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Holomorphic functions on Banach spaces

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Abstract. This is a survey about some problems from the theory of holomorphic functions on Banach spaces which have attracted the attention of many researchers during the last thirty years.

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1 Introduction

This is a survey about some problems from the theory of holomorphic functions on Banach spaces which have attracted the attention of many researchers during the last thirty years. The problems we selected, besides being related among themselves, are also closely connected with some celebrated problems from Banach space theory.

In Sections 2 and 3 we present some basic facts from the theory of polynomials and holomorphic mappings between Banach spaces.

Section 4 is devoted to the study of the space of holomorphic mappings of bounded type. The study of this space was one of the main motivations for the celebrated theorem of Josefson [40] and Nissenzweig [53].

Section 5 is devoted to the study of the space of bounded holomorphic mappings. The still unsolved problem as to whether the space of bounded holomorphic functions on the open unit disc has the approximation property, is equivalent to a problem of approximating certain bounded holomorphic mappings on the open unit disc with values in a Banach space.

Section 6 is devoted to the question of existence of an infinite dimensional Banach space for which all spaces of homogeneous polynomials are reflexive. The celebrated Banach space constructed by Tsirelson [61] was the first example of a space answering that question.

Sections 7 and 8 are devoted to the study of various weak continuity properties for polynomials or entire mappings between Banach spaces. The still unsolved problem as to whether every entire function which is weakly continuous on bounded sets, is necessarily weakly uniformly continuous on bounded sets, is discussed in detail. A pioneer result of Dineen [21], and the recent contribution of Carrión [12], are also presented here.

Section 9 is devoted to the still unsolved problem, raised by Michael [42] in 1952, as to whether every complex homomorphism on a commutative Fréchet algebra is necessarily continuous. It is shown that the Michael problem for arbitrary commutative Fréchet algebras, is equivalent to the corresponding problem for the Fréchet algebra of all entire functions of bounded type on some infinite dimensional Banach space.

The only new results in this survey appear in Section 9, where we present reformulations of the Michael problem in terms of algebras of entire functions on Tsirelson's space. The other results are usually given without proof. We have occasionally included a proof, or the sketch of a proof, when it is illuminating and technically simple.

This survey is based on the notes of a three-lecture minicourse delivered at the 56th Brazilian Analysis Seminar, held at the Universidade Federal Fluminense, in Niterói, Rio de Janeiro, during November 20-23, 2002. I wish to thank the organizers, and in particular Professor Chaim Hönig, for the kind invitation to deliver the minicourse.

2 Multilinear mappings and polynomials

$\mathbb{N}$ denotes the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}$ and $\mathbb{C}$ denote the field of all real numbers and all complex numbers, respectively.

The letters E and F always denote complex Banach spaces, not reduced to $\{0\}$. B_E denotes the closed unit ball of E, whereas S_E denotes the unit sphere of E.

$\mathcal{L}(E; F)$ denotes the Banach space of all continuous linear mappings from E into F, with the usual norm, that is

$$\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\}.$$

When $F = \mathbb{C}$ we write E' instead of $\mathcal{L}(E; \mathbb{C})$.

For each $m \in \mathbb{N}$ $\mathcal{L}(^m E; F)$ denotes the vector space of all m-linear mappings from E^m into F. $\mathcal{L}(^m E; F)$ is a Banach space under the norm

$$\|A\| = \sup\{\|A(x_1, \ldots, x_m)\| : \|x_k\| \leq 1\}.$$

Given $A \in \mathcal{L}(^m E; F)$ and $x \in E$, we define $Ax^m = A(x, \ldots, x)$. For convenience we also define $\mathcal{L}(^0 E; F) = F$ and $Ax^0 = A$ for each $A \in \mathcal{L}(^0 E; F)$ and $x \in E$. When $F = \mathbb{C}$ we write $\mathcal{L}(^m E)$ instead of $\mathcal{L}(^m E; \mathbb{C})$.

$\mathcal{L}^s(^m E; F)$ denotes the subspace of all $A \in \mathcal{L}(^m E; F)$ which are *symmetric*, that is $A(x_1, \ldots, x_m) = A(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ for each permutation σ of

$\{1, \ldots, m\}$. When $F = \mathbb{C}$, we write $\mathcal{L}^s({}^m E)$ instead of $\mathcal{L}^s({}^m E; \mathbb{C})$. S_m denotes the group of all permutations of $\{1, \ldots, m\}$.

1 Proposition. *$\mathcal{L}^s({}^m E; F)$ is a complemented subspace of $\mathcal{L}({}^m E; F)$. If we define*

$$A^s(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\sigma \in S_m} A(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$$

for every $A \in \mathcal{L}({}^m E; F)$, then the mapping $A \to A^s$ is a norm 1 projection from $\mathcal{L}({}^m E; F)$ onto $\mathcal{L}^s({}^m E; F)$.

2 Proposition (Newton's binomial formula)**.** *Given $A \in \mathcal{L}^s({}^m E; F)$ and $x, y \in E$, the following formula holds:*

$$A(x+y)^m = \sum_{k=0}^{m} A x^{m-k} y^k.$$

3 Proposition (polarization formula)**.** *Given $A \in \mathcal{L}^s({}^m E; F)$ and $x_1, \ldots, x_m \in E$, the following formula holds:*

$$A(x_1, \ldots, x_m) = \frac{1}{m! 2^m} \sum_{\theta_k = \pm 1} \theta_1 \ldots \theta_m A(\theta_1 x_1 + \cdots + \theta_m x_m)^m.$$

A mapping $P : E \to F$ is said to be a *continuous m-homogeneous polynomial* if there is an $A \in \mathcal{L}({}^m E; F)$ such that $P(x) = A x^m$ for every $x \in E$. In this case we write $P = \hat{A}$. $\mathcal{P}({}^m E; F)$ denotes the vector space of all continuous m-homogeneous polynomials from E into F. $\mathcal{P}({}^m E; F)$ is a Banach space under the norm

$$\|P\| = \sup\{\, \|P(x)\| : \|x\| \le 1 \,\}.$$

When $F = \mathbb{C}$ we write $\mathcal{P}({}^m E)$ instead of $\mathcal{P}({}^m E; \mathbb{C})$.

4 Proposition. *The mapping $A \to \hat{A}$ is a topological isomorphism between $\mathcal{L}^s({}^m E; F)$ and $\mathcal{P}({}^m E; F)$. For each $A \in \mathcal{L}^s({}^m E; F)$ the following inequalities hold:*

$$\|\hat{A}\| \le \|A\| \le \frac{m^m}{m!} \|\hat{A}\| \le e^m \|\hat{A}\|.$$

5 Proposition ([11])**.** *If $m \le n$, then $\mathcal{P}({}^m E; F)$ is isomorphic to a complemented subspace of $\mathcal{P}({}^n E; F)$.*

A mapping $P : E \to F$ is said to be a *continuous polynomial* if it can be represented as a sum $P = P_0 + P_1 + \cdots + P_m$, with $P_k \in \mathcal{P}({}^k E; F)$ for $k = 0, 1, \ldots, m$. $\mathcal{P}(E; F)$ denotes the vector space of all continuous polynomials from E into F. The representation $P = P_0 + P_1 + \cdots + P_m$ of each $P \in \mathcal{P}(E; F)$ is unique. When $F = \mathbb{C}$ we write $\mathcal{P}(E)$ instead of $\mathcal{P}(E; \mathbb{C})$.

A series of mappings of the form $\sum_{m=0}^{\infty} P_m(x-a)$, with $a \in E$ and $P_m \in \mathcal{P}(^m E; F)$ for every m, is said to be a *power series* from E into F. The *radius of convergence* of the power series is the supremum of all $r > 0$ such that the series converges uniformly on the ball $B(a;r)$.

6 Proposition (Cauchy-Hadamard formula)**.** *The radius of convergence of the power series $\sum_{m=0}^{\infty} P_m(x-a)$ is given by the formula*

$$\frac{1}{R} = \limsup_{m} \|P_m\|^{1/m}.$$

Proposition 5 is due to Aron and Schottenloher [11]. The other results in this section are well known and can be found in the books of Nachbin [51] or Mujica [44].

