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Transitive Partial Hyperbolic Flocks of Deficiency One

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Abstract. Partial hyperbolic flocks of deficiency one in $PG(3,q)$ are equivalent to translation planes with spreads in $PG(3,q)$ admitting Baer groups of order $q-1$. In this article, we completely classify the associated derivable translation planes of order p^4, for p a prime, inducing a collineation group transitive on the partial flock. There are exactly two possible non-linear flocks and planes, those whose planes are the semifield of order 16 with kernel $GF(4)$ and the plane of order 81, due to Johnson and Pomareda.

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1 Introduction

A flock of a hyperbolic quadric in $PG(3,q)$ is a covering of the quadric by mutually disjoint conics. By the 'Thas-Walker' construction in case (iii) (see e.g. [8], corresponding to a hyperbolic flock is a translation plane of order q^2, for q a prime power, with spread in $PG(3,q)$. The theorem of Thas/Bader-Lunardon completely classifies hyperbolic flocks.

1 Theorem. *[Thas [9] /Bader-Lunardon [1]] Let $\mathcal{H}$ be a flock of a hyperbolic quadric in $PG(3,q)$. Then $\mathcal{H}$ is one of the following types:*

(i) Linear,

(ii) a Thas flock, or

(iii) a Bader/ Baker-Ebert / Johnson flock for $q = 11, 23, 59$.

The corresponding translation planes are respectively Desarguesian (in case (i)), regular nearfield (in case (ii)) and irregular nearfield in case (iii). (see Gevaert and Johnson [3]).

Actually, Gavaert and Johnson [3], showed that any translation plane of order q^2 with spread in $PG(3,q)$ that admits an affine homology group of order $q-1$, one of whose component orbits union the axis and coaxis forms a regulus in $PG(3,q)$ determines a hyperbolic flock. Thus, the spread structure corresponding to a hyperbolic flock consists of $q+1$ reguli that mutually share two components. Suppose we derive one of these reguli. Then the affine homology group of order $q-1$ would be transformed into a Baer group (fixes a Baer subplane pointwise). The Baer group fixes another Baer subplane of the corresponding derived net and acts as a kernel homology group of the second Baer subplane. The Baer group now has q orbits of length $q-1$ (and $q+1$ fixed components; orbits of length 1). The set of q orbits union the Baer subplane pointwise fixed union the two Baer subplanes fixed by the Baer group forms a partial hyperbolic flock of deficiency one, so lacks one conic to be a hyperbolic flock.

2 Theorem. *[Johnson [5]] The set of partial flocks of deficiency one of a hyperbolic quadric in $PG(3,q)$ is equivalent to the set of translation planes of order q^2 with spread in $PG(3,q)$ that admits a Baer collineation group of order $q-1$.*

A partial hyperbolic flock of deficiency one may be extended uniquely to a hyperbolic flock if and only if the partial spread of degree $q+1$ of fixed components of the Baer group is a regulus in $PG(3,q)$.

In this note, we are interested in partial hyperbolic flocks of deficiency one that cannot be extended to a flock. There are not very many partial hyperbolic flocks of deficiency one, and it has been conjectured that there are only finitely many. However, there are no results on the non-existence of such structures. One way to begin to eliminate possible cases is to assume that the partial hyperbolic flock is transitive. We shall say that the partial flock is 'derivable' if in the derivable net defined by the associated Baer group is derivable. One very important derivable net is a regulus net (corresponds to a regulus in $PG(3,q)$). If we wish to determine the transitive partial flocks that are derivable by a regulus, we see by Theorems 1 and 2 that the translation plane is a regulus-derived nearfield plane and hence must be a Hall plane.

3 Remark. The translation plane corresponding to a transitive and regulus-derivable partial flock of a hyperbolic quadric is Hall (derived Desarguesian).

So, we wish to ask if there is a corresponding result for arbitrary transitive and derivable partial flocks of hyperbolic quadrics: Are the associated translation planes always Desarguesian? Our main result is a complete classification of transitive and derivable partial hyperbolic flocks of deficiency one and order p^2.

2 Transitive Partial Hyperbolic Flocks of Deficiency One

It is possible to give a general structure for the spreads of the translation planes that correspond to transitive partial hyperbolic flocks of deficiency one. There are two extremely unusual partial flocks that are both derivable and transitive whose translation planes are of orders 2^4 and 3^4. We consider such translation planes of arbitrary orders p^4, where p is a prime.

4 Lemma. *Let π be a translation plane of order p^4, for p a prime, that corresponds to a transitive partial hyperbolic flock of deficiency one.*

Then the spread may be represented as follows:

$$x = 0, y = x \begin{bmatrix} m(u) & bv^{-1} \\ v & u \end{bmatrix}; u, v \in GF(p^2),$$

and where m is an additive function of $GF(p^2)$ (where, $0^{-1} = 0$).

