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Stochastic processes in terms of inner premeasures

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Abstract. In a recent paper the author used his work in measure and integration to obtain the projective limit theorem of Kolmogorov in a comprehensive version in terms of inner premeasures. In the present paper the issue is the influence of the new theorem on the notion of stochastic processes. It leads to essential improvements in the foundation of special processes, of the Wiener process in the previous paper and of the Poisson process in the present one. But it also forms the basis for a natural redefinition of the entire notion. The stochastic processes in the reformed sense are in one-to-one correspondence with the traditional ones in case that the state space is a Polish topological space with its Borel σ algebra, but the sizes and procedures are quite different. The present approach makes an old idea of Kakutani come true, but with due adaptations.

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Dedicated to the Memory of Klaus Floret

In the book [11] and in the recent survey article [13] the present author developed a new structure in the domain of measure and integration. Its application in different branches led to essential improvements of the traditional substance. In [14] it has been applied to the projective limit theorems of the Prokhorov and Kolmogorov types. The method produced, thanks to its built-in powerful τ (=nonsequential) lift, the limit measure on an immense domain, which finished off the notorious trouble with far too small domains. Thus it produced the (one-dimensional) Wiener measure in an extensive version which could well be called the *true Wiener measure*. The introduction ended with the expectation that the paper will have quite some influence on the probabilistic concepts around stochastic processes.

The present paper will in fact be devoted to the notion of stochastic processes. On the one hand the author noted that the basic results of [14] can be invoked for the well-foundation of the Poisson process in much the same manner as for the Wiener process.

On the other hand the concern is the traditional overall notion of stochastic processes of our days, as presented in the treatises of Doob [7], Dellacherie-Meyer [3], Bauer [1], Hackenbroch-Thalmaier [9], Stromberg [15], and others. The definitive notion comes from the fundamental 1953 treatise [7] of Doob. The decades before saw most intensive efforts in order to master stochastic processes on *uncountable* time domains with the means of abstract measure theory which are of *countable* nature. The problems involved are made clear in the articles [4] and [5] section 2 and in the 1946 AMS address [6] of Doob. A typical unpleasant fact is in [5] theorem 2.1, attributed to Halmos. It seems that meanwhile such statements have fallen into oblivion. However, the facts remain inherent in our traditional notion of stochastic processes. There will be another intolerable example below (see theorem 4): In virtue of the definition a stochastic process is synonymous with the multitude of all its so-called *versions*). But this multitude turns out to contain a vast crowd of *pathological* members, manifested in the vast crowd of their absurd images in the path space, that is within the class of those subsets of the path space which are sometimes even called the *essential subsets* for the process. Thus it became clear that the multitude of *versions* of a stochastic process, that is the multitude of *probability measure extensions* of its native projective limit measure, needs a drastic reduction. But in the subsequent half century the traditional theory of stochastic processes did not produce definitive solutions to this end.

In particular Doob [6] described the idea of Kakutani to produce a *canonical* probability measure extension in the path space, say in case of a compact Polish state space, to its Borel σ algebra for the product topology, via outer regularity with respect to the lattice of open subsets. But Doob added at once that this idea likewise did not prove to be successful, say in the frame of Polish state spaces, and this remained so in the 1969 historical note in Bourbaki [2].

We turn to the new structure in measure and integration developed in [11] [13] and to the projective limit theorem of the Kolmogorov type obtained from it in [14]. We shall present this theorem below in a certain fortified version, as to the characterization of the class of projective limits. The τ version of the theorem furnishes an obvious counterpart to the traditional notion of stochastic processes, which is not less natural and simple and will be seen to be much more successful, as made explicit in the cases of the (one-dimensional) Wiener and Poisson processes. The decisive point is that the reformed notion offers a unified method to equip stochastic processes with canonical probability measures in the path space which have immense domains. In the particular case that the state space is a Polish topological space with its Borel σ algebra we shall exhibit a one-to-one correspondence between the two kinds of stochastic processes, under which the probability measure of the new process is an extension of the native projective limit measure of the traditional one. The new probability measure is of course the maximal inner τ extension of a certain inner τ premeasure. In the present particular case it need not be maximal Radon for the product topology of the path space, and its domain need not contain the full Borel σ algebra. To be sure, this holds true in case of a compact Polish state space, but also in a wider class of important cases which include the Wiener and Poisson processes with state space $\mathbb{R}$ (see corollary 14(3) below). The deviation from the Radon situation beyond that class amounts to a certain deviation from topology in the path space, and is due to an obvious and simple step in our approach (in the context of section 4 it is the step to pass from $\mathfrak{K}$ to $\mathfrak{K} \cup \{Y\}$). It is thanks to this step that our inner enterprise arrives at the success in the realm of Polish state spaces which had been denied to the outer attempt of Kakutani.

The paper consists of five sections. After a few preliminaries in section 1 we treat in section 2 the traditional notion of stochastic processes. In section 3 we produce the fortified version of the projective limit theorem in terms of inner premeasures, and on this basis then treat in section 4 the proposed reformed notion of stochastic processes and its relation to the previous one. At last section 5 will specialize to the Poisson process.

1 Preliminaries on probability measures

Our concern are the extensions and direct images of probability measures (*prob measures* for short). Much of the sequel can be found in Doob [4] [5] [6] in some form or other. Let X be a nonvoid set.

1 Lemma. *Let $\alpha : \mathfrak{A} \to [0, \infty]$ be a content on an algebra $\mathfrak{A}$ in X and $C \subset X$. Then the following are equivalent.*

(i) All $A \in \mathfrak{A}$ with $A \cap C = \varnothing$ have $\alpha(A) = 0$.

(ii) $\alpha(A) = \alpha^\star(A \cap C)$ for all $A \in \mathfrak{A}$.

In case $\alpha(X) < \infty$ also

(iii) $\alpha^\star(C) = \alpha(X)$.

Here as usual $\alpha^\star(E) = \inf\{\, \alpha(A) : A \in \mathfrak{A}$ with $A \supset E \,\}$ for $E \subset X$.

Proof. (i)⇒(ii). Fix $A \in \mathfrak{A}$. For $U \in \mathfrak{A}$ with $U \supset A \cap C$ then $U' \subset A' \cup C'$ and hence $U' \cap A \subset C'$, so that $\alpha(U' \cap A) = 0$. It follows that $\alpha(U) = \alpha(U) + \alpha(U' \cap A) = \alpha(U \cup A) \geqq \alpha(A)$. Thus $\alpha^\star(A \cap C) \geqq \alpha(A)$; the converse $\leqq$ is obvious. (ii)⇒(i) and (ii)⇒(iii) are obvious. (iii) and $\alpha(X) < \infty$ ⇒(i). For $A \in \mathfrak{A}$ with $A \cap C = \varnothing$ or $C \subset A'$ we have $\alpha(X) = \alpha^\star(C) \leqq \alpha(A') = \alpha(X) - \alpha(A)$ and hence $\alpha(A) = 0$. $\boxed{QED}$

Under these equivalent conditions the set C is called *thick* for α. In contrast, one defines α to *live* on $C \subset X$ iff all $A \subset X$ with $A \cap C = \varnothing$ fulfil $A \in \mathfrak{A}$ and $\alpha(A) = 0$. Then of course C is in $\mathfrak{A}$ and is thick for α. We have the consequences which follow. The proofs are routine.

2 Lemma. *Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a content on an algebra $\mathfrak{A}$ in X and $C \subset X$ be thick for α. Define*

$$\mathfrak{B} := \{ B \subset X : B \cap C \in \mathfrak{A} \sqcap C \} \text{ with } \mathfrak{A} \sqcap C := \{ A \cap C : A \in \mathfrak{A} \},$$
$$\beta : \mathfrak{B} \to [0,\infty] \text{ to be } \beta(B) = \alpha^\star(B \cap C) \text{ for } B \in \mathfrak{B}.$$

Then $\mathfrak{B}$ is an algebra in X with $\mathfrak{B} \supset \mathfrak{A}$ and $\mathfrak{B} \sqcap C = \mathfrak{A} \sqcap C$, and

(1) β is a content on $\mathfrak{B}$ which is an extension of α and lives on C. Moreover $\beta^\star(M) = \alpha^\star(M \cap C)$ for all $M \subset X$.

(2) If $\rho : \mathfrak{R} \to [0,\infty]$ is a content on an algebra $\mathfrak{R}$ in X which is an extension of α and lives on C then ρ is an extension of β.

(3) If α is a measure on the σ algebra $\mathfrak{A}$ then β is a measure on the σ algebra $\mathfrak{B}$.

Next we recall the notions of direct image formation as formulated in [14, section 3]. Let $K : \Omega \to X$ be a map defined on a nonvoid set Ω. For a σ algebra $\mathfrak{P}$ in Ω one defines the direct image $\vec{K}\mathfrak{P} := \{ A \subset X : K^{-1}(A) \in \mathfrak{P} \}$, which is a σ algebra in X. For a measure $P : \mathfrak{P} \to [0,\infty]$ on $\mathfrak{P}$ one defines the direct image $\vec{K}P : \vec{K}\mathfrak{P} \to [0,\infty]$ to be $\vec{K}P(A) = P(K^{-1}(A))$ for $A \in \vec{K}\mathfrak{P}$, which is a measure on $\vec{K}\mathfrak{P}$ with $\vec{K}P(X) = P(\Omega)$, hence a prob measure iff P is one. One verifies that $\vec{K}P$ lives on $K(\Omega) \subset X$. If $\mathfrak{A}$ is a σ algebra in X, then $\mathfrak{A} \subset \vec{K}\mathfrak{P}$ means that $K : \Omega \to X$ is measurable $\mathfrak{P} - \mathfrak{A}$ in the usual sense, and then $\alpha := \vec{K}P|\mathfrak{A}$ is the usual image measure $\alpha : \mathfrak{A} \to [0,\infty]$ of $P : \mathfrak{P} \to [0,\infty]$.

After this we start with some redefinitions of familiar notions. Let $\alpha : \mathfrak{A} \to [0,\infty[$ be a prob measure on a σ algebra $\mathfrak{A}$ in X. We define a *version* of α to be a map $K : (\Omega, \mathfrak{P}, P) \to X$ defined on a prob measure space $(\Omega, \mathfrak{P}, P)$ with $\mathfrak{A} \subset \vec{K}\mathfrak{P}$, so that K is measurable $\mathfrak{P} - \mathfrak{A}$, and $\alpha = \vec{K}P|\mathfrak{A}$. Thus a version $K : (\Omega, \mathfrak{P}, P) \to X$ of α produces, in form of

$$\vec{K}\mathfrak{P} =: \mathfrak{R} \text{ and } \vec{K}P =: \rho, \text{ and of } K(\Omega) =: C,$$

a prob measure extension $\rho : \mathfrak{R} \to [0,\infty[$ of α and a subset $C \subset X$ such that ρ lives on C. For the converse direction let $\rho : \mathfrak{R} \to [0,\infty[$ be a prob measure extension of α and $C \subset X$ be a subset such that ρ lives on C. Then the injection $J : (C, \mathfrak{R} \sqcap C, \rho|\mathfrak{R} \sqcap C) \to X$ is a version of α which produces

$$\overrightarrow{J}(\mathfrak{R} \sqcap C) = \mathfrak{R} \text{ and } \overrightarrow{J}(\rho|\mathfrak{R} \sqcap C) = \rho, \text{ and } J(C) = C.$$

Of course the identity map $I : (X, \mathfrak{R}, \rho) \to X$ is a version of α as well. It produces $\overrightarrow{I}\mathfrak{R} = \mathfrak{R}$ and $\overrightarrow{I}\rho = \rho$, but $I(X) = X$. Thus the simplest version of α is the identity map $I : (X, \mathfrak{A}, \alpha) \to X$, called the *standard version* of α.

We summarize the above with the somewhat pompous statement which follows.

3 Proposition. *Let $\alpha : \mathfrak{A} \to [0, \infty[$ be a prob measure on a measurable space $(X, \mathfrak{A})$ and $C \subset X$. Then the following are equivalent.*

(1) C is thick for α, that is $\alpha^\star(C) = 1$.

(2) α has a version $K : (\Omega, \mathfrak{P}, P) \to X$ with image $K(\Omega) \subset C$.

(3) α has a version $K : (\Omega, \mathfrak{P}, P) \to X$ with image $K(\Omega) = C$.

(4) α has a prob measure extension $\rho : \mathfrak{R} \to [0, \infty[$ with $C \in \mathfrak{R}$ and $\rho(C) = 1$.

(5) α has a prob measure extension $\rho : \mathfrak{R} \to [0, \infty[$ which lives on C.