3 Holomorphic mappings

The letter U always denotes a nonvoid open subset of E.

7 Theorem. *For a mapping $f : U \to F$, the following conditions are equivalent:*

(a) For each $a \in U$ there is an $A \in \mathcal{L}(E;F)$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - A(x-a)}{\|x-a\|} = 0.$$

(b) For each $a \in U$ there is a power series $\sum_{m=0}^{\infty} P_m(x-a)$ which converges to $f(x)$ uniformly on some ball $B(a;r) \subset U$.

(c) f is continuous and, for each $a \in U$, $b \in E$ and $\psi \in F'$, the function $\lambda \to \psi \circ f(a + \lambda b)$ is holomorphic in the usual sense on the open set $\{ \lambda \in \mathbb{C} : a + \lambda b \in U \}$.

A mapping $f : U \to F$ is said to be *holomorphic* if it verifies the equivalent conditions in Theorem 7. The mapping A in condition (a) is uniquely determined by f and a, and is called the *differential* of f at a. The power series $\sum_{m=0}^{\infty} P_m(x-a)$ in condition (b) is also uniquely determined by f and a, and is called the *Taylor series* of f at a. We denote by $P^m f(a)$ the polynomial P_m, and by $A^m f(a)$ the unique member of $\mathcal{L}^s(^m E; F)$ such that $[A^m f(a)]^\wedge = P^m f(a)$. $\mathcal{H}(U;F)$ denotes the vector space of all holomorphic mappings from U into F. When $F = \mathbb{C}$ we write $\mathcal{H}(U)$ instead of $\mathcal{H}(U;\mathbb{C})$. The Cauchy integral formulas, which we now state, are very important.

8 Theorem (Cauchy integral formula)**.** *Let $f \in \mathcal{H}(U;F)$, and let $a \in U$, $t \in E$ and $r > 0$ such that $a + \zeta t \in U$ for every $|\zeta| \leq r$. Then for every $|\lambda| < r$ the following formula holds:*

$$f(a + \lambda t) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(a + \zeta t)}{\zeta - \lambda} d\zeta.$$

9 Theorem (Cauchy integral formula)**.** *Let $f \in \mathcal{H}(U;F)$, and let $a \in U$, $t \in E$ and $r > 0$ such that $a + \zeta t \in U$ for every $|\zeta| \leq r$. Then for every $m \in \mathbb{N}_0$ the following formula holds:*

$$P^m f(a)(t) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(a + \zeta t)}{\zeta^{m+1}} d\zeta.$$

10 Theorem (Cauchy integral formula)**.** *Let $f \in \mathcal{H}(U;F)$, and let $a \in U$, $t_1, \ldots, t_n \in E$ and $r > 0$ such that $a + \zeta_1 t_1 + \cdots + \zeta_n t_n \in U$ whenever $|\zeta_k| \leq r$ for $k = 1, 2, \ldots, n$. Then for every $m \in \mathbb{N}_0$ and every multi-index $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ such that $\alpha_1 + \cdots + \alpha_n = m$, the following formula holds:*

$$\begin{aligned} &A^m f(a) t_1^{\alpha_1} \cdots t_n^{\alpha_n} \\ &\qquad = \frac{\alpha_1! \cdots \alpha_n!}{m!(2\pi i)^n} \int_{|\zeta_n|=r} \cdots \int_{|\zeta_1|=r} \frac{f(a + \zeta_1 t_1 + \cdots + \zeta_n t_n)}{\zeta_1^{\alpha_1+1} \cdots \zeta_n^{\alpha_n+1}} d\zeta_1 \cdots d\zeta_n. \end{aligned}$$

11 Theorem. *Let $f \in \mathcal{H}(U;F)$. If U is balanced, then the Taylor series of f at the origin converges to f uniformly on a suitable neighborhood of each compact subset of U.*

We recall that a set $A \subset E$ is said to be *balanced* if $\lambda A \subset A$ for every $|\lambda| \leq 1$.

Let $f \in \mathcal{H}(U;F)$, and let $a \in U$. $r_c f(a)$ denotes the radius of convergence of the Taylor series of f at a. $r_b f(a)$ denotes the supremum of all $r > 0$ such that $B(a;r) \subset U$ and f is bounded on $B(a;r)$. $r_c f(a)$ and $r_b f(a)$ are related as follows.

12 Theorem. *Let $f \in \mathcal{H}(U;F)$, and let $a \in U$. Then*

$$r_b f(a) = \min\{\, r_c f(a), d(a, E \setminus U)\,\}.$$

If X is a topological space, then $\mathcal{C}(X;F)$ denotes the vector space of all continuous mappings from X into F. $\mathcal{C}(X;F)$ is a locally convex space for the compact-open topology, that is the topology τ_c defined by the seminorms

$$p_K(f) = \sup\{\, \|f(x)\| : x \in K \,\},$$

where K varies over the compact subsets of X.

13 Proposition. *$\mathcal{H}(U;F)$ is a closed subspace of $\mathcal{C}(U;F)$.*

14 Proposition (Montel's theorem)**.** *Every bounded subset of $\mathcal{H}(U)$ is relatively compact.*

All the results in this section are well known and can be found in the books of Nachbin [51] or Mujica [44].

4 Holomorphic mappings of bounded type

A set $A \subset U$ is said to be U-*bounded* if A is bounded and $d(A, E \setminus U) > 0$. A mapping $f \in \mathcal{H}(U;F)$ is said to be of *bounded type* if f is bounded on all U-bounded sets. $\mathcal{H}_b(U;F)$ denotes the vector space of all holomorphic mappings of bounded type from U into F. Each of the sets

$$U_n = \{\, x \in U : \|x\| < n \text{ and } d(x, E \setminus U) > 1/n \,\}$$

is U-bounded, and each U-bounded set is contained in some U_n. $\mathcal{H}_b(U;F)$ is a Fréchet space for the topology defined by the seminorms

$$p_n(f) = \sup\{\, \|f(x)\| : x \in U_n \,\}.$$

When $F = \mathbb{C}$ we write $\mathcal{H}_b(U)$ instead of $\mathcal{H}_b(U;\mathbb{C})$.

If E is finite dimensional, then each U-bounded set is relatively compact in U, and hence $\mathcal{H}(U;F) = \mathcal{H}_b(U;F)$. Though it is far from obvious, the converse is also true, as we shall see. We begin with the following result of Dineen [19].

15 Theorem ([19])**.** *Suppose there exists a sequence $(\phi_m) \subset E'$ such that $\|\phi_m\| = 1$ for every m and $\lim_m \phi_m(x) = 0$ for every $x \in E$. Then $\mathcal{H}(U;F) \neq \mathcal{H}_b(U;F)$ for every open set $U \subset E$ and every Banach space F.*

Proof. Let $a \in U$ and $r > 0$ such that $B(a;2r) \subset U$. Let $b \in F$ with $\|b\| = 1$, and let $f : E \to F$ be defined by

$$f(x) = \sum_{m=1}^{\infty} r^{-m} [\phi_m(x-a)]^m b$$

for every $x \in E$. One can readily prove that $f \in \mathcal{H}(E;F)$, and therefore $f \in \mathcal{H}(U;F)$. On the other hand, it follows from the Cauchy-Hadamard formula 6 that $r_c f(a) = r$, and therefore $r_b f(a) = r < d(a, E \setminus U)$, by Theorem 12. Thus $f \notin \mathcal{H}_b(U;F)$. QED

The following result, a direct application of the Hahn-Banach theorem, is well known.

16 Proposition. *If E is a separable, infinite dimensional Banach space, then there exists a sequence $(\phi_m) \subset E'$ such that $\|\phi_m\| = 1$ for every m and $\lim_m \phi_m(x) = 0$ for every $x \in E$.*

Proof. Let (M_m) be an increasing sequence of finite dimensional subspaces of E such that $\cup_{m=1}^{\infty} M_m$ is dense in E. By the Hahn-Banach theorem there exists a sequence $(\phi_m) \subset E'$ such that $\|\phi_m\| = 1$ for every m and $\phi_m(x) = 0$ for every $x \in M_m$. Since $\cup_{m=1}^{\infty} M_m$ is dense in E, it follows easily that $\phi_m(x) \to 0$ for every $x \in E$. QED

Thus every separable, infinite dimensional Banach space verifies the hypothesis of Dineen's Theorem 15. This motivates the following problems.