Proof. All partial hyperbolic flocks of deficiency one for $q = 4$ are determined by computer in Royle [7]. For odd order p, we may apply fundamental results on transitive partial hyperbolic flocks of Biliotti-Johnson [2]. QED

Since the form of the spread is determined up to the function m, we ask what would occur if the associated translation plane is also derivable. By Johnson [4], any derivable partial spread for a translation plane of order p^4 may be represented in the following form:

$$x = 0, y = x \begin{bmatrix} u^\sigma & 0 \\ 0 & u \end{bmatrix}; u \in GF(p^2), \text{ where } \sigma = 1 \text{ or } p.$$

When $\sigma = 1$, the derivable partial spread is a regulus and we have seen in Remark 3 in this case, the translation plane is Hall. Hence, we may assume that $\sigma = p$.

2.1 Johnson's Partial Hyperbolic Flock of Order 4

Now assume that $p = 2$ and we have a derivable translation plane of order 2^4, which corresponds to a partial flock of a hyperbolic quadric of deficiency one. The translation planes of order 16 are completely determined and actually, all are derivable. We are interested here in one particular translation plane, the semifield plane of order 16 with kernel $GF(4)$. Here there is an automorphism group of order $4 - 1$ that fixes a subfield isomorphic to $GF(4)$ pointwise. What this means is that there is an associated Baer group of order $4 - 1$ and Theorem 2 shows that there is a partial hyperbolic flock. Since the translation plane is a semifield plane, there is an elation group of order 16, within which there is an elation group of order 4 that acts transitively on the 4 conics of the associated partial hyperbolic flock. Hence, we have an example of a transitive and derivable partial hyperbolic flock of deficiency one that is not classical. This is called the 'Johnson partial hyperbolic flock of order 4'.

2.2 Johnson-Pomareda Partial Hyperbolic Flock of Order 9

Now let $p = 3$ and consider the putative spread

$$x = 0, y = x \begin{bmatrix} u^3 & bv^{-1} \\ v & u \end{bmatrix}; u, v \in GF(3^2).$$

The question of the existence of a spread is equivalent to whether there is an element $b \in GF(9)$, such that the differences of the matrices are either non-singular or identically zero. Since the determinant of $\begin{bmatrix} u^3 & bv^{-1} \\ v & u \end{bmatrix}$ is $u^4 - b$, and u^4 is in $GF(3)$, it follows that b must be in $GF(9) - GF(3)$. Consider the difference

$$\begin{bmatrix} u^3 & bv^{-1} \\ v & u \end{bmatrix} - \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix},$$

which has determinant

$$u^4 - b(v^{-1} - 1)(v - 1).$$

Therefore, we must have

$$u^4 - b(v^{-1} - 1)(v - 1) = 0$$

if and only if $u = 0$, $v = 1$.

In the following section, we will consider the more general equation:

$$u^{p+1} - b(v^{-1} - 1)(v - 1) = 0$$

and show when $p = 3$, that b needs to avoid exactly 6 non-zero elements. Hence, b may be taken one of two ways to produce a transitive and derivable partial flock of deficiency one in $PG(3, 9)$. This construction is in [6] and is called the 'Johnson-Pomareda partial flock of order 9'.

Our main result completely determines the transitive and derivable partial hyperbolic flocks.

3 The Classification

5 Theorem. *Let $\mathcal{H}$ be a transitive and derivable partial hyperbolic flock of deficiency one in $PG(3, p^2)$. Then $\mathcal{H}$ is one of the following:*

(i) the 'derived' linear flock,

(ii) the Johnson partial flock of order 4 or

(iii) the Johnson-Pomareda partial flock of order 9.

Proof. Assume that we have a transitive and derivable partial flock of deficiency one in $PG(3, p^2)$. Then there is a corresponding translation plane of order $p^4 = q^2$, whose spread has the form

$$x = 0, y = x \begin{bmatrix} mu^p & bv^{-1} \\ v & u \end{bmatrix}; u, v \in GF(p^2).$$

If we choose $y = x$ to be a component, for $u = 1, v = 0$ (0^{-1} is agreed to be 0), then $m = 1$.