(6) α has a unique minimal prob measure extension $\rho : \mathfrak{R} \to [0, \infty]$ which lives on C.

Proof. On the one hand the last paragraph shows that (2)⇒(5)⇒(3), while (3)⇒(2) is obvious. On the other hand lemma 2 shows that (1)⇒(6), while (6)⇒(5)⇒ (4)⇒(1) are obvious. $\boxed{QED}$

2 The traditional notion of stochastic processes

We fix an infinite index set T called the *time domain*, and a measurable space $(Y, \mathfrak{B})$ with nonvoid Y called the *state space*. One forms the T-fold product set $X := Y^T$, the members of which are called the *paths* $x = (x_t)_{t \in T} : T \to Y$. For $t \in T$ let $H_t : X \to Y$ be the canonical projection $x \mapsto x_t$. In $X = Y^T$ one forms the finite-based product set system

$$\mathfrak{B}^{[T]} := \{ \prod_{t \in T} B_t : B_t \in \mathfrak{B} \; \forall \, t \in T \text{ with } B_t = Y \text{ for almost all } t \in T \},$$

and the generated σ algebra $\mathfrak{A} := \mathrm{A}\sigma(\mathfrak{B}^{[T]})$, which is the smallest σ algebra $\mathfrak{A}$ in X such that the $H_t : X \to Y$ for all $t \in T$ are measurable $\mathfrak{A} - \mathfrak{B}$. It is notorious that for uncountable T the formation $\mathfrak{A}$ appears to be too small, because its members $A \in \mathfrak{A}$ are of *countable type* in the sense that $A = \{x \in X : (x_t)_{t \in E} \in R\}$ for some $E \subset T$ countable $\neq \varnothing$ and some $R \subset Y^E$.

It is this situation where the traditional notion of stochastic processes comes into existence: A *stochastic process* with *time domain* T and *state space* $(Y, \mathfrak{B})$, for short for T and $(Y, \mathfrak{B})$, amounts to be a prob measure $\alpha : \mathfrak{A} \to [0, \infty[$ on the measurable space $(X, \mathfrak{A})$; explicit so for example in [9, section 2.1]. The versions $K : (\Omega, \mathfrak{P}, P) \to X$ of α in the above sense then become the *versions* of the stochastic process α in the traditional sense. We shall soon see the connection with other familiar terms.

First of all we want to show that the smallness of $\mathfrak{A}$ for uncountable T leads to unpleasant phenomena which are in drastic contrast to the intuition connected with the notion of stochastic processes.

4 Theorem. *Fix an arbitrary path $a = (a_t)_{t \in T} \in X$ and form*

$$C(a) := \{\, x \in X : x_t = a_t \text{ for all } t \in T \text{ except countably many ones} \}.$$

Then $C(a)$ is thick for all prob measures $\alpha : \mathfrak{A} \to [0, \infty[$ on $(X, \mathfrak{A})$. Thus after proposition 3 each such α has versions $K : (\Omega, \mathfrak{P}, P) \to X$ with $K(\Omega) \subset C(a)$ and prob measure extensions $\rho : \mathfrak{R} \to [0, \infty[$ which live on $C(a)$.

Note that $C(a)$ is $= X$ when T is countable, but is of obvious smallness when T is uncountable.

Proof. Fix $A \in \mathfrak{A}$ with $A \supset C(a)$. We prove that $A' = \varnothing$ and hence $A = X$. Let $A' = \{x \in X : (x_t)_{t \in E} \in R\}$ with $E \subset T$ countable $\neq \varnothing$ and $R \subset Y^E$, and assume that $A' \neq \varnothing$. Take $u = (u_t)_{t \in T} \in A'$, and define $x = (x_t)_{t \in T}$ to be $x_t = u_t$ for $t \in E$ and and $x_t = a_t$ for $t \in T \setminus E$. Then $x \in A' \subset (C(a))'$, whereas $x \in C(a)$ by definition. Thus we obtain a contradiction. $\boxed{QED}$

We recall another example which is in fact a famous one: Let $T = [0, \infty[$ and $Y = \mathbb{R}$ with $\mathfrak{B} = \mathrm{Bor}(\mathbb{R})$. Then $\mathrm{C}(T, \mathbb{R}) \notin \mathfrak{A}$. It is a famous result that the traditional Wiener measure $\alpha : \mathfrak{A} \to [0, \infty[$, that is the stochastic process of one-dimensional Brownian motion, has $\alpha^\star(\mathrm{C}(T, \mathbb{R})) = 1$. But one has also $\alpha^\star(X \setminus \mathrm{C}(T, \mathbb{R})) = 1$, which in the form $\alpha_\star(\mathrm{C}(T, \mathbb{R})) = 0$ is obvious, because each $A \in \mathfrak{A}$ with $A \subset \mathrm{C}(T, \mathbb{R})$ must be $A = \varnothing$.

The unpleasant smallness of $\mathfrak{A}$ appears to be unavoidable in view of its sensible ties to the finite subsets of T, in combination with the traditional methods of abstract measure theory which are of countable type. As a result a stochastic process has far too many thick subsets $C \subset X$. That means that one admits far too many versions $K : (\Omega, \mathfrak{P}, P) \to X$ and far too many prob measure extensions $\rho : \mathfrak{R} \to [0, \infty[$ of α. It sounds bizarre to name all these thick subsets $C \subset X$ in total the *essential subsets* for the stochastic process α, as for example in [1, section 38]. What is needed seems to be a drastic and clever reduction of the multitude of these companions. The ideal solution were a unique and universal prob measure extension $\Phi : \mathfrak{C} \to [0,\infty[$ of $\alpha : \mathfrak{A} \to [0,\infty[$, with an extensive domain in order that it be able to expose the full breadth of relevant features of the stochastic process. For example, the *true adequate* thick subsets for α should appear as those subsets $C \subset X$ on which the measure Φ lives. The traditional theory of stochastic processes does not produce such an answer. However, the present paper is an attempt to achieve this aim, and section 4 will be devoted to its answer. It is based on the new structure in measure and integration developed in [11] [13]. It will require a certain shift in the basic assumptions. A step into the right direction was the treatment of Brownian motion in Fremlin [8, 454–455], in that it has been done in the frame of topological measure theory.

The present section continues with the relevant points in the traditional theory. First of all we return to the ties of $\mathfrak{A}$ with the finite subsets of T. Define I to consist of the nonvoid finite subsets $p, q, \dots$ of T. For $p \in I$ one forms the product set Y^p, with $H_p : X \to Y^p$ the canonical projection $x \mapsto (x_t)_{t\in p}$, and also the canonical projections $H_{pq} : Y^q \to Y^p$ for the pairs $p \subset q$ in I. In Y^p one forms the usual product set system $\mathfrak{B}^p := \mathfrak{B} \times \cdots \times \mathfrak{B}$ and the generated σ algebra $\mathfrak{B}_p := \mathrm{A}\sigma(\mathfrak{B}^p)$. We have $\mathfrak{B}_p \subset \overrightarrow{H}_p\mathfrak{A}$ and $\mathfrak{B}_p \subset \overrightarrow{H}_{pq}\mathfrak{B}_q$, that means H_p is measurable $\mathfrak{A} - \mathfrak{B}_p$ and H_{pq} is measurable $\mathfrak{B}_q - \mathfrak{B}_p$.

Besides the prob measures $\alpha : \mathfrak{A} \to [0,\infty[$, called the stochastic processes for T and $(Y, \mathfrak{B})$, we consider the families $(\beta_p)_{p\in I}$ of prob measures $\beta_p : \mathfrak{B}_p \to [0,\infty[$ which are *projective* in the sense that $\beta_p = \overrightarrow{H}_{pq}\beta_q|\mathfrak{B}_p = \beta_q(H_{pq}^{-1}(\cdot))|\mathfrak{B}_p$ for all pairs $p \subset q$ in I, by [11, 3.1.σ] equivalent to

$$\beta_p\Big(\prod_{t\in p} B_t\Big) = \beta_q\Big(\prod_{t\in q} B_t\Big) \qquad \text{for } B_t \in \mathfrak{B}\ \forall\, t \in p \text{ and } B_t = Y\ \forall\, t \in q \setminus p.$$

Each prob measure $\alpha : \mathfrak{A} \to [0,\infty[$ produces such a projective family $(\beta_p)_{p\in I}$ via $\beta_p = \overrightarrow{H}_p\alpha|\mathfrak{B}_p = \alpha(H_p^{-1}(\cdot))|\mathfrak{B}_p$, by [11, 3.1.$\sigma$] equivalent to

$$\beta_p\Big(\prod_{t\in p} B_t\Big) = \alpha\Big(\prod_{t\in T} B_t\Big) \qquad \text{for } B_t \in \mathfrak{B}\ \forall\, t \in p \text{ and } B_t = Y\ \forall\, t \in T \setminus p.$$

Also [11, 3.1.σ] asserts that the correspondence $\alpha \mapsto (\beta_p)_{p\in I}$ is *injective.* But it need not be *surjective*; see for example [15, exercise 7.12]. The projective family $(\beta_p)_{p\in I}$ is called *solvable* iff it comes from some and hence from a unique prob measure $\alpha : \mathfrak{A} \to [0,\infty[$, called the *projective limit* of the family $(\beta_p)_{p\in I}$. Thus a stochastic process for T and $(Y, \mathfrak{B})$ can also be defined as a *solvable* *projective family* $(\beta_p)_{p\in I}$, called the family of *finite-dimensional distributions* of the process.

In the traditional theory there is a famous particular situation $(Y,\mathfrak{B})$ where all projective families $(\beta_p)_{p\in I}$ for all T are solvable: it is the situation that Y is a *Polish topological space* and $\mathfrak{B}=\mathrm{Bor}(Y)$. This is the projective limit theorem due to Kolmogorov [10, chapter III, section 4]. The situation will be contained in the development of section 4 as a basic special case.

We continue with the usual method to produce *versions* of a stochastic process $\alpha:\mathfrak{A}\to[0,\infty[$ with images in *prescribed* subsets $C\subset X$, which is in terms of so-called *modifications*. It will be invoked in the final remark 8 in section 5.

The maps $K:\Omega\to X=Y^T$ defined on a nonvoid set Ω are in one-to-one correspondence with the families $(K_t)_{t\in T}$ of maps $K_t:\Omega\to Y$ via $K_t=H_t\circ K$ or $K\omega=(K_t\omega)_{t\in T}$ for $\omega\in\Omega$. The relation

$$K^{-1}(A)=\bigcap_{t\in T}K_t^{-1}(B_t)\quad\text{for } A=\prod_{t\in T}B_t\in\mathfrak{B}^{[T]}$$

shows for $K:(\Omega,\mathfrak{P},P)\to X$ that K is measurable $\mathfrak{P}-\mathfrak{A}$ iff the K_t are measurable $\mathfrak{P}-\mathfrak{B}$ for all $t\in T$, and then via the classical uniqueness theorem [11, 3.1.σ] that K is a version of the prob measure $\alpha:\mathfrak{A}\to[0,\infty[$ iff

$$\alpha(A)=P\Big(\bigcap_{t\in T}K_t^{-1}(B_t)\Big)\quad\text{for } A=\prod_{t\in T}B_t\in\mathfrak{B}^{[T]}.$$

After this on defines the maps $K,L:(\Omega,\mathfrak{P},P)\to X=Y^T$ on a prob measure space $(\Omega,\mathfrak{P},P)$ to be *modifications* of each other iff for each $t\in T$ there exists an $F(t)\in\mathfrak{P}$ with $P(F(t))=1$ such that $K_t=L_t$ on $F(t)$.

5 Remark. If $K,L:(\Omega,\mathfrak{P},P)\to X$ are measurable $\mathfrak{P}-\mathfrak{A}$ and modifications of each other then $\vec{K}P|\mathfrak{A}=\vec{L}P|\mathfrak{A}$.

Proof. To be shown is that

$$\vec{K}P(A)=P(K^{-1}(A))\text{ and }\vec{L}P(A)=P(L^{-1}(A))$$

are equal for $A=\prod_{t\in T}B_t\in\mathfrak{B}^{[T]}$, that is for

$$B_t\in\mathfrak{B}\;\forall\, t\in p\text{ and }B_t=Y\;\forall t\in T\setminus p$$

with some nonvoid finite $p\subset T$. We have $K^{-1}(A)=\bigcap_{t\in p}K_t^{-1}(B_t)$. Now

$$F(t)\cap K_t^{-1}(B_t)\subset K_t^{-1}(B_t)\subset\big(F(t)\cap K_t^{-1}(B_t)\big)\cup(F(t))'\quad\text{for } t\in T,$$

$$\bigcap_{t\in p}F(t)\cap K_t^{-1}(B_t)\subset K^{-1}(A)\subset\Big(\bigcap_{t\in p}F(t)\cap K_t^{-1}(B_t)\Big)\cup\Big(\bigcup_{t\in p}(F(t))'\Big),$$

and hence $P(K^{-1}(A)) = P\big(\bigcap_{t\in p} F(t)\cap K_t^{-1}(B_t)\big)$. By the choice of the $F(t)$ $\forall\, t \in T$ this is also $= P(L^{-1}(A))$. $\boxed{QED}$

Now the method announced above can be formulated as follows.