17 Problem ([59]). If E is an infinite dimensional Banach space, does there exist a sequence $(\phi_m) \subset E'$ such that $\|\phi_m\| = 1$ for every m and $\lim_m \phi_m(x) = 0$ for every $x \in E$?

18 Problem ([55]). If E is an infinite dimensional Banach space, does E have a separable, infinite dimensional quotient space?

Problem 17 was raised by Thorp and Whitley in [59], whereas Problem 18 was raised by Rosenthal in [55]. It follows from Proposition 16 that an affirmative solution to Problem 18 would imply an affirmative solution to Problem 17, a result noticed by Thorp and Whitley in [60]. Problem 17 was solved independently by Josefson [40] and Nissenzweig [53], but Problem 18 remains unsolved.

19 Theorem ([40], [53]). *If E is an infinite dimensional Banach space, then there exists a sequence $(\phi_m) \subset E'$ such that $\|\phi_m\| = 1$ for every m and $\lim_m \phi_m(x) = 0$ for every $x \in E$.*

From Theorems 19 and 15 we immediately obtain the following corollary.

20 Corollary. *For a Banach space E, the following conditions are equivalent:*

(a) E is infinite dimensional.

(b) $\mathcal{H}(U;F) \neq \mathcal{H}_b(U;F)$ for every open set $U \subset E$ and every Banach space F.

(c) $\mathcal{H}(E) \neq \mathcal{H}_b(E)$.

We refer to Mujica [47] for a survey of results concerning Problem 18, and to Diestel [17] or Mujica [47], [50] for short proofs of the Josefson-Nissenzweig theorem.

5 Bounded holomorphic mappings

$\mathcal{H}^\infty(U;F)$ denotes the vector space of all $f \in \mathcal{H}(U;F)$ which are bounded on U. $\mathcal{H}^\infty(U;F)$ is a Banach space for the norm

$$\|f\| = \sup\{\, \|f(x)\| : x \in U \,\}.$$

When $F = \mathbb{C}$ we write $\mathcal{H}^\infty(U)$ instead of $\mathcal{H}^\infty(U;\mathbb{C})$.

$\mathcal{H}^\infty(U)$ is always a dual Banach space. This follows readily from the following useful result of Ng [52], which is an abstract version of an old result of Dixmier [23].

21 Theorem ([23], [52])**.** *If there exists a Hausdorff locally convex topology τ on E such that B_E is τ-compact, then E is isometrically isomorphic to a dual Banach space. More precisely, if F denotes the Banach space*

$$F = \{\, \phi \in E' : \phi|B_E \text{ is } \tau\text{-continuous} \,\},$$

then the evaluation mapping $J : E \to F'$ is an isometric isomorphism.

If τ_c denotes the compact-open topology on $\mathcal{H}^\infty(U)$, then we obtain the following proposition.

22 Proposition. *$\mathcal{H}^\infty(U)$ is isometrically isomorphic to a dual Banach space. More precisely, if $G^\infty(U)$ denotes the Banach space*

$$G^\infty(U) = \{\, \Phi \in \mathcal{H}^\infty(U)' : \Phi|B_{\mathcal{H}^\infty(U)} \text{ is } \tau_c\text{-continuous} \,\},$$

then the evaluation mapping $J : \mathcal{H}^\infty(U) \to G^\infty(U)'$ is an isometric isomorphism.

Proof. It follows from Proposition 14 that $B_{\mathcal{H}^\infty(U)}$ is τ_c-compact. Then an application of Theorem 21 completes the proof. QED

Proposition 22 has an interesting history, showing that mathematical discoveries are sometimes fortuitous. I was visiting Dublin in 1981, when Seán Dineen was writing his book [20]. When I saw the statement of Proposition 22 as Exercise 2.100 in the book, I asked Dineen how he proved the result. He answered that he didn't know, since he had only seen the result stated somewhere without proof. Shortly afterwards I came across the Dixmier-Ng theorem in a book of Holmes [37], and thus found the proof above. This story and some additional comments appear in Dineen's books [20, p. 417] and [22, pp. 238–239].

Proposition 22 may be regarded as a linearization theorem, since it identifies the bounded holomorphic functions on U with the continuous linear functionals on $G^\infty(U)$. This was the motivation for the following, much stronger, linearization theorem, due to the author [46].

23 Theorem ([46]). *Let $\delta_U : x \in U \to \delta_x \in G^\infty(U)$ denote the evaluation mapping, that is, $\delta_x(f) = f(x)$ for all $x \in U$ and $f \in \mathcal{H}^\infty(U)$. Then:*

(a) $\delta_U \in \mathcal{H}^\infty(U; G^\infty(U))$.

(b) For each Banach space F and each $f \in \mathcal{H}^\infty(U; F)$, there is a unique $T_f \in \mathcal{L}(G^\infty(U); F)$ such that $T_f \circ \delta_U = f$.

(c) The mapping $f \in \mathcal{H}^\infty(U; F) \to T_f \in \mathcal{L}(G^\infty(U); F)$ is an isometric isomorphism.

(d) f has a finite dimensional range if and only if T_f is a finite rank operator.

(e) f has a relatively compact (resp. weakly relatively compact) range if and only if T_f is a compact (resp. weakly compact) operator.

We recall that a Banach space E is said to have the *approximation property* if given a compact set $K \subset E$ and $\epsilon > 0$, there is a finite rank operator $T \in \mathcal{L}(E; E)$ such that $\|Tx - x\| < \epsilon$ for every $x \in K$. The approximation property was introduced by Grothendieck [33], who obtained the following theorems.

24 Theorem ([33]). *For a Banach space E, the following conditions are equivalent:*

(a) E has the approximation property.

(b) Given a Banach space F, an operator $S \in \mathcal{L}(F; E)$, a compact set $L \subset F$, and $\epsilon > 0$, there is a finite rank operator $T \in \mathcal{L}(F; E)$ such that $\|Ty - Sy\| < \epsilon$ for every $y \in L$.

(c) Given a Banach space F, an operator $S \in \mathcal{L}(E; F)$, a compact set $K \subset E$, and $\epsilon > 0$, there is a finite rank operator $T \in \mathcal{L}(E; F)$ such that $\|Tx - Sx\| < \epsilon$ for every $x \in K$.

(d) Given sequences $(x_n) \subset E$ and $(\phi_n) \subset E'$ such that $\sum_{n=1}^\infty \|\phi_n\| \|x_n\| < \infty$ and $\sum_{n=1}^\infty \phi_n(x) x_n = 0$ for every $x \in E$, we have that $\sum_{n=1}^\infty \phi_n(x_n) = 0$.

(e) Given a Banach space F, a compact operator $S \in \mathcal{L}(F; E)$, and $\epsilon > 0$, there is a finite rank operator $T \in \mathcal{L}(F; E)$ such that $\|T - S\| < \epsilon$.

25 Theorem ([33]). *E' has the approximation property if and only if, given a Banach space F, a compact operator $S \in \mathcal{L}(E; F)$, and $\epsilon > 0$, there is a finite rank operator $T \in \mathcal{L}(E; F)$ such that $\|T - S\| < \epsilon$.*

26 Theorem ([33]). *If E' has the approximation property, then E has the approximation property as well.*

After introducing an auxiliary topology on $\mathcal{H}^\infty(U;F)$, the author [46] used Theorems 23 and 24 to prove the following theorem.

27 Theorem ([46]). *If U is a balanced, bounded, open set in E, then $G^\infty(U)$ has the approximation property if and only if E has the approximation property.*

By using Theorems 23 and 25 the author obtained the following theorem.

