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On compact symmetric spaces associated to the exceptional Lie group E_6

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Abstract. We discuss cohomogeneity one isometric actions on the exceptional compact symmetric spaces $E_6/(SU(6) \times SU(2)/\mathbb{Z}_2)$, $E_6/(Spin(10) \times U(1)/\mathbb{Z}_4)$ and E_6/F_4. These symmetric spaces can be thought of as compact tubes, since the principal orbits of the considered isometric actions coincide with tubular hypersurfaces around totally geodesic singular orbits. We determine the radii of the tubes and the principal curvatures of the tubular hypersurfaces. Moreover, we compute the volumes of the principal orbits and the volumes of the symmetric spaces in terms of the maximal sectional curvature.

Keywords: Lie group, Riemannian symmetric space, isometric action, orbit, principal curvature, volume

MSC 2000 classification: primary 53C35, secondary 57S15, 53C40

1 Introduction

Take an isometric action $\lambda\colon F \times N \to N$ of a compact connected Lie group F on a complete Riemannian manifold N. Then the orbits of F are compact submanifolds of N. A closed connected submanifold Σ of N is called a section of λ if Σ intersects orthogonally all the orbits of F, and it is well-known that the sections are totally geodesic submanifolds of N. The isometric action λ is called hyperpolar if λ admits flat sections. Concerning discussions of hyperpolar actions on Riemannian manifolds, we refer to [4], [8] and [16].

A large class of hyperpolar actions on compact symmetric spaces has been discovered by Hermann in [11] and [12]. He has pointed out that if (G, K) is a Riemannian symmetric pair of compact type and F is another symmetric subgroup of G, then the natural action of F on G/K is hyperpolar. In the paper [9], Heintze, Palais, Terng and Thorbergsson gave some criterions for an isometric action to be hyperpolar, and these imply that the cohomogeneity one actions always admit flat sections. Then, the hyperpolar actions on irreducible compact symmetric spaces have been classified by Kollross in [15].

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The purpose of this paper is to discuss in detail cohomogeneity one isometric actions on three compact symmetric spaces associated to the exceptional Lie group E_6 and to show that each of them has got a well-defined tubular structure around a totally geodesic submanifold. A similar study of the non-exceptional irreducible compact symmetric spaces was presented in [17]. In Section 2, we summarize some basic results about cohomogeneity one actions on compact symmetric spaces. In Section 3, by means of the root space decomposition of the Lie algebra $\mathfrak{e}_6$ we construct some commuting involutions of E_6. These involutive automorphisms yield the four symmetric subgroups of the Lie group E_6.

In Sections 4 and 5, we study the isometric actions of the symmetric subgroup F_4 on the symmetric spaces $E_6/(SU(6)\times SU(2)/\mathbb{Z}_2)$ and $E_6/(Spin(10)\times U(1)/\mathbb{Z}_4)$. We prove that the principal orbits of these actions coincide with the tubular hypersurfaces around the totally geodesic orbits $F_4/(Sp(3)\times Sp(1)/\mathbb{Z}_2)$ and $F_4/Spin(9)$, respectively. This implies that the symmetric spaces above can be thought of as compact tubes. We determine the radii of the tubes and the principal curvatures of the principal orbits in terms of the maximal sectional curvature. Since the principal curvatures of the tubular orbits can explicitly be expressed, we compute the volumes of the principal orbits using some results of the paper [6] by Gray and Vanhecke. Hence, we obtain a simple method to calculate the volumes of the ambient symmetric spaces. We mention that using a quite different technique, Abe and Yokota also determined the volumes of these compact symmetric spaces in [1] and [2].

In Section 6, similar results are obtained for the isometric action of the Lie group $(SU(6) \times SU(2))/\mathbb{Z}_2$ on the symmetric space E_6/F_4. The orbits of this action are tubular surfaces around the totally geodesic orbit $SU(6)/Sp(3)$.

Throughout this paper $\langle\ ,\ \rangle$ denotes the Riemannian metric of the considered compact symmetric space N. The exponential map defined on the tangent bundle TN will be denoted by Exp and the Riemannian curvature tensor by R. κ will denote the maximal sectional curvature of N. We refer to [10] for basic facts on Lie groups and symmetric spaces. Concerning submanifolds, the basic concepts can be found in the books [7] and [14]. We always take the inherited Riemannian metrics on the submanifolds. If M is a submanifold of N, then the normal vector bundle of M will be denoted by $\nu(M)$. The restriction of Exp to the normal bundle $\nu(M)$ will be denoted by Exp_M.

2 Cohomogeneity one isometric actions on compact symmetric spaces

In this section, we recall some basic results concerning cohomogeneity one isometric actions on compact symmetric spaces. For details and proof we refer to [17].

Let (G, K) be a Riemannian symmetric pair of compact type, where the Lie group G is simply connected. Then there is a unique involution σ of G such that the subgroup K coincides with $G_\sigma = \{\, g \in G \mid \sigma(g) = g \,\}$. The tangent linear map $T_e\sigma$ of σ at the identity element e is an involution of the Lie algebra $\mathfrak{g}$ of G. Hence, the ± 1-eigenspaces of $T_e\sigma$ induce the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K.

Take the coset space $N = G/K$ and the smooth action $\lambda : G \times G/K \to G/K$ defined by $\lambda(g, hK) = ghK$ for $g,\ h \in G$. Select the point $o = eK$. We identify the tangent space T_oN of N at o with the subspace $\mathfrak{p}$ in the usual way. As well-known, the Killing form B of $\mathfrak{g}$ is negative definite. Endow $N = G/K$ with the Riemannian metric $\langle\ ,\ \rangle$ for which the above action λ is isometric and

$$\langle v_1, v_2 \rangle = -c^2 \cdot B(v_1, v_2) \tag{1}$$

holds for $v_1,\ v_2 \in \mathfrak{p}$ with a fixed positive number c. Then $N = G/K$ presents a simply connected compact symmetric space. Moreover, the Riemannian curvature tensor R at o satisfies

$$R(v_1, v_2)v_3 = -\big[[v_1, v_2], v_3\big] \tag{2}$$

for all $v_1,\ v_2,\ v_3 \in \mathfrak{p}$, where $[\ ,\ ]$ denotes the bracket operation in $\mathfrak{g}$.

Let ϱ be an involutive automorphism of G different from σ such that ϱ and σ commute. Take the symmetric subgroup $F = G_\varrho = \{\, g \in G \mid \varrho(g) = g \,\}$. Since G is simply connected, F is a connected compact Lie group. We note that the natural left action of F on N is called a Hermann action.

Consider now the other symmetric subgroup $H = \{\, g \in G \mid \varrho\sigma(g) = g \,\}$ defined by the involution $\varrho\sigma$ and the isometric action of H on N. Then the following statement is true.

1 Theorem. *The orbits $F(o)$ and $H(o)$ are totally geodesic submanifolds of N. If Σ is a maximal dimensional flat torus of the symmetric space $H(o)$ and Σ contains o, then Σ is a section of the isometric action $\lambda\colon F \times N \to N$.*

The above theorem implies that the rank of the symmetric space $H(o)$ is equal to the codimension of the principal orbits of F.