6 Proposition. *Let $\alpha : \mathfrak{A} \to [0,\infty[$ be a prob measure and $\rho : \mathfrak{R} \to [0,\infty[$ be a prob measure extension of α. Assume that $\varnothing \neq C \subset X$. Consider the properties*

(1) The identity map $I : (X,\mathfrak{R},\rho) \to X$ has a modification $J : (X,\mathfrak{R},\rho) \to X$ with $J(X) \subset C$ which is measurable $\mathfrak{R}-\mathfrak{A}$ and thus by remark 5 a version of α.

(2) The identity map $I : (X,\mathfrak{R},\rho) \to X$ has a modification $J : (X,\mathfrak{R},\rho) \to X$ with $J(X) \subset C$.

(3) For each nonvoid countable $U \subset T$ there exists an $R(U) \in \mathfrak{R}$ with $\rho(R(U)) = 1$ such that all $x \in R(U)$ have restrictions $x|U \in C|U$.

Then (1)⇒(2)⇒(3). Moreover (3)⇒(1) under the additional assumptions

(i) Y is a metric space and $\mathfrak{B} = \mathrm{Bor}(Y)$.

(ii) There exists a nonvoid countable $D \subset T$ such that each $t \in T$ has a sequence $(t(l))_l$ in D with $x_{t(l)} \to x_t$ for all $x \in C$.

In this case (3) is needed only for the $U = D \cup \{\, t \,\}$ with $t \in T$.

Proof. (1)⇒(2) is obvious. (2)⇒(3). For each $t \in T$ we have an $R(t) \in \mathfrak{R}$ with $\rho(R(t)) = 1$ such that $x_t = (Jx)_t$ for all $x \in R(t)$. For the $x \in R(U) := \bigcap_{t\in U} R(t) \in \mathfrak{R}$ therefore $x|U = (Jx)|U \in C|U$, so that $R(U)$ is as required.

(3)⇒(1).

(1) We start to note that for $u, v \in C$ one has $u|D = v|D \Rightarrow u = v$. This is clear from (ii).

(2) We define $J : X \to X$ with $J(X) \subset C$. In case $x \in R(D)$ we have a unique $u \in C$ with $x|D = u|D$, and define $Jx := u$. In case $x \in (R(D))'$ we define $Jx := c$ with an element $c \in C$ fixed in advance.

(3) $J : (X,\mathfrak{R},\rho) \to X$ is a modification of $I : (X,\mathfrak{R},\rho) \to X$. In fact, for fixed $t \in T$ we have the subset $R(D) \cap R(D \cup \{\, t \,\}) \in \mathfrak{R}$ with $\rho\big(R(D) \cap R(D \cup \{\, t \,\})\big) = 1$. For $x \in R(D) \cap R(D \cup \{\, t \,\})$ we have on the one hand $x|D = (Jx)|D$ with $Jx \in C$, and on the other hand $x|D \cup \{t\} = u|D \cup \{\, t \,\}$ for some $u \in C$. From (1) we see that $u = Jx$ and hence $x_t = u_t = (Jx)_t$.

(4) The map $J : X \to X$ is measurable $\mathfrak{R} - \mathfrak{A}$, that is the maps $J_t : X \to Y$ are measurable $\mathfrak{R} - \mathfrak{B}$ for all $t \in T$. In fact, we have $R(D) \in \mathfrak{R}$, and in case $t \in D$ we have $J_t|R(D) = I_t|R(D) = H_t|R(D)$ and $J_t|(R(D))' =$ const $= c_t$, so that J_t is measurable $\mathfrak{R} - \mathfrak{B}$. Then for arbitrary $t \in T$ the sequence $(t(l))_l$ in D asserted in (ii) furnishes $(Jx)_{t(l)} \to (Jx)_t$ for all $x \in X$, that is $J_{t(l)} \to J_t$ pointwise on X. It follows from (i) that J_t is measurable $\mathfrak{R} - \mathfrak{B}$.

$\boxed{QED}$

3 The fortified projective limit theorem in terms of inner premeasures

The remainder of the paper will be based on the new structure in measure and integration developed in [11] [13] and summed up in [14, sections 1 and 3]. The basic entities are the inner $\bullet$ premeasures and their maximal inner $\bullet$ extensions, for the three choices $\bullet = \star\sigma\tau$, here for the most part with $\bullet = \tau$. In the sequel we shall make free use of these matters.

The present section will be within the situation of [14, 5.3–5.4], which we start to recall. Let as before T be an infinite index set. For each $t \in T$ we assume a nonvoid set Y_t and a lattice $\mathfrak{K}_t$ in Y_t which contains the finite subsets of Y_t and is $\bullet$ compact. Then $\mathfrak{T}_t := \mathfrak{K}_t \cup \{Y_t\}$ is a lattice in Y_t with the obvious properties, in particular is $\bullet$ compact as well. We form the product set $X := \prod_{t\in T} Y_t$ and the set system

$$\mathfrak{S} := \{ \prod_{t\in T} T_t : T_t \in \mathfrak{T}_t \ \forall\, t \in T \text{ with } T_t = Y_t \text{ for almost all } t \in T \}^\star.$$

Thus $\mathfrak{S}$ is a lattice in X with $\varnothing, X \in \mathfrak{S}$ and is $\bullet$ compact. This formation is the decisive step in the new development.

Next as before define I to consist of the nonvoid finite subsets $p, q, \ldots$ of T. For $p \in I$ we form the product set $Y_p := \prod_{t\in p} Y_t$ and the set systems

$$\mathfrak{K}_p := \{ \prod_{t\in p} K_t : K_t \in \mathfrak{K}_t \ \forall\, t \in p \}^\star \quad \text{and} \quad \mathfrak{T}_p := \{ \prod_{t\in p} T_t : T_t \in \mathfrak{T}_t \ \forall\, t \in p \}^\star.$$

Thus $\mathfrak{K}_p$ and $\mathfrak{T}_p$ are lattices in Y_p which contain the finite subsets of Y_p and are $\bullet$ compact. One has $\mathfrak{K}_p \subset \mathfrak{T}_p \subset \mathfrak{K}_p \top \mathfrak{K}_p$. We also invoke the canonical projection $H_p : X \to Y_p$. One has the relations

$$\begin{aligned} H_p^{-1}(\mathfrak{T}_p) \subset \mathfrak{S} \quad &\text{and hence} \quad H_p^{-1}((\mathfrak{T}_p)_\bullet) \subset \mathfrak{S}_\bullet, \\ H_p(\mathfrak{S}) = \mathfrak{T}_p \quad &\text{and} \qquad\quad H_p(\mathfrak{S}_\bullet) \subset (\mathfrak{T}_p)_\bullet, \end{aligned}$$

of which the last one is nontrivial and follows from [14, 3.12].

After this we consider on the one hand the inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty[$, and on the other hand the families $(\varphi_p)_{p\in I}$ of inner $\bullet$ premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ which are *projective* in the sense that for all pairs $p \subset q$ in I one has

$$\varphi_p\Bigl(\prod_{t\in p} K_t\Bigr) = (\varphi_q)_\bullet\Bigl(\prod_{t\in q} K_t\Bigr) \quad \text{for } K_t \in \mathfrak{K}_t \ \forall\, t \in p \text{ and } K_t = Y_t \ \forall\, t \in q \setminus p.$$

Then the former [14, theorem 5.3] reads as follows.

7 Theorem. *For each projective family $(\varphi_p)_{p\in I}$ of inner $\bullet$ premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ with $\Phi_p = (\varphi_p)_\bullet|\mathfrak{C}((\varphi_p)_\bullet) < \infty$ there exists a unique inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ such that $\varphi_p = \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p$ and $\Phi_p(Y_p) = \Phi(X) = \varphi(X)$ for all $p \in I$. It fulfils*

$$\Phi(A) = \inf_{p\in I} \Phi_p(H_p(A)) \quad \textit{for } A \in \mathfrak{S}_\bullet.$$

Moreover $(\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))$ and $\Phi_p = \overrightarrow{H}_p\Phi$ for all $p \in I$.

The subsequent [14, theorem 5.4] then asserts that the inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty[$ which result under this procedure are precisely those which fulfil

$$(\bullet) \quad \varphi_\bullet(H_p^{-1}(\cdot)) : \mathfrak{P}(Y_p) \to [0,\infty[\text{ is inner regular } (\mathfrak{K}_p)_\bullet \text{ for all } p \in I.$$

In the present section we shall prove that this condition $(\bullet)$ is superfluous, because it is fulfilled for all inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty[$. The main step will be the lemma which follows.

8 Lemma. *Each inner $\bullet$ premeasure $\mathfrak{S} \to [0,\infty[$ fulfils*

$$(\circ) \quad \sup\{\, \varphi(H_p^{-1}(K)) : K \in \mathfrak{K}^p \,\} = \varphi(X) \quad \textit{for all } p \in I,$$

where $\mathfrak{K}^p := \{\, \prod_{t\in p} K_t : K_t \in \mathfrak{K}_t \ \forall\, t \in p \,\}$.

Proof. The proof will be via induction in card(p).

(1) Thus assume first that $p = \{s\}$ for some $s \in T$. Then the assertion reads $\sup\{\, \varphi(H_s^{-1}(K)) : K \in \mathfrak{K}_s \,\} = \varphi(X)$, where $H_s : X \to Y_s$ is the canonical projection. This is obvious when $Y_s \in \mathfrak{K}_s$, so assume that $Y_s \notin \mathfrak{K}_s$. We fix a nonvoid $A \in \mathfrak{K}_s$, and have $H_s^{-1}(A) \in \mathfrak{S}$ and $\varphi(X) = \varphi(H_s^{-1}(A)) + \varphi_\bullet(H_s^{-1}(A'))$ since $(H_s^{-1}(A))' = H_s^{-1}(A')$. Next fix $c < \varphi(X)$, and then $\varepsilon > 0$ such that $c+\varepsilon < \varphi(X)$. By definition there exists a set system $\mathfrak{M} \subset \mathfrak{S}$ nonvoid $\bullet$ which is downward directed $\downarrow$ some $D \in \mathfrak{S}_\bullet$ with $D \subset H_s^{-1}(A')$ such that $c + \varepsilon < \varphi(H_s^{-1}(A)) + \inf_{M\in\mathfrak{M}} \varphi(M)$, so that

$$c+\varepsilon < \varphi(H_s^{-1}(A))+\varphi(M) = \varphi\bigl(H_s^{-1}(A)\cup M\bigr)+\varphi\bigl(H_s^{-1}(A)\cap M\bigr) \quad \forall M \in \mathfrak{M}.$$

In view of $\inf_{M\in\mathfrak{M}} \varphi(H_s^{-1}(A)\cap M) = \varphi_\bullet(H_s^{-1}(A)\cap D) = 0$ and of directedness each $M\in\mathfrak{M}$ has an $N\in\mathfrak{M}$ with $N\subset M$ and $\varphi(H_s^{-1}(A)\cap N) < \varepsilon$ and hence $c < \varphi(H_s^{-1}(A)\cup N)$, so that

$$c < \varphi\big(H_s^{-1}(A)\cup M\big) \leqq \varphi\big(H_s^{-1}(A)\cup H_s^{-1}(H_s(M))\big) = \varphi\big(H_s^{-1}(A\cup H_s(M))\big).$$

Now the set system $\{\, H_s(M) : M\in\mathfrak{M}\,\}\subset\mathfrak{T}_s$ is nonvoid $\bullet$ and after [14, 3.12] downward directed $\downarrow H_s(D)\in(\mathfrak{T}_s)_\bullet$ with $H_s(D)\subset A'$. From $A\neq\varnothing$ we conclude that $H_s(M)\neq Y_s$ and hence $H_s(M)\in\mathfrak{K}_s$ for some $M\in\mathfrak{M}$. The assertion follows.

