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Wave stability for nearly-constrained materials in anisotropic generalized thermoelasticity

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Abstract. In generalized thermoelasticity Fourier's law of heat conduction in classical thermoleasticity is modified by introducing a relaxation time associated with the heat flux. Secular equations are derived for plane harmonic body waves propagating through anisotropic generalized thermoelastic materials subject to a thermomechanical near-constraint of an arbitrary nature connecting deformation with either temperature or entropy. The near-constraints are defined in such a way that as a certain parameter becomes infinite the constraint holds exactly. Therefore a nearly-constrained material is an unconstrained material. In an unconstrained material four stable thermoelastic waves may propagate in any direction but in a deformation-temperature constrained material one of these becomes unstable. The nature of this instability is explored in the passage to the limit of the constraint holding exactly. On the other hand, in a deformation-entropy constrained material only three waves may propagate in any direction but all are stable. The passage to the constrained limit illustrates this. Expansions are given for the wave speeds in terms of the relaxation time, constraint parameter and frequency. The two types of constraint are certainly not equivalent and yet a connection is demonstrated between the two near-constraints.

Keywords: Thermoelasticity, relaxation time, constraints, near-constraints, stability.

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In honour of Alan Day

1 Introduction

The idea of introducing a relaxation time associated with the heat flux into the theory of heat conduction is due to Cattaneo [3, 4] and an account of its applications in solid mechanics has been given by Leslie and Scott [10]. Manacorda [13, 14] was the first to consider the thermomechanical constraint of incompressibility at fixed temperature in isotropic thermoelasticity. Beevers [2] also considered wave propagation in such a constrained material. Both these authors noted the instability of one of the longitudinal waves. Chadwick and Scott [6] demonstrated the instability of one of the waves propagating through an arbitrarily deformation-constrained anisotropic thermoelastic material. In order to overcome the defect of an unstable wave theory Scott [17] posited a constraint in which the deformation is related to the entropy, rather than the temperature. He found that for this constraint only three waves may propagate in each direction but that all are stable. Leslie and Scott have discussed both types of thermomechanical constraint a series of papers, see [8] for classical isotropic thermoelasticity, [10] for generalized isotropic thermoelasticity and [12] for generalized anisotropic thermoelasticity.

The theory of near-constraints was developed in an attempt to understand the passage to the limit of a thermomechanical constraint holding exactly. A parameter is introduced, to be identified with an elastic modulus associated with the constraint, and the larger this parameter becomes the more nearly does the constraint hold exactly. Leslie and Scott studied the near-constraint of near-incompressibility in classical isotropic thermoelasticity [9] and in generalized isotropic thermoelasticity [11]. So far, the analysis of nearly-constrained materials has produced some interesting results. It demonstrates at which point stability is lost for a deformation-temperature near-constraint and shows that the two types of constraint are intimately related, despite not being equivalent.

In the present paper we extend the theory of thermomechanical constraints in generalized anisotropic thermoelasticity developed in [12] to consider thermomechanical near-constraints.

In Section 2 we derive the field equations and secular equations of constrained anisotropic thermoelasticity and of nearly-constrained anisotropic thermoelasticity. In Section 3 we derive low- and high-frequency limits and expansions for the deformation-temperature near-constraint and examine the stability properties. In Section 4 we do the same for the deformation-entropy near-constraint and in Section 5 we demonstrate the equivalence of these two near-constraints, in a certain sense, even though the constraints themselves are not equivalent.

2 Field and secular equations

We consider a material body $\mathcal{B}$ possessing an equilibrium configuration B_e with uniform density ρ_e, uniform temperature T_e and zero stress and heat flux. $B_{t'}$ is the configuration of $\mathcal{B}$ at time t'. In relation to fixed Cartesian coordinates, x'_i is the position vector of a material particle in $B_{t'}$ and $u'_i(x'_i, t')$ are the components of the displacement vector $\mathbf{u}'$. The components of the displacement gradient are given by $u'_{i,j'}$, where $(\ )_{,j'}$ denotes the spatial derivative $\partial(\ )/\partial x'_j$.

The field equations of classical linear thermoelasticity are

$$\sigma'_{ij,j'} = \rho_e \partial^2_{t'} u'_i, \qquad -q'_{i,i'} = \rho_e T_e \partial_{t'} \phi', \tag{1}$$

the first representing the balance of linear momentum in the absence of body force and the second representing the conservation of energy in the absence of heat supply. In (1), σ'_{ij} are the stress components, $\partial_{t'}$ is the time derivative, q'_i are the heat flux components and ϕ' is the entropy increment.

The Helmholtz free energy $A'(u'_{i,j'}, \theta')$ for an anisotropic thermoelastic body to quadratic order in the small quantities $u'_{i,j'}$ and θ', the temperature increment, is

$$A' = (2\rho_e)^{-1} \tilde{c}'_{ijkl} u'_{i,j'} u'_{k,l'} - \rho_e^{-1} \beta'_{ij} u'_{i,j'} \theta' - (2T_e)^{-1} c' \theta'^2, \tag{2}$$

in which $\tilde{c}'_{ijkl}$ are the components of the isothermal elastic modulus, β'_{ij} components of the temperature coefficients of stress and c' the specific heat at constant deformation. The free energy (2) acts as a potential for the stress components and entropy increment:

$$\sigma'_{ij} = \rho_e \frac{\partial A'}{\partial u'_{i,j'}} = \tilde{c}'_{ijkl} u'_{k,l'} - \beta'_{ij} \theta', \qquad \phi' = -\frac{\partial A'}{\partial \theta'} = \rho_e^{-1} \beta'_{ij} u'_{i,j'} + (T_e)^{-1} c' \theta'. \tag{3}$$

These are constitutive equations giving the stress components σ'_{ij} and entropy increment ϕ' as functions of the displacement gradient $u'_{i,j'}$ and temperature increment θ'.

In the theory of generalized thermoelasticity one further constitutive equation is adopted, one which links heat conduction q'_i with temperature gradient $\theta'_{,j'}$:

$$q'_i + \tau'_o \partial_{t'} q'_i = -k'_{ij} \theta'_{,j'}, \tag{4}$$

in which τ'_o is a positive relaxation time and k'_{ij} are the thermal conductivity components. Equation (4) is known as the modified Fourier law, with the standard Fourier law being retrieved upon putting $\tau'_o = 0$.

At this point it proves advantageous to replace all variables with dimensionless variables by introducing suitable scaling factors. We take a typical (non-zero) elastic modulus component $\tilde{c}'_{ijkl}$ as the stress scale γ, a typical (non-zero) component of the conductivity k'_{ij} as the conductivity scale k', the ambient temperature T_e as the temperature scale, and length scale l_0 and time scale t_0, which will be specified in the next paragraph. All dimensionless variables are defined in terms of these five scaling constants. In all cases the dimensional variable with a prime is replaced by a dimensionless variable without a prime:

$$\begin{gathered} x_i = \frac{x'_i}{l_0}, \quad t = \frac{t'}{t_0}, \quad u_i = \frac{u'_i}{l_0}, \quad \theta = \frac{\theta'}{T_e}, \\ \tilde{c}_{ijkl} = \frac{\tilde{c}'_{ijkl}}{\gamma}, \quad \beta_{ij} = \frac{T_e \beta'_{ij}}{\gamma}, \quad c = \frac{\rho_e T_e c'}{\gamma}, \\ A = \frac{\rho_e A'}{\gamma}, \quad \sigma_{ij} = \frac{\sigma'_{ij}}{\gamma}, \quad \phi = \frac{\rho_e T_e \phi'}{\gamma}, \\ k_{ij} = \frac{k'_{ij}}{k'}, \quad q_i = \frac{l_0 q'_i}{T_e k'}, \quad \tau = \frac{\tau'_0}{t_0}, \end{gathered} \tag{5}$$

in which the dimensionless specific heat c represents a ratio of heat energy to mechanical energy, τ is a dimensionless relaxation time and, if the constraint (14) is operating, α is the associated dimensionless thermal expansion coefficient.

We specify a velocity scale v_0 and the time scale t_0 by

$$v_0 = \sqrt{\left(\frac{\gamma}{\rho_e}\right)}, \quad t_0 = \frac{k'}{\gamma c'}, \tag{6}$$

so that the length scale is prescribed to be $l_0 = v_0 t_0$. The velocity v_0 is that of a typical elastic wave propagating in isothermal conditions. The frequency $\omega_0 = 1/t_0$ is the Debye frequency adopted for scaling purposes also by Chadwick [5] and, amongst others, by Leslie and Scott [12]. The Debye frequency is far higher than any likely to be excited in a thermoelastic material.