In what follows, we assume that $H(o)$ is a symmetric space of rank one and the codimension of the submanifold $F(o)$ in N is greater than one. We discuss the cohomogeneity one action $\lambda : F \times N \to N$, where $F(o)$ is a singular orbit.

Take the normal vector bundle $\nu(M)$ of the totally geodesic submanifold $M = F(o)$. Introduce the notation $M^t = \{\, \mathrm{Exp}_M(w) \mid w \in \nu(M),\ \|w\| = t \,\}$ for $t \geq 0$ and call M^t a tubular surface around M. For any $g \in F$, let λ_g denote the isometry of N defined by $\lambda_g(hK) = ghK$ for $hK \in N$. Applying geodesics which emanate perpendicularly from $M = F(o)$, we obtain that $\lambda_g \circ \mathrm{Exp}_M = \mathrm{Exp}_M \circ T\lambda_g$ is valid for all $g \in F$. Since $H(o)$ is a symmetric space of rank one, F acts transitively on the set of the unit vectors in $\nu(M)$. These facts imply that *the orbits of F are tubular surfaces around M.*

Take the orthogonal complementary subspaces T_oM and ν_oM in T_oN. Notice that $\nu_oM = T_oH(o)$ holds. Let u be a fixed unit vector of the normal subspace ν_oM and let $\gamma : \mathbb{R} \to N$ be the closed geodesic defined by $\gamma(0) = o$ and $\dot\gamma(0) = u$. Then $C = \gamma(\mathbb{R})$ intersects orthogonally all the orbits of F.

As well-known, the Jacobi operator $R_u : T_oN \to T_oN$ with respect to u is defined by the relation $R_u(v) = R(v,u)u$ for $v \in T_oN$. Theorem 1 implies that T_oM and ν_oM are invariant subspaces of the self-adjoint endomorphism R_u. Let a be the maximal eigenvalue of R_u on T_oM and let h denote the arc length of C. We need the number $r = \min \{ \pi/(2\sqrt{a}), \quad h/2 \}$, and the open tubular domain $\nu^r(M) = \{ w \in \nu(M) \mid \|w\| < r \}$ in $\nu(M)$. Then we can state the following theorem, which verifies that the symmetric space N can be thought of as a compact tube around $M = F(o)$ with radius r.

2 Theorem. *The restriction of the exponential map* Exp_M *to* $\nu^r(M)$ *is a diffeomorphism from* $\nu^r(M)$ *to the open domain* $\mathrm{Exp}_M(\nu^r(M))$ *in* N.

$M^r = F(\gamma(r))$ *is the other singular orbit of* F*, and* $\mathrm{Exp}_M(\nu^r(M)) \cup M^r = N$ *holds.*

3 Remark. It is clear that the orbits of F are also obtained as tubular surfaces around M^r. This implies that M^r is a minimal submanifold of N.

Consider the principal orbit $M^t = F(\gamma(t))$ for some $t,\ 0 < t < r$. Let ζ be the smooth unit normal vector field on M^t for which $\zeta(\gamma(t)) = \dot\gamma(t)$ holds. Observe that ζ is invariant under the action of F. Then the shape operator A_ζ of M^t with respect to ζ has constant eigenvalues, which are called principal curvatures of the hypersurface M^t in N.

We can express the principal curvatures of the principal orbits in explicit form as follows. Take the eigenvalues a_i $(i = 1, \ldots, s)$ of the Jacobi operator R_u on the invariant subspace T_oM with the multiplicities m_i $(i = 1, \ldots, s)$. Then, denote by χ the maximal sectional curvature of the compact symmetric space $H(o)$. Since the rank of $H(o)$ equals one, the eigenvalues of R_u on the invariant subspace ν_oM are $b_1 = \chi,\ b_2 = \chi/4$ and $b_3 = 0$ provided that $H(o)$ is not of constant curvature. Hereafter, we denote by k_j the multiplicity of the eigenvalue b_j of $R_u|\nu_oM$ for $j \in \{1, 2, 3\}$.

4 Theorem. *The constant principal curvatures of the principal orbit M^t are*

$$\mu_i(t) = \sqrt{a_i}\,\tan(\sqrt{a_i}\,t) \quad \text{and} \quad \hat\mu_j(t) = -\sqrt{b_j}\,\cot(\sqrt{b_j}\ t) \tag{3}$$

$(i = 1, \ldots, s;\ j = 1, 2)$ *with the corresponding multiplicities.*

Finally, we present a formula concerning volumes of principal orbits of F. For details and proof see the paper [6] and the book [7].

Denote by k the codimension of the submanifold $M = F(o)$ in N and by $\operatorname{Tr} A_{\dot\gamma(t)}$ the trace of the shape operator $A_{\dot\gamma(t)}$ of the tubular hypersurface M^t with respect to $\dot\gamma(t)$. We need the smooth function $\vartheta\colon [0,r) \to \mathbb{R}$ which satisfies the differential equation

$$\frac{\vartheta'(t)}{\vartheta(t)} = -\frac{k-1}{t} - \operatorname{Tr} A_{\dot\gamma(t)}$$

for $t \in (0,r)$ and for which $\vartheta(0) = 1$ is valid. By Theorem 4 we obtain that

$$\vartheta(t) = 2^{k_2}\, \chi^{\frac{1-k}{2}}\, t^{1-k} \cdot \sin^{k_1}(\sqrt{\chi}\, t) \cdot \sin^{k_2}(\frac{1}{2}\sqrt{\chi}\, t) \cdot \prod_{i=1}^{s} \cos^{m_i}(\sqrt{a_i}\, t) \qquad (4)$$

holds, where $k_1 + k_2 = k - 1$. Recall that $M = F(o)$ is a compact symmetric space. Denote by $\operatorname{Vol}(M)$ the volume of M and by $\operatorname{Vol}(S^{k-1})$ the volume of the unit sphere S^{k-1} of dimension $k-1$. Then, using Lemma 3.12 in [7], we can state the following theorem.

5 Theorem. *The relation*

$$\operatorname{Vol}(M^t) = \operatorname{Vol}(M) \cdot \operatorname{Vol}(S^{k-1}) \cdot t^{k-1}\, \vartheta(t) \qquad (5)$$

holds for the volumes of the principal orbits $M^t = F(\gamma(t)),\ \ 0 < t < r.$

6 Remark. Theorem 5 implies that there is a principal orbit of F which has the maximal volume. The mean curvature vector field of this tubular hypersurface vanishes, and therefore it is a minimal submanifold of N (for proof see Theorem 1 in [13]).

3 Symmetric subgroups of the exceptional compact Lie group E_6

Let us consider the simply connected compact Lie group E_6. We refer to [18] for details about E_6 and its symmetric subgroups. In this section, we construct some commuting involutions of the Lie algebra $\mathfrak{e}_6$ of E_6 by using its root space decomposition. Therefore we obtain some involutions of E_6 which provide the four symmetric subgroups of E_6.

