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On a paper of Dan Barbilian

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Abstract. We point out that the axiomatic analysis of the statement *The segments joining a point with the vertices of an equilateral triangle satisfy the (non-strict) triangle inequalities* in Barbilian's [1] misses the case in which the sum of the angles in a triangle is greater than 180°. We situate the statement correctly inside absolute geometry. We also point out that [1] contains the first proof that a Hilbert geometry with symmetric perpendicularity must be hyperbolic geometry, a proof commonly attributed to P. J. Kelly and L. J. Paige [5].

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MSC 2000 classification: 51F05, 51M04, 51F20, 51-03, 01A60.

1 The Möbius-Pompeiu theorem in absolute geometry

Pompeiu [10] published a proof, valid in Euclidean geometry, of a result that may be said to go back to Möbius (1852) [see [7]], stating that

> *The segments joining a point with the vertices of an equilateral triangle satisfy the (non-strict) triangle inequalities, i. e., if $S_1S_2S_3$ is an equilateral triangle and S a point in its plane, then*
> *(1)* $\sum_{i=1}^{3} SS_i \geq 2SS_k$ *for all* $k \in \{1, 2, 3\}$.

Barbilian [1] provided a very detailed axiomatic analysis of this statement, with the stated aim of finding out on which of Hilbert's [4] axioms it depends and on which it does not depend. Having showed in [1, 19°] that the statement remains true whenever the sum of the angles of a triangle is less than 180°, he states that it is independent of the axiom of parallelism, which would imply that the statement holds in absolute geometry, i. e. would hold in any geometry satisfying the plane axioms of the groups I, II, and III of axioms in [4] (models of those axioms are also called *Hilbert planes* in [9], where they are characterized algebraically). Although Barbilian [1, §3] showed that (1) does not hold in "the geometry of Riemann", he claimed that the assumption that the sum of the angles of a triangle be > 180° is not compatible with Hilbert's order axioms, and thus assumes that the angle sum of a triangle must be $\leq 180^\circ$ in all Hilbert planes. At the time, this was already known to be false, as shown by Dehn [3, Capitel III].

It is easy to see that (1) is false in all Hilbert planes, in which the sum sum of the angles of a triangle is $> 180^\circ$ (equivalently, in which the *metric constant* k is > 0 (see [9, Satz 5])). Let, in a Hilbert plane with $k > 0$, $S_1S_2S_3$ be an equilateral triangle (according to [8], in any Hilbert plane there is an equilateral triangle, so the assumption is never vacuous), and let P be the point of intersection of its perpendicular bisectors. Let S be the reflection of P in the side S_1S_2. The inequality in (1) amounts in this case to the inequality $S_1P \geq PS$, which does not hold, given that $S_1P \equiv S_1S$, $m(\angle S_1PS) = m(\angle S_1SP) = 60^\circ$, thus, since the sum of the angles of a triangle is $> 180^\circ$, $m(\angle PS_1S) > 60^\circ$, so $PS > S_1P$.

If we denote by $\mathcal{A}$ the theory axiomatized by the plane axioms of Hilbert's groups I, II, and III, by **MP** the Möbius-Pompeiu inequality (1), and by $k \leq 0$ the statement *If ABC is a triangle, M, N, and P the midpoints of the segments AB, AC, and BC, and D a point between B and C, such that MN is congruent to BD, then D is equal to P or lies between B and P*, then we have, by Barbilian's proof of **MP** in case the the sum of the angles of a triangle is less than 180° (i. e. the metric constant $k < 0$) and by the proofs that **MP** holds in case the sum of the angles of a triangle is 180° (i. e. the metric constant $k = 0$):

1 Theorem. $\mathcal{A} \vdash \boldsymbol{MP} \leftrightarrow k \leq 0$.

2 Symmetry of perpendicularity in a Hilbert geometry

Hilbert [4, Anh. 1] introduced a metric inside a domain bounded by a simple closed convex curve K of the real Euclidean plane by defining the length $h(A, B)$ of a segment AB — the rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ intersecting K in the points B' and A' — as the logarithm of the cross-ratio $[A, B, B', A']$ of A', A, B, B', in case $A \neq B$, and 0 otherwise. In case K is an ellipse, the resulting geometry is Klein's model of plane hyperbolic geometry.

If P and g are a point and a line respectively, then a point F on g is called a *foot of P on g* if $h(P, F) \leq h(P, X)$ for all points X on g. Line h, intersecting g in A, is said to be *perpendicular* to g (and we write $h \perp g$) if the foot of every point P on g is A, for all points P on h.

The fact that $h \perp g \leftrightarrow g \perp h$, for all lines g and h, implies that K is an ellipse, is commonly attributed to P. J. Kelly and L. J. Paige [5], who rely on a characterization of perpendicularity from [2] and on a characterization of the ellipse from [6].

However, this fact was first proved by D. Barbilian [1, 22°-25°], without relying on any facts from the literature, as he provides his own characterization of the ellipse in the process. The only difference lies in the assumptions on K: in [5] it is assumed that K contains at most one segment, whereas in [1] it is assumed that K contains no segment at all. The change in Barbilian's proof this weakened hypothesis would have required is minor. His result was forgotten even by the reviewer of [1] for *Zentralblatt*, the same who reviewed, 16 years and a World War later, [5] for *Mathematical Reviews*: Ruth Moufang.

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