(2) Assume that ($\circ$) holds true for some $p\in I$, and let $q = p\cup\{s\}$ for some $s\in T\setminus p$. Fix $\varepsilon>0$. Then there exist

$$\begin{aligned} A = \prod_{t\in p} K_t \in \mathfrak{K}^p \quad &\text{with} \quad \varphi(H_p^{-1}(A)) > \varphi(X)-\varepsilon,\\ B = K_s\in\mathfrak{K}_s \quad &\text{with} \quad \varphi(H_s^{-1}(B)) > \varphi(X)-\varepsilon. \end{aligned}$$

Now $A\times B = \prod_{t\in q} K_t\in\mathfrak{K}^q$ and $H_q^{-1}(A\times B) = H_p^{-1}(A)\cap H_s^{-1}(B)\in\mathfrak{S}$. It follows that

$$\begin{aligned} \varphi(X)+\varphi(H_q^{-1}(A\times B)) &\geqq \varphi\big(H_p^{-1}(A)\cup H_s^{-1}(B)\big)+\varphi\big(H_p^{-1}(A)\cap H_s^{-1}(B)\big)\\ &= \varphi(H_p^{-1}(A))+\varphi(H_s^{-1}(B)) > 2\varphi(X)-2\varepsilon, \end{aligned}$$

so that $\varphi(H_q^{-1}(A\times B)) > \varphi(X)-2\varepsilon$. Thus the assertion holds true for q.

$\boxed{QED}$

We turn to the improved version of [14, theorem 5.4]. Part of the deductions will be identical with the former ones, but we find it adequate to present the proof in its integrity.

9 Theorem. *The family of the maps*

$$\varphi\mapsto\varphi_p := \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p \quad \textit{for } p\in I$$

defines a one-to-one correspondence between the inner $\bullet$ premeasures $\varphi:\mathfrak{S}\to[0,\infty[$ with $\Phi=\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ and the projective families $(\varphi_p)_{p\in I}$ of inner $\bullet$ premeasures $\varphi_p:\mathfrak{K}_p\to[0,\infty[$ with $\Phi_p=(\varphi_p)_\bullet|\mathfrak{C}((\varphi_p)_\bullet)<\infty$. It fulfils

$$\Phi(A) = \inf_{p\in I}\Phi_p(H_p(A)) \quad \textit{for } A\in\mathfrak{S}_\bullet.$$

Moreover $(\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))$ and $\Phi_p = \overrightarrow{H}_p\Phi$ for all $p\in I$.

Proof.

(1) For each projective family $(\varphi_p)_{p\in I}$ of inner $\bullet$ premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ with $\Phi_p < \infty$ the above theorem 7 furnishes an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ which is related to $(\varphi_p)_{p\in I}$ via the listed properties, in particular $\varphi_p = \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p$ and even $(\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))$ for $p \in I$.

(2) Here we start from an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$. For the first three steps we fix some $p \in I$.

(i) The present initial recapitulation shows that the theorem [14, 3.10] on direct images of inner $\bullet$ premeasures can be applied to $H_p : X \to Y_p$ with $\mathfrak{S}$ and $\mathfrak{T}_p$. Its application to φ asserts that $\psi_p := \varphi_\bullet(H_p^{-1}(\cdot))|\mathfrak{T}_p = \varphi(H_p^{-1}(\cdot))|\mathfrak{T}_p \leqq \varphi(X) < \infty$ is an inner $\bullet$ premeasure $\psi_p : \mathfrak{T}_p \to [0,\infty[$ which fulfils $(\psi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))$.

(ii) We claim that $(\psi_p)_\bullet$ is inner regular $(\mathfrak{K}_p)_\bullet \subset (\mathfrak{T}_p)_\bullet$. To see this fix $A \subset Y_p$ and $c < (\psi_p)_\bullet(A)$, and then $B \in (\mathfrak{T}_p)_\bullet$ with $B \subset A$ such that $c < (\psi_p)_\bullet(B)$. The above lemma 8 furnishes some $K \in \mathfrak{K}^p$ with $\psi_p(K) = \varphi(H_p^{-1}(K)) > \varphi(X) - \big((\psi_p)_\bullet(B) - c\big)$. It follows that

$$\begin{aligned}\varphi(X) + (\psi_p)_\bullet(B\cap K) &\geqq (\psi_p)_\bullet(B\cup K) + (\psi_p)_\bullet(B\cap K)\\ &= (\psi_p)_\bullet(B) + (\psi_p)_\bullet(K) > \varphi(X) + c,\end{aligned}$$

and hence $(\psi_p)_\bullet(B\cap K) > c$. Now $B\cap K \in (\mathfrak{K}_p)_\bullet$ since $B \in (\mathfrak{T}_p)_\bullet \subset \mathfrak{K}_p \top (\mathfrak{K}_p)_\bullet$. Thus we have the assertion.

(iii) In view of (ii) the previous lemma [14, 1.6] can be applied to $\psi_p : \mathfrak{T}_p \to [0,\infty[$ and $\mathfrak{K}_p \subset \mathfrak{T}_p$. It asserts that $\varphi_p := \psi_p|\mathfrak{K}_p$ is an inner $\bullet$ premeasure $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ which fulfils $(\varphi_p)_\bullet = (\psi_p)_\bullet$. Thus we have $\varphi_p = \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p \leqq \varphi(X) < \infty$ and $(\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))$.

(iv) The family $(\varphi_p)_{p\in I}$ obtained in (iii) is projective. In fact, for $p \subset q$ in I and $A_t \subset Y_t\ \forall t \in q$ with $A_t = Y_t\ \forall t \in q \setminus p$ we have $H_p^{-1}\big(\prod_{t\in p} A_t\big) = H_q^{-1}\big(\prod_{t\in q} A_t\big)$, so that the last relation in (iii) furnishes $(\varphi_p)_\bullet\big(\prod_{t\in p} A_t\big) = (\varphi_q)_\bullet\big(\prod_{t\in q} A_t\big)$.

(3) The map $(\varphi_p)_{p\in I} \mapsto \varphi$ defined in (1) between the two prescribed domains is injective in view of $\varphi_p = \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p$ for $p \in I$. It remains to show that this map is surjective. Thus we start from an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$. Let $(\varphi_p)_{p\in I}$ be the family obtained from φ in (2), and then $\tilde{\varphi} : \mathfrak{S} \to [0,\infty[$ the inner $\bullet$ premeasure obtained from $(\varphi_p)_{p\in I}$ in (1). From (iii) and (1) then $\varphi_\bullet(H_p^{-1}(\cdot)) = (\varphi_p)_\bullet = \tilde{\varphi}_\bullet(H_p^{-1}(\cdot))$ for all $p \in I$. Now each $S \in \mathfrak{S}$ is of the form $S = A \times \prod_{t\in T\setminus p} Y_t = H_p^{-1}(A)$ for some $p \in I$ and $A \subset Y_p$. It follows that $\varphi = \tilde{\varphi}$.

$\boxed{QED}$

4 The reformed notion of stochastic processes

We turn to the reformed counterpart to the traditional situation of section 2. As before we fix an infinite set T called the *time domain.* But this time we assume the *state space* $(Y, \mathfrak{K})$ to consist of a nonvoid set Y and of a lattice $\mathfrak{K}$ in Y which contains the finite subsets of Y and is $\bullet$ compact. This amounts to the specialization $Y_t = Y$ and $\mathfrak{K}_t = \mathfrak{K}$ for all $t \in T$ in the situation of section 3. Thus $X = Y^T$, and $\mathfrak{S} = \big((\mathfrak{K} \cup \{Y\})^{[T]}\big)^\star$ in the notation used in section 2. Likewise for $p \in I$ we have $Y_p = Y^p$ and $\mathfrak{K}_p = (\mathfrak{K}^p)^\star$ with $\mathfrak{K}^p := \mathfrak{K} \times \cdots \times \mathfrak{K}$. For later use we insert a little remark.

10 Remark.

(i) $H_t(\mathfrak{S}) = \mathfrak{K} \cup \{ Y \}$ and $H_t(\mathfrak{S}_\bullet) = \mathfrak{K}_\bullet \cup \{ Y \}$ for $t \in T$. In fact, the first relation is obvious, and [14, 3.12] implies that $H_t(\mathfrak{S}_\bullet) \subset (H_t(\mathfrak{S}))_\bullet = \mathfrak{K}_\bullet \cup \{ Y \}$, while $H_t(\mathfrak{S}_\bullet) \supset \mathfrak{K}_\bullet \cup \{ Y \}$ is obvious.

(ii) It follows that

$$\{ S \in \mathfrak{S}_\bullet : H_t(S) \in \mathfrak{K}_\bullet \; \forall\, t \in T \} = \{ S \in \mathfrak{S}_\bullet : S \subset \text{ some } F \in \mathfrak{K}^T \},$$

which in the sequel will be called $\langle \mathfrak{S}_\bullet \rangle$. In particular $\langle \mathfrak{S}_\bullet \rangle = \mathfrak{S}_\bullet$ iff $Y \in \mathfrak{K}$.

After this we consider on the one hand the inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ and their maximal inner $\bullet$ extensions $\Phi := \varphi_\bullet | \mathfrak{C}(\varphi_\bullet)$ with $\Phi(X) = \varphi(X) = 1$ (the *inner $\bullet$ prob premeasures* for short), and on the other hand the projective families $(\varphi_p)_{p \in I}$ of inner $\bullet$ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0, \infty[$ and their Φ_p. These entities are the natural counterparts of the two kinds of prob measures which occur in the traditional situation of section 2. For them the new projective limit theorem 9 specializes as follows.

11 Theorem. *The family of the maps*

$$\varphi \mapsto \varphi_p := \varphi(H_p^{-1}(\cdot)) | \mathfrak{K}_p \quad \textit{for } p \in I$$

defines a one-to-one correspondence between the inner $\bullet$ prob premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ and the projective families $(\varphi_p)_{p \in I}$ of inner $\bullet$ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0, \infty[$. It fulfils

$$\Phi(A) = \inf_{p \in I} \Phi_p(H_p(A)) \quad \textit{for } A \in \mathfrak{S}_\bullet.$$

Moreover $(\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))$ and $\Phi_p = \overrightarrow{H}_p \Phi$ for all $p \in I$.

Thus the present situation appears to be much more favourable than the traditional one: This time *all* projective families $(\varphi_p)_{p\in I}$ deserve to be called *solvable*. Also the relations between these families $(\varphi_p)_{p\in I}$ and their projective limits φ look much deeper than before. But the main benefit compared with the traditional situation is that the resultant prob measure $\Phi=\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ on X has an immense domain, at least in case $\bullet=\tau$: In fact, even the most prominent subclass $\mathfrak{S}_\tau\subset\mathfrak{C}(\varphi_\tau)$ contains for example all $A\subset X$ of the form $A=\prod_{t\in T}K_t$ with $K_t\in\mathfrak{K}_\tau\cup\{\,Y\,\}\;\forall\,t\in T$, and hence reaches far beyond the class of subsets of countable type. Thus the present concept combines two properties which seemed to be incompatible in the traditional context: On the one hand to be rooted in the class of *finite* subsets of T (this is the assertion of the projective limit theorem), and on the other hand to be able to overcome the barrier of *countable type* in the subsets of X.

Thus we feel entitled to define a *stochastic process* with *time domain* T and *state space* $(Y,\mathfrak{K})$, for short for T and $(Y,\mathfrak{K})$, to be an inner τ prob premeasure $\varphi:\mathfrak{S}\to[0,\infty[$. We turn to the direct comparison with the traditional situation in the most fundamental particular case.