28 Theorem ([46]). *If U is an open subset of E, then $\mathcal{H}^\infty(U)$ has the approximation property if and only if, given a Banach space F, a mapping $f \in \mathcal{H}^\infty(U;F)$, with a relatively compact range, and $\epsilon > 0$, there is a mapping $g \in \mathcal{H}^\infty(U;F)$, with a finite dimensional range, such that $\|g - f\| < \epsilon$.*

Grothendieck [33] proved that several of the classical Banach spaces that occur in analysis have the approximation property, and raised the problem as to whether every Banach space has the approximation property. This problem remained open for nearly twenty years, and was finally solved by Enflo [26], who constructed the first example of a Banach space without the approximation property. Among the classical Banach spaces, the problems as to whether the space of operators $\mathcal{L}(\ell_2;\ell_2)$ and the space of holomorphic functions $\mathcal{H}^\infty(\Delta)$ (where Δ denotes the open unit disc) have the approximation property were pointed out as open in the book of Lindenstrauss and Tzafriri [41]. The problem concerning $\mathcal{L}(\ell_2;\ell_2)$ was solved by Szankowski [58], who proved that $\mathcal{L}(\ell_2;\ell_2)$ does not have the approximation property. But the problem concerning $\mathcal{H}^\infty(\Delta)$ remains unsolved. By Theorem 28 that problem is equivalent to the following problem.

29 Problem ([46]). Given a Banach space F, a mapping $f \in \mathcal{H}^\infty(\Delta;F)$, with a relatively compact range, and $\epsilon > 0$, does there exist a mapping $g \in \mathcal{H}^\infty(\Delta;F)$, with a finite dimensional range, such that $\|g - f\| < \epsilon$?

6 Reflexive spaces of homogeneous polynomials

By using topological tensor products, Ryan [56] proved that $\mathcal{P}(^mE)$ is always a dual Banach space. Ryan's result is also a direct consequence of the Dixmier-Ng Theorem 21.

30 Proposition ([56]). *$\mathcal{P}(^mE)$ is isometrically isomorphic to a dual Banach space. More precisely, if $Q(^mE)$ denotes the Banach space*

$$Q(^mE) = \{\, \Phi \in \mathcal{P}(^mE)' : \Phi|B_{\mathcal{P}(^mE)} \text{ is } \tau_c\text{-continuous} \,\},$$

then the evaluation mapping $J : \mathcal{P}(^mE) \to Q(^mE)'$ is an isometric isomorphism.

By using topological tensor products Ryan [56] obtained a linearization theorem for homogeneous polynomials. The following version of Ryan's linearization theorem in terms of $Q(^mE)$ is due to the author [46].

31 Theorem ([56], [46]). *Let $\delta_m : x \in E \to \delta_x \in Q(^mE)$ denote the evaluation mapping, that is $\delta_x(P) = P(x)$ for every $x \in E$ and $P \in \mathcal{P}(^mE)$. Then:*

(a) $\delta_m \in \mathcal{P}(^mE; Q(^mE))$.

(b) For each Banach space F and each $P \in \mathcal{P}(^mE; F)$, there is a unique $T_P \in \mathcal{L}(Q(^mE); F)$ such that $T_P \circ \delta_m = P$.

(c) The mapping $P \in \mathcal{P}(^mE; F) \to T_P \in \mathcal{L}(Q(^mE); F)$ is an isometric isomorphism.

(d) P has a finite dimensional range if and only if T_P is a finite rank operator.

(e) P is a compact (resp. weakly compact)polynomial if and only if T_P is a compact (resp weakly compact) operator.

We remark that $P \in \mathcal{P}(^mE; F)$ is said to be *compact* (resp. *weakly compact*) if $P(B_E)$ is a relatively compact (resp. weakly relatively compact) subset of F. The compact open topology on $\mathcal{P}(^mE)$ has the following important property.

32 Theorem ([43]). *The compact open topology τ_c is the finest topology on $\mathcal{P}(^mE)$ which coincides on each bounded set with the topology of pointwise convergence.*

33 Corollary ([46]). $Q(^mE) = (\mathcal{P}(^mE), \tau_c)'$.

Ryan [56] began a systematic study of the question of reflexivity of $\mathcal{P}(^mE)$. Before stating the next result, which collects results of several authors, but originated from Ryan's thesis [56], we introduce some terminology.

A polynomial $P \in \mathcal{P}(^mE; F)$ is said to be of *finite type* if it can be represented as a sum $P(x) = \sum_{k=1}^{p} [\phi_k(x)]^m b_k$, with $\phi_k \in E'$ and $b_k \in F$. $\mathcal{P}_f(^mE; F)$ denotes the subspace of all $P \in \mathcal{P}(^mE; F)$ of finite type. When $F = \mathbb{C}$ we write $\mathcal{P}_f(^mE)$ instead of $\mathcal{P}_f(^mE; \mathbb{C})$.

The following result gives some necessary and sufficient conditions for reflexivity of $\mathcal{P}(^mE)$.

34 Theorem ([56], [2], [49]). *For a reflexive Banach space E and $m \in \mathbb{N}$, consider the following conditions:*

(a) $\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\|\cdot\|}$.

(b) Each $P \in \mathcal{P}(^mE)$ is weakly continuous on bounded sets.

(c) Each $P \in \mathcal{P}(^mE)$ *is weakly sequentially continuous.*

(d) For each $P \in \mathcal{P}(^mE)$*, there is an* $x \in B_E$ *such that* $|P(x)| = \|P\|$.

(e) $\mathcal{P}(^mE)$ *is reflexive.*

(f) $(\mathcal{P}(^mE), \tau_c)' = (\mathcal{P}(^mE), \|\cdot\|)'$.

Then the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ *always hold. If* E *has the approximation property, then* $(f) \Rightarrow (a)$.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear.

$(c) \Rightarrow (d)$: Given $P \in \mathcal{P}(^mE)$, there is a sequence $(x_n) \subset B_E$ such that $\lim |P(x_n)| = \|P\|$. Since E is reflexive, B_E is weakly compact. By the Eberlein-Smulian theorem the sequence (x_n) has a subsequence (x_{n_k}) which converges weakly to some $x \in B_E$. Since P is weakly sequentially continuous, it follows that $|P(x)| = \lim |P(x_{n_k})| = \|P\|$.

$(d) \Rightarrow (e)$: By (d), given $P \in \mathcal{P}(^mE)$, there is an $x \in B_E$ such that

$$|T_P(\delta_x)| = |P(x)| = \|P\| = \|T_P\|.$$

Since the mapping $P \in \mathcal{P}(^mE) \to T_P \in Q(^mE)'$ is an isometric isomorphism, the James reflexivity theorem [38] guarantees that $Q(^mE)$ is reflexive. Hence $\mathcal{P}(^mE)$ is reflexive as well.

$(e) \Rightarrow (f)$: We know that

$$(\mathcal{P}(^mE), \tau_c)' = Q(^mE) \text{ and } Q(^mE)' = (\mathcal{P}(^mE), \|\cdot\|).$$

If $(\mathcal{P}(^mE), \|\cdot\|)$ is reflexive, then

$$(\mathcal{P}(^mE), \tau_c)' = Q(^mE) = (\mathcal{P}(^mE), \|\cdot\|)'.$$

$(f) \Rightarrow (a)$ when E has the approximation property: We first prove that $\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\tau_c}$. Indeed, let $P \in \mathcal{P}(^mE)$, let K be a compact subset of E, and let $\epsilon > 0$. Since P is continuous and K is compact, we can find $\delta > 0$ such that $|P(y) - P(x)| < \epsilon$ whenever $x \in K$ and $\|y - x\| < \delta$. Since E has the approximation property, there is a finite rank operator $T \in \mathcal{L}(E; E)$ such that $\|Tx - x\| < \delta$ for every $x \in K$, and therefore $|P \circ T(x) - P(x)| < \epsilon$ for every $x \in K$. Since T has finite rank, $P \circ T \in \mathcal{P}_f(^mE)$, and the claim has been proved.

