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Geometric structures arising from generalized j-planes

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Abstract. We study translation planes constructed by André net replacement on $jj\cdots j$-planes and derivation on $jj\cdots j$-planes. Then, we get to the conclusion that the family of non-André $jj\cdots j$-planes is new, and thus so are their replaced and derived planes.

We also study a new way to construct translation planes by putting together two 'halves' of planes that belong to two different $jj\cdots j$-planes. We show examples of planes of small order constructed this way.

Finally, we prove that using regular hyperbolic covers, $jj\cdots j$-planes induce partitions of Segre varieties by Veronesians (sometimes called flat flocks)

Keywords: Translation planes, André nets, derivable nets, glat flocks, generalized j-planes

MSC 2000 classification: Primary 51, 05; Secondary 20

1 Introduction

In [30] the author constructed and studied a family of translation planes that generalize the j-planes of Johnson, Pomareda and Wilke. These generalized j-planes are constructed from a field of matrices $F \cong GF(q)$ whenever the set

$$S = \{y = xf(M)M;\ M \in F\} \cup \{x = 0\}$$

defines a spread. Where

$$f : M_n(q) \to M_n(q)$$

is such that

(1) $f(MN) = f(M)f(N)$, for all $M, N \in F^*$,

(2) $f(M)^{q-1} = Id$ for all $M \in F^*$

In the same article it is shown that all generalized j-planes are isomorphic to $jj\cdots j$-planes, these planes are generalized j-planes with

$$f(M) = \mathrm{diag}(1, \det(M)^{j_2}, \det(M)^{j_3}, \ldots, \det(M)^{j_n})$$

where $j_2, j_3, \ldots, j_n$ are elements of $\{0, 1, 2, \ldots, q-2\}$

It is easy to see that these planes admit a homology group of order $(q^n - 1)/(q-1)$, thus their study is important as their existence would give an example of a non-André plane admitting such a group. The following results (which we will use later in this article) show how a homology group like this could determine the structure of a plane that admits it as a collineation group.

1 Theorem. *[Johnson–Pomareda [19]] Let π be a translation plane of order q^n and kernel containing $GF(q)$. Assume that the plane admits a cyclic homology group of order $(q^n - 1)/(q-1)$. Then one of the following must occur:*

(i) The axis and the coaxis of the group are both invariant or

(ii) The plane is André.

2 Theorem. *[Ostrom [26]] Let π be a translation plane of order q^n that admits a cyclic affine homology group H of order $(q^n - 1)/(q-1)$. Then any component orbit union the axis and coaxis of the group is a Desarguesian partial spread.*

3 Remark. In the study of $jj\cdots j$-planes, and the planes we will construct later in this article we will need a list of known planes of orders q^n for $n > 2$. This list has been collected from [4].

(1) Desarguesian.

(2) André.

(3) Nearfield.

(4) Generalized André.

(5) Semifield.

(6) Generalized Hall.

(7) Flag-transitive.

(8) $SL(2, q)$ plane (a plane that admits $SL(2, q)$ as a collineation group).

(9) Hiramine–Jha–Johnson [9].

(10) Symplectic (see, for example, [23], [3], [25] and [31]).

(11) Triangle transitive planes. See [29], for example.

(12) Culbert – Ebert planes of order q^3 [6].

(13) Cubic Figueroa planes [7].

(14) Johnson non-André hyper regulus replacement planes (see [13] and [14]).

(15) Derived algebraic lifted planes [17].

In [30] we showed that none of the planes listed above can be a (non-André) $jj\cdots j$-plane. Moreover, we studied what the transposed plane of a $jj\cdots j$-plane is.

4 Definition. Let $S = \{y = xM\} \cup \{(x = 0)\}$ be a spread. Then the spread given by $S^t = \{y = xM^t\} \cup \{x = 0\}$ is called the transposed spread of S. It is known that, after a change of basis, the collineations of S^t look like

$$(x, y) \to (x, y) \begin{bmatrix} D^t & B^t \\ C^t & A^t \end{bmatrix},$$

where

$$(x, y) \to (x, y) \begin{bmatrix} A & C \\ B & D \end{bmatrix}$$

is a collineation of S.

5 Lemma. *If Π be a $j_2 j_3 \cdots j_n$-plane, then its transposed plane is also a $j_2 j_3 \cdots j_n$-plane.*

Finally, we were able to get the full collineation group of a non-André $jj\cdots j$-plane.

6 Theorem. *The linear part of the translation complement of a non-André $jj\cdots j$-plane Π is isomorphic to the direct product of G and the kernel homologies.*

2 Replaced $jj\cdots j$-planes

I this section we will study a way to construct new planes from already known $jj\cdots j$-planes. The process used to achieve this is called André nets replacement. The reader is referred to [4] or [27] for more details on this process. We start with a short summary of the material that will be necessary to follow this section.

Every $jj\cdots j$-plane of order q^n admits a homology group of order $(q^n - 1)/(q-1)$:

$$H_y = \left\{ \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & M \end{bmatrix} \in G;\ \det(M) = 1 \right\}.$$

The orbits of lines of H_y define $q-1$ André nets on the plane. They all share the lines $y = 0$ and $x = 0$ and partition the rest of the lines.