First assume that Y is a Hausdorff topological space and $\mathfrak{K}=\mathrm{Comp}(Y)$. We equip Y^p for $p\in I$ with the product topology. Then $(\mathfrak{K}_p)_\tau=\mathrm{Comp}(Y^p)$ from [11, 21.3.2] and [12, 2.4.2], so that once more we are led to assume $\bullet=\tau$. We know from [13, 3.8.Inn] and [14, 1.4] that the inner τ prob premeasures $\varphi_p:\mathfrak{K}_p\to[0,\infty[$ are in one-to-one correspondence with the Radon prob premeasures $\phi_p:(\mathfrak{K}_p)_\tau=\mathrm{Comp}(Y^p)\to[0,\infty[$ via $(\varphi_p)_\tau=(\phi_p)_\tau$. Thus theorem 11 expresses a one-to-one correspondence between the inner τ prob premeasures $\varphi:\mathfrak{S}\to[0,\infty[$ and the families $(\beta_p)_{p\in I}$ of Borel-Radon prob measures $\beta_p:\mathrm{Bor}(Y^p)\to[0,\infty[$ which are *projective* in the sense that for all pairs $p\subset q$ in I one has

$$\beta_p\Big(\prod_{t\in p}K_t\Big)=\beta_q\Big(\prod_{t\in q}K_t\Big)\quad\text{for } K_t\in\mathfrak{K}\;\forall\,t\in p \text{ and } K_t=Y\;\forall\,t\in q\setminus p,$$

equivalent to $\beta_p(B)=\beta_q(H_{pq}^{-1}(B))$ for $B\in\mathrm{Bor}(Y^p)$. This result remains quite far from the traditional situation for $(Y,\mathfrak{B})$ with $\mathfrak{B}=\mathrm{Bor}(Y)$. Just note that the former $\mathfrak{B}_p=\mathrm{A}\sigma(\mathfrak{B}^p)$ is $\subset\mathrm{Bor}(Y^p)$, but in most cases is $\neq\mathrm{Bor}(Y^p)$ and need not even contain $(\mathfrak{K}_p)_\tau=\mathrm{Comp}(Y^p)$. What can be said is that

$$\mathfrak{B}^p\subset\mathfrak{B}_p\subset\mathrm{Bor}(Y^p)\subset\mathfrak{C}((\phi_p)_\tau)=\mathfrak{C}((\varphi_p)_\tau)=\vec{H}_p\mathfrak{C}(\varphi_\tau)$$

or $H_p^{-1}(\mathfrak{B}^p)\subset\mathfrak{C}(\varphi_\tau)$ for $p\in I$ implies that $\mathfrak{B}^{[T]}\subset\mathfrak{C}(\varphi_\tau)$ and hence $\mathfrak{A}\subset\mathfrak{C}(\varphi_\tau)$. Moreover $\mathfrak{K}\cup\{\,Y\,\}\subset\mathfrak{B}$ implies that $(\mathfrak{K}\cup\{\,Y\,\})^{[T]}\subset\mathfrak{B}^{[T]}$ and hence $\mathfrak{S}\subset\mathfrak{A}$.

The picture will be different when we assume that Y is a *Polish* topological space. As before let $\mathfrak{K}=\mathrm{Comp}(Y)$ and $\mathfrak{B}=\mathrm{Bor}(Y)$. For $p\in I$ then the product space Y^p is Polish as well and fulfils $\mathfrak{B}_p = \mathrm{Bor}(Y^p)$. We recall the fundamental fact that in Polish spaces all finite (and even all locally finite) Borel measures are Borel-Radon measures; see for example [11, 9.9.ii]. Thus the one-to-one correspondence $(\beta_p)_{p\in I} \mapsto (\phi_p)_{p\in I} \mapsto (\varphi_p)_{p\in I} \mapsto \varphi$ obtained above is now for the families $(\beta_p)_{p\in I}$ of *arbitrary* prob measures $\beta_p : \mathfrak{B}_p = \mathrm{Bor}(Y^p) \to [0,\infty[$ which are projective in the identical sense of the present section and of section 2. We combine this map with the injective correspondence $\alpha \mapsto (\beta_p)_{p\in I}$ of section 2 which sends each prob measure $\alpha : \mathfrak{A} \to [0,\infty[$ into the family of its finite-dimensional distributions $(\beta_p)_{p\in I}$, and at the end with the maps $\varphi \mapsto \Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$ and $\Phi \mapsto \Phi|\mathfrak{A}$. We claim that the total outcome

$$(\#) \quad \alpha \mapsto (\beta_p)_{p\in I} \mapsto (\phi_p)_{p\in I} \mapsto (\varphi_p)_{p\in I} \mapsto \varphi \mapsto \Phi \mapsto \Phi|\mathfrak{A}$$

is the identity map. In fact, for $A = \prod_{t\in T} B_t \in \mathfrak{B}^{[T]}$ there is some $p \in I$ with $B_t = Y\ \forall\, t \in T \setminus p$ and hence $A = H_p^{-1}(B)$ with $B = \prod_{t\in p} B_t \in \mathfrak{B}^p$. It follows that $\alpha(A) = \alpha(H_p^{-1}(B)) = \beta_p(B) = (\phi_p)_\tau(B) = (\varphi_p)_\tau(B) = \varphi_\tau(H_p^{-1}(B)) = \varphi_\tau(A) = \Phi(A)$, which combined with the uniqueness theorem [11, 3.1.σ] furnishes the assertion.

Now the individual maps which occur in (#) are all injective: this has been seen except for the last map $\Phi \mapsto \Phi|\mathfrak{A}$, and this one is injective because $\mathfrak{S} \subset \mathfrak{A}$ and because $\Phi|\mathfrak{S} = \varphi$ reproduces $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$. Therefore the little lemma which follows tells us that the individual maps in (#) are all surjective and hence one-to-one. In particular the first partial map $\alpha \mapsto (\beta_p)_{p\in I}$ is one-to one. Thus we have reobtained the projective limit theorem of Kolmogorov from our theorem 11.

12 Lemma. *Assume that $E_0, E_1, \ldots, E_n$ are nonvoid sets with $n \geqq 2$, and that the $\vartheta_l : E_{l-1} \to E_l$ $(l = 1, \ldots, n)$ are injective maps such that $\vartheta_n \circ \cdots \circ \vartheta_1$ is the identity map of $E_0 = E_n$. Then the maps $\vartheta_1, \ldots, \vartheta_n$ are surjective and hence one-to-one.*

Proof of lemma 12. It is clear that ϑ_n is surjective. Thus fix $1 \leqq l \leqq n-1$ and define $\theta_l : E_l \to E_l$ to be $\theta_l = \vartheta_l \circ \cdots \circ \vartheta_1 \circ \vartheta_n \circ \cdots \circ \vartheta_{l+1}$. Then θ_l is injective and fulfils $\theta_l \circ \theta_l = \theta_l$. Thus for $x \in E_l$ we have $\theta_l(\theta_l(x)) = \theta_l(x)$ and hence $\theta_l(x) = x$, so that θ_l is the identity map of E_l. It follows that ϑ_l is surjective. $\boxed{QED}$

We summarize the most important facts in the present situation.

13 Theorem. *Assume that Y is a Polish space with $\mathfrak{K} = \mathrm{Comp}(Y)$ and $\mathfrak{B} = \mathrm{Bor}(Y)$. We keep the previous notations. There is a one-to-one correspondence between the traditional stochastic processes $\alpha : \mathfrak{A} \to [0,\infty[$ for T and $(Y,\mathfrak{B})$, and the new stochastic processes $\varphi : \mathfrak{S} \to [0,\infty[$ for T and $(Y,\mathfrak{K})$.*

The correspondence satisfies $\mathfrak{S} \subset \mathfrak{A} \subset \mathfrak{C}(\varphi_\tau)$ and reads $\varphi = \alpha|\mathfrak{S}$ and $\alpha = \Phi|\mathfrak{A}$. Moreover $\varphi_\tau = (\alpha^\star|\mathfrak{S}_\tau)_\star$.

Proof. We know that $\mathfrak{S} \subset \mathfrak{A} \subset \mathfrak{C}(\varphi_\tau)$, and the one-to-one correspondence in question is expressed in the chain of maps (#). Thus $\alpha = \Phi|\mathfrak{A}$ and hence $\alpha|\mathfrak{S} = \Phi|\mathfrak{S} = \varphi$. It follows that

$$\varphi^\star \geqq \alpha^\star \geqq \Phi^\star \text{ and hence } \varphi^\star|\mathfrak{S}_\tau \geqq \alpha^\star|\mathfrak{S}_\tau \geqq \Phi^\star|\mathfrak{S}_\tau = \Phi|\mathfrak{S}_\tau = \varphi_\tau|\mathfrak{S}_\tau.$$

Now $\varphi^\star|\mathfrak{S}_\tau = \varphi_\tau|\mathfrak{S}_\tau$ because $\varphi_\tau|\mathfrak{S}_\tau$ is downward τ continuous. Therefore $\varphi_\tau|\mathfrak{S}_\tau = \alpha^\star|\mathfrak{S}_\tau$, and hence $\varphi_\tau = (\varphi_\tau|\mathfrak{S}_\tau)_\star = (\alpha^\star|\mathfrak{S}_\tau)_\star$ since φ_τ is inner regular $\mathfrak{S}_\tau$. $\boxed{QED}$

Thus we see that in the particular situation of theorem 13 each traditional stochastic process $\alpha : \mathfrak{A} \to [0, \infty[$ for T and $(Y, \mathfrak{B})$ possesses the canonical prob measure extension $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$. It can be described in simple and natural terms, and its domain $\mathfrak{C}(\varphi_\tau)$ is comprehensive enough to raise the hope that it will be able to fulfil the requirements expressed in section 2. This has been confirmed to quite some extent for the Wiener process of one-dimensional Brownian motion in [14, section 6]. Also it is simple to see for this process that the pathological thick subsets $C(a) \subset X$ of theorem 4 are measurable $\mathfrak{C}(\varphi_\tau)$ with measure $\Phi(C(a)) = 0$. The subsequent final section of the present paper has the aim to obtain a similar picture for the Poisson process.

14 Corollary. *Assume that Y is a Polish space as before, and let $X = Y^T$ be equipped with the product topology.*

(1) *We have* $\mathrm{Comp}(X) = \langle \mathfrak{S}_\tau \rangle \subset \mathfrak{S}_\tau \subset \mathrm{Cl}(X)$ *(:= the closed subsets of X). In particular* $\mathrm{Comp}(X) = \mathfrak{S}_\tau$ *iff Y is compact.*

(2) *Assume that Y is compact, and let $\varphi : \mathfrak{S} \to [0, \infty[$ be as before. Then [14, 1.4] asserts that $\phi := \varphi_\tau|\mathfrak{S}_\tau = \varphi_\tau|\,\mathrm{Comp}(X)$ is a Radon premeasure with $\phi_\tau = \varphi_\tau$. Hence $\Phi = \phi_\tau|\mathfrak{C}(\phi_\tau)$ is maximal Radon.*

(3) *Let $\varphi : \mathfrak{S} \to [0, \infty[$ be as before, and assume that*

$$\sup\{\, \Phi(S) : S \in \mathrm{Comp}(X) = \langle \mathfrak{S}_\tau \rangle \,\} = 1.$$

Then φ_τ is inner regular $\mathrm{Comp}(X) = \langle \mathfrak{S}_\tau \rangle$. *Thus [14, 1.6] asserts that $\phi := \varphi_\tau|\,\mathrm{Comp}(X)$ is a Radon premeasure with $\phi_\tau = \varphi_\tau$. Hence $\Phi = \phi_\tau|\mathfrak{C}(\phi_\tau)$ is maximal Radon.*

Proof.

(1) We have $\mathfrak{S} \subset \mathrm{Cl}(X)$ by definition and hence $\mathfrak{S}_\tau \subset \mathrm{Cl}(X)$. Then on the one hand $\mathrm{Comp}(X) \supset \langle \mathfrak{S}_\tau \rangle$, because $S \in \langle \mathfrak{S}_\tau \rangle$ is closed and by remark 10(ii) contained in some $F \in \mathrm{Comp}(X)$, so that $S \in \mathrm{Comp}(X)$. On the other hand $\mathrm{Comp}(X) \subset \mathfrak{S}_\tau$ in view of [12, 2.4.2] and hence $\mathrm{Comp}(X) \subset \langle \mathfrak{S}_\tau \rangle$, because $S \in \mathrm{Comp}(X)$ implies that $H_t(S) \in \mathrm{Comp}(Y) = \mathfrak{K}$ since H_t is continuous.

(2) is clear.

(3) To be shown is that φ_τ is inner regular $\mathrm{Comp}(X) = \langle\mathfrak{S}_\tau\rangle$. Fix $A \subset X$ and $c < \varphi_\tau(A)$, and then $S \in \mathfrak{S}_\tau$ with $S \subset A$ and $c < \varphi_\tau(S)$. By assumption there exists $E \in \langle\mathfrak{S}_\tau\rangle$ with $\varphi_\tau(E) > 1 - (\varphi_\tau(S) - c)$. Then $S \cap E \in \langle\mathfrak{S}_\tau\rangle$ with $S \cap E \subset A$ and

$$1 + \varphi_\tau(S \cap E) \geqq \varphi_\tau(S \cup E) + \varphi_\tau(S \cap E) = \varphi_\tau(S) + \varphi_\tau(E) > 1 + c,$$

and hence $\varphi_\tau(S \cap E) > c$ as required.

$\boxed{QED}$

We note that the assumption in (3) is fulfilled both for the (one-dimensional) Wiener process (in [14, 6.1] the subsets $E(\gamma, M))$ for $M > 0$ are in $\langle\mathfrak{S}_\tau\rangle$) and for the Poisson process (in theorem 27 below the subsets $E_m(T)$ for $m \in \mathbb{N}$ will be in $\langle\mathfrak{S}_\tau\rangle$). Thus we have confirmed the assertions made in the introduction.