As usual, considering an element X of $\mathfrak{e}_6$, the endomorphism $\operatorname{ad} X\colon \mathfrak{e}_6 \to \mathfrak{e}_6$ is defined by $\operatorname{ad} X(Y) = [X,Y]$ for $Y \in \mathfrak{e}_6$, where $[\ ,\ ]$ denotes the bracket operation in $\mathfrak{e}_6$. We denote by B the Killing form of $\mathfrak{e}_6$, which is negative definite.

Select a 6-dimensional Abelian subspace $\mathfrak{g}_0$ of $\mathfrak{e}_6$. Let β be a real-valued linear form on the linear space $\mathfrak{g}_0$. Take the subspace

$$\mathfrak{g}_\beta = \{\, Y \in \mathfrak{e}_6 \;\mid\; (\operatorname{ad} X)^2(Y) = -\beta(X)^2\, Y \quad \text{for all } X \in \mathfrak{g}_0 \,\}.$$

It is clear that $\mathfrak{g}_\beta = \mathfrak{g}_{-\beta}$ is valid. β is called a root if $\beta \neq 0$ and $\mathfrak{g}_\beta \neq \{0\}$ hold, and in this case $\mathfrak{g}_\beta$ is said to be the root subspace corresponding to β. Denote by Δ the set of the 72 roots with respect to $\mathfrak{g}_0$. We refer to Chapter X in [10] for details about the root system Δ. Select a basis $\alpha_1, \dots, \alpha_6$ of Δ the Dynkin diagram of which is represented in Figure 1. Let Δ^+ denote the set of

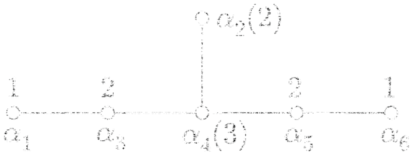

Figure 1. The Dynkin diagram of Δ with the coefficients of the highest root.

the positive roots with respect to the ordering defined by this basis. Then we obtain the root space decomposition

$$\mathfrak{e}_6 = \mathfrak{g}_0 + \textstyle\sum_{\beta \in \Delta^+} \mathfrak{g}_\beta, \tag{6}$$

which is orthogonal with respect to B. It is important to remark that the relation

$$[\mathfrak{g}_\beta, \mathfrak{g}_\delta] \subset \mathfrak{g}_{\beta+\delta} + \mathfrak{g}_{\beta-\delta} \tag{7}$$

is valid for any $\beta,\ \delta \in \mathfrak{g}_0^*$, where $\mathfrak{g}_0^*$ denotes the dual space of $\mathfrak{g}_0$.

As usual, we associate to each root $\beta \in \Delta$ the vector $H_\beta \in \mathfrak{g}_0$ for which $B(H_\beta, X) = \beta(X)$ holds for all $X \in \mathfrak{g}_0$. Then $\mathfrak{g}_\beta + \mathbb{R}H_\beta$ is a subalgebra of $\mathfrak{e}_6$ which is isomorphic to $\mathfrak{su}(2)$.

Take the basis $a_1, \dots, a_6$ of $\mathfrak{g}_0$ which is dual to the basis $\alpha_1, \dots, \alpha_6$ of $\mathfrak{g}_0^*$. Considering the parities of the integers $\beta(a_4),\ \beta \in \Delta$, we can divide Δ^+ into two disjoint subsets defined by

$$\Delta_K^+ = \{\, \beta \in \Delta^+ \mid \beta(a_4) \in 2\mathbb{Z} \,\} \quad \text{and} \quad \Delta_P^+ = \{\, \beta \in \Delta^+ \mid \beta(a_4) \in 2\mathbb{Z} + 1 \,\}.$$

Consider now the complementary subspaces

$$\mathfrak{k} = \mathfrak{g}_0 + \textstyle\sum_{\beta \in \Delta_K^+} \mathfrak{g}_\beta \qquad \text{and} \qquad \mathfrak{p} = \textstyle\sum_{\beta \in \Delta_P^+} \mathfrak{g}_\beta. \tag{8}$$

Using (7) and (8), we can show that $\mathfrak{k}$ is a subalgebra of $\mathfrak{e}_6$ and $\mathfrak{k}$ is isomorphic to $\mathfrak{su}(6)+\mathfrak{su}(2)$. In addition, the relations $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$ are valid. Hence, we find an involutive automorphism $\hat{\sigma}$ of $\mathfrak{e}_6$ defined by $\hat{\sigma}(Y+Z)=Y-Z$ for $Y\in\mathfrak{k}$ and $Z\in\mathfrak{p}$. This induces an involution $\sigma\colon E_6\to E_6$, where $\sigma\circ\exp=\exp\circ\hat{\sigma}$ holds with the exponential map $\exp\colon\mathfrak{e}_6\to E_6$. Then the subgroup $K=(E_6)_\sigma$ of the fixed elements of σ is isomorphic to $(SU(6)\times SU(2))/\mathbb{Z}_2$.

Select the element a_6 of $\mathfrak{g}_0$ and introduce the notation

$$\Delta_L^+=\{\,\beta\in\Delta^+\mid\beta(a_6)\in2\mathbb{Z}\,\},\qquad\Delta_M^+=\{\,\beta\in\Delta^+\mid\beta(a_6)\in2\mathbb{Z}+1\,\}.$$

Take the complementary subspaces

$$\mathfrak{l}=\mathfrak{g}_0+\textstyle\sum_{\beta\in\Delta_L^+}\mathfrak{g}_\beta,\qquad\mathfrak{m}=\textstyle\sum_{\beta\in\Delta_M^+}\mathfrak{g}_\beta\tag{9}$$

of $\mathfrak{e}_6$. In virtue of (7) and (9), it can be seen that the relations $[\mathfrak{l},\mathfrak{l}]\subset\mathfrak{l}$, $[\mathfrak{l},\mathfrak{m}]\subset\mathfrak{m}$ and $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{l}$ hold, and the Lie algebra $\mathfrak{l}$ is isomorphic to $\mathfrak{so}(10)+\mathbb{R}$. Therefore we find an involution $\hat{\tau}\colon\mathfrak{e}_6\to\mathfrak{e}_6$ defined by $\hat{\tau}(Y+Z)=Y-Z$ for $Y\in\mathfrak{l}$ and $Z\in\mathfrak{m}$. As in the previous case, $\hat{\tau}$ yields an involutive automorphism τ of the Lie group E_6, and the symmetric subgroup $L=(E_6)_\tau$ is isomorphic to $(Spin(10)\times U(1))/\mathbb{Z}_4$.