For $v \in GF(q)^*$, the nets are:

$$N_v = \{y = xS_vM;\ S_v = \Delta_L L \text{ and } \det(L) = v,\ M \in F^* \cap SL(n,q)\}\,.$$

The idea is to replace these nets by other sets of lines to create more translation planes. It was shown by Pomareda [28] that for any André net of this order there are exactly $n-1$ different replacements. In our case, they are:

$$N_v^{(k)} = \left\{y = (x^{q^k})S_vM;\ S_v = \Delta_L L \ \textit{and}\ \det(L) = v,\ M \in F^* \cap SL(n,q)\right\}$$

for $k = 1, 2, \ldots, n-1$.

It follows that, in order to know the replacements for N_v, we need to find a representation for the function $x \mapsto x^{q^k}$ for $k = 1, 2, \ldots, n-1$. We do this by identifying the elements of $GF(q^n)$ (the x's) with the elements of F (the matrices).

So, let $F = GF(q)(\beta)$, and let M_q be the matrix that represents $x \to x^q$ in the basis $\{1, \beta, \ldots, \beta^{n-1}\}$. Clearly M_q^k represents $x \to x^{q^k}$ for every k. Then the replacements for N_1 are:

$$N_1^{(k)} = \left\{y = xM_q^kM;\ M \in F^* \ \textit{and}\,\det(M) = 1\right\},$$

for $k = 1, 2, \ldots, n-1$.

In order to obtain the replacements for the other $q-2$ nets we change basis via $\gamma = \begin{bmatrix} S_v^{-1} & 0 \\ 0 & Id \end{bmatrix}$ to transform N_v into N_1. Then we replace N_1 and go back via γ^{-1}. The replacements for N_v are:

$$N_v^{(k)} = \left\{\ y = xS_vM_q^kM;\ M \in F^* \ \textit{and}\,\det(M) = 1\right\},$$

where $S_v = \Delta_L L$ is an arbitrary, but fixed.

7 Notation. We enumerate the nets of a given $jj\cdots j$-plane Π by saying that N_1 is the first net, N_α is the second and $N_{\alpha^{k-1}}$ is the k^{th} net of Π, where $GF(q)^* =< \alpha >$.

We use this enumeration to label the planes π that have been obtained via the net replacement of the nets; we will write $\pi = [n_1, n_2, \ldots, n_{q-1}]$ to mean that the i^{th} net has been replaced by using $x \to x^{q^{n_i}}$. In particular, $[0, 0, \ldots, 0]$ is the $jj\cdots j$-plane we started with.

8 Notation. In this section we will always consider non-Desarguesian $jj\cdots$ $\cdots j$-planes listed in remark 3.

9 Remark. Every replaced $jj\cdots j$-plane inherits the $((\infty), y=0)$-homology group of order $(q^n-1)/(q-1)$ from its associated $jj\cdots j$-plane. Also, it is easy to see that, just like $jj\cdots j$-planes, replaced $jj\cdots j$-planes have kernel $GF(q)$ and spreads in $PG(2n-1,q)$.

It is possible that a replaced $jj\cdots j$-plane is also a $jj\cdots j$-plane. In order for this to happen, the plane must, at least, admit a cyclic group of order $q-1$ that commutes with H_y and thus it permutes the André nets. In [30] we showed that in a $jj\cdots j$-plane this group induces a $((0),\infty)$-homology group of order $q-1$ called H_x. Moreover, in [30] we proved that an André $jj\cdots j$-plane is a nearfield plane by usingH_x. As it is fairly easy to construct André planes that are not $jj\cdots j$-planes, then we are only allowed to say that there might be André replaced $jj\cdots j$-planes that are not nearfield.

10 Lemma. *Let $\pi_1=[n_1,n_2,\ldots,n_{q-1}]$ and $\pi_2=[m_1,m_2,\ldots,m_{q-1}]$ be two planes obtained by net replacement from the same $jj\cdots j$-plane Π. If there is a k, $1\le k\le q-1$, such that $n_i\equiv m_{i+k} \bmod n$ for every i, then the planes are isomorphic.*

Proof. Let G be the group of order q^n-1 that is associated to Π. Note that the subgroup of order $q-1$ of G acts transitively on the nets of Π. This is the group that induces the isomorphism between π_1 and π_2. $\boxed{QED}$

11 Corollary. *Let $\pi_1=[n_1,n_2,\ldots,n_{q-1}]$ and $\pi_2=[m_1,m_2,\ldots,m_{q-1}]$ be two planes obtained by net replacement from the same $jj\cdots j$-plane Π.*

If, for a fixed k and every i, $n_i\equiv m_i+k \bmod n$, then the planes are isomorphic.

Proof. It is clear that $\Phi:=\begin{bmatrix} M_q^{-1} & 0\\ 0 & Id\end{bmatrix}$ maps N_v into $N_v^{(1)}$ and, in general, $N_v^{(k)}$ into $N_v^{(k+1)}$ for every $v\in GF(q)^*$. It follows that Φ induces an isomorphism between π_1 and π_2. $\boxed{QED}$

12 Remark. Using Φ we can always work with planes of the form $[0,n_2,\ldots$ $\ldots,n_{q-1}]$. Thus, in the search for new planes we do not need to replace all nets in a $jj\cdots j$-plane, it is enough to replace all but the first net.

