges adjacent to a given vertex are all of a different level. Especially the degree of each vertex is ≤ 3.

A finite tree of long exact sequences gives rise to a spectral sequence. This will be shown in the sequel. I will only consider spectral sequences starting with the E_1-term and confined to merely finitely many columns (the analogous result for the E_2-term and spectral sequences confined to finitely many lines arises by shearing along the diagonal).

2 Proposition. *Let $E_1^{p,q} = A^{p,q} \Rightarrow B^{p+q}$ be a spectral sequence, such that $A^{p,q} = 0$ for $p < 0$ or $p > n$. Then there are two spectral sequences converging to 0, whose E_1-term has the form:*

$$\begin{array}{cccc} \dots & & & \\ A^{0,q+1} & \dots & A^{n,q+1} & B^{n+q+1} \\ A^{0,q} & \dots & A^{n,q} & B^{n+q} \\ A^{0,q-1} & \dots & A^{n,q-1} & B^{n+q-1} \\ \dots & & & \end{array} \quad \textit{or} \quad \begin{array}{cccc} \dots & & & \\ B^{q+1} & A^{0,q+1} & \dots & A^{n,q+1} \\ B^{q} & A^{0,q} & \dots & A^{n,q} \\ B^{q-1} & A^{0,q-1} & \dots & A^{n,q-1} \\ \dots & & & \end{array}$$

The additional maps are given by the filtration on the B^i.

Proof. There are filtrations on the $A^{r,s}$, decomposing each module into parts (composition factors). The maps of the spectral sequence identify various such parts:

part	n	of $A^{r,s}$	maps isomorphically to part		0	of	$A^{r+1,s}$
part	$n-1$	of $A^{r,s}$	maps isomorphically to part		1	of	$A^{r+2,s-1}$
			...				
part	$r+1$	of $A^{r,s}$	maps isomorphically to part		$n-r-1$	of	$A^{n,-n+s+r+1}$

---

part	0	of $A^{r,s}$	is the isomorphic image of part	n	of	$A^{r-1,s}$
part	1	of $A^{r,s}$	is the isomorphic image of part	$n-1$	of	$A^{r-2,s+1}$
			...			
part	$r-1$	of $A^{r,s}$	is the isomorphic image of part	$n-r+1$	of	$A^{0,s+r-1}$

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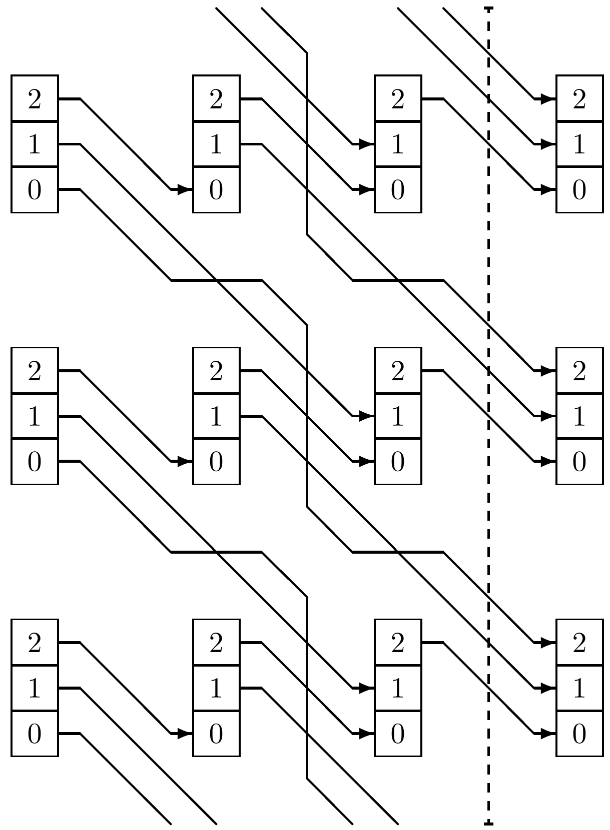
part r of $A^{r,s}$ maps isomorphically to part $n-r$ of B^{r+s}.



Putting the column with the B as described in the theorem in front of or after the other columns of the original spectral sequence, will produce a spectral sequence converging to 0. $\boxed{QED}$

In this theorem a cyclic symmetry appears between the columns of the spectral sequence together with the convergence target. It is possible to rotate the entries of such a spectral sequence by column (with the correct shift), while the formation remains a spectral sequence: $(E_1^{p,q})_{0\leq p\leq n} \Rightarrow 0$ implies $(\tilde{E}_1^{p,q}) \Rightarrow 0$ with $\tilde{E}_1^{p,q} = E_1^{p-1,q}$ for $p \neq 0$ and $\tilde{E}_1^{0,q} = E_1^{n,q-n+1}$.