Let us take the 4-dimensional subspace $A=\sum_{j=2}^5\mathbb{R}\alpha_j$ of $\mathfrak{g}_0^*$, and denote by $A^\perp$ its orthogonal complement in $\mathfrak{g}_0^*$. Let Δ_A^+ be the set of those positive roots which are contained in A. This means that each element β of Δ_A^+ can be expressed in the form $\beta=\sum_{j=2}^5n_j\alpha_j$, where $n_j\in\mathbb{Z}$ and $n_j\geq0$. Consider now the subspace $\mathfrak{a}_0=\sum_{j=2}^5\mathbb{R}H_{\alpha_j}$ of $\mathfrak{g}_0$, and denote by $\mathfrak{a}_0^\perp$ the orthogonal complement of $\mathfrak{a}_0$ in $\mathfrak{g}_0$. It is easy to show that $\mathfrak{a}=\mathfrak{a}_0+\sum_{\beta\in\Delta_A^+}\mathfrak{g}_\beta$ presents a subalgebra of $\mathfrak{e}_6$ which is isomorphic to $\mathfrak{so}(8)$.

Let $r\colon\mathfrak{g}_0^*\to\mathfrak{g}_0^*$ be the orthogonal reflection of $\mathfrak{g}_0^*$ on the two-dimensional subspace $A^\perp$. It can be seen that

$$r(\alpha_1)=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5,\qquad r(\alpha_6)=\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$$

and $r(\alpha_j)=-\alpha_j$ $(j=2,3,4,5)$ hold. Using the notation $\Delta_R^+=\Delta^+\setminus\Delta_A^+$, we obtain that $r(\Delta_R^+)=\Delta_R^+$. Since the roots β and $r(\beta)$ are orthogonal for $\beta\in\Delta_R^+$, the relation $[\mathfrak{g}_\beta,\mathfrak{g}_{r(\beta)}]=0$ holds provided that $\beta\in\Delta_R^+$. Then, this reflection r induces an involution $\hat{\varrho}\colon\mathfrak{e}_6\to\mathfrak{e}_6$ such that the following assertions are true:

(1) The Lie algebra of the elements left fixed by $\hat{\varrho}$ is isomorphic to $\mathfrak{f}_4$.

(2) $\hat{\varrho}(Y)=Y$ and $\hat{\varrho}(Z)=-Z$ hold for all $Y\in\mathfrak{a}$ and $Z\in\mathfrak{a}_0^\perp$.

(3) $\hat{\varrho}(\mathfrak{g}_\beta)=\mathfrak{g}_{r(\beta)}$ is valid for each $\beta\in\Delta_R^+$.

Hereafter, we denote by $\mathfrak{f}_4$ the subalgebra of the fixed elements of $\hat{\varrho}$. Take the involutive automorphism ϱ of E_6 determined by $\hat{\varrho}$. Clearly, the subgroup $(E_6)_\varrho$ is isomorphic to the exceptional compact Lie group F_4.

The involution $\hat{\varrho}$ yields the decomposition $\mathfrak{e}_6 = \mathfrak{f}_4 + \mathfrak{n}$, where $\mathfrak{n} = \{\, Z \in \mathfrak{e}_6 \mid \hat{\varrho}(Z) = -Z \,\}$. Since $\mathfrak{g}_\beta + \mathfrak{g}_{r(\beta)}$ is an invariant subspace of $\hat{\varrho}$, it is reasonable to introduce the notation $\mathfrak{q}_{\beta,r(\beta)} = \{\, Y \in \mathfrak{g}_\beta + \mathfrak{g}_{r(\beta)} \mid \hat{\varrho}(Y) = Y \,\}$ and $\mathfrak{q}^{\perp}_{\beta,r(\beta)} = \{\, Z \in \mathfrak{g}_\beta + \mathfrak{g}_{r(\beta)} \mid \hat{\varrho}(Z) = -Z \,\}$. Then $\mathfrak{f}_4$ and $\mathfrak{n}$ can be expressed as the direct sums of orthogonal subspaces in the following way:

$$\mathfrak{f}_4 = \mathfrak{a}_0 + \textstyle\sum_{\beta\in\Delta_A^+} \mathfrak{g}_\beta + \sum_{\beta\in\Delta_R^+} \mathfrak{q}_{\beta,r(\beta)}, \qquad \mathfrak{n} = \mathfrak{a}_0^{\perp} + \sum_{\beta\in\Delta_R^+} \mathfrak{q}^{\perp}_{\beta,r(\beta)}. \tag{10}$$

We remark that for each $\beta \in \Delta_R^+$, the subspace $\mathfrak{q}_{\beta,r(\beta)} + \mathbb{R}(H_\beta + H_{r(\beta)})$ is a subalgebra of $\mathfrak{e}_6$ which is isomorphic to $\mathfrak{su}(2)$, and $\mathfrak{q}^{\perp}_{\beta,r(\beta)}$ is a Lie triple system.

Observe that $\hat{\varrho}(\mathfrak{k}) = \mathfrak{k}$ and $\hat{\varrho}(\mathfrak{p}) = \mathfrak{p}$ hold. It follows from this that the involutions $\hat{\varrho}$ and $\hat{\sigma}$ commute. Consider now the symmetric subalgebra $\mathfrak{h} = \{\, Y \in \mathfrak{e}_6 \mid \hat{\varrho}\hat{\sigma}(Y) = Y \,\}$. Using the relation $\mathfrak{h} = \mathfrak{k} \cap \mathfrak{f}_4 + \mathfrak{p} \cap \mathfrak{n}$ and the above decompositions, we obtain that $\mathfrak{h}$ is isomorphic to the Lie algebra $\mathfrak{sp}(4)$. $\hat{\varrho}\hat{\sigma}$ induces the involutive automorphism $\varrho\sigma$ of E_6, and the subgroup H of the fixed elements of $\varrho\sigma$ is isomorphic to $Sp(4)/\mathbb{Z}_2$.

Finally, we can show that $\hat{\tau}$ commutes with $\hat{\varrho}$ and $\hat{\sigma}$. Consequently, $\varrho\tau$ and $\sigma\tau$ are also involutions of E_6. It can be seen that the symmetric subgroups $(E_6)_{\varrho\tau}$ and $(E_6)_{\sigma\tau}$ are isomorphic to F_4 and $(SU(6) \times SU(2))/\mathbb{Z}_2$, respectively.

4 The isometric action of F_4 on $E_6/(SU(6)\times SU(2)/\mathbb{Z}_2)$

Let us consider the compact symmetric space E_6/K, where the symmetric subgroup K defined by the involution σ is isomorphic to $(SU(6) \times SU(2))/\mathbb{Z}_2$. In this section, we also denote the symmetric space E_6/K by N. Using the relations (2) and (8), it can be seen that the maximal sectional curvature κ of E_6/K is equal to $1/(c^2 \cdot 12)$. We mention that the value of κ can also be obtained by means of Theorem 11.1 of Chapter VII in [10].

Take the symmetric subgroups $F_4 = (E_6)_\varrho$ and $H = (E_6)_{\varrho\sigma}$ and their natural isometric actions on $N = E_6/K$. Recall that H is isomorphic to the Lie group $Sp(4)/\mathbb{Z}_2$. Concerning the totally geodesic orbits of the point $o = eK$ under these actions, we can state the following assertion.