13 Lemma. *Let π be a replaced $jj\cdots j$-plane. Then π is an André plane if and only if Π is an André plane.*

Proof. Clearly, the replaced planes obtained from the André $jj\cdots j$-plane are also André.

Assume π is an André plane. Because of a result by Foulser [8], one of the two symmetric homology groups of π that have been inherited from some Desarguesian plane and the homology group H_y inherited from Π are the same.

This implies that, by reversing one of the nets $N_v^{(k)}$ for $v \in GF(q)^*$, we obtain more André planes. In particular, Π is André. $\boxed{QED}$

14 Remark. Using the same type of argument used in the previous lemma, one may show that replaced $jj \cdots j$-planes are Desarguesian, if and only if, they yield from a Desarguesian plane or an André plane.

Since we know what happens when we replace nets on an André plane, then we will only have left to study replaced $jj \cdots j$-planes that are non-André.

15 Lemma. *A non-André replaced $jj \cdots j$-plane that is not an $SL(2, q)$-plane or a derived lifted plane cannot be isomorphic to any plane listed in remark 3.*

Proof. This proof is essentially the same as the long series of lemmas and remarks in [30] starting in remark 8 until corollary 8, as those proofs just needed the existence of a homology group of size $(q^n - 1)/(q - 1)$ and that the plane had kernel containing $GF(q)$. Both of these hypothesis hold in replaced $jj \cdots j$-planes, thus the result follows. $\boxed{QED}$

16 Remark. As mentioned in definition 4, if one transposes all the matrices in the spreadset of a translation plane π, then one obtains the spreadset of the transposed plane of π. In lemma 5 we showed that a $jj \cdots j$-plane is isomorphic to its transposed plane, this result is not necessarily valid for replaced $jj \cdots j$-planes. However, it is easy to see, from definition 4, that the transposed of a replaced $jj \cdots j$-plane will admit a cyclic homology group of order $(q^n-1)/(q-1)$ and will have kernel containing $GF(q)$, thus the previous lemma also applies to these planes.

We still need to learn more about the collineation group of a replaced $jj \cdots j$-plane to assure that it cannot be an $SL(2, q)$-plane or a derived lifted plane.

17 Lemma. *The translation complement of a non-André replaced plane π fixes the lines $x = 0$ and $y = 0$.*

Proof. This is a corollary of theorem 1. $\boxed{QED}$

18 Theorem. *Let π be a plane obtained by net replacement from the $jj \cdots j$-plane Π. Then, the linear part of the translation complement of π is the group inherited from Π.*

Proof. Recall that we are considering replaced planes with the net N_1 left unreplaced. Let Ψ be an element of the translation complement of π. Because of the previous lemma we can assume that Ψ fixes $x = 0$ and $y = 0$.

By hypothesis Ψ fixes the net N_1, then by using G', the group induced in π by G, we can assume that Ψ fixes $y = x$, and it follows that, as a block matrix, $\Psi = \mathrm{diag}(A, A)$, where A is some invertible matrix. Since Ψ fixes N_1, the matrix A normalizes F, just as in the proof of theorem 7 in [30]. Then, by theorem 6, this element was already considered as induced by the collineation group of Π. $\boxed{QED}$

19 Remark. If Ψ does not fix N_1. Consider the homology group $H = \Phi^{-1}H_y\Phi$. Since F and $\Phi^{-1}F\Phi$ are both fields of order q^n, then either they are the same or they share at most $q^{\frac{n}{k}}$ elements, where k is the smallest prime dividing n.

If the fields are not the same, then $|H \cap H_y| \leq q^{\frac{n}{k}}$ and this implies that the order of the homology group HH_y is at least $(q^n-1)^2/(q-1)^2 q^{\frac{n}{k}}$, which is larger than $q^n - 1$. Contradiction.

Hence, H_y is normal in the translation complement of Π.

Since Ψ fixes $x = 0$ and $y = 0$, then we can represent it as $\Phi = \mathrm{diag}(A, B)$, where $A, B \in GL(n, q)$. Moreover, the fact that Φ normalizes H_y implies that B normalizes F, and thus $B \in F$. It follows that the image of some $M \in N_1$ under Φ is $\Phi(y = xM) = (y = xA^{-1}MB)$, where $MB \in F$, and $\det MB = \det B$, then $A^{-1} = M_{q^j}\Delta_B$ for some j.

If p divided the order of Φ, then Φ^i would have order p for some i. Since the order of B divides q^n-1, then $\Phi^i = \mathrm{diag}(P, Id)$ for some matrix P. However, it is not hard to see that if $B^i = Id$, then $P = M_{q^{ij}}$, which cannot have order p.

20 Corollary. *A non-André replaced $jj\cdots j$-plane π cannot be an $SL(2,q)$-plane or a lifted derived plane.*

Proof. This follows immediately from the translation complement of π containing no p elements. $\boxed{QED}$

21 Remark. The same argument used in remark 16 shows that the transposed of a replaced $jj\cdots j$-plane cannot be an SL(2,q) plane or a lifted derived plane.