The components of the dimensionless displacement gradient are given by

$$u_{i,j} = u'_{i,j'},$$

where $(\ )_{,j}$ denotes the dimensionless spatial derivative $\partial(\ )/\partial x_j$. On denoting the dimensionless time derivative $\partial/\partial t$ by a superposed dot, we find that the field equations (1)–(4) may be written in terms of dimensionless variables as

$$\begin{gathered} \sigma_{ij,j} = \ddot{u}_i, \qquad -q_{i,i} = \frac{1}{c}\dot{\phi}, \\ A = \tfrac{1}{2}\tilde{c}_{ijkl} u_{i,j} u_{k,l} - \beta_{ij} u_{i,j}\theta - \tfrac{1}{2}c\theta^2, \\ \sigma_{ij} = \frac{\partial A}{\partial u_{i,j}} = \tilde{c}_{ijkl} u_{k,l} - \beta_{ij}\theta, \qquad \phi = -\frac{\partial A}{\partial \theta} = \beta_{ij} u_{i,j} + c\theta, \\ q_i + \tau \dot{q}_i = -k_{ij}\theta_{,j}. \end{gathered} \tag{7}$$

We may eliminate the quantities σ_{ij}, q_i and ϕ from (7) to obtain the four field equations of generalized thermoelasticity in dimensionless form for the four quantities $u_i(\mathbf{x}, t)$ and $\theta(\mathbf{x}, t)$:

$$\begin{aligned} \tilde{c}_{ijkl} u_{k,jl} - \beta_{ij}\theta_{,j} &= \ddot{u}_i, \\ k_{ij}\theta_{,ij} &= \dot{\theta} + \frac{1}{c}\beta_{ij}\dot{u}_{i,j} + \tau\left(\ddot{\theta} + \frac{1}{c}\beta_{ij}\ddot{u}_{i,j}\right), \end{aligned} \tag{8}$$

first obtained by Scott [15] in dimensional form. On taking $\tau = 0$ these equations reduce to the field equations of classical thermoelasticity given by Chadwick [5]. If the temperature coefficient of stress vanishes, i.e. if $\beta_{ij} = 0$, these equations decouple into purely elastic equations for the displacements u_i, and a pure diffusion equation for the temperature θ in the classical case $\tau = 0$.

We now seek complex exponential solutions of the field equations (8) in the form of a plane harmonic wave,

$$\{\mathbf{u}, \theta\} = \{\mathbf{U}, \Theta\} \exp\left\{\mathrm{i}\omega'(\mathbf{n}\cdot\mathbf{x}'/v' - t')\right\} = \{\mathbf{U}, \Theta\} \exp\left\{\mathrm{i}\omega(\mathbf{n}\cdot\mathbf{x}/v - t)\right\}, \tag{9}$$

in which $\mathbf{U}$ and Θ are dimensionless complex constant amplitudes, ω' is the real frequency, $\mathbf{n}$ is the real unit wave normal vector and v' is the complex wave velocity. In terms of the time scale t_0 and velocity scale v_0 already defined the dimensionless frequency ω and dimensionless wave speed v are defined by

$$\omega = \frac{\omega'}{\omega_0} = \omega' t_0, \qquad v = \frac{v'}{v_0}.$$

Substituting (9) into (8) and eliminating the amplitudes $\mathbf{U}$ and Θ between the resulting equations yields the secular equation [15, (4)], now given in terms of dimensionless variables:

$$w \det(w\mathbf{1} - \hat{\mathbf{Q}}) + \frac{\mathrm{i}\omega}{1 - \mathrm{i}\omega\tau} \det(w\mathbf{1} - \tilde{\mathbf{Q}}) = 0, \tag{10}$$

in which we have taken $k' = k'_{ij} n_i n_j$ in (5), so that $k_{ij} n_i n_j = 1$. The dimensionless squared wave speed is denoted by

$$w = v^2,$$

and $\hat{\mathbf{Q}}$ and $\tilde{\mathbf{Q}}$ are the dimensionless isentropic and isothermal acoustical tensors:

$$\hat{Q}_{ik} = \hat{c}_{ijkl} n_j n_l, \qquad \tilde{Q}_{ik} = \tilde{c}_{ijkl} n_j n_l. \tag{11}$$

We denote by $\hat{c}_{ijkl}$ the dimensionless isentropic elastic modulus

$$\hat{c}_{ijkl} = \tilde{c}_{ijkl} + \frac{1}{c}\beta_{ij}\beta_{kl}. \tag{12}$$

Using (12) and (11) we see that the acoustical tensors are related by

$$\hat{\mathbf{Q}} = \tilde{\mathbf{Q}} + \frac{1}{c}\,\mathbf{b} \otimes \mathbf{b}, \tag{13}$$

where $\mathbf{b} := \boldsymbol{\beta}\mathbf{n}$ is a dimensionless vector. On taking $\tau = 0$ the secular equation (10) of generalized thermoelasticity reduces to that of classical thermoelasticity, see [5]. The stability of all solutions of (10) has been demonstrated in the generalized case in [15] and in the classical case in [16].

Deformation-temperature constraint

If a constraint operates that relates displacement gradient to temperature then its linearized form is given in dimensional variables by

$$\tilde{N}_{ij} u'_{i,j'} - \alpha'\theta' = 0, \tag{14}$$

where $\tilde{N}_{ij}$ is a dimensionless symmetric tensor and α' a thermal expansion coefficient. The constraint (14) may be replaced by the dimensionless form

$$\tilde{N}_{ij} u_{i,j} - \alpha\theta = 0, \tag{15}$$

in which $\alpha := T_e \alpha'$ is dimensionless.

It has been shown that one way of obtaining the field equations of deformation-temperature constrained thermoelasticity is to replace the Helmholtz free energy A by $A^\dagger$:

$$A^\dagger = A + \tilde{\eta}(\tilde{N}_{ij} u_{i,j} - \alpha\theta),$$

where $\tilde{\eta}(\mathbf{x}, t)$ is an unknown Lagrange multiplier, and then to replace A by $A^\dagger$ in (7). This leads to the following expressions for stress and entropy:

$$\sigma_{ij} = \tilde{c}_{ijkl} u_{k,l} - \beta_{ij}\theta + \tilde{\eta}\tilde{N}_{ij}, \qquad \phi = \beta_{ij} u_{i,j} + c\theta + \alpha\tilde{\eta},$$

in which we see that $\tilde{\eta}\tilde{\mathbf{N}}$ is a reaction stress due to the constraint and $\alpha\tilde{\eta}$ is a reaction entropy. Then we find that the field equations (15) are replaced by

$$\begin{aligned} &\tilde{c}_{ijkl} u_{k,jl} - \beta_{ij}\theta_{,j} + \tilde{N}_{ij}\tilde{\eta}_{,j} = \ddot{u}_i, \\ &k_{ij}\theta_{,ij} = \dot{\theta} + \frac{1}{c}\beta_{ij}\dot{u}_{i,j} + \frac{\alpha}{c}\,\dot{\tilde{\eta}} + \tau\left(\ddot{\theta} + \frac{1}{c}\beta_{ij}\ddot{u}_{i,j} + \frac{\alpha}{c}\,\ddot{\tilde{\eta}}\right), \end{aligned} \tag{16}$$

which, together with the constraint equation (15), furnish a system of five field equations for the five quantities u_i, θ and $\tilde{\eta}$. We again seek complex exponential solutions of the form (9), together with $\tilde{\eta} = \tilde{H}\exp\{\mathrm{i}\omega(\mathbf{n}\cdot\mathbf{x}/v - t)\}$, and eliminate the amplitudes $\mathbf{U}$, Θ and $\tilde{H}$ to obtain the dimensionless form of the secular equation of deformation-temperature constrained generalized thermoelasticity first obtained by Leslie and Scott [12]:

$$w \det\left(w\mathbf{1} - \tilde{\mathbf{P}}\right) + \frac{\mathrm{i}\omega}{1 - \mathrm{i}\omega\tau}\tilde{\sigma}^2\,\tilde{\mathbf{c}}\cdot\left(w\mathbf{1} - \tilde{\mathbf{P}}\right)^{\mathrm{adj}}\tilde{\mathbf{c}} = 0, \tag{17}$$

in which we define

$$\tilde{\sigma}^2 := \frac{c}{\alpha^2}, \quad \tilde{\mathbf{c}} := \tilde{\mathbf{N}}\mathbf{n},$$

and

$$\tilde{\mathbf{P}} := \hat{\mathbf{Q}} - \frac{1}{c}\left(\mathbf{b} + \frac{c}{\alpha}\,\tilde{\mathbf{c}}\right)\otimes\left(\mathbf{b} + \frac{c}{\alpha}\,\tilde{\mathbf{c}}\right). \tag{18}$$

The dimensionless parameter $\tilde{\sigma}$ is a measure of the relative importance of the mechanical and thermal components of the constraint (15). The special case $\tilde{\sigma} \to 0$, i.e. $\alpha \to \infty$, corresponds to a purely isothermal constraint and the special case $\tilde{\sigma} \to \infty$, i.e. $\alpha \to 0$, corresponds to a purely mechanical constraint.

