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Rotation hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

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Abstract. We introduce the notion of rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and we prove a criterium for a hypersurface of one of these spaces to be a rotation hypersurface. Moreover, we classify minimal rotation hypersurfaces, flat rotation hypersurfaces and rotation hypersurfaces which are normally flat in the Euclidean resp. Lorentzian space containing $\mathbb{S}^n \times \mathbb{R}$ resp. $\mathbb{H}^n \times \mathbb{R}$.

Keywords: rotation hypersurface, flat, minimal

MSC 2000 classification: 53B25

1 Introduction

In [3] the classical notion of a rotation surface in $\mathbb{E}^3$ was extended to rotation hypersurfaces of real space forms of arbitrary dimension. Motivated by the recent study of hypersurfaces of the Riemannian products $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, see for example [2], [4], [5] and [6], we will extend the notion of rotation hypersurfaces to these spaces. Starting with a curve α on a totally geodesic cylinder $\mathbb{S}^1 \times \mathbb{R}$ resp. $\mathbb{H}^1 \times \mathbb{R}$ and a plane containing the axis of the cylinder, we will construct such a hypersurface and we will compute its principal curvatures. Moreover, we will prove a criterium for a hypersurface of $\mathbb{S}^n \times \mathbb{R}$ resp. $\mathbb{H}^n \times \mathbb{R}$ to be a rotation hypersurface and we will end the paper with some applications, including a classification of minimal rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

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2 Preliminaries

Denote by $\mathbb{E}^{n+2}$ the Euclidean space of dimension $n+2$ and by $\mathbb{L}^{n+2}$ the Lorentzian space of dimension $n+2$, equipped with the metric $ds^2 = -dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2$. In order to study the spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, we use the following models:

$$\begin{aligned} \mathbb{S}^n \times \mathbb{R} &= \{(x_1, \ldots, x_{n+2}) \in \mathbb{E}^{n+2} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}, \\ \mathbb{H}^n \times \mathbb{R} &= \{(x_1, \ldots, x_{n+2}) \in \mathbb{L}^{n+2} \mid -x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = -1,\ x_1 > 0\}. \end{aligned}$$

From now on, we denote by $\mathbb{M}$ either $\mathbb{S}$ or $\mathbb{H}$ and we set $\varepsilon = 1$ in the first case and $\varepsilon = -1$ in the second case. Remark that $\xi = (x_1, \ldots, x_{n+1}, 0)$ is a normal vector field on $\mathbb{M}^n \times \mathbb{R}$ with $\langle \xi, \xi \rangle = \varepsilon$ and that the Levi Civita connection $\widetilde{\nabla}$ of $\mathbb{M}^n \times \mathbb{R}$ is given by

$$\widetilde{\nabla}_X Y = D_X Y + \varepsilon \langle X_{\mathbb{M}^n}, Y_{\mathbb{M}^n} \rangle \xi,$$

where D is the covariant derivative in $\mathbb{E}^{n+2}$ resp. $\mathbb{L}^{n+2}$ and $X_{\mathbb{M}^n}$ and $Y_{\mathbb{M}^n}$ denote the projections of X and Y on the tangent space to $\mathbb{M}^n$. The curvature tensor $\widetilde{R}$ of $\mathbb{M}^n \times \mathbb{R}$ is given by

$$\langle \tilde{R}(X,Y)Z, W \rangle = \varepsilon(\langle Y_{\mathbb{M}^n}, Z_{\mathbb{M}^n} \rangle \langle X_{\mathbb{M}^n}, W_{\mathbb{M}^n} \rangle - \langle X_{\mathbb{M}^n}, Z_{\mathbb{M}^n} \rangle \langle Y_{\mathbb{M}^n}, W_{\mathbb{M}^n} \rangle).$$