However, there are natural cases where the assumption in (3) is violated. We insert a simple example (we note that the example makes sense in the full frame of the present section and can also be extended to that of section 3.

15 Example. Let the $\vartheta_t : \mathfrak{K} \to [0,\infty[$ for $t \in T$ be inner τ prob premeasures, and the $\varphi_p = \prod_{t\in p} \vartheta_t$ for $p \in I$ be their products in the sense of [12, section 1]. Thus the $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ are inner τ prob premeasures with

$$(\varphi_p)_\tau\big(\prod_{t\in p} A_t\big) = \prod_{t\in p} (\vartheta_t)_\tau(A_t) \quad \text{for } A_t \subset Y \ \forall t \in p,$$

and hence form a projective family $(\varphi_p)_{p\in I}$. Let $\varphi : \mathfrak{S} \to [0,\infty[$ with $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$ be its projective limit in the sense of theorem 11. We claim that if T is uncountable and $\vartheta_t < 1$ for all $t \in T$ then $\Phi|\langle\mathfrak{S}_\tau\rangle = 0$, so that the assumption in (3) is violated. In fact, for $S \in \langle\mathfrak{S}_\tau\rangle$ we have $S \subset$ some $F \in \mathfrak{K}^T$, that is $F = \prod_{t\in T} K_t$ with $K_t \in \mathfrak{K}\ \forall\, t \in T$. For $p \in I$ thus

$$\Phi(S) \leqq \Phi(F) \leqq \Phi\big(\prod_{t\in p} K_t \times Y^{T\setminus p}\big) = \varphi_p\big(\prod_{t\in p} K_t\big) = \prod_{t\in p} \vartheta_t(K_t).$$

Now there exists an uncountable $M \subset T$ such that $\vartheta_t(K_t) \leqq$ some $c < 1$ for all $t \in M$. It follows that $\Phi(S) \leqq c^{\mathrm{card}(p)}$ for all $p \subset M$ and hence $\Phi(S) = 0$. $\boxed{QED}$

After this we return to the overall situation. There are two choices for the definition of the notion of stochastic processes. On the one hand the traditional notion for T and $(Y, \mathfrak{B})$ as the prob measures $\alpha : \mathfrak{A} \to [0,\infty[$, or as the solvable projective families $(\beta_p)_{p\in I}$ of prob measures $\beta_p : \mathfrak{B}_p \to [0,\infty[$. On the other hand the reformed notion for T and $(Y, \mathfrak{K})$ as the inner τ prob premeasures

$\varphi : \mathfrak{S} \to [0, \infty[$, or as the projective families $(\varphi_p)_{p \in I}$ of inner τ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0, \infty[$.

We have seen that the two classes of stochastic processes are in one-to-one correspondence in the particular case that Y is a Polish space with $\mathfrak{B} = \mathrm{Bor}(Y)$ and $\mathfrak{K} = \mathrm{Comp}(Y)$. The traditional notion has of course the benefit of *lead*, in that the entire immeasurable literature on stochastic processes is written in its terms. But otherwise it seems that the benefits in basic structure and procedures are more on the other side. The decisive point is that the new notion offers, in sharp contrast to the former wild collections of versions and prob measure extensions, a unified method to produce canonical prob measures on immense domains.

It must of course be clarified whether the new concept will keep what it promises, on the whole and beyond the two particular processes under consideration. That is above all - in traditional terms - that it *confirms the good ones* out of the crowd of all thick subsets, and *rejects the bad ones.* The present author is not an expert in stochastics. But a decade of work with the new structure in measure and integration which forms the basis raised his confidence that the structure will be able to cope with the present challenge in stochastics as well.

5 The Poisson process in terms of inner premeasures

In the present section we assume $T = [0, \infty[$ and the Polish space $Y = \mathbb{R}$ with $\mathfrak{K} = \mathrm{Comp}(\mathbb{R})$ and $\mathfrak{B} = \mathrm{Bor}(\mathbb{R})$, as in [14, section 6]. Also as before we fix a family $(\gamma_t)_{t \in T}$ of Radon prob premeasures $\gamma_t : \mathfrak{K} \to [0, \infty[$ with $\gamma_0 = \delta_0 | \mathfrak{K}$ which under convolution fulfils $\gamma_s \star \gamma_t = \gamma_{s+t}$ for $s, t \in T$, and construct its projective family $(\varphi_p)_{p \in I}$ of inner τ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0, \infty[$ and the resultant inner τ prob premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ as in [14, 6.5] and theorem 11 above. We start with a little addendum to [14, 6.5], in that we write down the adequate form of the usual independence relation. Hereafter we shall specialize $(\gamma_t)_{t \in T}$ to the Poisson semigroup.

16 Proposition. *Let* $p = \{t(1), \ldots, t(n)\} \in I$ *with* $0 =: t(0) \leqq t(1) < \cdots < t(n)$*. For* $B_1, \ldots, B_n \subset \mathbb{R}$ *then*

$$\varphi_\tau \left(\bigcap_{l=1}^{n} [H_{t(l)} - H_{t(l-1)} \in B_l] \right) = \prod_{l=1}^{n} \varphi_\tau \left([H_{t(l)} - H_{t(l-1)} \in B_l] \right).$$

Proof.

(1) We note that $\varphi_\tau(A \cap N) = \varphi_\tau(A)$ for all $A \subset X$ and $N \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(N) = 1$. In fact, we have $\varphi_\tau(A \cap N) + \varphi_\tau(A \cup N) = \varphi_\tau(A) + \varphi_\tau(N)$ from [11, 4.12.4], and $\varphi_\tau(N) = \varphi_\tau(A \cup N) = 1$.

(2) We can assume that $n \geqq 2$. In view of (1) with $N := [H_{t(0)} = 0] \in \mathfrak{S}$ the assertion reads

$$\varphi_\tau \left(\bigcap_{l=2}^{n} [H_{t(l)} - H_{t(l-1)} \in B_l] \cap [H_{t(1)} \in B_1] \right)$$
$$= \prod_{l=2}^{n} \varphi_\tau \left([H_{t(l)} - H_{t(l-1)} \in B_l] \right) \varphi_\tau \left([H_{t(1)} \in B_1] \right).$$

After [14, 6.5] the right side is $= \prod_{l=1}^{n} \left(\gamma_{t(l)-t(l-1)} \right)_\tau (B_l)$, which in view of the definition of γ_p and of [12, 1.3.0] is $= (\gamma_p)_\tau (B_1 \times \cdots \times B_n)$. The left side is in the former notations

$$= \varphi_\tau \big(\{ x \in X : x_{t(1)} \in B_1 \text{ and } x_{t(l)} - x_{t(l-1)} \in B_l \ (l = 2, \ldots, n) \} \big)$$
$$= \varphi_\tau \left(H_p^{-1}(B) \right) = (\varphi_p)_\tau (B) = (\gamma_p)_\tau \left(G_p^{-1}(B) \right)$$

with

$$B = \{ z \in \mathbb{R}^p : z_{t(1)} \in B_1 \text{ and } z_{t(l)} - z_{t(l-1)} \in B_l \ (l = 2, \ldots, n) \},$$

and hence $= (\gamma_p)_\tau (B_1 \times \cdots \times B_n)$.

$\boxed{QED}$

In the remainder of the section we assume that

$$\gamma_t = e^{-t} \sum_{l=0}^{\infty} \left(\frac{t^l}{l!} \right) (\delta_l | \mathfrak{K}) \quad \text{for } t > 0 \quad \text{and} \quad \gamma_0 = \delta_0 | \mathfrak{K}.$$

Note that the same formulae hold true for $(\gamma_t)_\tau$. It is well-known that the present assumptions are fulfilled; see for example [15, 7.12.2(b)]. Under this particular choice the above $(\varphi_p)_{p \in I}$ and $\varphi : \mathfrak{S} \to [0, \infty[$ with $\Phi = \varphi_\tau | \mathfrak{C}(\varphi_\tau)$ correspond to the traditional *Poisson process* $\alpha : \mathfrak{A} \to [0, \infty[$. Our aim is to construct a subset $E \in \mathfrak{C}(\varphi_\tau)$ of $X = \mathbb{R}^T$ with $\Phi(E) = 1$ which fulfils the traditional requirements. This will be a kind of counterpart to the former [14, theorem 6.1] for the Wiener process.

17 Remark. For real $t > 0$ and for $n \geqq 0$ we have

$$e^{-t} \sum_{l=n}^{\infty} \left(\frac{t^l}{l!} \right) \leqq e^{-t} \sum_{l=n}^{\infty} \left(\frac{t^n}{n!} \, \frac{t^{l-n}}{(l-n)!} \right) = \frac{t^n}{n!}.$$

18 Lemma. *Let $D \subset T$ be countable and dense with $0 \in D$. Then there exists an $A \in \mathfrak{A}$ with $\alpha(A) = 1$ such that all $x \in A$ have restrictions $x|D$ with values in $\mathbb{N} \cup \{0\} =: \mathbb{N}0$ and $x_0 = 0$ which are monotone increasing and continuous.*

We recall from [14, 6.5] for $0 \leqq s < t$ that

$$\varphi_\tau([H_t - H_s \in B]) = (\gamma_{t-s})_\tau(B) \qquad \text{for } B \subset \mathbb{R},$$

in particular

$$\alpha([H_t - H_s \in B]) = (\gamma_{t-s})_\tau(B) \qquad \text{for } B \in \mathfrak{B},$$

and hence $\alpha([H_t - H_s \in \mathbb{N}0]) = 1$.

Proof of lemma 18.

(1) In view of $\alpha([H_0 = 0]) = 1$ it follows that the countable intersection

$$B = [H_0 = 0] \cap \bigcap_{s<t \text{ in } D} [H_t - H_s \in \mathbb{N}0]$$

is a set $B \in \mathfrak{A}$ with $\alpha(B) = 1$ such that the $x \in B$ have restrictions $x|D$ with values in $\mathbb{N}0$ and $x_0 = 0$ which are monotone increasing.

(2) Next fix an $s \in D$, and take a sequence $(s(l))_l$ in D such that $s < s(l) \downarrow s$. For $x \in B$ then

$$x_s \leqq \inf\{\, x_t : t \in D \text{ with } t > s \,\} = \lim_{l\to\infty} x_{s(l)} =: F_s(x),$$

which produces a function $F_s : B \to \mathbb{N}0$ with $H_{s(l)}|B \downarrow F_s \geqq H_s|B$. It follows that $[H_{s(l)} - H_s \geqq 1] \cap B \downarrow [F_s - H_s \geqq 1] \in \mathfrak{A}$. Now remark 17 shows that

$$\alpha\big([H_{s(l)} - H_s \geqq 1] \cap B\big) \leqq \alpha\big([H_{s(l)} - H_s \geqq 1]\big) = (\gamma_{s(l)-s})_\tau([1,\infty[) \leqq s(l) - s.$$

Therefore $\alpha([F_s - H_s \geqq 1]) = 0$. Thus $V_s := [F_s = H_s|B] \subset B$ is a member of $\mathfrak{A}$ with $\alpha(V_s) = 1$ such that all $x \in V_s$ are right continuous at s.

(3) In case $0 < s \in D$ the same method furnishes a member $U_s \subset B$ of $\mathfrak{A}$ with $\alpha(U_s) = 1$ such that all $x \in U_s$ are left continuous at s. We set $U_0 := B$. It follows that $A := \bigcap_{s\in D} U_s \cap V_s \in \mathfrak{A}$ has $\alpha(A) = 1$ and is as required.

$\boxed{QED}$

As in [14, section 6] we define $\mathbb{D} \subset T$ to consist of the dyadic rationals $\geqq 0$ and $\mathbb{D}(n) := \{\, t \in T : 2^n t \in \mathbb{N}0 \text{ and } t \leqq n \,\}$ for $n \in \mathbb{N}$. Thus $\mathbb{D}(n) \uparrow \mathbb{D}$.

We start the first part of our construction. We define $B_n \subset X$ for $n \in \mathbb{N}$ to consist of the $x \in X$ such that $x|\mathbb{D}(n)$ has values in $\mathbb{N}0$ with $x_0 = 0$ and is monotone increasing, and such that $x_t - x_s \leqq 1$ for all $s < t$ in $\mathbb{D}(n)$ with $t - s \leqq 2/2^n$.

19 Remark.

(i) Each $x \in B_n$ fulfils $x_t \leqq n2^{n-1}$ for all $t \in \mathbb{D}(n)$.

(ii) $B_n \in \mathfrak{S}_\sigma \subset \mathfrak{A}$.

(iii) $\alpha(B'_n) \leqq 4n2^{-n}$.

Proof.