Deformation-temperature near-constraint

In order to model a material in which the constraint (15) holds exactly as some sort of limiting case of an unconstrained material we take the dimensionless Helmholtz free energy in the form

$$A^{\dagger\dagger} = A + \tfrac{1}{2}\tilde{\chi}(\tilde{N}_{ij}u_{i,j} - \alpha\theta)^2, \tag{19}$$

in which the positive quantity $\tilde{\chi}$ may be regarded as a dimensionless elastic modulus associated with the constraint. The limit $\tilde{\chi} \to 0$ corresponds to the unconstrained material already considered and the limit $\tilde{\chi} \to \infty$ corresponds to the constraint holding exactly, provided that the limit is taken such that

$$\tilde{\chi} \to \infty, \quad \tilde{N}_{ij}u_{i,j} - \alpha\theta \to 0, \quad \tilde{\eta} := \tilde{\chi}(\tilde{N}_{ij}u_{i,j} - \alpha\theta) \quad \text{remains bounded.}$$

The various terms of (19) may be recombined into the single quadratic form

$$A^{\dagger\dagger} = \tfrac{1}{2}\tilde{c}^{\dagger\dagger}_{ijkl}u_{i,j}u_{k,l} - \beta^{\dagger\dagger}_{ij}u_{i,j}\theta - \tfrac{1}{2}c^{\dagger\dagger}\theta^2,$$

where

$$\begin{aligned} \tilde{c}^{\dagger\dagger}_{ijkl} &= \tilde{c}_{ijkl} + \tilde{\chi}\tilde{N}_{ij}\tilde{N}_{kl}, \\ \beta^{\dagger\dagger}_{ij} &= \beta_{ij} + \alpha\tilde{\chi}\tilde{N}_{ij}, \\ c^{\dagger\dagger} &= \left(1 - \tilde{\chi}/\tilde{\sigma}^2\right)c. \end{aligned} \tag{20}$$

A necessary criterion for thermoelastic stability is that the specific heat must be positive, see [16]. If $\tilde{\chi}$ becomes large enough ($\tilde{\chi} > \tilde{\sigma}^2$) the specific heat $c^{\dagger\dagger}$ defined by $(20)_3$ becomes negative thus explaining the instability.

Since a nearly-constrained material is, in fact, an unconstrained material we may obtain the field equations of nearly-constrained thermoelasticity simply by replacing the appropriate moduli in the field equations (15) by the quantities defined at (20). We may then obtain the corresponding secular equation directly from the secular equation (10) of unconstrained generalized thermoelasticity:

$$w\,\det(w\mathbf{1}-\hat{\mathbf{Q}}^{\dagger\dagger})+\frac{\mathrm{i}\omega}{1-\mathrm{i}\omega\tau}.\frac{1}{c^{\dagger\dagger}/c}.\det(w\mathbf{1}-\tilde{\mathbf{Q}}^{\dagger\dagger})=0 \tag{21}$$

in which, by analogy with (13), we may define

$$\hat{\mathbf{Q}}^{\dagger\dagger}=\tilde{\mathbf{Q}}^{\dagger\dagger}+\frac{1}{c^{\dagger\dagger}}\,\mathbf{b}^{\dagger\dagger}\otimes\mathbf{b}^{\dagger\dagger} \tag{22}$$

with $\mathbf{b}^{\dagger\dagger}:=\boldsymbol{\beta}^{\dagger\dagger}\mathbf{n}$. Using (20) and (22) we find that explicitly in terms of $\tilde{\chi}$

$$\begin{aligned}\tilde{\mathbf{Q}}^{\dagger\dagger}&=\tilde{\mathbf{Q}}+\tilde{\chi}\tilde{\mathbf{c}}\otimes\tilde{\mathbf{c}},\\ \hat{\mathbf{Q}}^{\dagger\dagger}&=\hat{\mathbf{Q}}+\frac{\tilde{\chi}}{(1-\tilde{\chi}/\tilde{\sigma}^2)}\left(\frac{\alpha}{c}\,\mathbf{b}+\tilde{\mathbf{c}}\right)\otimes\left(\frac{\alpha}{c}\,\mathbf{b}+\tilde{\mathbf{c}}\right).\end{aligned} \tag{23}$$

Despite the apparently complicated nature of its dependence on $\tilde{\chi}$, coming from (23), we find that the secular equation (21) of nearly-constrained thermoelasticity may be expressed in a form that is simply linear in $\tilde{\chi}$:

$$\begin{aligned}&w\,\det(w\mathbf{1}-\hat{\mathbf{Q}})+\frac{\mathrm{i}\omega}{1-\mathrm{i}\omega\tau}\det(w\mathbf{1}-\tilde{\mathbf{Q}})\\ &\quad-\frac{\tilde{\chi}}{\tilde{\sigma}^2}\left\{w\det\left(w\mathbf{1}-\tilde{\mathbf{P}}\right)+\frac{\mathrm{i}\omega}{1-\mathrm{i}\omega\tau}\tilde{\sigma}^2\,\tilde{\mathbf{c}}\cdot\left(w\mathbf{1}-\tilde{\mathbf{P}}\right)^{\mathrm{adj}}\tilde{\mathbf{c}}\right\}=0,\end{aligned} \tag{24}$$

with $\tilde{\mathbf{P}}$ defined, as before, by (18). As $\tilde{\chi}\to 0$ this secular equation reduces to that of unconstrained thermoelasticity and as $\tilde{\chi}\to\infty$ it reduces to that of deformation-temperature constrained thermoelasticity, both limits being expected.

Deformation-entropy constraint

For a thermoelastic material subject to a constraint connecting deformation with entropy we must employ the entropy ϕ as an independent variable in place of temperature θ. We therefore employ the internal energy $U(u_{i,j},\phi)$ as a potential function in place of the Helmholtz free energy A. In dimensionless form

$$U=\tfrac{1}{2}\hat{c}_{ijkl}u_{i,j}u_{k,l}-c^{-1}\beta_{ij}u_{i,j}\phi+\tfrac{1}{2}c^{-1}\phi^2, \tag{25}$$

all quantities having been previously defined. The dimensionless stress and temperature are given in terms of (25) by

$$\sigma_{ij} = \frac{\partial U}{\partial u_{i,j}} = \hat{c}_{ijkl} u_{k,l} - \frac{1}{c}\beta_{ij}\phi, \qquad \theta = \frac{\partial U}{\partial \phi} = -\frac{1}{c}\beta_{ij}u_{i,j} + \frac{1}{c}\phi \tag{26}$$

and, with the use of (7), the four field equations for $u_i(\mathbf{x}, t)$ and $\phi(\mathbf{x}, t)$ are found to be, in dimensionless form,

$$\begin{aligned} \hat{c}_{ijkl} u_{k,\,jl} - \frac{1}{c}\beta_{ij}\phi_{,\,j} &= \ddot{u}_i\,, \\ k_{ij}\left(\phi_{,\,ij} - \beta_{kl}u_{k,\,ijl}\right) &= \dot{\phi} + \tau\ddot{\phi}. \end{aligned} \tag{27}$$

These are equivalent to the field equations (8) for $u_i(\mathbf{x}, t)$ and $\theta(\mathbf{x}, t)$ and so give rise to the same secular equation (10) for plane harmonic waves.

A linearized form of the constraint relating displacement gradient to entropy is given by

$$\hat{N}_{ij} u'_{i,\,j'} - \nu'\phi' = 0, \tag{28}$$

where $\hat{N}_{ij}$ is a symmetric constraint tensor and ν' a positive constant. These quantities play a similar role to $\tilde{N}_{ij}$ and α' in the case of a deformation-temperature constraint. The dimensionless form of the constraint (28) is

$$\hat{N}_{ij} u_{i,\,j} - \nu\phi = 0, \tag{29}$$

in which $\nu := \gamma\nu'/\rho_e T_e$ is dimensionless. We replace the internal energy U by

$$U^* = U + \hat{\eta}\left(\hat{N}_{ij}u_{i,\,j} - \nu\phi\right),$$

where $\hat{\eta}(\mathbf{x}, t)$ is a Lagrange multiplier, in (26) and eventually obtain the field equations of deformation-entropy constrained thermoelasticity in the form

$$\begin{aligned} \hat{c}_{ijkl} u_{k,\,jl} - \frac{1}{c}\beta_{ij}\phi_{,\,j} + \hat{N}_{ij}\hat{\eta}_{,\,j} &= \ddot{u}_i\,, \\ k_{ij}\left(\phi_{,\,ij} - \beta_{kl}u_{k,\,ijl} + \nu c\hat{\eta}_{,\,ij}\right) &= \dot{\phi} + \tau\ddot{\phi}. \end{aligned} \tag{30}$$

These equations are not equivalent to the field equations (16) of deformation-temperature constrained thermoelasticity. The secular equation of deformation-entropy constrained thermoelasticity is shown from (30) to take the form first obtained by Leslie and Scott [12]:

$$\hat{\sigma}^2 w \hat{\mathbf{c}} \cdot (w\mathbf{1} - \hat{\mathbf{P}})^{\mathrm{adj}}\,\hat{\mathbf{c}} - \frac{\mathrm{i}\omega}{1 - \mathrm{i}\omega\tau}\det(w\mathbf{1} - \hat{\mathbf{P}}) = 0,$$

in which we define

$$\hat{\sigma}^2 := \frac{c}{\nu^2}, \qquad \hat{\mathbf{c}} := \hat{\mathbf{N}}\mathbf{n},$$

and

$$\hat{\mathbf{P}} := \tilde{\mathbf{Q}} + \frac{1}{c}\left(\mathbf{b} - \frac{c}{\nu}\,\hat{\mathbf{c}}\right) \otimes \left(\mathbf{b} - \frac{c}{\nu}\,\hat{\mathbf{c}}\right).$$

The quantity $\hat{\sigma}$ plays a role for deformation-entropy constraints similar to that played by $\tilde{\sigma}$ for deformation-temperature constraints. In fact, $\nu\hat{\sigma} = \alpha\tilde{\sigma}$.