The next proposition shows, that it is possible to combine two spectral sequences coinciding in one column into a single spectral sequence which will no longer contain the doubly appearing column (splicing or, in the chemical analogon, binding).

3 Proposition. *Let $E_1^{p,q} = (A^{p,q})_{0\leq p\leq n} \Rightarrow 0$ and $\bar{E}_2^{p,q} = (B^{p,q})_{0\leq p\leq m} \Rightarrow 0$ be two spectral sequences converging to 0. Furthermore let $A^{n,q} = B^{0,q}$ for all q. Then there is a spectral sequence converging to 0 with E_1-term:*

$$\begin{array}{ccccccc} A^{0,q+1} & \dots & A^{n-1,q+1} & B^{1,q+1} & \dots & B^{m,q+1} \\ A^{0,q} & \dots & A^{n-1,q} & B^{1,q} & \dots & B^{m,q} \\ A^{0,q-1} & \dots & A^{n-1,q-1} & B^{1,q-1} & \dots & B^{m,q-1} \end{array}$$

The maps within the A-section of the spectral sequence stay the same as before. The same goes for the B-section. The maps from the A-section to the B-section arise essentially by composition of the respective maps of the two original spectral sequences.

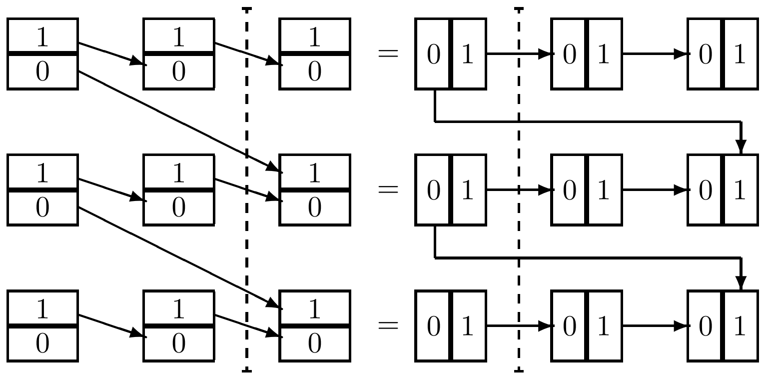
Proof. Explicitly there are defined on each $A = A^{n,q} = B^{0,q}$ two filtrations $0 \subseteq N_1 \subseteq \ldots \subseteq N_n = A$ and $0 \subseteq M_1 \subseteq \ldots \subseteq M_m = A$ (one from each spectral sequence). These induce the filtration $((N_i \cap M_j) + N_{i-1})/N_{i-1}$ on N_i/N_{i-1} and $((M_j \cap N_i) + M_{j-1})/M_{j-1}$ on M_j/M_{j-1}. The quotients of these filtrations are canonically isomorphic to $N_i \cap M_j/N_{i-1} \cap M_{j-1}$. These isomorphisms will produce the desired maps:

The filtration on the $A^{r,s}$ and the $B^{t,u}$ decomposes these into n parts (part 1,..., part n) respectively m parts (part 1,..., part m).

In the composed spectral sequence the parts of $A^{r,s}$ for $r < n$ are mapped as before, except part $r+1$. This originally was mapped to part $n-r$ of $A^{n,-n+r+s+1}$ (namely N_{n-r}/N_{n-r-1}). Now this further decomposes by the M-filtration into m subparts.

The parts of $B^{t,u}$ are also mapped as before except part t. This one originally came from part $m-t+1$ of $B^{0,t+u-1}$ (namely M_{m-t+1}/M_{m-t}). Now this further decomposes by the N-filtration into n finer subparts.

In the composed spectral sequence now subpart t of part $r+1$ of $A^{r,s}$ is defined to map to subpart $r+1$ of part t of $B^{t,u}$. These are isomorphic according to the above discussion. $\boxed{QED}$



The long exact sequences $\to A^i \to A^{i+1} \to A^{i+2} \to$ exactly correspond to those E_1-spectral sequences which consist of three columns and converge to 0, i. e.:

$$\begin{pmatrix} \dots & & \\ A^{i+3} & A^{i+4} & A^{i+5} \\ A^{i} & A^{i+1} & A^{i+2} \\ \dots & & \end{pmatrix}$$

The coinciding of two long exact sequences at every third entry exactly corresponds to the coinciding of the associated spectral sequences in one column.

By repeated splicing and rotating according to the above propositions every tree of bonded long exact sequences can be transformed into a spectral sequence. The ordering of columns will be uniquely determined up to cyclic shift and does not depend on the sequence of the splicing along the graph, which can be seen as follows: At first make every existing vertex of the graph 3-valued by adjoining new edges and as vertices the third part of an exact sequence. Number the edges of the new graph Γ with the level 0, 1, 2 of the bond. Now every such graph can be embedded uniquely into the plane such that the respective three levels of a vertex are given the same predetermined orientation. The loose ends of edges, respectively the third parts of exact sequences are the columns of the spectral sequence according to the cyclic order in which they appear when looked at from one vertex of Γ.