7 Proposition. *$F_4(o)$ is isometric to the symmetric space $F_4/(Sp(3) \times Sp(1)/\mathbb{Z}_2)$ with maximal sectional curvature κ, and $H(o)$ is isometric to the quaternion projective space $\mathbb{H}P^3$ with maximal sectional curvature $\kappa/2$.*

Proof. It is clear that the isotropy subgroup of F_4 at o coincides with $F_4 \cap K$. Since ϱ and σ commute, it can be seen that $\sigma(F_4) = F_4$ and $F_4 \cap K = \{ g \in F_4 \mid \sigma(g) = g \}$ are valid. This implies that $F_4 \cap K$ is a symmetric subgroup of F_4 and the orbit $F_4(o)$ is isometric to the symmetric space $F_4/(F_4 \cap K)$. By the decompositions (8) and (10) we obtain that $\mathfrak{f}_4 \cap \mathfrak{k}$ is isomorphic to the Lie algebra $\mathfrak{sp}(3) + \mathfrak{sp}(1)$. Hence, $F_4 \cap K$ is isomorphic to $(Sp(3) \times Sp(1))/\mathbb{Z}_2$.

Consider the exponential map $\mathrm{Exp}_o : T_oN \to N$ on the tangent space T_oN of N at o, which is identified with the subspace $\mathfrak{p}$ of $\mathfrak{e}_6$. Then the totally geodesic orbit $F_4(o)$ coincides with $\mathrm{Exp}_o(\mathfrak{f}_4 \cap \mathfrak{p})$. The Lie triple system $\mathfrak{f}_4 \cap \mathfrak{p}$ contains the two-dimensional root subspace $\mathfrak{g}_{\alpha_4}$ of $\mathfrak{e}_6$, and it follows from this that the maximal sectional curvature of $F_4(o)$ equals κ.

Take the other totally geodesic orbit $H(o) = \mathrm{Exp}_o(\mathfrak{h} \cap \mathfrak{p})$. Since the relation $\mathfrak{h} \cap \mathfrak{k} = \mathfrak{f}_4 \cap \mathfrak{k}$ holds, the isotropy subgroup $H \cap K$ of H at o is also isomorphic to $(Sp(3) \times Sp(1))/\mathbb{Z}_2$. Consequently, the symmetric space $H(o) = H/(H \cap K)$ is isometric to $\mathbb{H}P^3$. We can verify that there is no root subspace of $\mathfrak{e}_6$ which is included in the Lie triple system $\mathfrak{h} \cap \mathfrak{p}$. On the other hand, select the root $\beta = \alpha_1 + \alpha_3 + \alpha_4$ which is contained in $\Delta_P^+ \cap \Delta_R^+$. Then the two-dimensional subspace $\mathfrak{q}^{\perp}_{\beta,r(\beta)}$ is a Lie triple system included in $\mathfrak{h} \cap \mathfrak{p}$, and the surface $\mathrm{Exp}_o(\mathfrak{q}^{\perp}_{\beta,r(\beta)})$ is a totally geodesic submanifold of N. It can be seen that the sectional curvature with respect to the plane $\mathfrak{q}^{\perp}_{\beta,r(\beta)}$ equals $\kappa/2$. This implies that the maximal sectional curvature of $H(o)$ is $\kappa/2$. $\boxed{QED}$

Hereafter, we study the isometric action $\lambda\colon F_4 \times N \to N$ of F_4 on N. It follows from Theorem 1 and Proposition 7 that the cohomogeneity of λ is equal to one. Hence, the orbits of F_4 are tubular surfaces around the totally geodesic submanifold $M = F_4(o)$.

Take the orthogonal decomposition $T_oN = T_oM + \nu_oM$, where $\nu_oM = T_oH(o)$. Fix a unit vector u of the subspace ν_oM and take the closed geodesic $\gamma : \mathbb{R} \to N$ defined by $\gamma(t) = \mathrm{Exp}_o(tu)$. Then the circle $C = \gamma(\mathbb{R})$ intersects orthogonally all the orbits of F.

Consider now the Jacobi operator $R_u : T_oN \to T_oN$. Since $H(o)$ is isometric to $\mathbb{H}P^3$ with maximal sectional curvature $\kappa/2$, the eigenvalues of the restriction of R_u to ν_oM (and their multiplicities) are

$$b_1 = \kappa/2 \ \ (k_1 = 3), \quad b_2 = \kappa/8 \ \ (k_2 = 8), \quad b_3 = 0 \ \ (k_3 = 1). \tag{11}$$

In order to describe the shape operators of the principal orbits and to compute their volumes we need the eigenvalues of the other restricted endomorphism $R_u|T_oM$.

8 Proposition. *The eigenvalues of R_u on the invariant subspace T_oM (and their multiplicities) are*

$$a_1 = \kappa/2 \ \ (m_1 = 5), \quad a_2 = \kappa/8 \ \ (m_2 = 8), \quad a_3 = 0 \ \ (m_3 = 15). \tag{12}$$

Proof. We shall apply the restricted root space decomposition of $\mathfrak{p} = T_oN$ (for details see Chapter VII in [10]). As well-known, the rank of E_6/K equals 4. Let $\mathfrak{p}_0$ be a four-dimensional Abelian subspace of $\mathfrak{p}$. Take a real-valued linear form α on $\mathfrak{p}_0$ and the subspace $\mathfrak{p}_\alpha$ defined by

$$\mathfrak{p}_\alpha = \{\, Y \in \mathfrak{p} \mid (\operatorname{ad} X)^2(Y) = -\alpha(X)^2\, Y \text{ for all } X \in \mathfrak{p}_0 \,\}.$$

Recall that α is a restricted root if $\alpha \neq 0$ and $\mathfrak{p}_\alpha \neq \{0\}$ hold, and the dimension of $\mathfrak{p}_\alpha$ is called the multiplicity of α. Let $\mathcal{R}$ denote the set of the restricted roots. We can take the orthogonal decomposition $\mathfrak{p} = \mathfrak{p}_0 + \sum_{\alpha\in\mathcal{R}^+} \mathfrak{p}_\alpha$, where $\mathcal{R}^+$ denotes the set of the positive restricted roots with respect to an ordering in the dual space of $\mathfrak{p}_0$. Using the Killing form B of $\mathfrak{e}_6$, we associate to each $\alpha \in \mathcal{R}$ the vector $X_\alpha \in \mathfrak{p}_0$ for which $B(X_\alpha, X) = \alpha(X)$ holds for all $X \in \mathfrak{p}_0$. It is well-known that the restricted root system $\mathcal{R}$ of E_6/K is of type $\mathcal{F}_4$ (for details see Chapter X in [10]). Hence, the elements of $\mathcal{R}^+$ are ε_i $(1 \le i \le 4)$, $\varepsilon_j \pm \varepsilon_k$ $(1 \le j < k \le 4)$ and $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$, where the roots ε_i $(1 \le i \le 4)$ are orthogonal and all the combinations of signs are admissible. The multiplicity of the long roots equals one, and the multiplicity of the short roots equals two.