We have an orbit of length $q(q-1)$ so to check the conditions for a partial spread, we form

$$\begin{bmatrix} u^p & bv^{-1} \\ v & u \end{bmatrix} - \begin{bmatrix} [c]cc0 & b \\ 1 & 0 \end{bmatrix}$$

and taking the determinant we obtain:

$$u^{p+1} - b(v-1)(v^{-1} - 1) \neq 0, \ \forall u, v \in GF(p^2).$$

Since $u^{p+1} \in GF(p)$, we need to be able to choose b in $GF(p^2) - \{0\}$, so that

$$b \notin \left\{(2 - (v + v^{-1})GF(p)^*; v \in GF(p)\right\} = S.$$

Note that, in particular, b cannot be in $GF(p)$. Henceforth, take v non-zero.

Furthermore, we may assume that p is odd, since the situation for $p = 2$ is known by the previous remarks. Choose a basis for $GF(p^2)$ as $\{1, t\}$ such that $t^2 = \gamma$, a non-square. Then, for $v = t\alpha + \beta$; $\alpha, \beta \in GF(p)$, an easy calculation shows that

$$(*)\ (2 - (v + v^{-1})) = t\left(-\alpha\left(1 + \frac{1}{\Delta}\right)\right) + 2 - \beta\left(1 - \frac{1}{\Delta}\right),$$
$$\Delta = \alpha^2\gamma - \beta^2.$$

To determine S, we consider the following cases:

Case I. $\beta = 0$

So

$$S = \{t(-\alpha\left(1+\frac{1}{\alpha^2\gamma}\right)) + 2 : \alpha \in GF(p)\}$$

Case II. $\Delta = 1$,
so

$$S = \{t(-2\alpha) + 2 : \alpha \in GF(p)\}$$

Case III. $2 - \beta\left(1 - \frac{1}{\Delta}\right) = 0$
So

$$S = \{t(-\alpha\left(1+\frac{1}{\Delta}\right)) : \alpha \in GF(p)\}$$

First choosing $\beta = 0$, we obtain elements

$$t(-\alpha\left(1+\frac{1}{\alpha^2\gamma}\right)) + 2 = t(-(\alpha + \alpha^{-1}\gamma^{-1})) + 2,$$

for all $\alpha \neq 0$.

We claim that the number of such non-zero elements $(\alpha+\alpha^{-1}\gamma^{-1})$ is $(p-1)/2$. The first question is when

$$\alpha + \alpha^{-1}\gamma^{-1} = 0.$$

This is true if and only if

$$\alpha^2 = -\gamma^{-1}$$

if and only if -1 is a non-square.

We now check intersections:

Assume that

$$\alpha + \alpha^{-1}\gamma^{-1} = \delta + \delta^{-1}\gamma^{-1}. \text{ for some } \delta \in GF(p)$$

We obtain

$$(\alpha - \delta) = \gamma^{-1}(\alpha - \delta)/\alpha\delta.$$

So, for $\alpha \neq \delta$.

$$\alpha\delta = \gamma^{-1}.$$

Hence, we obtain exactly $(p-1)/2$ such elements, as claimed, of which there are $(p-1)/2 - 1$ non-zero elements when -1 is a non-square.

To summarize, if $\beta = 0$ we have $(p-3)/2$ non-zero elements when -1 is a non-square and $(p-1)/2$ non-zero elements when -1 is a square.

Now choose $\Delta = 1$, to obtain from $(*)$,

$$t(-2\alpha) + 2.$$

There are a total of $p+1$, solutions (α, β) to $\Delta = 1$. We claim that there are $(p-1)/2$ nonzero solutions for α, when -1 is a square and $(p+1)/2$ non-zero solutions for α, when -1 is a nonsquare.

Case 1: First assume that -1 is a square in $GF(p)$.

Then, it is possible that $\alpha = 0$. Taking out the two solutions $(0, \pm\beta_0)$, leaves a total of $(p-1)$ remaining solutions. For the remaining solutions, since now $\alpha\beta \neq 0$, once given a solution (α, β), we obtain four solutions $(\pm\alpha, \pm\beta)$, where the plus/minus symbols are independent of each other.

This gives $2(p-1)/4 = (p-1)/2$ non-zero solutions for α.

Case 2: Now assume that -1 is a non-square in $GF(p)$, then $\alpha\beta \neq 0$. Hence, the previous argument shows that there are $(p+1)/2$ non-zero solutions for α.

We now compare the two sets of elements of the form $t\delta + 2$:

$$\left\{t(-(\alpha + \alpha^{-1}\gamma^{-1})) + 2\right\} \text{ and } \{t(-2\alpha) + 2;\ \Delta = 1\}.$$

To check intersections, suppose that

$$(\alpha + \alpha^{-1}\gamma^{-1}) = 2\tau, \text{ such that } \tau^2\gamma - \beta^2 = 1.$$

We obtain a quadratic

$$\alpha^2 - 2\tau\alpha + \gamma^{-1} = 0,$$

whose discriminant is

$$4(\tau^2 - \gamma^{-1}),$$

which we require to be a square (since it is never zero). This then is equivalent to having

$$\tau^2\gamma - 1 = \beta^2 \text{ a non-square},$$

which is a contradiction. Hence, the two sets are disjoint.