Since $(\mathcal{P}(^mE), \tau_c)' = (\mathcal{P}(^mE), \|\cdot\|)'$, it follows that

$$\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\tau_c} = \overline{\mathcal{P}_f(^mE)}^{\|\cdot\|},$$

completing the proof. QED

The implications $(b) \Rightarrow (d) \Rightarrow (e)$ are due to Ryan [56], and the equivalence $(a) \Leftrightarrow (e)$, when E has the approximation property, is due to Alencar [2]. In his thesis Ryan [56] claimed that conditions (b), (d) and (e) are always equivalent, but his proof of the implication $(e) \Rightarrow (b)$ was incomplete. Condition (f) was introduced by the author in [49], where it was shown that $(e) \Rightarrow (f)$, and $(f) \Rightarrow (a)$ when E has the approximation property. In [49] it was also shown that $(f) \Rightarrow (b)$ when E has the *compact approximation property*, a property that we have not defined here, and which is weaker than the approximation property. But it seems to be unknown whether the conditions in Theorem 34 are equivalent without some sort of approximation hypothesis on E. In particular the following problem remains open.

35 Problem. Let E be a Banach space such that $\mathcal{P}(^m E)$ is reflexive.

(a) Does $\mathcal{P}(^m E) = \overline{\mathcal{P}_f(^m E)}^{\|\cdot\|}$?

(b) Is every $P \in \mathcal{P}(^m E)$ weakly continuous on bounded sets?

We remark that Alencar's proof in [2] of the equivalence $(a) \Leftrightarrow (e)$, when E has the approximation property, is completely different from the one presented here. His proof is based on some duality results of Gupta [34] and Dineen [18], and on another result of Alencar [1] of coincidence of certain classes of homogeneous polynomials.

The following problem was raised by Ryan [56].

36 Problem ([56])**.** Does there exist an infinite dimensional Banach space E such that $\mathcal{P}(^m E)$ is reflexive for every $m \in \mathbb{N}$?

Before presenting the solution to this problem, we give some additional necessary conditions for $\mathcal{P}(^m E)$ to be reflexive for every $m \in \mathbb{N}$.

37 Proposition. *$\mathcal{P}(^m \ell_p)$ contains a subspace isometrically isomorphic to ℓ_∞ whenever $m \geq p$. In particular $\mathcal{P}(^m \ell_p)$ is not reflexive whenever $m \geq p$.*

Proof. Given $a = (\alpha_k) \in \ell_\infty$, we define $P_a \in \mathcal{P}(^m \ell_p)$ by

$$P_a(x) = \sum_{k=1}^{\infty} \alpha_k \xi_k^m$$

for every $x = (\xi_k) \in \ell_p$. Then one can readily verify that the mapping

$$a \in \ell_\infty \to P_a \in \mathcal{P}(^m \ell_p)$$

is an isometric embedding. QED

38 Corollary. *If E has a quotient isomorphic to ℓ_p, then $\mathcal{P}(^m E)$ contains a subspace isomorphic to ℓ_∞ whenever $m \geq p$. In particular $\mathcal{P}(^m E)$ is not reflexive whenever $m \geq p$.*

Proof. If $S \in \mathcal{L}(E; \ell_p)$ is a surjective operator, then the mapping

$$P \in \mathcal{P}(^m\ell_p) \to P \circ S \in \mathcal{P}(^m E)$$

is an isomorphic embedding. Then the conclusion follows from Proposition 37. QED

39 Corollary ([4]). *If $\mathcal{P}(^m E)$ is reflexive for every $m \in \mathbb{N}$, then E' is reflexive and contains no subspace isomorphic to any ℓ_p.*

Proof. E' is reflexive, being isomorphic to a complemented subspace of $\mathcal{P}(^m E)$, by Proposition 5. If E' contains a subspace isomorphic to some ℓ_p, then $E'' = E$ has a quotient isomorphic to $\ell'_p = \ell_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Thus, by Corollary 38, $\mathcal{P}(^m E)$ is not reflexive whenever $m \geq q$. QED

According to Ryan [56], Proposition 37 is due to Richard Aron. Corollary 38 can be found in the book of Dineen [22], and Corollary 39 is due to Alencar, Aron and Dineen [4].

Corollary 39 shows that Problem 36 is connected with the following old problem from Banach space theory.

40 Problem. Does every infinite dimensional Banach space contain a subspace isomorphic to either c_0 or ℓ_p, for some $1 \leq p < \infty$?

Problem 40 was solved in the negative by B. Tsirelson [61], who constructed a reflexive, infinite dimensional Banach space X, with an unconditional Schauder basis (e_n), which contains no subspace isomorphic to any ℓ_p. Shortly afterwards Figiel and Johnson [28] proved that the dual X' of X is also a reflexive, infinite dimensional Banach space, with an unconditional Schauder basis, which contains no subspace isomorphic to any ℓ_p. Thus both X and X' verify the necessary conditions in Corollary 39. But Tsirelson [61] proved that X has the following additional property:

$$\sup_{n+1 \leq k \leq 2n} |\lambda_k| \leq \| \sum_{k=n+1}^{2n} \lambda_k e_k \| \leq 2 \sup_{n+1 \leq k \leq 2n} |\lambda_k|$$

for every $(\lambda_k) \subset \mathbb{C}$ and every $n \in \mathbb{N}$. By using this property of X, Alencar, Aron and Dineen [4] solved Problem 36 as follows.

41 Theorem ([4]). *If X denotes Tsirelson's space, then:*

(a) For each $m \in \mathbb{N}$, every $P \in \mathcal{P}(^m X)$ is weakly sequentially continuous.

(b) $\mathcal{P}(^m X)$ is reflexive for every $m \in \mathbb{N}$.

Alencar, Aron and Dineen [4] actually proved more. They proved that *if U is a balanced open subset of Tsirelson's space X, then $\mathcal{H}(U)$ is a reflexive locally convex space for the Nachbin compact-ported topology τ_ω*, a topology that we have not defined here, and which is stronger that the compact-open topology. They also proved that $\mathcal{P}(^m X')$ is not reflexive whenever $m \geq 2$.

Following Farmer [27] we say that a Banach space E is *polynomially reflexive* if $\mathcal{P}(^m E)$ is reflexive for every $m \in \mathbb{N}$. Farmer [27] observed that *every quotient space of a polynomially reflexive Banach space is polynomially reflexive as well.* Indeed if $S \in \mathcal{L}(E; F)$ is a surjective operator, then the mapping $P \in \mathcal{P}(^m F) \to P \circ S \in \mathcal{P}(^m E)$ is an isomorphic embedding, and the desired conclusion follows.

For additional results on the reflexivity of $\mathcal{P}(^m E)$ we refer to a survey of Gamelin [30].

In this section we have restricted our attention to the study of reflexivity of spaces of scalar-valued homogeneous polynomials. Several authors have devoted their attention to the study of reflexivity of spaces of vector-valued homogeneous polynomials. We mention Alencar [3], Alencar, Aron and Fricke [5], Alencar and Floret [6], [7], Gonzalo and Jaramillo [32], Jaramillo and Moraes [39] and Mujica [49].

7 Weakly continuous polynomials

Weak continuity properties of polynomials appeared already in the preceding section in connection with the reflexivity of $\mathcal{P}(^m E)$. This section is devoted to a systematic study of weak continuity properties of polynomials.

$\mathcal{P}_{wsc}(^m E; F)$ denotes the subspace of all $P \in \mathcal{P}(^m E; F)$ which are weakly sequentially continuous. $\mathcal{P}_w(^m E; F)$ (resp. $\mathcal{P}_{wu}(^m E; F)$) denotes the subspace of all $P \in \mathcal{P}(^m E; F)$ which are weakly continuous (resp. weakly uniformly continuous) on each bounded subset of E. Clearly we have the following inclusions:

$$\overline{\mathcal{P}_f(^m E; F)}^{\|\cdot\|} \subset \mathcal{P}_{wu}(^m E; F) \subset \mathcal{P}_w(^m E; F) \subset \mathcal{P}_{wsc}(^m E; F) \subset \mathcal{P}(^m E; F).$$

$\mathcal{P}_k(^m E; F)$ denotes the subspace of all $P \in \mathcal{P}(^m E; F)$ which are compact, that is $P(B_E)$ is relatively compact in F. The following results are due to Aron and Prolla [10].