Notice that $R_u = -(\operatorname{ad} u)^2$ holds for the Jacobi operator R_u. Since the arc length of C equals $(2\sqrt{2}\,\pi)/\sqrt{\kappa}$, we assume that u is parallel to the root vector X_{ε_1}. In this case $\mathfrak{p}_{\varepsilon_1}$ and $\mathfrak{p}_{\varepsilon_1\pm\varepsilon_j}$ $(j = 2, 3, 4)$ are eigenspaces of R_u with the eigenvalue $\kappa/2$. This means that $\kappa/2$ is an eigenvalue of R_u with the multiplicity 8. Analogously, considering the two-dimensional eigenspaces $\mathfrak{p}_{\frac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)}$, we obtain that $\kappa/8$ is another eigenvalue of R_u with the multiplicity 16. It is clear that R_u vanishes on the other root subspaces. Then, the relation (11) presenting the multiplicities of the eigenvalues of $R_u|\nu_oM$ implies that Proposition 8 is true. QED

9 Remark. By Theorem 2 the symmetric space $N = E_6/K$ can be thought of as a compact tube around $M = F_4(o)$. It follows from Proposition 8 that the radius of this tube is $r = \pi/\sqrt{2\kappa}$. Moreover, by means of the relations (11), (12) and (3) we get all the principal curvatures of the principal orbits M^t, $0 < t < r$.

Since $M = F_4(o)$ is isometric to $F_4/(Sp(3) \times Sp(1)/\mathbb{Z}_2)$,

$$\operatorname{Vol}(M) = \frac{2^{23}}{3^5 \cdot 5^3 \cdot 7^2 \cdot 11}\,\pi^{14}\,\kappa^{-14}$$

is valid (see [5] for proof). Considering the well-known equality $\operatorname{Vol}(S^{11}) = \frac{1}{60}\pi^6$, Theorem 5 implies the following statement.

10 Corollary. *The relation*

$$\operatorname{Vol}(M^t) = \operatorname{Vol}(M) \cdot \tfrac{1}{60}\pi^6 \cdot (\kappa/2)^{-\frac{11}{2}} \cdot \sin^{11}(\sqrt{\kappa/2}\ t)\cos^5(\sqrt{\kappa/2}\ t)$$

holds for the volumes of the principal orbits $M^t = F_4(\gamma(t)), \quad 0 < t < r.$

Using Theorem 2 and Corollary 10, we obtain that

$$\mathrm{Vol}(N) = \int_0^r \mathrm{Vol}(M^t)\,dt = \frac{2^{23}}{3^7 \cdot 5^4 \cdot 7^3 \cdot 11}\,\pi^{20}\,\kappa^{-20}$$

holds for the volume of the compact symmetric space $N = E_6/K$.

5 The isometric action of F_4 on $E_6/(Spin(10)\times U(1)/\mathbb{Z}_4)$

Recall that the symmetric subgroup L defined by the involution τ is isomorphic to $(Spin(10) \times U(1))/\mathbb{Z}_4$. Consider now the compact symmetric space $N = E_6/L$ of rank two. Then the tangent space T_oN of N at the point $o = eL$ is identified with the subspace $\mathfrak{m}$ of $\mathfrak{e}_6$. By means of (2) and (9) it can be seen that the maximal sectional curvature κ of $N = E_6/L$ is equal to $1/(c^2 \cdot 12)$.

Take the symmetric subgroups $F_4 = (E_6)_\varrho$ and $H = (E_6)_{\varrho\tau}$. Using the decompositions (9) and (10), we obtain that in this case H is isomorphic to F_4. We can characterize the orbits of the point o under the isometric actions of F_4 and H on N.

11 Proposition. *The totally geodesic orbits* $F_4(o)$ *and* $H(o)$ *are isometric to the Cayley projective plane* $\mathbb{O}P^2$ *with maximal sectional curvature* $\kappa/2$.

Proof. As in the proof of Proposition 7, first we take the isotropy subgroup $F_4 \cap L$ of F_4 at o. Since the involutions ϱ and τ commute, $\tau(F_4) = F_4$ and $F_4 \cap L = \{\, g \in F_4 \mid \tau(g) = g \,\}$ are valid. Consequently, $F_4 \cap L$ is a symmetric subgroup of F_4 and the orbit $F_4(o)$ is isometric to the symmetric space $F_4/(F_4 \cap L)$. The relations (9) and (10) imply that $\mathfrak{f}_4 \cap \mathfrak{l}$ is isomorphic to the Lie algebra $\mathfrak{so}(9)$. It follows from this that $F_4 \cap L$ is isomorphic to $Spin(9)$. Therefore we obtain that $F_4(o)$ coincides with the Cayley projective plane $\mathbb{O}P^2$.

The tangent space $T_oF_4(o)$ of the totally geodesic orbit $F_4(o)$ at o equals the Lie triple system $\mathfrak{f}_4 \cap \mathfrak{m}$. However, there is no root subspace of $\mathfrak{e}_6$ which is contained in $\mathfrak{f}_4 \cap \mathfrak{m}$. On the other hand, $\mathfrak{q}_{\alpha_6, r(\alpha_6)}$ presents a two-dimensional Lie triple system which is included in $\mathfrak{f}_4 \cap \mathfrak{m}$. The sectional curvature with respect to the plane $\mathfrak{q}_{\alpha_6, r(\alpha_6)}$ equals $\kappa/2$. Consequently, the maximal sectional curvature of the symmetric space $F_4(o)$ is $\kappa/2$.

Finally, notice that since H is isomorphic to F_4, the totally geodesic orbit $H(o)$ is isometric to $F_4(o)$. $\boxed{QED}$

In what follows, we discuss the isometric action $\lambda\colon F_4 \times N \to N$ *of* F_4 *on* N. It is clear that the principal orbits are tubular hypersurfaces around the singular orbit $M = F_4(o)$.

Fix a unit vector u of the normal subspace $\nu_o M$. We shall need the closed geodesic $\gamma : \mathbb{R} \to N$ defined by $\gamma(t) = \mathrm{Exp}_o(tu)$. Take the Jacobi operator R_u on $T_o N$. Proposition 11 implies that the eigenvalues of the restriction of R_u to $\nu_o M$ (and their multiplicities) are

$$b_1 = \kappa/2 \quad (k_1 = 7), \quad b_2 = \kappa/8 \quad (k_2 = 8), \quad b_3 = 0 \quad (k_3 = 1). \tag{13}$$

Concerning the eigenvalues of the other restricted endomorphism $R_u|T_o M$, we state the following assertion.

12 Proposition. *The eigenvalues of R_u on the subspace $T_o M$ (and their multiplicities) are*

$$a_1 = \kappa/2 \quad (m_1 = 1), \quad a_2 = \kappa/8 \quad (m_2 = 8), \quad a_3 = 0 \quad (m_3 = 7). \tag{14}$$

Proof. As in the proof of Proposition 8, we apply the restricted root space decomposition of $\mathfrak{m} = T_o N$. Take a two-dimensional Abelian subspace $\mathfrak{m}_0$ of $\mathfrak{m}$ and the set $\mathcal{R}$ of the restricted roots in the dual space of $\mathfrak{m}_0$. Considering the root subspaces $\mathfrak{m}_\alpha$, $\alpha \in \mathcal{R}$, we obtain the orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_0 + \sum_{\alpha \in \mathcal{R}^+} \mathfrak{m}_\alpha$. It is well-known that the restricted root system $\mathcal{R}$ of the symmetric space E_6/L is of type $\mathcal{BC}_2$, which is represented in Figure 2. In this case the positive roots of $\mathcal{R}$ are $\varepsilon_1 \pm \varepsilon_2$, ε_i and $2\varepsilon_i$ $(i = 1, 2)$. The multiplicities of the roots are 6, 8 and 1, respectively.