An easy corollary of the previous results summarizes this section

22 Corollary. *A non-André replaced $jj\cdots j$-plane is new, it has kernel $GF(q)$ and spread in $PG(2n-1, q)$. However, it might be isomorphic to a $jj\cdots j$-plane.*

More new planes might be obtained by transposing replaced $jj\cdots j$-planes.

Using the restriction given in lemma 10 and remark 12 we shortly survey the non-André planes obtained by net replacement on the $jj\cdots j$-planes of small order listed in section 6 of [30], paper to which the reader is referred for more information on the $jj\cdots j$-planes used here.

Planes of order 4^3 were studied in detail in [22]. In short, the non-André planes found (a $0,1$-plane and a $2,2$-plane) were constructed using the same field F, an extension of $GF(4) = GF(2)(\alpha)$ using the polynomial $p(x) = x^3 - \alpha$. Moreover, in [22] we showed that.

23 Lemma. *Let* Π *be the* $(0,1)$*-plane described above, and let* $\pi = [0,2,1]$*. Then* π *is isomorphic to the* $2,2$*-plane mentioned above.*

This lemma allows us to restrict our study non-André jj-planes of order 4^3 to the family of planes obtained from the $0,1$-plane Π.

24 Remark. Using lemma 10 and corollary 11, we can reduce the number of possible distinct isomorphism classes of replaced planes to 5; these are:

$$\begin{gathered}[0,0,0],\\ [0,1,1],\\ [0,1,2],\\ [0,2,1],\\ [0,2,2].\end{gathered}$$

In this case, it is possible to learn what the transposed planes of replaced jj-planes of order 4^3 are.

25 Lemma. *Let* π *be a transposed replaced* jj*-plane. Then,*

(1) $[0,1,1]^t \cong [0,2,2]$,

(2) $[0,2,2]^t \cong [0,1,1]$,

(3) $[0,1,2]^t \cong [0,2,1]$,

(4) $[0,2,1]^t \cong [0,1,2]$.

Note that the previous lemma says that replaced jj-planes of order 4^3 are not self-transposed.

For order 4^4 we found two more non-André jjj-planes, for either one of them, the replaced planes obtained after using lemma 10 and remark 12 are:

$$\begin{gathered}[0,0,1]\\ [0,0,2]\\ [0,0,3]\\ [0,1,2]\\ [0,2,1]\end{gathered}$$

Thus we have at most 10 non-isomorphic planes of order 4^4

26 Remark. Not much is known about the replaced planes of $jj\cdots j$-planes of orders 7^3, 3^4, and 5^4 besides the fact that the study of these planes yields dozens of new planes, even after obtaining several isomorphisms by using lemma 10 and remark 12. It is our intent to investigate these planes in the near future.

3 Derived $jj\cdots j$-planes

In this section we will study planes that can be obtained from $jj\cdots j$-planes or replaced $jj\cdots j$-planes by the process of derivation, see [24] or [4] for more information about this well-known process

A derivable net in a translation plane of order q^n needs to be of size $q^{\frac{n}{2}}+1$ and has to be coordinatized by a field. Such a derivable net can be replaced by a different type of subspaces than the ones used to replace André nets. The lines of the (new) derived plane are the lines in Π that are not in the set to replace plus a set of affine translation subplanes of Π, of order $q^{\frac{n}{2}}$, that cover the lines of the set to be replaced. These new line-sets are called Baer subplanes. Note that an affine plane of order $q^{\frac{n}{2}}$ has q^n points, just like a regular line of Π.

Any plane obtained by net derivation in a (possibly replaced) $jj\cdots j$-plane will be called 'derived $jj\cdots j$-plane'. We have found two different situations that assure the existence of these nets in (possibly replaced) $jj\cdots j$-planes.

27 Notation. For the rest of this section, let Π be a plane obtained by replacing the derivable net D in the (possibly replaced) $jj\cdots j$-plane Π_0. The derived net of D will be called D'.

A first case we will study is when the homology group H_y in a (possibly replaced) $jj\cdots j$-plane has a subgroup of size $h-1$, where $h^2 = q^n$. These conditions will be called **hypothesis H1**.

When hypothesis $H1$ holds, the orbits of lines under this group look, in some basis, like $\{\, y = xm;\ m \in GF(h)^* \,\}$. Since our plane has order h^2, each of these orbits union the lines $x = 0$ and $y = 0$ forms a derivable net (see, e.g., [11]).

28 Remark. Note that we can derive only one of these nets at a time as once one of them is derived, the lines $x = 0$ and $y = 0$ are not there anymore and thus all the other derivable nets are incomplete now. Also, note that each of these nets is contained in exactly one hyper-regulus, thus we could, after the derivation has been done, replace any of the other $q-2$ hyper-reguli that do not contain the derived net. Also, we could derive a plane that has already been replaced.

In order to know all the planes obtained by derivation on (possibly replaced) $jj\cdots j$-planes that satisfy $H1$, it is enough to consider the case of planes that have been derived after all the hyper-reguli replacements have been performed.