Not every spectral sequence arises in this manner from a compound of long exact sequences, but all of practical importance: For a complex C filtered by subcomplexes $0 = C_0 \subseteq C_1 \subseteq C_2 \ldots \subseteq C_n = C$ there is a well-known construction leading to a spectral sequence. In our setting this arises as follows: There are short exact sequences of complexes

$$\begin{array}{ccccccccccc}
\dots & & & & & & & & & & \\
0 & \to & C_i & \to & C_i+1 & \to & C_{i+1}/C_i & \to & 0 & & \\
 & & & & \| & & & & & & \\
 & & 0 & \to & C_{i+1} & \to & C_{i+2} & \to & C_{i+2}/C_{i+1} & \to & 0 \\
 & & & & & & & & \dots & &
\end{array}$$

These are bonded such that the corresponding long exact homology sequences are bonded in the above sense, giving a linear tree of long exact sequences and by the above argument a spectral sequence.

The construction of a spectral sequence by an exact couple is also contained in the above. An exact couple is a special linear tree of long exact sequences.

4 Example. As an example the spectral sequence of Grothendieck for the derived functors of a composition of two functors will be established. Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be two left exact functors of abelian categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, where $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and such that F maps injective objects to G-acyclic objects.

Let $0 \to M \to I \to Q \to 0$ the first step of an injective resolution of $M \in \mathrm{Ob}(\mathcal{A})$.

Then $0 \to F(I)/F(M) \to F(Q) \to R^1F(M) \to 0$ is exact in $\mathcal{B}$ (F-sequence). The associated G-Sequence is the line in the following commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
0 \to GF(I)/GF(M) & = & GF(I)/GF(M) & & \\
\downarrow & & \downarrow & & \\
0 \to G(F(I)/F(M)) & \longrightarrow & GF(Q) & \to & G(R^1F(M)) \to R^1G(F(I)/F(M)) \\
\downarrow & & \downarrow & & \\
R^1G(F(M)) & --\to & R^1(GF)(M) & & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

The first column is the G-sequence for $0 \to F(M) \to F(I) \to F(I)/F(M) \to 0$. This furthermore gives isomorphisms $R^iG(F(I)/F(M)) = R^{i+1}G(F(M))$ for $i \geq 1$. The second column is the GF-sequence for $0 \to M \to I \to Q \to 0$. The snake lemma for the two columns gives an identification of the two last cokernels. This establishes the long exact sequence

$$\begin{aligned}
0 \to R^1G(F(M)) \to R^1(GF)(M) \to G(R^1F)(M) \to \\
R^2G(F(M)) \to R^1G(F(Q)) \to (R^1G)(R^1F)(M) \to
\end{aligned}$$

$$R^3G(F(M)) \to R^2G(F(Q)) \to (R^2G)(R^1F)(M) \to \dots$$

At every third place we have the entry $R^iG(F(M))$ and similarly $R^iG(F(Q))$. If the same sequence is written for a resolution of Q instead of M, the two long exact sequences are bonded. Let $Q_0 = M$ and $Q_1 = Q$ and $0 \to Q_i \to I_i \to Q_{i+1} \to 0$ be exact. Then this leads to a compound of long exact sequences. Schematically this looks as follows:

$$\begin{tikzcd}
0 \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \\
& & & & 0 \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \\
& & & & & & & & 0 \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

The long exact sequences are independent of the choice of injective resolution of M. This follows from the homotopy of any two resolutions.

To make it into a proper compound of long exact sequences add further objects and isomorphisms as follows:

$$\begin{tikzcd}
0 \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \\
& & \bullet \arrow[r, equal] & \bullet \arrow[d, equal] & 0 \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \\
& & & \bullet \arrow[r, equal] & \bullet \arrow[d, equal] & & \bullet \arrow[r, equal] & \bullet \arrow[d, equal] & 0 \arrow[r] & \bullet \arrow[r] & \bullet \arrow[d, equal] \\
& & & & {} & & & {} & & & {}
\end{tikzcd}$$

The respective third entries which are unmatched are in sequence number i:

$$R^qG(R^1F)(Q_i) = R^qG(R^{i+1}F)(M), q \geq 0$$

The additional unmatched entries in sequence number 1 are $R^qG(F(M)),\ q \geq 0$. The sequence of unmatched entries in the last sequence starts as $R^1(GF)(Q_i), i \geq 0$ or $R^q(GF)(M),\ q \geq 1$.

To form an E_2-spectral sequence these will have to be the lines of the spectral sequence. If one writes the spectral sequence as

$$\mathrm{E}_2^{p,q} = R^pG(R^qF(M)) \Rightarrow R^{p+q}(GF)(M)$$

one easily checks that the lines are in the correct relative position.

References

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