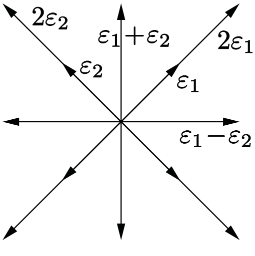

Figure 2. The root system of type $\mathcal{BC}_2$ and its positive roots.

Since the arc length of $C = \gamma(\mathbb{R})$ equals $(2\sqrt{2}\,\pi)/\sqrt{\kappa}$, we assume that u is parallel to the root vector $X_{\varepsilon_1+\varepsilon_2}$ in $\mathfrak{m}_0$. Then the root subspaces $\mathfrak{m}_{\varepsilon_1+\varepsilon_2}$, $\mathfrak{m}_{2\varepsilon_1}$ and $\mathfrak{m}_{2\varepsilon_2}$ are eigenspaces of the Jacobi operator $R_u = -(\mathrm{ad}\, u)^2$ with the eigenvalue $\kappa/2$. Consequently, $\kappa/2$ is an eigenvalue of R_u with the multiplicity 8. It is clear that the 8-dimensional subspaces $\mathfrak{m}_{\varepsilon_1}$ and $\mathfrak{m}_{\varepsilon_2}$ are also eigenspaces of R_u with the eigenvalue $\kappa/8$, and R_u vanishes on $\mathfrak{m}_{\varepsilon_1-\varepsilon_2}$. Hence, the relation

(13) which gives the multiplicities of the eigenvalues of $R_u|\nu_o M$ implies that Proposition 12 is true. $\boxed{QED}$

13 Remark. The symmetric space $N = E_6/L$ can be thought of as a compact tube around $M = F_4(o)$. Proposition 12 implies that the radius of this tube equals $r = \pi/\sqrt{2\kappa}$. Using the relations (13), (14) and (3), we obtain the principal curvatures of the tubular hypersurfaces M^t, $0 < t < r$.

Since the symmetric space $M = F_4(o)$ is isometric to the Cayley projective plane $\mathbb{O}P^2$, $\mathrm{Vol}(M) = \dfrac{2^9}{3^3 \cdot 5^2 \cdot 7 \cdot 11}\, \pi^8\, (\kappa/2)^{-8}$ holds, where κ denotes the maximal sectional curvature of N. Moreover, it follows from Theorem 5 that the following assertion is true.

14 Corollary. *Concerning the volumes of the principal orbits* $M^t = F_4(\gamma(t))$,

$$\mathrm{Vol}(M^t) = \mathrm{Vol}(M) \cdot \mathrm{Vol}(S^{15}) \cdot (\kappa/2)^{-\frac{15}{2}} \cdot \sin^{15}(\sqrt{\kappa/2}\; t) \cos(\sqrt{\kappa/2}\; t)$$

is valid, where $\mathrm{Vol}(S^{15}) = 2\pi^8/7!\,$.

As in the previous case, we can compute the volume of the compact symmetric space $N = E_6/L$. We obtain

$$\mathrm{Vol}(N) = \int_0^r \mathrm{Vol}(M^t)\, dt = \frac{2^{18}}{3^5 \cdot 5^3 \cdot 7^2 \cdot 11}\, \pi^{16}\, \kappa^{-16}$$

by using the relation in Corollary 14.

6 The isometric action of $(SU(6) \times SU(2))/\mathbb{Z}_2$ on E_6/F_4

Let us take the compact symmetric space $N = E_6/F_4$ of rank two, where $F_4 = (E_6)_\varrho$. Select the point $o = eF_4$ of N. In this case the tangent space T_oN of N at o is identified with the subspace $\mathfrak{n}$ of $\mathfrak{e}_6$. Observe that there is no root subspace of $\mathfrak{e}_6$ which is included in $\mathfrak{n}$. Therefore the relations (2) and (10) imply that the maximal sectional curvature κ of $N = E_6/F_4$ is equal to $1/(c^2 \cdot 24)$.

Recall that the symmetric subgroup K defined by the involution σ is isomorphic to $(SU(6) \times SU(2))/\mathbb{Z}_2$ and $H = (E_6)_{\varrho\sigma}$ is isomorphic to $Sp(4)/\mathbb{Z}_2$. Consider the natural isometric actions of K and H on $N = E_6/F_4$.

15 Proposition. *The totally geodesic orbits $K(o)$ and $H(o)$ are isometric to the symmetric spaces $SU(6)/Sp(3)$ and $\mathbb{H}P^3$ with maximal sectional curvature κ, respectively.*

Proof. In the proof of Proposition 7, we obtained that $K\cap F_4$ is isomorphic to the Lie group $(Sp(3)\times Sp(1))/\mathbb{Z}_2$. Hence, the totally geodesic singular orbit $K(o)=K/(K\cap F_4)$ is isometric to the symmetric space $SU(6)/Sp(3)$.

On the other hand, since $H\cap F_4$ is also isomorphic to $(Sp(3)\times Sp(1))/\mathbb{Z}_2$, the orbit $H(o)=H/(H\cap F_4)$ is isometric to the quaternion projective space $\mathbb{H}P^3$. $\boxed{QED}$

Hereafter, we discuss the isometric action $\lambda\colon K\times N\to N$ *of* K *on* N*.* Proposition 15 implies that the cohomogeneity of λ is equal to one. Hence, the orbits of K are tubular surfaces around the totally geodesic submanifold $M=K(o)$.

Consider the orthogonal decomposition $T_oN=T_oM+\nu_oM$, where $\nu_oM=T_oH(o)$. Fix a unit vector u of ν_oM and take the closed geodesic $\gamma:\mathbb{R}\to N$ defined by $\gamma(t)=\mathrm{Exp}_o(tu)$ for $t\in\mathbb{R}$.

It follows from Proposition 15 that the eigenvalues of the Jacobi operator R_u on the normal subspace ν_oM (and their multiplicities) are

$$b_1=\kappa\ \ (k_1=3),\quad b_2=\kappa/4\ \ (k_2=8),\quad b_3=0\ \ (k_3=1). \tag{15}$$

Moreover, we can state the following assertion.