Deformation-entropy near-constraint

In order to model a material in which the constraint holds exactly as a limiting case we consider the internal energy in the form

$$U^{**} = U + \tfrac{1}{2}\hat{\chi}(\hat{N}_{ij}u_{i,j} - \nu\phi)^2, \tag{31}$$

where $\hat{\chi}$ is interpreted as a dimensionless elastic modulus associated with the constraint. The limit $\hat{\chi} \to 0$ corresponds to an unconstrained material and $\hat{\chi} \to \infty$ to the constraint holding exactly, this limit behaving such that

$$\hat{\chi} \to \infty, \;\; \hat{N}_{ij}u_{i,j} - \nu\phi \to 0, \;\; \hat{\eta} := \hat{\chi}(\hat{N}_{ij}u_{i,j} - \nu\phi) \;\text{ remains bounded.}$$

The various terms of (31) may be recombined into the single quadratic form

$$U^{**} = \tfrac{1}{2}\hat{c}^{**}_{ijkl}u_{i,j}u_{k,l} - (c^{**})^{-1}\beta^{**}_{ij}u_{i,j}\phi - \tfrac{1}{2}(c^{**})^{-1}\phi^2,$$

where

$$\begin{aligned} \hat{c}^{**}_{ijkl} &= \hat{c}_{ijkl} + \hat{\chi}\hat{N}_{ij}\hat{N}_{kl}, \\ \beta^{**}_{ij} &= (1 + \nu^2 c\hat{\chi})^{-1}(\beta_{ij} + \nu c\hat{\chi}\hat{N}_{ij}), \\ c^{**} &= (1 + \nu^2 c\hat{\chi})^{-1}\, c. \end{aligned} \tag{32}$$

The specific heat c^{**} remains positive for all $\hat{\chi}$, which explains why stability is not lost in the limit of a deformation-entropy constraint. However, $c^{**} \to 0$ in the limit of the constraint and this must be regarded as anomalous.

Since a nearly-constrained material is, in fact, an unconstrained material we may obtain the field equations of nearly-constrained thermoelasticity simply by replacing the appropriate moduli in the field equations (15) by the quantities defined at (32). We may then obtain the corresponding secular equation directly from the secular equation (10) of unconstrained generalized thermoelasticity:

$$w\det(w\mathbf{1} - \hat{\mathbf{Q}}^{**}) + \frac{\mathrm{i}\omega}{1 - \mathrm{i}\omega\tau}.\frac{1}{c^{**}/c}.\det(w\mathbf{1} - \tilde{\mathbf{Q}}^{**}) = 0 \tag{33}$$

in which, by analogy with (13), we may define

$$\hat{\mathbf{Q}}^{**} = \tilde{\mathbf{Q}}^{**} + \frac{1}{c^{**}}\,\mathbf{b}^{**}\otimes\mathbf{b}^{**} \tag{34}$$

with $\mathbf{b}^{**} := \boldsymbol{\beta}^{**}\mathbf{n}$. Using (32) and (34) we find that explicitly in terms of $\hat{\chi}$

$$\begin{aligned}\tilde{\mathbf{Q}}^{**} &= \tilde{\mathbf{Q}} + \frac{\hat{\chi}}{1+\nu^2 c\hat{\chi}}\,(\nu\mathbf{b}-\tilde{\mathbf{c}})\otimes(\nu\mathbf{b}-\hat{\mathbf{c}})\,,\\ \hat{\mathbf{Q}}^{**} &= \hat{\mathbf{Q}} + \hat{\chi}\hat{\mathbf{c}}\otimes\hat{\mathbf{c}}.\end{aligned} \tag{35}$$

Despite the apparently complicated nature of its dependence on $\hat{\chi}$, coming from (35), we find that the secular equation (33) of nearly-constrained thermoelasticity may be expressed in a form that is simply linear in $\hat{\chi}$:

$$\begin{aligned}&w\det(w\mathbf{1}-\hat{\mathbf{Q}}) + \frac{\mathrm{i}\omega}{1-\mathrm{i}\omega\tau}\det(w\mathbf{1}-\tilde{\mathbf{Q}})\\ &\quad-\nu^2 c\hat{\chi}\left\{\hat{\sigma}^2 w\hat{\mathbf{c}}\cdot(w\mathbf{1}-\hat{\mathbf{P}})^{\mathrm{adj}}\,\hat{\mathbf{c}} - \frac{\mathrm{i}\omega}{1-\mathrm{i}\omega\tau}\det(w\mathbf{1}-\hat{\mathbf{P}})\right\} = 0.\end{aligned} \tag{36}$$

As $\hat{\chi}\to 0$ this secular equation reduces to that of unconstrained thermoelasticity and as $\hat{\chi}\to\infty$ it reduces to that of deformation-entropy constrained thermoelasticity.

3 Deformation-temperature near-constraints

3.1 Low- and high-frequency limits

Low-frequency limit

Taking the low-frequency limit $\omega\to 0$, the secular equation (24) reduces to the form

$$w\det(w\mathbf{1}-\hat{\mathbf{Q}}) - \frac{\tilde{\chi}}{\tilde{\sigma}^2}w\det(w\mathbf{1}-\tilde{\mathbf{P}}) = 0, \tag{37}$$

or from (21)

$$w\det(w\mathbf{1}-\hat{\mathbf{Q}}^{\dagger\dagger}) = 0, \tag{38}$$

which is independent of τ, so that analysis of this limit yields the same results as the case $\tau = 0$. The low frequency roots are $w = 0, \hat{q}_1^{\dagger\dagger}, \hat{q}_2^{\dagger\dagger}, \hat{q}_3^{\dagger\dagger}$, where $\hat{q}_i^{\dagger\dagger}$, $i = 1,2,3$, denote the eigenvalues of $\hat{\mathbf{Q}}^{\dagger\dagger}$. With $\tilde{p}_i, \hat{q}_i$, $i = 1,2,3$, denoting the eigenvalues of $\tilde{\mathbf{P}}$ and $\hat{\mathbf{Q}}$, respectively, ordered such that

$$\tilde{p}_1 < \tilde{p}_2 < \tilde{p}_3, \quad \hat{q}_1 < \hat{q}_2 < \hat{q}_3,$$

in which strict inequalities have been assumed, we may use the methods of Leslie and Scott [12] to show that the eigenvalues $\hat{q}_i^{\dagger\dagger}$ may be ordered such that:
(i) for $0 < \tilde{\chi} < \tilde{\sigma}^2$,

$$\tilde{p}_1 < \hat{q}_1 \leq \hat{q}_1^{\dagger\dagger} < \tilde{p}_2 < \hat{q}_2 \leq \hat{q}_2^{\dagger\dagger} < \tilde{p}_3 < \hat{q}_3 \leq \hat{q}_3^{\dagger\dagger}, \tag{39}$$

(ii) for $\tilde{\chi} > \tilde{\sigma}^2$,

$$\hat{q}_3^{\dagger\dagger} \leq \tilde{p}_1 < \hat{q}_1 < \hat{q}_1^{\dagger\dagger} \leq \tilde{p}_2 < \hat{q}_2 < \hat{q}_2^{\dagger\dagger} \leq \tilde{p}_3 < \hat{q}_3, \tag{40}$$

where $\hat{q}_1 > 0$ but $\tilde{p}_1$ may take either sign. Note that if $\tilde{p}_1 \geq 0$ it is possible for $\hat{q}_3^{\dagger\dagger} = 0$ for some value of $\tilde{\chi}$, say $\tilde{\chi}_c$, such that

$$\tilde{\chi}_c = \frac{\hat{q}_1 \hat{q}_2 \hat{q}_3}{\tilde{p}_1 \tilde{p}_2 \tilde{p}_3}. \tag{41}$$

It may be shown further that as $\tilde{\chi}$ increases each of the roots $\hat{q}_i^{\dagger\dagger}$, $i = 1, 2, 3$, is monotonically increasing, such that $\hat{q}_i^{\dagger\dagger}$ increase from $\hat{q}_i$ to $\tilde{p}_{i+1}$, $i = 1, 2$, and $\hat{q}_3^{\dagger\dagger}$ increases from $\hat{q}_3$ to $+\infty$, for $0 \leq \tilde{\chi} < \tilde{\sigma}^2$, then from $-\infty$ to $\tilde{p}_1$, for $\tilde{\sigma}^2 < \tilde{\chi} < \infty$.