Let $f : M^n \to \mathbb{M}^n \times \mathbb{R}$ be a hypersurface with unit normal N. Let T denote the projection of $\frac{\partial}{\partial x_{n+2}}$ on the tangent space to M^n and denote by θ an angle function such that $\cos\theta = \langle N, \frac{\partial}{\partial x_{n+2}} \rangle$. This means that

$$\frac{\partial}{\partial x_{n+2}} = f_* T + \cos\theta\, N.$$

Let ∇ and R denote the Levi Civita connection and the Riemann Christoffel curvature tensor of M^n respectively and let S be the shape operator of the hypersurface. Then the equations of Gauss and Codazzi are given by

$$\begin{aligned} \langle R(X,Y)Z, W \rangle &= \langle SX, W \rangle \langle SY, Z \rangle - \langle SX, Z \rangle \langle SY, W \rangle \\ &\quad + \varepsilon(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \\ &\quad + \langle Y, T \rangle \langle W, T \rangle \langle X, Z \rangle + \langle X, T \rangle \langle Z, T \rangle \langle Y, W \rangle \\ &\quad - \langle X, T \rangle \langle W, T \rangle \langle Y, Z \rangle - \langle Y, T \rangle \langle Z, T \rangle \langle X, W \rangle), \end{aligned} \tag{1}$$

$$\nabla_X SY - \nabla_Y SX - S[X,Y] = \varepsilon \cos\theta (\langle Y, T \rangle X - \langle X, T \rangle Y), \tag{2}$$

where X, Y, Z, W are vector fields tangent to M^n. Moreover, by using the fact that $\frac{\partial}{\partial x_{n+2}}$ is parallel in $\mathbb{M}^n \times \mathbb{R}$, we obtain

$$\nabla_X T = \cos\theta\, SX, \qquad X[\cos\theta] = -\langle SX, T \rangle. \tag{3}$$

3 Definition and calculation of the principal curvatures

Consider a three-dimensional subspace P^3 of $\mathbb{E}^{n+2}$ resp. $\mathbb{L}^{n+2}$, containing the x_{n+2}-axis. Then $(\mathbb{M}^n \times \mathbb{R}) \cap P^3 = \mathbb{M}^1 \times \mathbb{R}$. Let P^2 be a two-dimensional subspace of P^3, also through the x_{n+2}-axis. Denote by $\mathcal{I}$ the group of isometries of $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$, which leave $\mathbb{M}^n \times \mathbb{R}$ globally invariant and which leave P^2 pointwise fixed. Finally, let α be a curve in $\mathbb{M}^1 \times \mathbb{R}$ which does not intersect P^2.

1 Definition. The rotation hypersurface M^n in $\mathbb{M}^n \times \mathbb{R}$ with profile curve α and axis P^2 is defined as the $\mathcal{I}$-orbit of α.

Remark that rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ are foliated by spheres. Rotation hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ are foliated by spheres if P^2 is Lorentzian, by hyperbolic spaces if P^2 is Riemannian and by horospheres if P^2 is degenerate. It is clear from the definition that the velocity vector of α is proportional to T, unless α lies in a plane orthogonal to $\frac{\partial}{\partial x_{n+2}}$, in which case $T = 0$.

We will now construct an explicit parametrisation for a rotation hypersurface M^n. To do this, we distinguish four cases. In all cases we will assume that P^3 is spanned by $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_{n+1}}$ and $\frac{\partial}{\partial x_{n+2}}$.

1 Case. $\mathbb{M}^n = \mathbb{S}^n$

We may assume that P^2 is spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_{n+2}}$. First, we consider the case that the profile curve is not a vertical line on $\mathbb{S}^1 \times \mathbb{R}$. Then it can be parametrized as follows:

$$\alpha(s) = (\cos(s), 0, \dots, 0, \sin(s), a(s)),$$

for a certain function a. Since α should not intersect P^2, one has to choose the parametrisation interval such that $\sin(s)$ never vanishes.