42 Theorem ([10]). *For Banach spaces E and F we have that:*

(a) $\mathcal{P}_{wu}(^m E; F) \subset \mathcal{P}_k(^m E; F)$ *for every* $m \in \mathbb{N}$.

(b) $\mathcal{P}_{wu}(^m E; F) = \mathcal{P}_k(^m E; F)$ *for* $m = 1$.

Aron and Prolla [10] actually proved more. They proved that *if* $f : E \to F$ *is any mapping which is weakly uniformly continuous on bounded sets, then* f *maps bounded subsets of* E *onto relatively compact subsets of* F.

43 Example ([10])**.** Let $P \in \mathcal{P}(^2\ell_2)$ be defined by

$$P(x) = \sum_{k=1}^{\infty} \xi_k^2$$

for every $x = (\xi_k) \in \ell_2$. Clearly $P \in \mathcal{P}_k(^2\ell_2)$, but $P \notin \mathcal{P}_w(^2\ell_2)$, since $e_n \to 0$ weakly, but $P(e_n) = 1 \not\to 0$. Thus the conclusion of Theorem 42 (b) need not be true when $m > 1$.

44 Theorem ([10])**.** *For a Banach space* E *the following conditions are equivalent:*

(a) E' *has the approximation property.*

(b) $\mathcal{P}_{wu}(^1E;F) = \overline{\mathcal{P}_f(^1E;F)}^{\|\cdot\|}$ *for every Banach space* F.

(c) $\mathcal{P}_{wu}(^mE;F) = \overline{\mathcal{P}_f(^mE;F)}^{\|\cdot\|}$ *for every Banach space* F *and every* $m \in \mathbb{N}$.

Conditions (a) and (b) are equivalent by Theorems 25 and 42. Clearly (c) implies (b). Finally (c) follows from (b) by induction on m with the aid of the following lemma.

45 Lemma ([10])**.** *Let* $P = \hat{A}$, *with* $A \in \mathcal{L}^s(^mE;F)$ *and* $m \geq 2$. *Let*

$$\tilde{P} \in \mathcal{L}(E;\mathcal{P}(^{m-1}E;F))$$

be defined by

$$\tilde{P}(x)(t) = A(x,t,\ldots,t)$$

for all $x, t \in E$. *If* $P \in \mathcal{P}_{wu}(^mE;F)$, *then* $\tilde{P}$ *is a compact operator, and its image is contained in* $\mathcal{P}_{wu}(^{m-1}E;F)$.

Clearly $\mathcal{P}_w(^mE;F) = \mathcal{P}_{wu}(^mE;F)$ when E is reflexive. But the following problem was raised by Aron and Prolla [10].

46 Problem ([10])**.** Does the equality $\mathcal{P}_w(^mE;F) = \mathcal{P}_{wu}(^mE;F)$ hold for all Banach spaces E and F, and all $m \in \mathbb{N}$?

Problem 46 was solved by Aron, Hervés and Valdivia [9] as follows.

47 Theorem ([9])**.** $\mathcal{P}_w(^mE;F) = \mathcal{P}_{wu}(^mE;F)$ *for all Banach spaces* E *and* F, *and all* $m \in \mathbb{N}$.

The spaces $\mathcal{P}_{wsc}(^mE;F)$ and $\mathcal{P}_w(^mE;F)$ are related as follows.

48 Theorem ([9], [35]). *For a Banach space E the following conditions are equivalent:*

(a) E contains no subspace isomorphic to ℓ_1.

(b) $\mathcal{P}_{wsc}(^mE;F) = \mathcal{P}_w(^mE;F)$ for every Banach space F and every $m \in \mathbb{N}$.

(c) $\mathcal{P}_{wsc}(^mE) = \mathcal{P}_w(^mE)$ for some $m \geq 2$.

The implication $(a) \Rightarrow (b)$ is due to Aron, Hervés and Valdivia [9]. The implication $(b) \Rightarrow (c)$ is obvious, and the implication $(c) \Rightarrow (a)$ is due to Gutiérrez [35].

For variants of the results in this section we refer to the aforementioned survey of Gamelin [30]. For other weak continuity properties of polynomials we refer to a survey of Gutiérrez, Jaramillo and Llavona [36] and the references there.

8 Weakly continuous entire mappings

$\mathcal{H}_{wsc}(E;F)$ denotes the subspace of all $f \in \mathcal{H}(E;F)$ which are weakly sequentially continuous. $\mathcal{H}_w(E;F)$ (resp. $\mathcal{H}_{wu}(E;F)$) denotes the subspace of all $f \in \mathcal{H}(E;F)$ which are weakly continuous (resp. weakly uniformly continuous) on each bounded subset of E. Clearly we have the following inclusions:

$$\mathcal{H}_{wu}(E;F) \subset \mathcal{H}_w(E;F) \subset \mathcal{H}_{wsc}(E;F) \subset \mathcal{H}(E;F).$$

The following result is a holomorphic version of Theorem 48.

49 Proposition ([9]). *If E is separable and contains no subspace isomorphic to ℓ_1, then $\mathcal{H}_{wsc}(E;F) = \mathcal{H}_w(E;F)$ for every Banach space F.*

50 Corollary ([9]). *If E has a separable dual, then $\mathcal{H}_{wsc}(E;F) = \mathcal{H}_w(E;F)$ for every Banach space F.*

If $f \in \mathcal{H}_w(E;F)$, then it follows from the Cauchy integral formula 9 that $P^m f(0) \in \mathcal{P}_w(^mE;F)$ for every $m \in \mathbb{N}$. By using this fact, Theorem 47 and the remark after Theorem 42, Aron, Hervés and Valdivia [9] established the following relationship between $\mathcal{H}_w(E;F)$ and $\mathcal{H}_{wu}(E;F)$.

51 Proposition ([9]). *$\mathcal{H}_{wu}(E;F) = \mathcal{H}_w(E;F) \cap \mathcal{H}_b(E;F)$.*

Clearly $\mathcal{H}_w(E;F) = \mathcal{H}_{wu}(E;F)$ when E is reflexive. The first example of a non-reflexive Banach space for which this equality holds is due to Seán Dineen [21], who proved that $\mathcal{H}_{wsc}(c_0;F) = \mathcal{H}_w(c_0;F) = \mathcal{H}_{wu}(c_0;F)$ for every Banach space F. These results led Aron, Hervés and Valdivia [9] to pose the following problems.

52 Problem ([9]). Does the equality $\mathcal{H}_{wsc}(E;F) = \mathcal{H}_w(E;F)$ hold for every Banach space F if and only if E contains no subspace isomorphic to ℓ_1?

53 Problem ([9]). Does the equality $\mathcal{H}_{wsc}(E;F) = \mathcal{H}_{wu}(E;F)$ hold for every Banach space F when E has a separable dual?

54 Problem ([9]). Does the equality $\mathcal{H}_w(E;F) = \mathcal{H}_{wu}(E;F)$ hold for all Banach spaces E and F?

Problem 52 was solved by Joaquín Gutiérrez [35], who improved Proposition 49 as follows.

55 Theorem ([35]). *For a Banach space E the following conditions are equivalent:*

(a) E contains no subspace isomorphic to ℓ_1.

(b) $\mathcal{H}_{wsc}(E;F) = \mathcal{H}_w(E;F)$ for every Banach space F.

(c) $\mathcal{H}_{wsc}(E) = \mathcal{H}_w(E)$.

Humberto Carrión [12] recently solved Problem 53 and, in so doing, obtained a partial solution to Problem 54. Before presenting a more detailed description of the aforementioned results of Dineen [21] and Carrión [12], we introduce some additional terminology.

$\mathcal{H}_{bk}(E;F)$ denotes the subspace of all $f \in \mathcal{H}(E;F)$ which are bounded on the weakly compact subsets of E. Clearly we have the inclusions

$$\mathcal{H}_b(E;F) \subset \mathcal{H}_{bk}(E;F) \text{ and } \mathcal{H}_{wsc}(E;F) \subset \mathcal{H}_{bk}(E;F).$$

With this notation Dineen [21] obtained the following theorem.