29 Theorem. *Hypothesis $H1$ holds only in the following cases:*

i. Π *is a plane of order* q^{2k} *where* $q = 2$ *or* 3. *In this case* $h = q^k$.

ii. Π *is a plane of order* 4^n *and* n *is odd. In this case* $h = 2^n$.

Proof. Assume $h-1$ divides $\frac{q^n-1}{q-1}$, then $q-1$ divides $\frac{q^n-1}{h-1}$. Hence, since $h^2=q^n$, we get $q-1$ divides $h+1$. Let us say that $q=p^\alpha$ and $h=p^\beta$.

If $2\beta=n\alpha$, then $\beta=k\alpha+r$, with $r<\alpha$ or $r=0$.

Assume $p^\alpha-1$ divides $p^\beta+1$. Then,

$$\begin{aligned} p^\beta+1 &= p^\beta-p^r+p^r+1 \\ &= p^r(p^{\beta-r}-1)+p^r+1 \\ &= p^r(p^{k\alpha}-1)+p^r+1 \end{aligned}$$

It follows that $p^\alpha-1$ divides p^r+1. Since $r<\alpha$ or $r=0$ we take cases.

If $r=0$ we get $p^\alpha-1$ divides 2, which forces $p^\alpha-1=1,2$. Hence, $p^\alpha=2,3$, $\alpha=1$ and $\beta=k$. This proves case i.

If $0\neq r<\alpha$, then $p^\alpha-1$ divides p^r+1 only can happen if $p^\alpha-1\leq p^r+1$. This only can occur when $p^r(p^{\alpha-r}-1)\leq 2$, then $p=2$, $r=1$ and $\alpha=1,2$. This gives us case ii. $\boxed{QED}$

30 Remark. The homology group of order $h-1$ of Π_0 becomes a Baer group of order $h-1$ in Π because the line $(y=0)$ (of Π_0), which is fixed pointwise by H_y, becomes a Baer subplane in Π.

A Baer group is a group that fixes a Baer subplane pointwise.

There is a second way for a (possibly replaced) $jj\cdots j$-plane to have a derivable net. We use a theorem by Jha and Johnson to prove the existence of these second type of derivable nets.

31 Theorem. *[Jha–Johnson [11]] Let π be a finite translation plane of order q^2 which admits a homology group H of order $q+1$. If H is cyclic then any H-orbit of components defines a derivable net.*

32 Corollary. *Let Π be a (possibly replaced) $jj\cdots j$-plane of order q^n. If $n=2k$, then the subgroup H of H_y of order q^k+1 defines q^k-1 disjoint derivable nets on Π.*

We will say these derivable nets are of 'second type'.

In this case, we can derive up to q^k-1 nets at a time. Since these nets partition the set of hyper-reguli, we could, after the derivation has been done, replace any of the hyper-reguli that do not contain the derived nets.

Note that, in order to know all the planes obtained by derivation on (possibly replaced) $jj\cdots j$-planes with derivable nets of second type, it is enough to consider the case of planes that have been derived after all the hyper-reguli replacements have been performed.

Finally, since the derivable nets of second type do not contain the axis and coaxis of H_y, then, when the order of the plane is q^{2k}, it is possible to derive (possibly replaced) $jj\cdots j$-planes using derivable nets of both types. Also, the homology group of order q^k+1 used to obtain the derivable nets is inherited by the new plane as a homology group, not as a Baer group as in the previous case.

33 Remark. Each derivable net of a $jj\cdots j$-plane has been defined using a suitable cyclic homology group, this group is inherited by the derived plane. That is, the homology group is a collineation group of the new plane.

In some cases, the inherited group is the full collineation group of a derived plane. We will now see that some derived $jj\cdots j$-planes have this property. This will depend on the order of the plane.

34 Theorem. *[Johnson–Ostrom [18]] Let Π_0 be a translation plane of order >16. Let D be a derivable net and let K be the kernel of Π_0. Let Π be the plane obtained by deriving D.*

If the Baer subplanes of D incident with the zero vector are not all K-subspaces, then the full group of Π is the inherited group.

35 Corollary. *Let Π be a plane of order q^n, with n odd, obtained by derivation on Π_0. Then, the collineation group of Π is inherited from Π_0. Thus, the collineation group of Π is given by the stabilizer of D in Π_0.*

Proof. If n is odd, then the order of the plane is 4^n and the order of the Baer subplanes that cover D is 2^n. As n is odd, the kernel of Π cannot be $GF(4)$, thus it must be $GF(2)$. The result follows from the previous theorem. $\boxed{QED}$

What follows is a sequence of results describing the structure of derived $jj\cdots j$-planes of order q^n for n odd. These results can also be generalized to any derived $jj\cdots j$-plane satisfying the hypothesis of theorem 2.

36 Lemma. *Let Π be a non-André plane of order q^n, with n odd, obtained by derivation on Π_0, a (possibly replaced) non-André $jj\cdots j$-plane. Then, Π is neither one of the planes listed in remark 3.*

Proof. Since the collineation group of Π is inherited from Π_0, it has to stabilize the derivable net. It follows that Π cannot be Desarguesian, André, triangle transitive or flag transitive (and thus nor Figueroa, see [5]). Similarly, in a semifield plane, there is an orbit of size q^n in ℓ_∞ (given by an elation group), thus a derived $jj\cdots j$-plane cannot be a semifield plane. For a similar reason, as generalized André, planes admit symmetric homology groups, Π cannot be one of these planes.