An explicit parametrisation of the rotation hypersurface is given by

$$f(s, t_1, \dots, t_{n-1}) = (\cos(s), \sin(s)\varphi_1(t_1, \dots, t_{n-1}), \dots, \sin(s)\varphi_n(t_1, \dots, t_{n-1}), a(s)),$$

where $\varphi = (\varphi_1, \dots, \varphi_n)$ is an orthogonal parametrisation of the unit sphere $\mathbb{S}^{n-1}(1)$ in $\mathbb{E}^n$, i.e. $\varphi_1^2 + \dots + \varphi_n^2 = 1$ and $\frac{\partial \varphi_1}{\partial t_i}\frac{\partial \varphi_1}{\partial t_j} + \dots + \frac{\partial \varphi_n}{\partial t_i}\frac{\partial \varphi_n}{\partial t_j} = \delta_{ij}\left\|\frac{\partial \varphi}{\partial t_i}\right\|^2$. Remark that

$$\begin{aligned}
\frac{\partial f}{\partial s} &= (-\sin(s), \cos(s)\varphi_1, \dots, \cos(s)\varphi_n, a'(s)), \\
\frac{\partial f}{\partial t_i} &= (0, \sin(s)\frac{\partial \varphi_1}{\partial t_i}, \dots, \sin(s)\frac{\partial \varphi_n}{\partial t_i}, 0), \\
\xi &= (\cos(s), \sin(s)\varphi_1, \dots, \sin(s)\varphi_n, 0).
\end{aligned}$$

Hence the unit normal vector field N on M^n, tangent to $\mathbb{M}^n \times \mathbb{R}$ is given by

$$N = \frac{1}{\sqrt{1+a'(s)^2}}(-\sin(s)a'(s), \cos(s)a'(s)\varphi_1, \ldots, \cos(s)a'(s)\varphi_n, -1).$$

We will now compute the shape operator S of M^n. Observe that for X, Y tangent to M^n

$$\langle SX, Y\rangle = \langle -\widetilde{\nabla}_X N, Y\rangle = \langle \widetilde{\nabla}_X Y, N\rangle = \langle D_X Y, N\rangle,$$

where $\widetilde{\nabla}$ is the Levi Civita connection of $\mathbb{M}^n \times \mathbb{R}$ and D that of $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$. Now using the fact that φ is an orthogonal parametrisation of a unit sphere, we find

$$\begin{aligned} \langle S\frac{\partial f}{\partial t_i}, \frac{\partial f}{\partial t_j}\rangle &= \langle \frac{\partial^2 f}{\partial t_i \partial t_j}, N\rangle = 0, \\ \langle S\frac{\partial f}{\partial t_i}, \frac{\partial f}{\partial s}\rangle &= \langle \frac{\partial^2 f}{\partial t_i \partial s}, N\rangle = 0. \end{aligned}$$

This implies that the basis $\left\{\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t_1}, \ldots, \frac{\partial f}{\partial t_{n-1}}\right\}$ diagonalizes S. We compute the principal curvatures as follows:

$$\begin{aligned} \lambda &= \langle S\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\rangle \frac{1}{\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\rangle} \\ &= \langle \frac{\partial^2 f}{\partial s^2}, N\rangle \frac{1}{1+a'(s)^2} \\ &= -\frac{a''(s)}{(1+a'(s)^2)^{3/2}}, \\ \mu_i &= \langle S\frac{\partial f}{\partial t_i}, \frac{\partial f}{\partial t_i}\rangle \frac{1}{\langle \frac{\partial f}{\partial t_i}, \frac{\partial f}{\partial t_i}\rangle} \\ &= \langle \frac{\partial^2 f}{\partial t_i^2}, N\rangle \frac{1}{\sin(s)^2 \left\| \frac{\partial \varphi}{\partial t_i} \right\|^2} \\ &= -\frac{a'(s)\cot(s)}{(1+a'(s)^2)^{1/2}}. \end{aligned}$$

Since μ_i is independent of i, we denote it by μ.

If α is a vertical line $\alpha(s) = (\cos(c), 0, \ldots, 0, \sin(c), s)$, where c is a real constant such that $\sin(c) \neq 0$, we obtain from an analogous calculation

$$\begin{aligned} \lambda &= 0, \\ \mu &= -\cot(c). \end{aligned}$$

We conclude that the shape operator S has at most two distinct eigenvalues and if there are exactly two, one of them has multiplicity 1 and the corresponding eigenspace is spanned by T.