Therefore, we obtain for -1 non-square a total of $(p-3)/2 + (p+1)/2 = (p-1)$ and for -1 square, we obtain a total of $(p-1)/2 + (p-1)/2 = (p-1)$ elements.

That is, we obtain a set of $p-1$ elements

$$\{t\theta + 2; \forall\theta \in GF(p)^*\} \text{ (see } (*)).$$

Now assume that there are elements α and β such that Δ is not ± 1 and $\alpha\beta \neq 0$ such that

$$(**)\ 2 - \beta\left(1 - \frac{1}{\Delta}\right) = 0.$$

If this is the case, then we would obtain elements of the form $t(-\alpha\left(1 + \frac{1}{\Delta}\right))$. Hence, the set of elements that we would need to avoid would include:

$$\{t\theta + 2; \forall\theta \in GF(p)^*\}\, GF(p)^* \cup \left\{t(-\alpha\left(1 + \frac{1}{\Delta}\right))GF(p)^*\right\},$$

where the α is not specified, but note that the coefficient of t is non-zero. Hence, we obtain

$$\{t\theta + 2; \forall \theta \in GF(p)^*\} \, GF(p)^* \cup \ \{t\delta; \delta \in GF(p)^*\}\,.$$

Note that we have a total of $(p-1)^2 + (p-1) = p^2 - p$ distinct elements all of which are not in $GF(p)$. However, this is precisely the possible number of elements of $GF(p^2) - GF(p)$, which means that it is not possible to avoid an element b, so that there are no possible planes.

Note that when $p = 3$, $\Delta = \pm 1$, so $(**)$ has no solution and the number of elements to avoid then will be 4 since $3^2 - 2 = 6$, a plane may be constructed as noted above (as is known previously in Johnson and Pomareda [6], where it turns out that there is a unique such plane).

So assume that $p > 3$. Consider the equation

$$(***)\ 2 - (v + v^{-1}) = t\delta, \text{ for some } \delta \in GF(p)^*.$$

Assume that there is a non-zero solution for v then the corresponding $\Delta \neq -1$ or 1. However, also $\alpha\beta \neq 0$ in the equation.

$$(*)\ (2 - (v + v^{-1})) = t(-\alpha \left(1 + \frac{1}{\Delta}\right)) + 2 - \beta \left(1 - \frac{1}{\Delta}\right),$$
$$\Delta = \alpha^2\gamma - \beta^2.$$

Hence, there exist α and β such that Δ is not ± 1 , $\alpha\beta \neq 0$, and such that

$$(**)\ 2 - \beta \left(1 - \frac{1}{\Delta}\right) = 0,$$

if and only if $(***)$ has a solution. Equation $(***)$ leads to the following quadratic:

$$v^2 + (t\delta - 2)v + 1 = 0.$$

The discriminant is

$$(t\delta - 2)^2 - 4,$$

so we have a solution if and only if there exists a δ such that

$$\left(\frac{t\delta - 2}{2}\right)^2 - 1 \text{ is a square.}$$

Clearly, there are $(p-1)/2$ possible distinct elements $(\frac{t\delta-2}{2})^2 - 1$, as δ varies over $GF(p)$. We need one of these to be a square. Hence, assume that all are non-squares ρ, that is, we have for each non-square ρ, there exists an element δ so that

$$\left(\frac{t\delta - 2}{2}\right)^2 - 1 \ = \rho.$$

This means that

$$\rho + 1 \text{ is a square}$$

for all ρ non-square. So, there is an element $\rho \neq 0$ so that $\rho + 1 = 1$, a contradiction.

This proves the theorem. $\boxed{QED}$

The proof of the previous theorem actually proves the following corollary.

6 Corollary. *If $q = p^r$, for p odd then*

$$x = 0, y = x \begin{bmatrix} u^q & bv^{-1} \\ v & u \end{bmatrix}; u, v \in GF(q^2),$$

is a spread if and only if $q = p$ and $p = 2$ or 3.

This completes the proof of our main result. There are other ways to consider the exceptional flocks of Johnson order 4 and Johnson-Pomareda order 9. For example, one could ask if there are transitive and derivable partial flocks of any order q or merely if there could be transitive partial flocks that are not derivable. We conjecture that there are no other possible partial flocks. We leave these as open questions for the consideration of the reader.

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