The orbits on ℓ_∞ in a Hiramine–Jha–Johnson plane are inconsistent with a derived $jj\cdots j$-plane. Also, a nearfield plane of order q^n, with $n>3$ must be André. Thus, Π is not nearfield.

As Johnson non-André hyper-regulus replacement planes have a collineation group induced from the plane they were constructed from, but related to a net of size $(q^n-1)/(q-1)$, not of size $q^{n/2}+1$ as derived planes, then these planes cannot be derived $jj\cdots j$-planes.

As the collineation group of a $jj\cdots j$-plane has no p elements, then neither does a derived $jj\cdots j$-plane, it follows that derived $jj\cdots j$-planes cannot be derived lifted planes or $SL(2,q)$-planes. A similar argument rules out Culbert–Ebert planes (see [30] to see why these planes are not $jj\cdots j$-planes).

Finally, in [21] it was shown that the derivation of a symplectic plane produces a set-transpose plane (which is a plane with spread S such that $S^t = S$). Also, a set-transpose plane admitting a homology group H with axis $\mathrm{y}=0$ and coaxis $\mathrm{x}=0$ then it must also admit a homology group with axis $\mathrm{x}=0$ and coaxis $\mathrm{y}=0$ that is isomorphic to H. Thus, if a derived $jj\cdots j$-plane is symplectic, then there is a set-transpose $jj\cdots j$-plane, which forces two symmetric homology groups of order $(q^n-1)/(q-1)$, forcing this plane to be André. $\boxed{QED}$

37 Remark. We note from Johnson [15] that the sequences of construction processes 'transpose—derive' and 'derive—transpose' produce the same plane. Hence, the transposes of two derived planes are isomorphic if and only if the transposes of the two corresponding planes from which the indicated planes are derived are isomorphic. Hence, no additional planes are obtained from the transpose of a derived plane.

Finally, note that a derived $jj\cdots j$-plane Π of order q^n, with n odd, cannot be isomorphic to a (possibly replaced) $jj\cdots j$-plane because the derived plane does not have an orbit of size q^n-1 in ℓ_∞.

Thus, putting together the results in the first two sections in tis article and the results in [30] we have shown.

38 Theorem. *The family of non-André planes of order q^n, formed by $jj\cdots j$-planes, derived $jj\cdots j$-plane, and replaced $jj\cdots j$-plane is new.*

39 Remark. If a derivable net of a plane Π can be sent to another derivable net of Π by some collineation Φ, then the corresponding derived planes are isomorphic. It is easy to see that the converse holds as well.

We close this section giving some information about certain derived $jj\cdots j$-planes of small order. These planes arise from the examples discussed in section 6 in [30].

The study of derived jj-planes of order 4^3 (see [22] for more details) yields exactly 9 non-isomorphic derived jj-planes of order 4^3.

Also, these planes are new, their full collineation groups are inherited from the plane they were derived from, this follows from the fact that the kernel of a derived jj-plane of order 4^3 is isomorphic to $GF(2)$, not $GF(4)$.

Planes of order 7^3 admit derivable nets of both types. On the contrary, planes of order 4^4 and 5^4 admit derivable nets of second type only.

The myriad of planes obtained by derivation will be studied in a future article.

4 One-half j-planes

In this section we will discuss a construction method that might produce unknown planes. The idea is to take two distinct $jj\cdots j$-planes constructed over the same field and then collect close to half of the lines in the first plane with close to half of the lines in the second plane to create a new plane.

Firstly, consider two jj-planes of order 4^3 constructed over the same field F (just like the ones found and discussed in [22]). So, let $\mathbb{D}$ be the Desarguesian plane with spread the field F. We partition this spread into nets using the determinant of its matrices, that is

$$F = N_1 \cup N_\alpha \cup N_{\alpha^2}$$

where

$$N_\theta = \{M \in F;\ \det(M) = \theta\}$$

We now consider a j_1, k_1-plane Π and a j_2, k_2-plane π. They can be represented as

$$\Pi = N_1 \cup \begin{bmatrix} 1 & & \\ & \alpha^{j_1} & \\ & & \alpha^{k_1} \end{bmatrix} N_\alpha \cup \begin{bmatrix} 1 & & \\ & \alpha^{2j_1} & \\ & & \alpha^{2k_1} \end{bmatrix} N_{\alpha^2}$$

and

$$\pi = N_1 \cup \begin{bmatrix} 1 & & \\ & \alpha^{j_2} & \\ & & \alpha^{k_2} \end{bmatrix} N_\alpha \cup \begin{bmatrix} 1 & & \\ & \alpha^{2j_2} & \\ & & \alpha^{2k_2} \end{bmatrix} N_{\alpha^2}$$

Now consider the set

$$S = N_1 \cup \begin{bmatrix} 1 & & \\ & \alpha^{j_1} & \\ & & \alpha^{k_1} \end{bmatrix} N_\alpha \cup \begin{bmatrix} 1 & & \\ & \alpha^{2j_2} & \\ & & \alpha^{2k_2} \end{bmatrix} N_{\alpha^2}$$

which is formed by the common net N_1 and one net from each plane Π and π.