56 Theorem ([21]). *For every Banach space F we have that:*

(a) $\mathcal{H}_{bk}(c_0;F) = \mathcal{H}_b(c_0;F)$.

(b) $\mathcal{H}_{wsc}(c_0;F) = \mathcal{H}_w(c_0;F) = \mathcal{H}_{wu}(c_0;F)$.

The bulk of Dineen's paper [21] is devoted to the proof of (a). (b) follows from (a) with the aid of Proposition 51 and Corollary 50.

By adapting Dineen's method of proof, Carrión was able to show that the conclusion of Theorem 56 remains valid when c_0 is replaced by a Banach space with a shrinking, unconditional Schauder basis. By a clever refinement of that proof he was able to delete the hypothesis of unconditionality of the basis. Thus he obtained the following theorem.

57 Theorem ([12]). *Let E be a Banach space with a shrinking Schauder basis. Then for each Banach space F we have that:*

(a) $\mathcal{H}_{bk}(E;F) = \mathcal{H}_b(E;F)$.

(b) $\mathcal{H}_{wsc}(E;F) = \mathcal{H}_w(E;F) = \mathcal{H}_{wu}(E;F)$.

This theorem has very important consequences.

58 Corollary ([12])**.** *If E has a separable dual, then for each Banach space F we have that:*

(a) $\mathcal{H}_{bk}(E;F) = \mathcal{H}_b(E;F)$.

(b) $\mathcal{H}_{wsc}(E;F) = \mathcal{H}_w(E;F) = \mathcal{H}_{wu}(E;F)$.

59 Corollary ([12])**.** *If each separable subspace of E has a separable dual, then for each Banach space F we have that:*

(a) $\mathcal{H}_{bk}(E;F) = \mathcal{H}_b(E;F)$.

(b) $\mathcal{H}_{wsc}(E;F) = \mathcal{H}_w(E;F) = \mathcal{H}_{wu}(E;F)$.

Corollary 58(a) follows from Theorem 57(a) with the aid of a result of Davis, Figiel, Johnson and Pelczynski [16], which asserts that *each Banach space with a separable dual is a quotient of a Banach space with a shrinking Schauder basis.* Corollary 59(a) follows easily from Corollary 58(a). In Theorem 57 and in Corollary 58, (b) follows from (a) with the aid of Proposition 51 and Corollary 50. In Corollary 59, (b) follows from (a) with the aid of Proposition 51 and Theorem 55.

60 Example ([12])**.** Since every weakly compact subset of ℓ_1 is norm compact, it follows that $\mathcal{H}_{bk}(\ell_1;F) = \mathcal{H}(\ell_1;F) \neq \mathcal{H}_b(\ell_1;F)$ for every Banach space F. Thus the conclusion of Corollary 59(a) is not valid for $E = \ell_1$.

61 Note (Note added in proof)**.** Manuel Valdivia has observed that there is a subtle gap in the proof of a result of Humberto Carrión [12, Theorem 11] (Theorem 57 in this survey), which he has recognized. At present it is not clear whether this result is correct or not. I am indebted to Pilar Rueda for informing me of Valdivia's observation.

9 Topological algebras of entire functions

A *topological algebra* is a complex algebra and a topological vector space in which ring multiplication is continuous. All topological algebras are assumed to be Hausdorff and to have a unit element. A topological algebra A is said to be *locally m-convex* if its topology is defined by a family of seminorms p with the property that $p(xy) \leq p(x)p(y)$ for all $x, y \in A$. A complete, metrizable, locally m-convex algebra is called a *Fréchet algebra.*

Every Banach algebra is a Fréchet algebra. If U is an open subset of a Banach space E, then $\mathcal{H}(U)$ is an example of a commutative, complete, locally m-convex algebra, whereas $\mathcal{H}_b(U)$ is an example of a commutative Fréchet algebra.

Locally m-convex algebras were systematically studied by Michael [42], who proved that every complete, locally m-convex algebra is topologically isomorphic to a projective limit of Banach algebras. In this way many properties of complete locally m-convex algebras can be derived from the corresponding properties of Banach algebras. But some important properties of Banach algebras are not shared by complete locally m-convex algebras. Indeed it is well known that every complex homomorphism on a Banach algebra is automatically continuous. But this need not be true for complete locally m-convex algebras, as the following example shows.

62 Example ([42])**.** Let W be the set of all countable ordinals, with the order topology, that is

$$W = \{\, \alpha : \alpha \text{ is an ordinal and } \alpha < \omega_1 \,\},$$

where ω_1 denotes the first uncountable ordinal. If we set

$$W^* = W \cup \{\, \omega_1 \,\},$$

then W is a countably compact topological space which is not compact, and W^* is the one-point compactification of W. Then $C(W)$ is a commutative, complete, locally m-convex algebra. Each $f \in C(W)$ is eventually constant, and has a unique extension $f^* \in C(W^*)$. It follows that the mapping $f \in C(W) \to f^* \in C(W^*)$ is an algebra isomorphism, and the functional $f \in C(W) \to f^*(\omega_1) \in \mathbb{C}$ is a discontinuous homomorphism. Since the space W is countably compact, but not compact, it follows that W is not metrizable. For proofs of these assertions we refer to the book of Gillman and Jerison [31, Chapter 5].

Despite several claims to the contrary, the following problems, raised by Michael [42] in 1952, remain unsolved.

63 Problem ([42])**.** If A is a commutative Fréchet algebra, is every homomorphism $\phi : A \to \mathbb{C}$ necessarily continuous?

64 Problem ([42])**.** If A is a commutative, complete, locally m-convex algebra, is every homomorphism $\phi : A \to \mathbb{C}$ necessarily *bounded*, that is bounded on all bounded subsets of A?

Clearly an affirmative solution to Problem 64 would imply an affirmative solution to Problem 63. But Dixon and Fremlin [25] proved that the reverse implication is also true. Thus Problems 63 and 64 are equivalent.

I. Craw [14], D. Clayton [13] and M. Schottenloher [57] proved that to solve the Michael problem for an arbitrary commutative Fréchet algebra, it is sufficient to solve the corresponding problem for certain Fréchet algebras of holomorphic functions. By a refinement of their techniques we proved that to solve the Michael problem for an arbitrary commutative Fréchet algebra, it is sufficient to solve the corresponding problem for the Fréchet algebra $\mathcal{H}_b(E)$, where E is some infinite dimensional Banach space. Our proof is based on the following theorem, which is also due to the author [44].

65 Theorem ([44]). *Let E be a Banach space with a normalized Schauder basis (e_n), and let $(\phi_n) \subset E'$ be the corresponding sequence of coordinate functionals. Let A be a commutative, complete, locally m-convex algebra. Let (a_n) be a sequence in A such that $\sum_{n=1}^{\infty} \sqrt{p(a_n)} < \infty$ for every continuous seminorm p on A. Then there is a continuous algebra homomorphism $T : \mathcal{H}(E) \to A$ such that $T(1) = 1$ and $T(\phi_n) = a_n$ for every $n \in \mathbb{N}$.*

66 Theorem ([44], [45]). *The following assertions are equivalent:*

(a) For each commutative, complete, locally m-convex algebra A, every homomorphism $\phi : A \to \mathbb{C}$ is bounded.

(b) For each commutative Fréchet algebra A, every homomorphism $\phi : A \to \mathbb{C}$ is continuous.

(c) There is an infinite dimensional Banach space E such that every homomorphism $\Phi : \mathcal{H}(E) \to \mathbb{C}$ is bounded.

(d) There is an infinite dimensional Banach space E such that every homomorphism $\Phi : \mathcal{H}_b(E) \to \mathbb{C}$ is continuous.

Proof. The implications $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$ and $(b) \Rightarrow (d)$ are clear. To complete the proof we show the implications $(c) \Rightarrow (a)$ and $(d) \Rightarrow (a)$ at the same time.

Suppose that (a) is false. Then there exist a commutative, complete, locally m-convex algebra A and an unbounded homomorphism $\psi : A \to \mathbb{C}$. Hence there is a bounded sequence (b_n) in A such that $|\psi(b_n)| > 8^n$ for every n. Let $a_n = 4^{-n} b_n$ for every n. Then for each continuous seminorm p on A there is a constant $c > 0$ such that $p(a_n) \leq 4^{-n} c$ for every n. Thus $\sum_{n=1}^{\infty} \sqrt{p(a_n)} < \infty$ for each continuous seminorm p on A.