High-frequency limit

On taking the high-frequency limit, $\omega \to \infty$, the secular equation (24) reduces to

$$\det(w\mathbf{1} - \tilde{\mathbf{Q}}) - \tilde{\chi}\tilde{\mathbf{c}} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\mathrm{adj}}\, \tilde{\mathbf{c}} - \tau w \left[\det(w\mathbf{1} - \hat{\mathbf{Q}}) - \frac{\tilde{\chi}}{\tilde{\sigma}^2} \det(w\mathbf{1} - \tilde{\mathbf{P}})\right] = 0, \tag{42}$$

a quartic equation in w with roots denoted by $\bar{q}_i^{\dagger\dagger}$, $i = 1, 2, 3, 4$. Alternatively, from (21) the form

$$\det(w\mathbf{1} - \tilde{\mathbf{Q}}^{\dagger\dagger}) - \tau(1 - \tilde{\chi}/\tilde{\sigma}^2) w \det(w\mathbf{1} - \hat{\mathbf{Q}}^{\dagger\dagger}) = 0 \tag{43}$$

may be obtained. We rewrite (42) in the form

$$\det(w\mathbf{1} - \tilde{\mathbf{Q}}) - \tau w \det(w\mathbf{1} - \hat{\mathbf{Q}}) + \frac{\tilde{\chi}}{\tilde{\sigma}^2} \left[\tau w \det(w\mathbf{1} - \tilde{\mathbf{P}}) - \tilde{\sigma}^2 \tilde{\mathbf{c}} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\mathrm{adj}}\, \tilde{\mathbf{c}}\right] = 0 \tag{44}$$

and observe that from (43) we may determine the values of $\bar{q}_i^{\dagger\dagger}$ in the various limits of τ and from (44), the values of $\bar{q}_i^{\dagger\dagger}$ in the various limits of $\tilde{\chi}$. We define the polynomials

$$\begin{aligned} f(w) &:= \det(w\mathbf{1} - \hat{\mathbf{Q}}) \equiv \prod_{n=1}^{3} (w - \hat{q}_n), \\ g(w) &:= \det(w\mathbf{1} - \tilde{\mathbf{Q}}) \equiv \prod_{n=1}^{3} (w - \tilde{q}_n), \\ \tilde{G}(w) &:= \tilde{\mathbf{c}} \cdot (w\mathbf{1} - \tilde{\mathbf{P}})^{\mathrm{adj}}\, \tilde{\mathbf{c}} \equiv \tilde{\mathbf{c}} \cdot \tilde{\mathbf{c}} \prod_{n=1}^{2} (w - \tilde{W}_n), \end{aligned} \tag{45}$$

in which $\tilde{q}_i$, $i = 1, 2, 3$, denote the eigenvalues of $\tilde{\mathbf{Q}}$, and $\tilde{W}_i$, $i = 1, 2$, denote the zeros of $\tilde{G}(w)$.

From (43) and the interlacing results of Scott [16], we find that the interlacing properties of the eigenvalues of $\tilde{\mathbf{Q}}^{\dagger\dagger}$ and $\hat{\mathbf{Q}}^{\dagger\dagger}$, denoted by $\tilde{q}_i^{\dagger\dagger}$ and $\hat{q}_i^{\dagger\dagger}$, $i = 1, 2, 3$, respectively, are such that:

(i) $0 < \tilde{\chi} < \tilde{\sigma}^2$,

$$0 < \tilde{q}_1^{\dagger\dagger} < \hat{q}_1^{\dagger\dagger} < \tilde{q}_2^{\dagger\dagger} < \hat{q}_2^{\dagger\dagger} < \tilde{q}_3^{\dagger\dagger} < \hat{q}_3^{\dagger\dagger}, \tag{46}$$

(ii) $\tilde{\sigma}^2 < \tilde{\chi} < \tilde{\chi}_c$,

$$\hat{q}_3^{\dagger\dagger} < 0 < \tilde{q}_1^{\dagger\dagger} < \hat{q}_1^{\dagger\dagger} < \tilde{q}_2^{\dagger\dagger} < \hat{q}_2^{\dagger\dagger} < \tilde{q}_3^{\dagger\dagger}, \tag{47}$$

(iii) $\tilde{\chi} > \tilde{\chi}_c$,

$$0 < \hat{q}_3^{\dagger\dagger} < \tilde{q}_1^{\dagger\dagger} < \hat{q}_1^{\dagger\dagger} < \tilde{q}_2^{\dagger\dagger} < \hat{q}_2^{\dagger\dagger} < \tilde{q}_3^{\dagger\dagger}, \tag{48}$$

in which strict inequalities have been assumed.

Using the inequalities (46)−(48) we can determine that the roots of (43), $\bar{q}_i^{\dagger\dagger}$, $i = 1, 2, 3, 4$, are such that for fixed $\tilde{\chi}$:

(i) $0 < \tilde{\chi} < \tilde{\sigma}^2$,

$$0 \leq \bar{q}_1^{\dagger\dagger} \leq \tilde{q}_1^{\dagger\dagger} < \hat{q}_1^{\dagger\dagger} \leq \bar{q}_2^{\dagger\dagger} \leq \tilde{q}_2^{\dagger\dagger} < \hat{q}_2^{\dagger\dagger} \leq \bar{q}_3^{\dagger\dagger} \leq \tilde{q}_3^{\dagger\dagger} < \hat{q}_3^{\dagger\dagger} \leq \bar{q}_4^{\dagger\dagger}, \tag{49}$$

(ii) $\tilde{\sigma}^2 < \tilde{\chi} < \tilde{\chi}_c$,

$$\bar{q}_4^{\dagger\dagger} \leq \hat{q}_3^{\dagger\dagger} < 0 \leq \bar{q}_1^{\dagger\dagger} \leq \tilde{q}_1^{\dagger\dagger} < \hat{q}_1^{\dagger\dagger} \leq \bar{q}_2^{\dagger\dagger} \leq \tilde{q}_2^{\dagger\dagger} < \hat{q}_2^{\dagger\dagger} \leq \bar{q}_3^{\dagger\dagger} \leq \tilde{q}_3^{\dagger\dagger}, \tag{50}$$

(iii) $\tilde{\chi} > \tilde{\chi}_c$,

$$\bar{q}_4^{\dagger\dagger} \leq 0 < \hat{q}_3^{\dagger\dagger} \leq \bar{q}_1^{\dagger\dagger} \leq \tilde{q}_1^{\dagger\dagger} < \hat{q}_1^{\dagger\dagger} \leq \bar{q}_2^{\dagger\dagger} \leq \tilde{q}_2^{\dagger\dagger} < \hat{q}_2^{\dagger\dagger} \leq \bar{q}_3^{\dagger\dagger} \leq \tilde{q}_3^{\dagger\dagger}. \tag{51}$$

As $\tau \to 0$, $\bar{q}_i^{\dagger\dagger} \to \tilde{q}_i^{\dagger\dagger}$, $i = 1, 2, 3, 4$ where $\tilde{q}_4^{\dagger\dagger}$ is defined to be infinitely large (positive for $0 < \tilde{\chi} < \tilde{\sigma}^2$, and negative for $\tilde{\chi} > \tilde{\sigma}^2$). As $\tau \to \infty$, for (i) and (ii), $\bar{q}_i^{\dagger\dagger} \to \hat{q}_{i-1}^{\dagger\dagger}$, $i = 1, 2, 3, 4$, where $\hat{q}_0 := 0$, but for (iii) $\bar{q}_i^{\dagger\dagger} \to \hat{q}_{i-1}^{\dagger\dagger}$, $i = 2, 3$, whilst $\bar{q}_1^{\dagger\dagger} \to \hat{q}_3^{\dagger\dagger}$ and $\bar{q}_4^{\dagger\dagger} \to 0$. It may be demonstrated further that as τ increases $\bar{q}_i^{\dagger\dagger}$, $i = 1, 2, 3$, decrease monotonically and $\bar{q}_4^{\dagger\dagger}$ decreases monotonically for $\tilde{\chi} < \tilde{\sigma}^2$ and increases monotonically for $\tilde{\chi} > \tilde{\sigma}^2$.

However, the situation is complicated by the fact that $\tilde{q}_i^{\dagger\dagger}$ and $\hat{q}_i^{\dagger\dagger}$ are all dependent on $\tilde{\chi}$. The eigenvalues $\tilde{q}_i^{\dagger\dagger}$ interlace with $\tilde{q}_i$ and $\tilde{W}_i$, see (45), according to

$$0 < \tilde{q}_1 \leq \tilde{q}_1^{\dagger\dagger} \leq \tilde{W}_1 < \tilde{q}_2 \leq \tilde{q}_2^{\dagger\dagger} \leq \tilde{W}_2 < \tilde{q}_3 \leq \tilde{q}_3^{\dagger\dagger},$$

such that $\tilde{q}_i^{\dagger\dagger} \to \tilde{q}_i$ as $\tilde{\chi} \to 0$, $i = 1, 2, 3$, and $\tilde{q}_i^{\dagger\dagger} \to \tilde{W}_i$ as $\tilde{\chi} \to \infty$, $i = 1, 2, 3$, where $\tilde{W}_3$ is defined to be infinitely large. The eigenvalues $\hat{q}_i^{\dagger\dagger}$ have already been examined, see (39) and (40). It is easily shown that the eigenvalues $\tilde{q}_i^{\dagger\dagger}$ and $\hat{q}_i^{\dagger\dagger}$ increase monotonically with $\tilde{\chi}$.