16 Proposition. *The eigenvalues of* R_u *on the invariant subspace* T_oM *(and their multiplicities) are*

$$a_1=\kappa\ \ (m_1=5),\quad a_2=\kappa/4\ \ (m_2=8),\quad a_3=0\ \ (m_3=1). \tag{16}$$

Proof. We use the same technique as in the proof of Proposition 8. Take a two-dimensional Abelian subspace $\mathfrak{n}_0$ of $\mathfrak{n}$ and the restricted roots in the dual space of $\mathfrak{n}_0$. Recall that the restricted root system $\mathcal{R}$ of E_6/F_4 is of type $\mathcal{A}_2$. This means that $\mathcal{R}$ consists of only six roots. Let $\alpha,\ \beta$ be a basis of the root system $\mathcal{R}$ represented in Figure 3. Then we get the orthogonal decomposition $\mathfrak{n}=\mathfrak{n}_0+\mathfrak{n}_\alpha+\mathfrak{n}_\beta+\mathfrak{n}_{\alpha+\beta}$ of $\mathfrak{n}$, where the root subspaces are 8-dimensional.

Since the arc length of $C=\gamma(\mathbb{R})$ equals $(2\,\pi)/\sqrt{\kappa}$, we assume that u is parallel to the root vector $X_{\alpha+\beta}$ in $\mathfrak{n}_0$. Then $R_u(v)=\kappa\, v$ and $R_u(w)=(\kappa/4)\,w$ hold for all $v\in\mathfrak{n}_{\alpha+\beta}$ and $w\in\mathfrak{n}_\alpha+\mathfrak{n}_\beta$. Therefore we obtain that the eigenvalues of R_u on T_oN are $\kappa,\ \kappa/4$ and 0 with the multiplicities $8,\ 16$ and 2, respectively. Consequently, the relation (15) verifies that Proposition 16 is true. $\boxed{QED}$

17 Remark. Let us consider the symmetric space $N=E_6/F_4$ as a compact tube around $M=K(o)$. It follows from Proposition 16 that the radius of this tube equals $r=\pi/(2\sqrt{\kappa})$. Moreover, by means of the relations (15), (16) and (3) we can express the principal curvatures of the principal orbits $M^t,\ 0<t<r$.

Since $M=K(o)$ is isometric to $SU(6)/Sp(3)$, we obtain that $\mathrm{Vol}(M)=\frac{\sqrt{3}}{24}\,\pi^8\,\kappa^{-7}$ is valid. Hence, Theorem 5 implies the following statement.

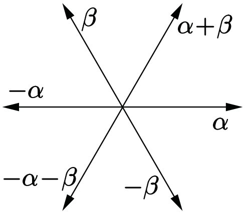

Figure 3. The root system of type $\mathcal{A}_2$.

18 Corollary. *The relation*

$$\mathrm{Vol}(M^t) = \mathrm{Vol}(M) \cdot \mathrm{Vol}(S^{11}) \cdot \kappa^{-\frac{11}{2}} \cdot \sin^{11}(\sqrt{\kappa}\, t) \cos^5(\sqrt{\kappa}\, t)$$

holds for the volumes of the principal orbits $M^t = K(\gamma(t)), \quad 0 < t < r$.

Using Corollary 18, we get

$$\mathrm{Vol}(N) = \int_0^r \mathrm{Vol}(M^t)\, dt = \frac{\sqrt{3}}{2^9 \cdot 3^3 \cdot 5 \cdot 7}\, \pi^{14}\, \kappa^{-13}\,.$$

for the volume of the compact symmetric space $N = E_6/F_4$.

19 Remark. Finally, let us consider the compact symmetric space $N = E_6/H$, where the subgroup $H = (E_6)_{\varrho\sigma}$ is isomorphic to $Sp(4)/\mathbb{Z}_2$. Take the isometric actions of $F_4 = (E_6)_\varrho$ and $K = (E_6)_\sigma$ on N. Select the point $o = eH$ of N. We have shown that the isotropy subgroup $F_4 \cap H = K \cap H$ at o is isomorphic to $(Sp(3) \times Sp(1))/\mathbb{Z}_2$. Hence, the totally geodesic orbits $F_4(o)$ and $K(o)$ are isometric to $F_4/(Sp(3) \times Sp(1)/\mathbb{Z}_2)$ and $SU(6)/Sp(3)$, respectively. Since $K(o)$ is a symmetric space of rank two, it follows from Theorem 1 that the cohomogeneity of the hyperpolar action of F_4 on $E_6/(Sp(4)/\mathbb{Z}_2)$ equals two.

References

[1] K. Abe, I. Yokota: *Realization of spaces $E_6/(U(1)Spin(10))$, $E_7/(U(1)E_6)$, $E_8/(U(1)E_7)$ and their volumes*, Tokyo J. Math., **20** (1997), 73–86.

[2] K. Abe, I. Yokota: *Volumes of compact symmetric spaces*, Tokyo J. Math., **20** (1997), 87–105.

[3] L. Conlon: *Remarks on commuting involutions*, Proc. Amer. Math. Soc., **22** (1969), 255–257.

[4] L. Conlon: *Variational completeness and K-transversal domains*, J. Differential Geom., **5** (1971), 135–147.

[5] B. Csikós, L. Verhóczki: *Tubular structures of compact symmetric spaces associated with the exceptional Lie group F_4*, Geom. Dedicata, **109** (2004), 239–252.

[6] A. Gray, L. Vanhecke: *The volumes of tubes in a Riemannian manifold*, Rend. Sem. Mat. Univ. Politec. Torino, **39** (1981), 1–50.

[7] A. Gray: Tubes, Addison–Wesley, Redwood City, (1990).

[8] E. Heintze, R. S. Palais, C. L. Terng, G. Thorbergsson: *Hyperpolar actions and k-flat homogeneous spaces*, J. Reine Angew. Math., **454** (1994), 163–179.

[9] E. Heintze, R. S. Palais, C. L. Terng, G. Thorbergsson: Hyperpolar actions on symmetric spaces. In: Geometry, topology and physics for Raoul Bott, Conf. Proc. Lecture Notes Geom. Topology IV (edited by S. T. Yau), Int. Press, Cambridge, (1995), 214–245.

[10] S. Helgason: Differential geometry, Lie groups, and symmetric spaces, Academic Press, New York, (1978).

[11] R. Hermann: *Variational completeness for compact symmetric spaces*, Proc. Amer. Math. Soc., **11** (1960), 544–546.

[12] R. Hermann: *Totally geodesic orbits of groups of isometries*, Nederl. Akad. Wetensch. Proc. Ser. A, **65** (1962), 291–298.

[13] W. Y. Hsiang, H. B. Lawson Jr.: *Minimal submanifolds of low cohomogeneity*, J. Differential Geom., **5** (1971), 1–38.

[14] S. Kobayashi, K. Nomizu: Foundations of differential geometry II, Interscience Publishers, New York, (1969).

[15] A. Kollross: *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc., **354** (2002), 571–612.

[16] J. Szenthe: *Orthogonally transversal submanifolds and the generalizations of the Weyl group*, Period. Math. Hungar., **15** (1984), 281–299.

[17] L. Verhóczki: *Special cohomogeneity one isometric actions on irreducible symmetric spaces of types I and II*, Beiträge Algebra Geom., **44** (2003), 57–74.

[18] I. Yokota: *Simply connected compact simple Lie group $E_{6(-78)}$ of type E_6 and its involutive automorphisms*, J. Math. Kyoto Univ., **20** (1980), 447–473.