(i) It suffices to estimate x_n. To this end note that $x_t - x_s \leqq 1$ for each consecutive pair $s < t$ in the sequence of the $2l/2^n$ $(l = 0, 1, \ldots, n2^{n-1})$ which starts with 0 and ends with n.

(ii) For $p = \mathbb{D}(n)$ we have $B_n = K \times \mathbb{R}^{T\setminus p}$ for some $K \subset \mathbb{R}^p$ which is closed by definition and hence compact by (i). Thus $K \in \mathrm{Comp}(\mathbb{R}^p) = (\mathfrak{K}_p)_\sigma = ((\mathfrak{K}^p)^\star)_\sigma$ and hence $B_n = K \times \mathbb{R}^{T\setminus p} \in \mathfrak{S}_\sigma$.

(iii) We have

$$B_n = [H_0 = 0] \cap \bigcap_{s<t \text{ in } \mathbb{D}(n)} [H_t - H_s \in \mathbb{N}0] \cap \bigcap_{0<t-s\leqq 2/2^n \text{ in } \mathbb{D}(n)} [H_t - H_s \leqq 1],$$

$$B'_n = [H_0 \neq 0] \cup \bigcup_{s<t \text{ in } \mathbb{D}(n)} [H_t - H_s \notin \mathbb{N}0] \cup \bigcup_{0<t-s\leqq 2/2^n \text{ in } \mathbb{D}(n)} [H_t - H_s > 1].$$

Thus $\alpha(B'_n)$ is $\leqq$ the sum of the measures of all these subsets. The subsets of the first two kinds have measure 0, and for the last ones we have $\alpha\big([H_t - H_s > 1]\big) \leqq (1/2)(t-s)^2 \leqq 2/2^{2n}$ from remark 17. Since the number of terms of the last kind is $\leqq 2n2^n$, it follows that $\alpha(B'_n) \leqq 4n2^{-n}$.

$\boxed{QED}$

We complete the first part of the construction with $A_m := \bigcap_{n=m}^{\infty} B_n$ for $m \in \mathbb{N}$, and with $A := \bigcup_{m=1}^{\infty} A_m$. Thus $A_m \uparrow A$. For the sequel we define for $t \in \mathbb{R}$ as usual $[t]$ to be the largest integer $\leqq t$ and $\{\,t\,\}$ to be the smallest integer $\geqq t$.

20 Remark.

(1) The $x \in A$ have restrictions $x|\mathbb{D}$ with values in $\mathbb{N}0$ and $x_0 = 0$ which are monotone increasing.

(2) Each $x \in A_m$ fulfils $x_t \leqq n2^{n-1}$ for all $n \geqq m$ and $t \in \mathbb{D}$ with $t \leqq n$.

(3) $A_m \in \mathfrak{S}_\sigma \subset \mathfrak{A}$ for $m \in \mathbb{N}$ and hence $A \in \mathfrak{A}$, and $\alpha(A) = 1$.

(4) For each $x \in A_m$ and all $n \geqq m$ we have

$$x_t - x_s \leqq \{(1/2)\big(\{2^n t\} - [2^n s]\big)\} \quad \text{for } s, t \in \mathbb{D} \text{ with } s < t \leqq n.$$

PROOF.

(1) For $x \in A_m$ and $s \leqq t$ in $\mathbb{D}$ there is an $n \geqq m$ such that $s, t \in \mathbb{D}(n)$. Thus the assertions are clear from $x \in B_n$.

(2) In view of (1) it suffices to prove $x_n \leqq n2^{n-1}$ for $n \geqq m$. But this follows from remark 19(i) since $x \in B_n$.

(3) The first assertion follows from remark 19(ii). From remark 19(iii) now $\sum_{n=1}^{\infty} \alpha(B'_n) < \infty$, so that $\alpha(A'_m) \leqq \sum_{n=m}^{\infty} \alpha(B'_n)$ implies that $\alpha(A'_m) \downarrow 0$. Thus $\alpha(A) = \lim\limits_{m\to\infty} \alpha(A_m) = 1$.

(4) Fix $s, t \in \mathbb{D}$ with $s < t \leqq n$. Then

$$0 \leqq p := [2^n s] \leqq 2^n s < 2^n t \leqq \{2^n t\} =: q \leqq 2^n n.$$

Let $r = \{(1/2)(q-p)\}$, that is $r \in \mathbb{N}$ with $r - 1 < (1/2)(q-p) \leqq r$ or $p + 2(r-1) < q \leqq p + 2r$. We obtain for $u := p/2^n \leqq s < t \leqq q/2^n =: v \leqq n$ from the last condition in the definition of $x \in B_n$, applied to the consecutive pairs in $(p+2l)/2^n$ $(l = 0, \ldots, r-1)$ and to the pair $(p + 2(r-1))/2^n < q/2^n$, and from (1) that $x_t - x_s \leqq x_v - x_u \leqq r$. This is the assertion.

$\boxed{QED}$

We turn to the second part of our construction. In contrast to the first part it will exceed the frame of $\mathfrak{A}$.

We start to define $F_n(U) \subset X$ for $0 \in U \subset T$ and $n \in \mathbb{N}$ to consist of the $x \in X$ such that $x|U \cap [0,n]$ has values in $\mathbb{N}0$ with $x_0 = 0$ and is monotone increasing, and is such that

$$x_t - x_s \leqq \{(1/2)\big(\{2^n t\} - [2^n s]\big)\} \quad \text{for } s, t \in U \text{ with } s < t \leqq n.$$

21 Remark.

(i) Each $x \in F_n(U)$ fulfils $x_t \leqq n2^{n-1}$ for all $t \in U$ with $t \leqq n$.

(ii) $F_n(U)$ is antitone in U, and $F_n(U) = \bigcap\limits_{p \in I \text{ with } 0 \in p \subset U} F_n(p)$.

(iii) $F_n(U) \in \mathfrak{S}_\tau$.

PROOF.

(i) For $t \in U$ with $t \leqq n$ we have $x_t \leqq \{(1/2)\{2^n n\}\} = n2^{n-1}$.

(ii) Is obvious from the definition.

(iii) In view of (ii) it suffices to prove that $F_n(p) \in \mathfrak{S}_\sigma$ for $0 \in p \in I$. In fact, we have $F_n(p) = K \times \mathbb{R}^{T \setminus p \cap [0,n]}$ for some $K \subset \mathbb{R}^{p \cap [0,n]}$ which is closed by definition and hence compact by (i). Thus $F_n(p) \in \mathfrak{S}_\sigma$ as in the proof of remark 19(ii).

$\boxed{QED}$

We complete the second part of the construction with $E_m(U) := \bigcap_{n=m}^{\infty} F_n(U)$ for $0 \in U \subset T$ and $m \in \mathbb{N}$, and with $E(U) := \bigcup_{m=1}^{\infty} E_m(U)$. Thus $E_m(U) \uparrow E(U)$.

22 Remark.

(1) The $x \in E(U)$ have restrictions $x|U$ with values in $\mathbb{N}0$ and $x_0 = 0$ which are monotone increasing.

(2) Each $x \in E_m(U)$ fulfils $x_t \leqq n2^{n-1}$ for all $n \geqq m$ and $t \in U$ with $t \leqq n$.

(3) The $E_m(U)$ and $E(U)$ are antitone in U.

(4) $E_m(U) \in \mathfrak{S}_\tau$ and hence $E(U) \in (\mathfrak{S}_\tau)^\sigma \subset \mathfrak{C}(\varphi_\tau)$.

(5) $A_m \subset E_m(\mathbb{D})$ for $m \in \mathbb{N}$ and hence $A \subset E(\mathbb{D})$. Therefore $\Phi(E(\mathbb{D})) = 1$.

Proof. (1) For $x \in E_m(U)$ and $s \leqq t$ in U there is an $n \geqq m$ with $s \leqq t \leqq n$. Thus the assertions are clear from $x \in F_n(U)$. (2) (3) (4) are clear from remark 21(i)(ii)(iii). (5) For $x \in A_m$ and $n \geqq m$ we see from remark 5(1) and remark 5(4) that $x \in F_n(\mathbb{D})$. Thus $x \in E_m(\mathbb{D})$. The last assertion then follows from remark 5(3). $\boxed{QED}$

After this construction our procedure will be quite close to the previous proof of [14, theorem 6.1].

23 Lemma. *Let $U \subset T$ be dense with $0 \in U$. Then $\Phi(E_m(U \cup p)) = \Phi(E_m(U))$ for all $p \in I$ and $m \in \mathbb{N}$.*

Proof. By remark 22(3) it suffices to prove that

$$\Phi(E_m(U \cup \{s\})) \geqq \Phi(E_m(U))$$

for $s \in T \setminus U$. Thus fix $s \in T \setminus U$ and $m \in \mathbb{N}$. Note that $s > 0$.

(1) Let $D \subset U \subset T$ be a countable dense subset with $0 \in D$. From lemma 18 applied to $D \cup \{ s \}$ we obtain an $A \in \mathfrak{A}$ with $\alpha(A) = 1$ such that all $x \in A$ have restrictions $x|D \cup \{ s \}$ with values in $\mathbb{N}0$ which are monotone increasing and continuous. Thus for each $x \in A$ there exists $0 < \varepsilon(x) < s$ such that x is constant $= x_s \in \mathbb{N}0$ on $(D \cup \{ s \}) \cap]s - \varepsilon(x), s + \varepsilon(x)[$.

(2) We claim that $E_m(U) \cap A \subset E_m(U \cup \{s\})$, which at once implies the assertion. Thus fix $x \in E_m(U) \cap A$. Then (1) combined with remark 22(1) implies that x is constant $= x_s \in \mathbb{N}0$ on $(U \cup \{s\}) \cap]s - \varepsilon(x), s + \varepsilon(x)[$.

(3) To be shown is $x \in F_n(U \cup \{s\})$ for $n \geqq m$, and what remains is

$$x_v - x_u \leqq \{(1/2)\big(\{2^n v\} - [2^n u]\big)\} \quad \text{for } u, v \in U \cup \{s\} \text{ with } u < v \leqq n.$$

This is clear when $u, v \in U$. In case $u = s < v \in U$ one applies the relation to $t, v \in U$ with $u = s < t < v$ and $t < s + \varepsilon(x)$, and obtains

$$x_v - x_u = x_v = x_t \leqq \{(1/2)\big(\{2^n v\} - [2^n t]\big)\} \leqq \{(1/2)\big(\{2^n v\} - [2^n u]\big)\}.$$

The same procedure works in case $u < v = s$.

$\boxed{QED}$

24 Lemma. *Assume that $\mathbb{D} \subset U \subset T$ and $m \in \mathbb{N}$. Fix $x \in E_m(U)$ and define $y \in X$ to be $y_t = \inf\{\, x_s : s \in U$ with $s \geqq t \,\}$ for $t \in T$. Then $y \in E_m(T)$ and $y|U = x|U$.*

Proof. The above $y \in X$ has values in $\mathbb{N}0$ with $y_0 = 0$ and is monotone increasing, and of course $y|U = x|U$. To be shown is $y \in F_n(T)$ for $n \geqq m$, and what remains is

$$y_v - y_u \leqq \{(1/2)\big(\{2^n v\} - [2^n u]\big)\} \quad \text{for } u, v \in T \text{ with } u < v \leqq n.$$

This is clear when $u, v \in U$. We proceed to settle the other cases.

(1) The definition of y combined with remark 22(1) implies for each $t \in T \setminus U$ the existence of an $\varepsilon(t) > 0$ such that $y_t = x_s$ for all $s \in U \cap]t, t + \varepsilon(t)[$.

(2) Now assume first $u < v \leqq n$ with $u \in U$ and $v \in T \setminus U$. Then $v \notin \mathbb{D}$, so that $v < n$ and $2^n v \notin \mathbb{N}$. Thus for $s \in U \cap]v, v + \varepsilon(v)[$ sufficiently close to v we have $s < n$ and $\{2^n s\} = \{2^n v\}$, besides $x_s = y_v$ from (1). It follows that

$$y_v - y_u = x_s - x_u \leqq \{(1/2)\big(\{2^n s\} - [2^n u]\big)\} = \{(1/2)\big(\{2^n v\} - [2^n u]\big)\}.$$

Thus we have the assertion for all $u < v \leqq n$ with $u \in U$.

(3) Now let $u < v \leqq n$ with $u \in T \setminus U$. For $s \in U \cap]u, u + \varepsilon(u)[$ sufficiently close to u we have $s < v$ and $[2^n s] = [2^n u]$ as before, besides $y_s = x_s = y_u$ from (1). It follows from (2) that

$$y_v - y_u = y_v - y_s \leqq \{(1/2)\big(\{2^n v\} - [2^n s]\big)\} = \{(1/2)\big(\{2^n v\} - [2^n u]\big)\}.$$

The proof is complete. $\boxed{QED}$

25 Proposition. *$\Phi(E_m(\mathbb{D})) = \Phi(E_m(T))$ for $m \in \mathbb{N}$ and hence $\phi(E(\mathbb{D})) = \Phi(E(T))$. Therefore $\Phi(E(T)) = 1$.*

Proof.