We now examine the roots in the high-frequency limit as functions of $\tilde{\chi}$ and so return to the form (44) of the high-frequency secular equation. In terms of the real quartic polynomials

$$\begin{aligned} h(w) &:= w\det(w\mathbf{1}-\hat{\mathbf{Q}}) - \tau^{-1}\det(w\mathbf{1}-\tilde{\mathbf{Q}}) \equiv \prod_{n=1}^{4}(w-\bar{q}_n), \\ \tilde{h}(w) &:= w\det(w\mathbf{1}-\tilde{\mathbf{P}}) - \tilde{\sigma}^2\tau^{-1}\tilde{\mathbf{c}}\cdot(w\mathbf{1}-\tilde{\mathbf{P}})^{\mathrm{adj}}\,\tilde{\mathbf{c}} \equiv \prod_{n=1}^{4}(w-\tilde{h}_n), \end{aligned} \tag{52}$$

equation (44) may be rewritten as

$$h(w) - \frac{\tilde{\chi}}{\tilde{\sigma}^2}\,\tilde{h}(w) = 0. \tag{53}$$

In the limit $\tilde{\chi} \to 0$, the roots of (53) are the zeros of $h(w)$, $w = \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4$, and in the limit $\tilde{\chi} \to \infty$, the roots are the zeros of $\tilde{h}(w)$, $w = \tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4$. Again, the analysis is not quite so simple since each of $\bar{q}_i$ and $\tilde{h}_i$, $i = 1, 2, 3, 4$, depends on τ. We find that $\bar{q}_i$ vary such that as $\tau \to 0$, $\bar{q}_i \to \tilde{q}_i$, $i = 1, 2, 3, 4$, where $\tilde{q}_4$ is defined to be infinitely large, and as $\tau \to \infty$, $\bar{q}_i \to \hat{q}_{i-1}$, $i = 1, 2, 3, 4$, where $\hat{q}_0 := 0$, and we have the interlacings:

$$0 \le \bar{q}_1 \le \tilde{q}_1 < \hat{q}_1 \le \bar{q}_2 \le \tilde{q}_2 < \hat{q}_2 \le \bar{q}_3 \le \tilde{q}_3 < \hat{q}_3 \le \bar{q}_4.$$

It may be shown that each $\bar{q}_i$ decreases monotonically between its limits as τ increases and also that the roots $\tilde{h}_i$ are such that as $\tau \to 0$, $\tilde{h}_i \to \tilde{W}_{i-1}$, $i = 2, 3$, $\tilde{h}_1 \to -\infty$ and $\tilde{h}_4 \to +\infty$, and as $\tau \to \infty$, $\tilde{h}_i \to \tilde{p}_{i-1}$, $i = 1, 2, 3, 4$, where $\tilde{p}_0 := 0$ (assuming $\tilde{p}_1 > 0$, otherwise $\tilde{h}_1 \to \tilde{p}_1$, and $\tilde{h}_2 \to 0$), and we have the interlacings:

$$\tilde{h}_1 \le 0 < \tilde{p}_1 \le \tilde{h}_2 \le \tilde{W}_1 < \tilde{p}_2 \le \tilde{h}_3 \le \tilde{W}_2 < \tilde{p}_3 \le \tilde{h}_4,$$

or

$$\tilde{h}_1 \le \tilde{p}_1 < 0 \le \tilde{h}_2 \le \tilde{W}_1 < \tilde{p}_2 < \tilde{h}_3 \le \tilde{W}_2 < \tilde{p}_3 \le \tilde{h}_4,$$

depending upon whether $\tilde{p}_1$ is positive or negative. Again, each root varies monotonically between its limits; $\tilde{h}_i$, $i = 2, 3, 4$ decreasing and $\tilde{h}_1$ increasing.

The roots of (53) for $\tilde{\chi}$ finite interlace with the roots of this equation in the 0 and ∞ limits of $\tilde{\chi}$, i.e. with $\bar{q}_i, \tilde{h}_i$, $i = 1, 2, 3, 4$. But precisely how they interlace depends upon the interlacing between $\bar{q}_i$ and $\tilde{h}_i$ and at present it is unclear how these two quantities interact. By analysis of the interlacings obtained so far we determine that

$$\tilde{h}_2 < \bar{q}_3, \quad \bar{q}_1 < \tilde{h}_3 < \bar{q}_4, \quad \bar{q}_2 < \tilde{h}_4,$$

and $\tilde{h}_1 < 0$. Knowledge of the interlacing between $\bar{q}_i$ and $\tilde{h}_i$ can be taken further by considering the limit $\tau \to 0$ and investigating the interlacing between the quantities $\bar{q}_i^{\dagger\dagger}, \tilde{q}_i^{\dagger\dagger}, \hat{q}_i^{\dagger\dagger}$. Using this information it may be shown for sufficiently small values of τ that the quantities $\tilde{h}_i$ and $\bar{q}_i$ interlace such that

$$\tilde{h}_1 < 0 < \bar{q}_1 < \tilde{h}_2 < \bar{q}_2 < \tilde{h}_3 < \bar{q}_3 < \tilde{h}_4 < \bar{q}_4. \tag{54}$$

It has not been proved that this ordering holds for all τ but we shall assume that it does.

By evaluating the secular equation (53) of the high-frequency limit at $\bar{q}_i$ and $\tilde{h}_i$, $i = 1, 2, 3, 4$, and using (54) we can deduce that the roots in the high-frequency limit are such that:
(i) for $0 < \tilde{\chi} < \tilde{\sigma}^2$,

$$\tilde{h}_1 < 0 < \bar{q}_1 \leq \bar{q}_1^{\dagger\dagger} < \tilde{h}_2 < \bar{q}_2 \leq \bar{q}_2^{\dagger\dagger} < \tilde{h}_3 < \bar{q}_3 \leq \bar{q}_3^{\dagger\dagger} < \tilde{h}_4 < \bar{q}_4 \leq \bar{q}_4^{\dagger\dagger}, \tag{55}$$

(ii) for $\tilde{\chi} > \tilde{\sigma}^2$,

$$\bar{q}_4^{\dagger\dagger} \leq \tilde{h}_1 < 0 < \bar{q}_1 < \bar{q}_1^{\dagger\dagger} \leq \tilde{h}_2 < \bar{q}_2 < \bar{q}_2^{\dagger\dagger} \leq \tilde{h}_3 < \bar{q}_3 < \bar{q}_3^{\dagger\dagger} \leq \tilde{h}_4 < \bar{q}_4. \tag{56}$$

It can be shown that each of the roots $\bar{q}_i^{\dagger\dagger}$ in the high-frequency limit is monotonically increasing on the range $0 < \tilde{\chi} < \tilde{\sigma}^2$.

3.2 Low- and high-frequency expansions

We have examined the secular equation in the frequency limits and now investigate it as each limit is approached. For this we use the form of the secular equation for a thermoelastic material subject to a deformation-temperature near-constraint given by (21), which has the same form as that of an unconstrained material, see [15].

Before continuing we need to examine the interlacing between the eigenvalues $\hat{q}_i^{\dagger\dagger}$ and $\tilde{q}_i^{\dagger\dagger}$, $i = 1, 2, 3$, and the interlacing between $\hat{q}_i^{\dagger\dagger}$, $i = 1, 2, 3$, and the roots in the high-frequency limit, $\bar{q}_i^{\dagger\dagger}$, $i = 1, 2, 3, 4$. Using the relationship between $\hat{\mathbf{Q}}^{\dagger\dagger}$ and $\tilde{\mathbf{Q}}^{\dagger\dagger}$, as given by (22), there exist three possible interlacings between $\hat{q}_i^{\dagger\dagger}$ and $\tilde{q}_i^{\dagger\dagger}$, $i = 1, 2, 3$, depending on $\tilde{\chi}$. These have already been determined as (46)$-$(48). The interlacings between $\hat{q}_i^{\dagger\dagger}$, $i = 1, 2, 3$, and $\bar{q}_i^{\dagger\dagger}$, $i = 1, 2, 3, 4$, have also already been determined, again with three regions of $\tilde{\chi}$ to be considered, see (49)$-$(51).

By defining the polynomials

$$\begin{aligned} f^{\dagger\dagger}(w) &:= w \prod_{n=1}^{3} (w - \hat{q}_n^{\dagger\dagger}), \\ g^{\dagger\dagger}(w) &:= \prod_{n=1}^{3} (w - \tilde{q}_n^{\dagger\dagger}), \end{aligned} \tag{57}$$

the secular equation (21) can be written in the form

$$f^{\dagger\dagger}(w) + \frac{\mathrm{i}\omega}{1-\mathrm{i}\omega\tau}\cdot\frac{1}{1-\tilde{\chi}/\tilde{\sigma}^2}\cdot g^{\dagger\dagger}(w) = 0, \tag{58}$$

and we now proceed to determine low- and high-frequency approximations for the four branches of (58).

Low-frequency expansions

For low frequencies the secular equation (58) gives the expansions

$$w_i(\omega) = \hat{q}_i^{\dagger\dagger} - \frac{\mathrm{i}\omega}{1-\tilde{\chi}/\tilde{\sigma}^2}\,\frac{g^{\dagger\dagger}(\hat{q}_i^{\dagger\dagger})}{f^{\dagger\dagger\prime}(\hat{q}_i^{\dagger\dagger})} + \mathrm{O}(\omega^2). \tag{59}$$

The parameter τ does not appear until terms of the order ω^2 and therefore conclusions are independent of τ and results may be taken directly from those of the classical case $\tau = 0$, see [7].

For stability we require Im $w_i(\omega) < 0$, which is so for sufficiently small frequencies provided the coefficient of $\mathrm{i}\omega$ is negative. By examining the sign of this coefficient we find that for $0 < \tilde{\chi} < \tilde{\sigma}^2$ all four branches are stable but for $\tilde{\chi} > \tilde{\sigma}^2$ three branches are stable and one unstable. (For $\tilde{\chi} > \tilde{\sigma}^2$, the unstable branch emanates from $\hat{q}_3^{\dagger\dagger}$ when $\tilde{\sigma}^2 < \tilde{\chi} < \tilde{\chi}_c$ and from the origin when $\tilde{\chi} > \tilde{\chi}_c$, where $\tilde{\chi}_c$ is given by (41). The second possibility can only occur if $\tilde{p}_1 > 0$.)