Since the net N_1 belongs to both Π and π then S will define a plane iff

$$\det\left(\begin{bmatrix} 1 & & \\ & \alpha^{j_1} & \\ & & \alpha^{k_1} \end{bmatrix} M - \begin{bmatrix} 1 & & \\ & \alpha^{2j_2} & \\ & & \alpha^{2k_2} \end{bmatrix} N \right) \neq 0$$

for all $M, N \in F$ with $\det(M) = \alpha$ and $\det(N) = \alpha^2$.

But this can be re-written as

$$\det\left(Id - \begin{bmatrix}1 & & \\ & \alpha^{2j_2-j_1} & \\ & & \alpha^{2k_2-k_1}\end{bmatrix} NM^{-1}\right) \neq 0$$

for all $M, N \in F$ with $\det(M) = \alpha$ and $\det(N) = \alpha^2$.

Since $\det(NM^{-1}) = \alpha$, then the determinant above would be non-zero if there were a j,k-plane constructed over the same field F, with $j = 2j_2 - j_1$ and $k = 2k_2 - k_1$.

Now consider the j,k-planes of order 4^3 with $(j,k) = (0,1), (1,0), (1,2), (2,1)$ $(2,2)$ and $(0,0)$ (a Desarguesian plane). Then the following table (computed mod 3, as we are dealing with exponents in a field of order 4)

(j_1, k_1)	(j_2, k_2)	$(2j_2 - j_1, 2k_2 - k_1)$
$(0,1)$	$(1,0)$	$(2,2)$
$(0,1)$	$(2,2)$	$(1,0)$
$(1,0)$	$(2,2)$	$(0,1)$
$(1,2)$	$(2,1)$	$(0,0)$

yields four one-half planes.

So, we have constructed a few planes, but they are not new as the matrix M_4, needed to replace André nets (see section 2) is

$$M_4 = \begin{bmatrix}1 & & \\ & \alpha & \\ & & \alpha^2\end{bmatrix}$$

Thus, all planes found here are replaced $jj\cdots j$-planes.

We now consider $j,0,j$-planes of order 5^4 (these examples have been discussed in [30]) with field F given by an extension of $GF(5)$ by a root α of $p(x) = x^4 - 2x^2 - 2$ or $q(x) = x^4 - x^2 - 3$. In any case, F is partitioned into nets as

$$F = N_1 \cup N_2 \cup N_4 \cup N_3$$

where the subindices indicate the determinant of the matrices in that net.

So, the putative one-half j-plane is formed from a $j,0,j$-plane Π and the $k,0,k$-plane π

$$S = N_1 \cup \begin{bmatrix}1 & & & \\ & 2^k & & \\ & & 1 & \\ & & & 2^k\end{bmatrix} N_2 \cup \begin{bmatrix}1 & & & \\ & 4^j & & \\ & & 1 & \\ & & & 4^j\end{bmatrix} N_4 \cup \begin{bmatrix}1 & & & \\ & 3^k & & \\ & & 1 & \\ & & & 3^k\end{bmatrix} N_3$$

Analyzing this set as we did with the planes of order 4^3 we get that S determines a plane if there is a $r, 0, r$-plane, with $r = 3j + 2k$ (modulo 4). But, since there are $j, 0, j$-planes for $j = 0, 1, 2, 3$ (see [30]), then all the candidates to be one-half j-planes actually are translation planes.

Note that the partition into nets is completely irrelevant for this construction, as any partition into nets from a $j, 0, j$-plane Π and a $k, 0, k$-plane π will need the existence of an $r, 0, r$-plane ($r = \alpha j + \beta k$ mod 4 for some α, β) to create a new plane. Hence, as above, since there are $j, 0, j$-planes of order 5^4 for $j = 0, 1, 2, 3$, then we always obtain a translation plane.

Whether or not there are one-half j-planes of any order is a question that is related to the existence of $jj \cdots j$-planes of any order, which is an open problem.

5 Flat flocks

We finish this article by mentioning one of the odd connections of $jj \cdots j$-planes. These planes are related to a generalization of certain partitions of Segre varieties by Veroneseans. The idea of connecting these apparently completely different ideas comes from a theorem by Johnson.

40 Theorem. *[Johnson [16]] The set of translation planes of order q^2 with spread in $PG(3, q)$ that admit cyclic affine homology groups of order $q + 1$ is equivalent to the set of flocks of a quadratic cone.*

Again, when $n = 2$, since every flock of a quadratic cone gives at least one translation plane with the required homology group, there are tremendous varieties of such translation planes. In particular 'j-planes' of order q^2 admit a cyclic collineation group of order $q^2 - 1$, of which there is an affine homology subgroup of order $q + 1$. Hence, in particular, j-planes correspond to flocks of quadratic cones.

Hence, as $jj \cdots j$-planes and replaced $jj \cdots j$-planes of order q^n that admit an affine homology group of order $(q^n - 1)/(q - 1)$, for $n > 2$, then we will try to connect them with some type of partition that reminds of a flock.