(1) We claim that $E_m(\mathbb{D} \cup p) \subset H_p^{-1}\big(H_p(E_m(T))\big)$ for all $p \in I$ and $m \in \mathbb{N}$. In fact, for $x \in E_m(\mathbb{D} \cup p)$ we obtain from lemma 24 some $y \in E_m(T)$ with $x|\mathbb{D} \cup p = y|\mathbb{D} \cup p$, in particular $x|p = y|p$. Thus $H_p(x) = H_p(y) \in H_p(E_m(T))$, which is the assertion.

(2) From remark 22(4) we know that $E_m(T) \in \mathfrak{S}_\tau$. Thus theorem 11 implies that
$$\Phi(E_m(T)) = \inf_{p \in I} \Phi_p\big(H_p(E_m(T))\big).$$
Now once more from theorem 11 and then from (1) and lemma 23
$$\begin{aligned}\Phi_p\big(H_p(E_m(T))\big) &= (\varphi_p)_\tau\big(H_p(E_m(T))\big) = \varphi_\tau\big(H_p^{-1}\big(H_p(E_m(T))\big)\big)\\ &\geqq \varphi_\tau(E_m(\mathbb{D} \cup p)) = \Phi(E_m(\mathbb{D} \cup p)) = \Phi(E_m(\mathbb{D})).\end{aligned}$$
It follows that $\Phi(E_m(T)) \geqq \Phi(E_m(\mathbb{D}))$ and hence $\Phi(E_m(T)) = \Phi(E_m(\mathbb{D}))$ in view of remark 22(3). The last assertion then results from remark 22(5).

$\boxed{QED}$

We have thus obtained the counterpart of [14, theorem 6.1]. We continue with another fundamental point.

We know from remark 22(1) that the $x \in E(T)$ have values in $\mathbb{N}0$ with $x_0 = 0$ and are monotone increasing. This implies the existence of the one-sided limits
$$\begin{aligned} x_t^+ &= \lim_{s \downarrow t} := \inf\{\, x_s : s \in T \text{ with } s > t \,\} \in \mathbb{N}0 \quad \text{for } t \in T,\\ x_t^- &= \lim_{s \uparrow t} := \sup\{\, x_s : s \in T \text{ with } s < t \,\} \in \mathbb{N}0 \quad \text{for } 0 < t \in T,\end{aligned}$$
with in addition $x_0^- := x_0 = 0$. It is clear that
$$x \text{ is constant} = x_t^+ \text{ on some }]t, t + \varepsilon(x,t)[\quad \text{for } t \in T,$$
$$x \text{ is constant} = x_t^- \text{ on some }]t - \varepsilon(x,t), t[\quad \text{for } 0 < t \in T,$$
with $\varepsilon(x,t) > 0$. Of course $x_t^- \leqq x_t \leqq x_t^+$. We prove that a famous feature of the Poisson process holds true on $E(T)$.

26 Proposition. *We have $x_t^+ - x_t^- \leqq 1$ for all $x \in E(T)$ and $t \in T$. Thus at least one of the equalities $x_t = x_t^-$ and $x_t = x_t^+$ holds true.*

PROOF. Fix $x \in E(T)$ and $t \in T$, and then $m \in \mathbb{N}$ with $x \in E_m(T)$ and $n \geqq m$ with $n > t$. Thus $x \in F_n(T)$, in particular

$$x_v - x_u \leqq \{(1/2)\big(\{2^n v\} - [2^n u]\big)\} \quad \text{for all } 0 \leqq u < v \leqq n.$$

We consider three cases.

(1) The case $t = 0$. For $0 = u < v \leqq 2/2^n$ and hence $\leqq n$ we have $x_v \leqq \{(1/2)\{2^n v\}\} \leqq \{(1/2)2\} = 1$. Hence $x_0^+ \leqq 1$.

(2) The case $t > 0$ with $2^n t \in \mathbb{N}$. For $0 \leqq u < v \leqq n$ and $t - 2^{-n} \leqq u < t < v \leqq t + 2^{-n}$ we have $2^n t - 1 \leqq 2^n u < 2^n t < 2^n v \leqq 2^n t + 1$. Thus $\{2^n v\} \leqq 2^n t + 1$ and $[2^n u] \geqq 2^n t - 1$, so that $x_v - x_u \leqq \{(1/2)2\} = 1$. Hence $x_t^+ - x_t^- \leqq 1$.

(3) The case $t > 0$ with $2^n t \notin \mathbb{N}$. Here $[2^n t] < 2^n t < \{2^n t\}$ with $\{2^n t\} - [2^n t] = 1$. For $0 \leqq u < v \leqq n$ and $[2^n t]/2^n \leqq u < t < v \leqq \{2^n t\}/2^n$ we have $[2^n t] \leqq 2^n u < 2^n t < 2^n v \leqq \{2^n t\}$. Thus $\{2^n v\} = \{2^n t\}$ and $[2^n u] = [2^n t]$, so that $x_v - x_u \leqq \{(1/2)1\} = 1$. Hence $x_t^+ - x_t^- \leqq 1$.

$\boxed{QED}$

After this we collect the most important properties of the subset $E(T) \subset X$.

27 Theorem.

(1) $E_m(T) \in \mathfrak{S}_\tau$ for $m \in \mathbb{N}$ and $E_m(T) \uparrow E(T)$. Thus $E(T) \in (\mathfrak{S}_\tau)^\sigma \subset \mathfrak{C}(\varphi_\tau)$.

(2) $\Phi(E(T)) = 1$.

(3) The members $x \in E(T)$ have values in $\mathbb{N}0$ with $x_0 = 0$ and are monotone increasing with $x_t^+ - x_t^- \leqq 1$ for all $t \in T$. Moreover the members $x \in E_m(T)$ fulfil for $n \geqq m$ the estimation

$$x_t - x_s \leqq \{(1/2)\big(\{2^n t\} - [2^n s]\big)\} \quad \textit{for all } 0 \leqq s < t \leqq n.$$

This theorem is, after the treatment of the Wiener process in [14, section 6], the second concrete evidence in favor of our systematic enterprise in the domain of stochastic processes. Also it is not hard to see that as before the pathological thick subsets $C(a) \subset X$ of theorem 4 are measurable $\mathfrak{C}(\varphi_\tau)$ with measure $\Phi(C(A)) = 0$. To appreciate the present achievement we note that the traditional treatment of the Poisson process as a rule starts from paths with integer values, that is uses the state space $(\mathbb{Z}, \mathfrak{P}(\mathbb{Z}))$ instead of $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}))$. The author does not see how the present development could be performed without the new structure in measure and integration.

However, we have to realize that the subset $E(T) \subset X$ does not favor the right continuous paths above the left continuous ones, nor vice versa.

28 Proposition. *Assume that $x \in E_m(T)$. Then $E_m(T)$ likewise contains the paths $y \in X$ with $y_t = x_t$ for $t \in \mathbb{D}$ and $y_t \in \{x_t^-, x_t, x_t^+\}$ for $t \in T \setminus \mathbb{D}$.*

PROOF. Fix some $y \in X$ of this kind. Then y has values in $\mathbb{N}0$ with $y_0 = 0$ and is monotone increasing. To be shown is $y \in F_n(T)$ for $n \geqq m$, and what remains is

$$y_t - y_s \leqq \{(1/2)\big(\{2^n t\} - [2^n s]\big)\} \quad \text{for all } 0 \leqq s < t \leqq n.$$

We consider the different cases.

$s \in \mathbb{D}$: For $u := s$ we have $0 \leqq u \leqq s$ with $[2^n u] = [2^n s]$ and $x_u = x_s = y_s$.

$s \notin \mathbb{D}$: There exists $0 \leqq u < s$ with $[2^n u] = [2^n s]$ and $x_u = x_s^- \leqq y_s$.

$t \in \mathbb{D}$: For $v := t$ we have $t \leqq v \leqq n$ with $\{2^n v\} = \{2^n t\}$ and $x_v = x_t = y_t$.

$t \notin \mathbb{D}$: There exists $t < v \leqq n$ with $\{2^n v\} = \{2^n t\}$ and $x_v = x_t^+ \geqq y_t$.

It follows that

$$y_t - y_s \leqq x_v - x_u \leqq \{(1/2)\big(\{2^n v\} - [2^n u]\big)\} = \{(1/2)\big(\{2^n t\} - [2^n s]\big)\}$$

under all combinations of these cases. $\boxed{QED}$

29 Remark. Let $C \subset X$ consist of the $x \in X$ with values in $\mathbb{N}0$ and $x_0 = 0$ which are monotone increasing and *right continuous*. It seems that C does not have its proper place in the present frame, at least it resisted so far the author's efforts to prove that $E(T) \cap C = \{\, x \in E(T) : x_t^+ = x_t \text{ for all } t \in T \,\}$ is in $\mathfrak{C}(\varphi_\tau)$. It is known that (at least under the restriction mentioned above) the subset $C \subset X$ is thick for $\alpha : \mathfrak{A} \to [0, \infty[$, so that in virtue of proposition 3 there exist prob measure extensions $\rho : \mathfrak{R} \to [0, \infty[$ of α which live on C; see for example [1, section 41]. It seems that these ρ are quite different from our $\Phi = \varphi_\tau | \mathfrak{C}(\varphi_\tau)$. Also the author thinks that the rôle of right continuous will be less important as soon as a domain like the present $\mathfrak{C}(\varphi_\tau)$ becomes available. In any case, we want to deduce from the present results that the subset $E(T) \cap C$ is *thick* for α.

PROOF.

(1) We shall invoke proposition 6 implication (3)⇒(1) for $\Phi : \mathfrak{C}(\varphi_\tau) \to [0,\infty[$ and $E(T) \cap C$. To this end we note that the present T and $(Y, \mathfrak{B})$ and our $E(T) \cap C$ fulfil the assumptions (i) (ii) in proposition 6. We have to prove that for each nonvoid countable $U \subset T$ there exists an $R(U) \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(R(U)) = 1$ such that all $x \in R(U)$ have restrictions $x|U \in (E(T) \cap C)|U$. We can pass from U to $U \cup \mathbb{D}$ and hence assume that $\mathbb{D} \subset U \subset T$.

(2) From lemma 18 applied to U we obtain an $A(U) \in \mathfrak{A}$ with $\alpha(A(U)) = 1$ such that all $x \in A(U)$ have continuous restrictions $x|U$. We let $R(U) := E(T) \cap A(U)$, so that $R(U) \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(R(U)) = 1$.

(3) Now fix $x \in R(U)$ and $m \in \mathbb{N}$ such that $x \in E_m(T) \subset E_m(U)$. We take $y \in E_m(T) \subset E(T)$ as formed in lemma 24. Then first of all $x|U = y|U$. Thus it remains to prove that $y \in E(T) \cap C$, that is that $y_t^+ = y_t$ for all $t \in T$. But from the definitions

$$y_t^+ = \inf\{\, y_u : u \in T \text{ with } u > t \,\} = \inf\{\, x_s : s \in U \text{ with } s > t \,\}.$$

In case $t \in U$ this is $= x_t$ since $x|U$ is continuous after (2) and hence $= y_t$, and in case $t \notin U$ it is $= y_t$ from the definition.

$\boxed{QED}$

In conclusion we remark that the two well-known assertions on continuous and discontinuous behavior which follow hold true for the present $\Phi : \mathfrak{C}(\varphi_\tau) \to [0,\infty[$ and $E(T) \subset X$. The usual proofs combined with proposition 13 and lemma 18 will do; see for example [1, 41.3].

30 Proposition.

(1) *Let $U \subset T$ be countable $\neq \varnothing$. Then there exists an $A \in \mathfrak{A}$ with $\alpha(A) = 1$ such that all $x \in E(T) \cap A$ are continuous in the points of U.*

(2) *There exists an $A \in \mathfrak{A}$ with $\alpha(A) = 1$ such that each $x \in E(T) \cap A$ fulfils $x_t^+ - x_t^- = 1$ for infinitely many $t \in T$.*

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31 Note. (added 25 April 2006). In connection with remark 29 the author wants to refer to two subsequent articles, which can be obtained as preprints under `http://www.math.uni-sb.de/PREPRINTS/preprint117.pdf` and `http://www.math.uni-sb.de/PREPRINTS/preprint118.pdf`.

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