2 Case. $\mathbb{M}^n = \mathbb{H}^n$ and P^2 is Lorentzian

In this case we may assume again that P^2 is spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_{n+2}}$. Starting with the curve

$$\alpha(s) = (\cosh(s), 0, \ldots, 0, \sinh(s), a(s)),$$

with $s \neq 0$, we can perform exactly the same calculation as in case 1, yielding

$$\begin{aligned} \lambda &= -\frac{a''(s)}{(1+a'(s)^2)^{3/2}}, \\ \mu &= -\frac{a'(s)\coth(s)}{(1+a'(s)^2)^{1/2}}. \end{aligned}$$

If α is a vertical line $\alpha(s) = (\cosh(c), 0, \ldots, 0, \sinh(c), s)$, where $c \neq 0$ is a real constant, we obtain

$$\begin{aligned} \lambda &= 0, \\ \mu &= -\coth(c). \end{aligned}$$

3 Case. $\mathbb{M}^n = \mathbb{H}^n$ and P^2 is Riemannian

We may suppose that P^2 is spanned by $\frac{\partial}{\partial x_{n+1}}$ and $\frac{\partial}{\partial x_{n+2}}$ and that the profile curve is given by

$$\alpha(s) = (\cosh(s), 0, \ldots, 0, \sinh(s), a(s)).$$

Remark that α does not intersect P^2.

An explicit parametrisation of the rotation hypersurface is given by

$$f(s, t_1, \ldots, t_{n-1}) = (\cosh(s)\varphi_1(t_1, \ldots, t_{n-1}), \ldots, \cosh(s)\varphi_n(t_1, \ldots, t_{n-1}), \sinh(s), a(s)),$$

where $\varphi = (\varphi_1, \ldots, \varphi_n)$ is an orthogonal parametrisation of the hyperbolic space $\mathbb{H}^{n-1}(-1)$ in $\mathbb{L}^n$. This means $-\varphi_1^2 + \varphi_2^2 + \cdots + \varphi_n^2 = -1$, $\varphi_1 > 0$ and $-\frac{\partial \varphi_1}{\partial t_i}\frac{\partial \varphi_1}{\partial t_j} + \cdots + \frac{\partial \varphi_n}{\partial t_i}\frac{\partial \varphi_n}{\partial t_j} = \delta_{ij}\left\|\frac{\partial \varphi}{\partial t_i}\right\|^2$. Hence we obtain

$$\begin{aligned} \frac{\partial f}{\partial s} &= (\sinh(s)\varphi_1, \ldots, \sinh(s)\varphi_n, \cosh(s), a'(s)), \\ \frac{\partial f}{\partial t_i} &= (\cosh(s)\frac{\partial \varphi_1}{\partial t_i}, \ldots, \cosh(s)\frac{\partial \varphi_n}{\partial t_i}, 0, 0), \\ \xi &= (\cosh(s)\varphi_1, \ldots, \cosh(s)\varphi_n, \sinh(s), 0), \\ N &= \frac{1}{\sqrt{1+a'(s)^2}}(\sinh(s)a'(s)\varphi_1, \ldots, \sinh(s)a'(s)\varphi_n, \cosh(s)a'(s), -1). \end{aligned}$$

It turns out that the basis $\left\{\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t_1}, \ldots, \frac{\partial f}{\partial t_{n-1}}\right\}$ diagonalizes the shape operator and the principal curvatures can be computed in the same way as above, yielding

$$S = \begin{pmatrix} \lambda & & & \\ & \mu & & \\ & & \ddots & \\ & & & \mu \end{pmatrix},$$

with $ST = \lambda T$ and

$$\begin{aligned} \lambda &= -\frac{a''(s)}{(1+a'(s)^2)^{3/2}}, \\ \mu &= -\frac{a'(s)\tanh(s)}{(1+a'(s)^2)^{1/2}}. \end{aligned}$$