When $\hat{q}_3^{\dagger\dagger} = 0$ the expansions (59) break down for $i = 0, 3$. This happens at the value $\tilde{\chi} = \tilde{\chi}_c$ given by (41), see [7]. Summarizing, the expansions are found to be

$$w_{0,3}(\omega) = \pm\frac{\sqrt{2}}{2}(1-\mathrm{i})\frac{\tilde{q}_1^{\dagger\dagger}\tilde{q}_2^{\dagger\dagger}\tilde{q}_3^{\dagger\dagger}}{\hat{q}_1^{\dagger\dagger}\hat{q}_2^{\dagger\dagger}}\omega^{\frac{1}{2}} + \mathrm{O}(\omega), \tag{60}$$

with $w_i(\omega)$, $i = 1, 2$, given by (59) unchanged. From (60) it may be determined that for $\tilde{\chi} = \tilde{\chi}_c$ stability conclusions for frequencies sufficiently small remain unchanged, with $w_{0,3}(\omega)$ representing one stable and one unstable branch. The only effect of this case is that these branches approach their low-frequency limits with arguments $\frac{-\pi}{4}$ and $\frac{3\pi}{4}$.

High-frequency expansions

We define the quartic polynomial $\tilde{h}^{\dagger\dagger}(w)$ by

$$\tilde{h}^{\dagger\dagger}(w) := \prod_{n=1}^{4}(w - \bar{q}_n^{\dagger\dagger}) \tag{61}$$

and recall that the roots $\bar{q}_i^{\dagger\dagger}$ have been found to interlace with $\hat{q}_i^{\dagger\dagger}$ and $\tilde{q}_i^{\dagger\dagger}$ according to (49)−(51). For high frequencies the roots of the secular equation (21) take the approximate form

$$w_i(\omega) = \bar{q}_i^{\dagger\dagger} - \frac{f^{\dagger\dagger}(\bar{q}_i^{\dagger\dagger})}{\tilde{h}^{\dagger\dagger\prime}(\bar{q}_i^{\dagger\dagger})}(\mathrm{i}\omega\tau)^{-1} + \mathrm{O}(\omega^{-2}), \quad i = 1, 2, 3, 4. \tag{62}$$

The stability condition Im $w_i < 0$ holds for sufficiently large frequencies provided that

$$\frac{f^{\dagger\dagger}(\bar{q}_i^{\dagger\dagger})}{\tilde{h}^{\dagger\dagger\prime}(\bar{q}_i^{\dagger\dagger})} < 0, \quad i = 1, 2, 3, 4. \tag{63}$$

From the interlacing properties (49)−(51), we find that (63) holds for all four branches if $0 < \tilde{\chi} < \tilde{\sigma}^2$ but for $\tilde{\chi} > \tilde{\sigma}^2$ only three of the branches are stable with the branch terminating at $\bar{q}_4^{\dagger\dagger}$ now unstable. The parameter τ is significant in the high-frequency expansions but does not affect the stability.

3.3 Stability

We have proved that in the low- and high-frequency limits the branches of the secular equation are: (i) for $0 < \tilde{\chi} < \tilde{\sigma}^2$ all stable, and (ii) for $\tilde{\chi} > \tilde{\sigma}^2$ three are stable and one unstable. However, this does not ensure stability for the entire frequency range. To prove that a branch that is stable/unstable in the frequency limits remains so for the entire frequency range we must show that it cannot cross the real axis for $0 < \omega < \infty$. Thus we must prove that the secular equation can have real roots only in the frequency limits. Rearranging the secular equation into the form

$$\frac{f^{\dagger\dagger}(w)}{g^{\dagger\dagger}(w)} = -\frac{\mathrm{i}\omega}{(1 - \tilde{\chi}/\tilde{\sigma}^2)(1 - \mathrm{i}\omega\tau)}$$

shows that real w and real ω cannot coexist, except possibly in the frequency limits, as we would be seeking to equate a real number with a complex number. (This assumes $\tilde{\chi} \neq \tilde{\sigma}^2$ and τ finite). Thus we have proved that a branch that is stable/unstable in the frequency limits will remain so for the entire frequency range $0 < \omega < \infty$.

4 Deformation-entropy near-constraints

4.1 Low- and high-frequency limits

Low-frequency limit

In the low-frequency limit the secular equation (36) reduces to the form

$$w \det(w\mathbf{1} - \hat{\mathbf{Q}}) - \nu^2 c\hat{\chi} w\hat{\sigma}^2 \hat{\mathbf{c}} \cdot (w\mathbf{1} - \hat{\mathbf{Q}})^{\mathrm{adj}}\, \hat{\mathbf{c}} = 0, \tag{64}$$

or from (33)

$$w \det(w\mathbf{1} - \hat{\mathbf{Q}}^{**}) = 0, \tag{65}$$

which are independent of τ. The low-frequency limit roots are $w = 0$ and $w = \hat{q}_i^{**}$, $i = 1, 2, 3$, where $\hat{q}_i^{**}$, $i = 1, 2, 3$, denote the eigenvalues of $\hat{\mathbf{Q}}^{**}$.

We define the polynomial

$$\hat{G}(w) := \hat{\mathbf{c}} \cdot (w\mathbf{1} - \hat{\mathbf{Q}})^{\mathrm{adj}}\, \hat{\mathbf{c}} \equiv \hat{\mathbf{c}} \cdot \hat{\mathbf{c}} \prod_{n=1}^{2} (w - \hat{W}_n), \tag{66}$$

and may derive the interlacing properties

$$0 < \hat{q}_1 \le \hat{q}_1^{**} \le \hat{W}_1 < \hat{q}_2 \le \hat{q}_2^{**} \le \hat{W}_2 < \hat{q}_3 \le \hat{q}_3^{**}, \tag{67}$$

with each root monotonically increasing as $\hat{\chi}$ increases, noting that $\hat{q}_i^{**} \to \hat{q}_i$ as $\hat{\chi} \to 0$, and $\hat{q}_i^{**} \to \hat{W}_i$, $i = 1, 2$, $\hat{q}_3^{**} \to \infty$ as $\hat{\chi} \to \infty$.

High-frequency limit

In the high-frequency limit the secular equation (36) reduces to

$$\begin{aligned} &-\tau w \det(w\mathbf{1} - \hat{\mathbf{Q}}) + \det(w\mathbf{1} - \tilde{\mathbf{Q}}) \\ &\quad + \nu^2 c\hat{\chi} \left\{ \det(w\mathbf{1} - \hat{\mathbf{P}}) + \tau\hat{\sigma}^2 w \hat{\mathbf{c}} \cdot (w\mathbf{1} - \hat{\mathbf{P}})^{\mathrm{adj}}\, \hat{\mathbf{c}} \right\} = 0, \end{aligned} \tag{68}$$

a quartic equation in w with roots denoted by $\bar{q}_i^{**}$, $i = 1, 2, 3, 4$. Alternatively, using (33) this may be written

$$\det(w\mathbf{1} - \tilde{\mathbf{Q}}^{**}) - \tau(1 + \nu^2 c\hat{\chi})^{-1} w \det(w\mathbf{1} - \hat{\mathbf{Q}}^{**}) = 0. \tag{69}$$

From (68) we can examine the roots $\bar{q}_i^{**}$ as functions of $\hat{\chi}$, and from (69) as functions of τ.

From (69) and the interlacing properties of the eigenvalues of $\hat{\mathbf{Q}}^{**}$ and $\tilde{\mathbf{Q}}^{**}$, see [16], denoted by $\hat{q}_i^{**}$ and $\tilde{q}_i^{**}$, respectively, we find that

$$0 < \tilde{q}_1^{**} < \hat{q}_1^{**} < \tilde{q}_2^{**} < \hat{q}_2^{**} < \tilde{q}_3^{**} < \hat{q}_3^{**},$$

from which it may be determined that the roots $\bar{q}_i^{**}$ interlace with $\hat{q}_i^{**}$ and $\tilde{q}_i^{**}$, for fixed $\hat{\chi}$, according to

$$0 \leq \bar{q}_1^{**} \leq \tilde{q}_1^{**} < \hat{q}_1^{**} \leq \bar{q}_2^{**} \leq \tilde{q}_2^{**} < \hat{q}_2^{**} \leq \bar{q}_3^{**} \leq \tilde{q}_3^{**} < \hat{q}_3^{**} \tag{70}$$

with $\bar{q}_i^{**} \to \tilde{q}_i^{**}$, $i = 1, 2, 3, 4$, for $\tau \to 0$ and $\bar{q}_i^{**} \to \hat{q}_{i-1}^{**}$, $i = 1, 2, 3, 4$, where $\hat{q}_0^{**} := 0$. It may be shown further that the roots $\bar{q}_i^{**}$ monotonically decrease as τ increases. In the limits of τ, $\bar{q}_i^{**}$ have been found to be $\tilde{q}_i^{**}$ and $\hat{q}_i^{**}$, which depend on $\hat{\chi}$ such that

$$0 < \tilde{q}_1 \leq \tilde{q}_1^{**} \leq \hat{p}_1 < \tilde{q}_2 \leq \tilde{q}_2^{**} \leq \hat{p}_2 < \tilde{q}_3 \leq \tilde{q}_3^{**} \leq \hat{p}_3,$$

where $\hat{p}_i$ denote the eigenvalues of $\hat{\mathbf{P}}$. We find that $\tilde{q}_i^{**} \to \tilde{q}_i$, $i = 1, 2, 3$, as $\hat{\chi} \to 0$, $\tilde{q}_i^{**} \to \hat{p}_i$, $i = 1, 2, 3$, as $\hat{\chi} \to \infty$, each monotonically increasing with $\hat{\chi}$. The behaviour of $\hat{q}_i^{**}$ is as given by (67), again each monotonically increasing with $\hat{\chi}$.