We claim that (c) and (d) are false. Indeed let E be any infinite dimensional Banach space. By a classical result of Mazur (see Diestel's book [17, Chapter V]), E contains a closed, infinite dimensional subspace M with a normalized Schauder basis (e_n). Let $(\phi_n) \subset M'$ be the corresponding sequence of coordinate functionals. By Theorem 65 there is an algebra homomorphism $T : \mathcal{H}(M) \to A$ such that $T(1) = 1$ and $T(\phi_n) = a_n$ for every n. Let $R : \mathcal{H}(E) \to \mathcal{H}(M)$ be the restriction mapping. Since the sequence (ϕ_n) is bounded in M', by the Hahn-Banach theorem there is a bounded sequence $(\tilde{\phi}_n)$ in E' such that $R(\tilde{\phi}_n) = \phi_n$ for every n. Thus $\psi \circ T \circ R$ is a complex homomorphism on $\mathcal{H}(E)$, which is unbounded, since $|\psi \circ T \circ R(\tilde{\phi}_n)| = |\psi(a_n)| > 2^n$ for every n. For the same reason, the restriction of $\psi \circ T \circ R$ to $\mathcal{H}_b(E)$ is an unbounded, and therefore discontinuous complex homomorphism on $\mathcal{H}_b(E)$. This shows that (c) and (d) are false, and completes the proof of the theorem. QED

Observe that Theorem 66 gives another proof of the equivalence of Problems 63 and 64.

We next characterize the complex homomorphisms and the continuous complex homomorphisms on the Fréchet algebra $\mathcal{H}_b(E)$, where E varies over a class of Banach spaces that contains Tsirelson's space.

67 Theorem ([48]). *Let E be a reflexive Banach space such that $\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\|\cdot\|}$ for every $m \in \mathbb{N}$. Then each continuous homomorphism $\Phi : \mathcal{H}_b(E) \to \mathbb{C}$ is an evaluation, that is, there is an $a \in E$ such that $\Phi(f) = f(a)$ for every $f \in \mathcal{H}_b(E)$.*

68 Theorem ([48]). *Let E be a reflexive Banach space such that $\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\|\cdot\|}$ for every $m \in \mathbb{N}$. Then given $f_1, \ldots, f_p \in \mathcal{H}_b(E)$, without common zeros, there are $g_1, \ldots, g_p \in \mathcal{H}_b(E)$ such that $\sum_{k=1}^p f_k g_k = 1$.*

69 Theorem. *Let E be a reflexive Banach space such that*

$$\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\|\cdot\|}$$

for every $m \in \mathbb{N}$. Then each homomorphism $\Phi : \mathcal{H}_b(E) \to \mathbb{C}$ is a local evaluation, that is, given $f_1, \ldots, f_p \in \mathcal{H}_b(E)$, there is an $a \in E$ such that $\Phi(f_k) = f_k(a)$ for $k = 1, \ldots, p$.

Proof. If there were no $a \in E$ such that $\Phi(f_k) = f_k(a)$ for $k = 1, \ldots, p$, then the functions $f_1 - \Phi(f_1), \ldots, f_p - \Phi(f_p)$ would have no common zeros. By Theorem 68 there would exist $g_1, \ldots, g_p \in \mathcal{H}_b(E)$ such that $\sum_{k=1}^p (f_k - \Phi(f_k))g_k = 1$. By applying Φ we would get $0 = 1$, absurd. QED

Given a Banach space E, we consider the mapping

$$\sigma_b : x \in E \to (f(x))_{f \in \mathcal{H}_b(E)} \in \mathbb{C}^{\mathcal{H}_b(E)}.$$

The mapping σ_b is clearly injective. If we use Definition 7(c) to define holomorphic mappings between locally convex spaces, then σ_b is easily seen to be holomorphic. Furthermore, σ_b maps bounded sets onto bounded sets. Thus, with the obvious notation, $\sigma_b \in \mathcal{H}_b(E; \mathbb{C}^{\mathcal{H}_b(E)})$. With this notation, from Theorems 67 and 69 we easily get the following theorem.

70 Theorem. *Let E be a reflexive Banach space such that*

$$\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\|\cdot\|}$$

for every $m \in \mathbb{N}$. Then for a function $\Phi : \mathcal{H}_b(E) \to \mathbb{C}$ we have that:

(a) Φ is a continuous algebra homomorphism if and only if $(\Phi(f))_{f\in\mathcal{H}_b(E)} \in \sigma_b(E)$.

(b) Φ is an algebra homomorphism if and only if $(\Phi(f))_{f\in\mathcal{H}_b(E)} \in \overline{\sigma_b(E)}$.

We next characterize the complex homomorphisms and the bounded complex homomorphisms of the locally m-convex algebra $\mathcal{H}(E)$, where E varies over a class of Banach spaces that contains Tsirelson's space.

71 Theorem ([43]). *Let E be a separable Banach space such that $\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\tau_c}$ for every $m \in \mathbb{N}$. Then each bounded homomorphism $\Phi : \mathcal{H}(E) \to \mathbb{C}$ is an evaluation.*

72 Theorem ([48]). *Let E be a reflexive Banach space such that $\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\|\cdot\|}$ for every $m \in \mathbb{N}$. Then given $f_1, \ldots, f_p \in \mathcal{H}(E)$, without common zeros, there are $g_1, \ldots, g_p \in \mathcal{H}(E)$ such that $\sum_{k=1}^p f_k g_k = 1$.*

73 Theorem. *Let E be a reflexive Banach space such that*

$$\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\|\cdot\|}$$

for every $m \in \mathbb{N}$. Then each homomorphism $\Phi : \mathcal{H}(E) \to \mathbb{C}$ is a local evaluation.

The proof of Theorem 73 is similar to that of Theorem 69, but uses Theorem 72 instead of Theorem 68.

Given a Banach space E we consider the mapping

$$\sigma : x \in E \to (f(x))_{f\in\mathcal{H}(E)} \in \mathbb{C}^{\mathcal{H}(E)}.$$

Then the mapping σ is injective and is holomorphic, that is $\sigma \in \mathcal{H}(E; \mathbb{C}^{\mathcal{H}(E)})$. With this notation, from Theorems 71 and 73 we easily get the following theorem.

74 Theorem. *Let E be a separable and reflexive Banach space such that $\mathcal{P}(^mE) = \overline{\mathcal{P}_f(^mE)}^{\|\cdot\|}$ for every $m \in \mathbb{N}$. Then for a function $\Phi : \mathcal{H}(E) \to \mathbb{C}$ we have that:*

(a) Φ is a bounded homomorphism if and only if $(\Phi(f))_{f\in\mathcal{H}(E)} \in \sigma(E)$.

(b) Φ is a homomorphism if and only if $(\Phi(f))_{f\in\mathcal{H}(E)} \in \overline{\sigma(E)}$.

It follows from Theorems 34 and 41 that Tsirelson's space X verifies the hypotheses of Theorems 70 and 74. Thus it follows from Theorems 66, 70 and 74 that Problems 63 and 64 are equivalent to the following problems.

75 Problem. Does the mapping $\sigma_b : X \to \mathbb{C}^{\mathcal{H}_b(X)}$ have a closed image?

76 Problem. Does the mapping $\sigma : X \to \mathbb{C}^{\mathcal{H}(X)}$ have a closed image?

By using Theorem 41 we can prove that the image of the mapping $\sigma_b : X \to \mathbb{C}^{\mathcal{H}_b(X)}$ is sequentially closed. By using a result of Petunin and Savkin [54] (see also [29]) we can prove that the image of the mapping $\sigma : X \to \mathbb{C}^{\mathcal{H}(X)}$ is sequentially closed too. But I have unable to determine whether these sets are closed.

For other reductions of the Michael problem we refer to an article of Dixon and Esterle [24]. For other results in this direction we refer to a survey of Dales [15] on automatic continuity.

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