In the case that α is a vertical line $\alpha(s) = (\cosh(c), 0, \ldots, 0, \sinh(c), s)$, with $c \in \mathbb{R}$, we have

$$\begin{aligned} \lambda &= 0, \\ \mu &= -\tanh(c). \end{aligned}$$

4 Case. $\mathbb{M}^n = \mathbb{H}^n$ and P^2 is degenerate

In this case we work with the following pseudo-orthonormal basis for $\mathbb{L}^{n+2}$:

$$e_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{n+1}}\right), \quad e_{n+1} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{n+1}}\right), \quad e_k = \frac{\partial}{\partial x_k}$$

for $k \in \{2, \ldots, n, n+2\}$ and we may assume that P^2 is spanned by e_{n+1} and e_{n+2}. Remark that $\langle e_1, e_1\rangle = \langle e_{n+1}, e_{n+1}\rangle = 0$ and $\langle e_1, e_{n+1}\rangle = 1$. If α is not a vertical line, we may assume that it is given by

$$\alpha(s) = (s, 0, \ldots, 0, -\frac{1}{2s}, a(s))$$

with respect to the basis $\{e_1, \ldots, e_{n+2}\}$.

In [3], it was proven that the group $\mathcal{I}$ consists in this case of transformations of the form $A_{(t,i)}$, with $t \in \mathbb{R}$, $i \in \{2, \ldots, n\}$, whose action on α is given by

$$A_{(t,i)}\alpha(s) = (s, 0, \ldots, 0, \underbrace{ts}_{i}, 0, \ldots, 0, -\frac{1}{2s} - s\frac{t^2}{2}, a(s)).$$

This means that a parametrisation of the rotation hypersurface is given by

$$f(s, t_2, \ldots, t_n) = (s, st_2, \ldots, st_n, -\frac{1}{2s} - \frac{s}{2}\sum_{i=2}^{n} t_i^2, a(s)).$$

Hence we obtain

$$\frac{\partial f}{\partial s} = (1, t_2, \ldots, t_n, \frac{1}{2s^2} - \frac{1}{2}\sum t_i^2, a'(s)),$$
$$\frac{\partial f}{\partial t_i} = (0, 0, \ldots, 0, \underbrace{s}_{i}, 0, \ldots, 0, -st_i, 0),$$
$$\xi = (s, st_2, \ldots, st_n, -\frac{1}{2s} - \frac{s}{2}\sum t_i^2, 0),$$
$$N = \frac{1}{\sqrt{\frac{1}{s^2} + a'(s)^2}}(sa'(s), sa'(s)t_2, \ldots, sa'(s)t_n, \frac{1}{2s}a'(s) - \frac{s}{2}a'(s)\sum t_i^2, -\frac{1}{s}).$$

The principal curvatures can be computed in an analogous way as before:

$$\lambda = -\frac{sa'(s) + s^2 a''(s)}{(1 + s^2 a'(s)^2)^{3/2}},$$
$$\mu = -\frac{sa'(s)}{(1 + s^2 a'(s)^2)^{1/2}},$$

where λ has, in general, multiplicity 1 and T is an eigenvector with eigenvalue λ.

If α is a vertical line $\alpha(s) = (c, 0, \ldots, 0, -\frac{1}{2c}, s)$, with $c \in \mathbb{R}$, we obtain

$$\lambda = 0,$$
$$\mu = -1.$$

4 Criterium

We prove the following criterium for a hypersurface of $\mathbb{M}^n \times \mathbb{R}$ to be a rotation hypersurface:

2 Theorem. *Take $n \geq 3$ and let $f : M^n \to \mathbb{M}^n \times \mathbb{R}$ be a hypersurface with shape operator*

$$S = \begin{pmatrix} \lambda & & & \\ & \mu & & \\ & & \ddots & \\ & & & \mu \end{pmatrix},$$

with $\lambda \neq \mu$ and suppose that $ST = \lambda T$. Assume moreover that there is a functional relation $\lambda(\mu)$. Then M^n is an open part of a rotation hypersurface.