In order to examine $\bar{q}_i^{**}$ as a function of $\hat{\chi}$, we define by analogy with $(52)_2$ the cubic polynomial

$$\hat{h}(w) := (1+\tau\hat{\sigma}^2\hat{\mathbf{c}}\cdot\hat{\mathbf{c}})^{-1}\{\det(w\mathbf{1}-\hat{\mathbf{P}})+\tau\hat{\sigma}^2 w\hat{\mathbf{c}}\cdot(w\mathbf{1}-\hat{\mathbf{P}})^{\mathrm{adj}}\,\hat{\mathbf{c}}\} \equiv \prod_{n=1}^{3}(w-\hat{h}_n), \tag{71}$$

with zeros denoted by $\hat{h}_i$, $i = 1, 2, 3$. Comparable with (53) in the deformation-temperature case, the secular equation (68) in the high-frequency limit of the deformation-entropy near-constraint can written as

$$h(w) - \nu^2 c\hat{\chi}\tau^{-1}(1 + \tau\hat{\sigma}^2\hat{\mathbf{c}}\cdot\hat{\mathbf{c}})\hat{h}(w) = 0, \tag{72}$$

so that clearly the roots in the limit $\hat{\chi} \to 0$ are the zeros of $h(w)$, $\bar{q}_i$, $i = 1, 2, 3, 4$, and in the limit $\hat{\chi} \to \infty$ are the zeros of $\hat{h}(w)$, $\hat{h}_i$, $i = 1, 2, 3$, together with a root infinitely large. Each of these zeros varies with τ. We find that $\bar{q}_i$, $i = 1, 2, 3, 4$, behave according to

$$0 \leq \bar{q}_1 \leq \tilde{q}_1 < \hat{q}_1 \leq \bar{q}_2 \leq \tilde{q}_2 < \hat{q}_2 \leq \bar{q}_3 \leq \tilde{q}_3 < \hat{q}_3 \leq \bar{q}_4, \tag{73}$$

and that $\bar{q}_i \to \tilde{q}_i$, $i = 1, 2, 3, 4$, as $\tau \to 0$, where $\tilde{q}_4 \to \infty$, and $\bar{q}_i \to \hat{q}_{i-1}$, $i = 1, 2, 3, 4$, as $\tau \to \infty$, where $\hat{q}_0 := 0$. Each root monotonically decreases as τ increases.

We find that $\hat{h}_i$, $i = 1, 2, 3$, behave like

$$0 \leq \hat{h}_1 \leq \hat{p}_1 < \hat{W}_1 \leq \hat{h}_2 \leq \hat{p}_2 < \hat{W}_2 \leq \hat{h}_3 < \hat{p}_3, \tag{74}$$

with $\hat{h}_i \to \hat{p}_i$, $i = 1, 2, 3$, as $\tau \to 0$, and $\hat{h}_i \to \hat{W}_i$, $i = 1, 2, 3$, as $\tau \to \infty$, where $\hat{W}_3$ is defined to be infinitely large. Again, each root is monotonically decreasing as τ increases.

The roots $\bar{q}_i^{**}$, $i = 1, 2, 3, 4$, interlace with the values obtained in the limits of $\hat{\chi}$, i.e. $\bar{q}_i$ and $\hat{h}_i$, but how they interlace depends upon the interlacings between $\bar{q}_i$ and $\hat{h}_i$. From exact analysis, we have so far determined only the following restrictions on the orderings between $\bar{q}_i$ and $\hat{h}_i$:

$$\hat{h}_1 < \bar{q}_3, \quad \bar{q}_1 < \hat{h}_2 < \bar{q}_4, \quad \bar{q}_2 < \hat{h}_3. \tag{75}$$

However, by examining the limit $\tau \to 0$ we may show further that

$$0 < \bar{q}_1 < \hat{h}_1 < \bar{q}_2 < \hat{h}_2 < \bar{q}_3 < \hat{h}_3 < \bar{q}_4. \tag{76}$$

This follows from the fact that $\bar{q}_i = \tilde{q}_i$, $i = 1, 2, 3, 4$, where $\tilde{q}_4 \to \infty$, and $\hat{h}_i = \hat{p}_i$ in this limit and the interlacings between $\tilde{q}_i$ and $\hat{p}_i$ are easily determined. Thus it follows that the interlacing (76) must hold for sufficiently small values of τ. Assuming that (76) holds, by evaluating the high-frequency limit of the secular equation (72) at $\bar{q}_i$, $i = 1, 2, 3, 4$, and $\hat{h}_i$, $i = 1, 2, 3$, and also at infinity, it can be shown that

$$0 < \bar{q}_1 \le \bar{q}_1^{**} \le \hat{h}_1 < \bar{q}_2 \le \bar{q}_2^{**} \le \hat{h}_2 < \bar{q}_3 \le \bar{q}_3^{**} \le \hat{h}_3 < \bar{q}_4 \le \bar{q}_4^{**}. \tag{77}$$

It can be shown that each of the roots $\bar{q}_i^{**}$ in the high-frequency limit is monotonically increasing with $\hat{\chi}$.

4.2 Low- and high-frequency expansions

We now look in more detail at the low- and high-frequency limits of the secular equation by taking power series expansions about each of the roots in these limits. To do this we cast the secular equation into a similar form to that for an unconstrained material and follow the method used for that situation.

Define $f^{**}(w)$ and $g^{**}(w)$ by

$$\begin{aligned} f^{**}(w) &:= w \prod_{n=1}^{3} (w - \hat{q}_n^{**}), \\ g^{**}(w) &:= \prod_{n=1}^{3} (w - \hat{q}_n^{**}), \end{aligned} \tag{78}$$

so that the secular equation (33) may be written

$$f^{**}(w) + \frac{\mathrm{i}\omega}{1 - \mathrm{i}\omega\tau} \cdot \frac{1}{1 + \nu^2 c\hat{\chi}} \cdot g^{**}(w) = 0. \tag{79}$$

Low-frequency expansions

Low-frequency expansions give the same results as the non-generalized case due to τ not appearing in power series expansions until terms of the order ω^2. The low-frequency limit roots are the zeros of $f^{**}(w)$ and expansions about each of these roots prove that for sufficiently small frequencies each of the branches of the secular equation is stable.

High-frequency expansions

In the high-frequency limit the roots of the secular equation (68) have been denoted by $\bar{q}_i^{**}$, $i = 1, 2, 3, 4$, and shown to interlace with $\hat{q}_i^{**}$ and $\tilde{q}_i^{**}$ according to (70). We define

$$h^{**}(w) := \prod_{n=1}^{4} (w - \bar{q}_n^{**}). \tag{80}$$

and observe that the secular equation (33) may be cast into the form

$$f^{**}(w) - \mathrm{i}\omega\tau h^{**}(w) = 0 \tag{81}$$

from which we see that the high-frequency roots may be approximated by

$$w_i(\omega) = \bar{q}_i^{**} + \frac{f^{**}(\bar{q}_i^{**})}{h^{**\prime}(\bar{q}_i^{**})}(\mathrm{i}\omega\tau)^{-1} + O(\omega^{-2}), \quad i = 1, 2, 3, 4.$$

From the interlacing properties (70) we see that for sufficiently large frequencies the stability condition $\mathrm{Im}\ w_i(\omega) < 0$ holds. Thus all four branches of the secular equation are stable at high frequencies.

So far we have shown only that each of the branches of the secular equation is stable for sufficiently large or small frequencies. The same methods that were used in the deformation-temperature case suffice to show that all four branches of the secular equation of a material subject to a deformation-entropy near-constraint are stable for all frequencies.

5 A connection between deformation-temperature and deformation-entropy near-constraints

We find that deformation-temperature and deformation-entropy near-constraints are intimately related despite their non-equivalence. The closest possible correspondence between the two near-constraints is obtained by making the identifications

$$\begin{aligned}
\hat{\mathbf{c}} &= \tilde{\mathbf{c}} + \frac{\alpha}{c}\,\mathbf{b},\\
\nu &= \frac{\alpha}{c},\\
\hat{\chi} &= \frac{\tilde{\chi}}{1 - \tilde{\chi}/\tilde{\sigma}^2},\\
\hat{\eta} &= \tilde{\eta},
\end{aligned} \tag{82}$$

see [18], which imply $\hat{\sigma} = c\tilde{\sigma}$.

The finite limit $\tilde{\chi} \to \tilde{\sigma}^2$, approached from below, corresponds to the limit $\hat{\chi} \to \infty$ so that the deformation-temperature constraint holding approximately, with $\tilde{\chi} = \tilde{\sigma}^2$, corresponds to the deformation-entropy constraint holding exactly. Alts [1] suggested that the undesirable instabilities of the deformation-temperature constraint may be avoided by allowing the constraint to hold only approximately. Our analysis supports this view. For larger values of $\tilde{\chi}$, $\hat{\chi}$ becomes negative, further exposing the deficiencies of the deformation-temperature constraint.

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