Recently, Bader, Cossidente and Lunardon [1, 2] have generalized the idea of a flock of a hyperbolic quadric of $PG(3, q)$ to flat flocks of the Segre variety $\mathcal{S}_{n,n}$. They also provided an equivalence between flat flocks and the class of translation planes that admit an (A, B)-regular spread.

The following two definitions may be found in [10, Chapter 25].

41 Definition. Consider two projective spaces $PG(n_1, K)$ and $PG(n_2, K)$ with $n_i \geq 1$.

Let η be a bijection between $\{0, 1, \ldots, n_1\} \times \{0, 1, \ldots, n_2\}$ and $\{0, 1, \ldots, m\}$, with $m + 1 = (n_1 + 1)(n_2 + 1)$.

Then the Segre variety of the 2 given projective spaces is the variety $\mathcal{S}_{n_1,n_2}$ given by

$$\Big\{(x_0, x_1, \ldots, x_m);\ x_{\eta(i_1,i_2)} = x_{i_1}^{(1)} x_{i_2}^{(2)} \text{ with } (x_0^{(i)}, x_1^{(i)}, \ldots, x_{n_i}^{(i)}) \in PG(n_i, K)\Big\}$$

of $PG(m, K)$.

42 Definition. The Veronesean variety of all quadrics of $PG(n, K)$, $n \geq 1$, is the variety $\mathcal{V}_n$ given by

$$\{(x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, \ldots, x_0x_n, x_1x_2, \ldots, x_{n-1}x_n);\ (x_0, x_1, \ldots, x_n) \in PG(n, K)\}$$

of $PG(N, K)$ with $N = n(n + 3)/2$.

43 Definition. A flock of $\mathcal{S}_{n,n}$ is a partition of it into caps of size $(q^n - 1)/(q - 1)$.

If the caps are Veronesean varieties obtained as sections of $\mathcal{S}_{n,n}$ by linear subspaces of the projective space $PG(n^2+2n, q)$ in which $\mathcal{S}_{n,n}$ resides, then the flock is called a flat flock.

The flat flock is linear if all the subspaces of its Veronesean members share an n-dimensional subspace of $PG(n^2 + 2n, q)$.

44 Remark. The smallest Segre variety $\mathcal{S}_{n,n}$ is $Q^+(3, q) = \mathcal{S}_{1,1}$ and the smallest Veronesean variety is $\mathcal{V}_1$, an oval in $PG(2, q)$. This explains why flat flocks can be considered as a generalization of flocks of hyperbolic quadrics in $PG(3, q)$.

45 Definition. Let A and B be members of a spread S of $PG(2n + 1, q)$. We say S is (A, B)-regular if for every component $C \in S \setminus (A, B)$, the regulus generated by $\{A, B, C\}$ is contained in S.

46 Theorem. *[Bader, Cossidente, Lunardon [2]] Flat flocks of $\mathcal{S}_{n,n}$ and (A, B)-regular spreads in $PG(2n + 1, q)$ are equivalent. Moreover, the Veronese varieties correspond to $GF(q)$-reguli.*

47 Definition. Let R be a net of degree $q + 1$ corresponding to a partial spread in $PG(2n + 1, K)$, where $K \cong GF(q)$.

i. If R contains a Desarguesian subplane of order q, R is said to be a "rational net". The associated partial spread is called a "rational partial spread".

ii. If R is a rational net that may be embedded in a Desarguesian affine plane, the partial spread is called a "rational Desarguesian net". The associated partial spread is called a "rational Desarguesian partial spread".

A 'hyperbolic cover of order q' of a spread S in $PG(2n + 1, K)$ is a set of $(q^{n+1} - 1)/(q - 1)$ rational Desarguesian partial spreads each of degree $q + 1$ that share two components of S and whose union is S.

If the rational Desarguesian partial spreads are all K-reguli, we call the hyperbolic cover a 'regulus hyperbolic cover'.

48 Theorem. *[Jha–Johnson [12]] Flat flocks of $\mathcal{S}_{n,n}$ are equivalent to translation planes of order q^{n+1} that admit a regulus hyperbolic cover.*

Some examples of flat flocks may be found in [2], [12] and [13]. These flat flocks are related to planes that are Desarguesian, semifield, regular nearfield $N(n+1,q)$ or André.

49 Corollary. *Every $jj\cdots j$-plane of order q^n induces a flat flock. Also, when $q-1$ divides n, every replaced $jj\cdots j$-plane induces a flat flock.*

Proof. The homology group H_x (of order $q-1$) induces a regulus hyperbolic cover of the plane.

Since replaced $jj\cdots j$-planes that are not $jj\cdots j$-planes do not admit the homology group H_x, then we use that $q-1$ divides n to get:

$$gcd(q-1,(q^n-1)/(q-1)) = gcd(q-1,n) = q-1$$

Thus, the homology subgroup of H_y of order $q-1$ induces the desired regulus hyperbolic cover of the plane. $\boxed{QED}$

50 Remark. Note that the non-André $jj\cdots j$-planes induce new flat flocks. The ones induced by André $jj\cdots j$-planes have been studied in [2]. However, there might be some kind of isomorphism between these new flocks and the ones already studied.

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