Proof. Let D_λ and D_μ be the distributions spanned by the eigenspaces of λ and μ respectively. These distributions are involutive. For D_λ this is clear, since it is one-dimensional. For D_μ, we use the equation of Codazzi (2). Take linearly independent vector fields X and Y in D_μ. Then

$$\begin{aligned} S[X,Y] &= \nabla_X SY - \nabla_Y SX - \varepsilon\cos\theta(\langle Y,T\rangle X - \langle X,T\rangle Y) \\ &= \nabla_X(\mu Y) - \nabla_Y(\mu X) \\ &= X[\mu]Y - Y[\mu]X + \mu[X,Y]. \end{aligned}$$

Now $X[\mu]Y - Y[\mu]X \in D_\mu$, whereas $(S-\mu\,\mathrm{id})[X,Y] \in D_\lambda$, since $(S-\lambda\,\mathrm{id})(S-\mu\,\mathrm{id}) = 0$. This implies that $X[\mu] = Y[\mu] = (S-\mu\,\mathrm{id})[X,Y] = 0$. Hence D_μ is involutive and μ is constant along the leaves of D_μ. Due to the relation $\lambda(\mu)$, we find that λ is also constant along the leaves of D_μ.

Fix a point $p \in M^n$ and denote by $M_\lambda(p)$ and $M_\mu(p)$ the leaves of D_λ and D_μ through p. On a neighbourhood of p in M^n we choose coordinates $(t, u_1, \dots, u_{n-1})$ such that $T = \frac{\partial}{\partial t}$ and such that $(u_1, \dots, u_{n-1})$ are local coordinates on $M_\mu(p)$. Let $U_i = \frac{\partial}{\partial u_i}$ for $i = 1, \dots, n-1$ and denote by N a unit normal on the hypersurface.

First we will show that $M_\mu(p)$ is totally umbilical in $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$. Denote by D the covariant derivative in $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$ and by $\widetilde{\nabla}$ the Levi Civita connection of $\mathbb{M}^n \times \mathbb{R}$. Then

$$D_{U_i}N = \widetilde{\nabla}_{U_i}N = -\mu U_i \tag{4}$$

$$D_T N = \widetilde{\nabla}_T N = -\lambda T. \tag{5}$$

Denoting by λ' and μ' the derivatives of λ and μ with respect to t, we find

$$\begin{aligned} 0 &= D_T D_{U_i} N - D_{U_i} D_T N - D_{[U_i,T]} N \\ &= D_T(-\mu U_i) - D_{U_i}(-\lambda T) \\ &= -\mu' U_i - \mu D_T U_i + \lambda D_{U_i} T \\ &= -\mu' U_i + (\lambda - \mu) D_{U_i} T, \end{aligned}$$

from which

$$D_{U_i}T = \frac{\mu'}{\lambda - \mu} U_i. \tag{6}$$

Finally, we have

$$D_{U_i}\xi = (U_i)_{\mathbb{M}^n} = U_i. \tag{7}$$

Equations (4), (6) and (7) yield that $M_\mu(p)$ is totally umbilical in $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$. This implies that $M_\mu(p) \subset P^n(p)$, where $P^n(p)$ is an n-dimensional affine subspace of $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$. We will now show that these subspaces are parallel for different leaves of D_μ, i.e. if we vary the point p.

Consider the following vector field along $M_\mu(p)$:

$$X = \frac{\mu'}{\lambda - \mu} N + \mu T.$$

Remark that $\langle U_i, X\rangle = 0$ and $D_{U_i} X = 0$. This means that X is a constant vector field along $M_\mu(p)$, orthogonal to $P^n(p)$. Next consider

$$Y = \xi + \frac{1}{\mu} N.$$

Again, we observe $\langle U_i, Y\rangle = 0$ and $D_{U_i} Y = 0$, such that Y is also constant along $M_\mu(p)$ and orthogonal to $P^n(p)$. Since X and Y are linearly independent, we can consider the plane $\pi(p)$ spanned by $X(p)$ and $Y(p)$, which is the orthogonal complement of $P^n(p)$ for every point $p \in M^n$. To prove the parallelism of the subspaces $P^n(p)$, it suffices to prove the parallelism of the planes $\pi(p)$. Therefore, we have to show that $D_T X$ and $D_T Y$ are in the direction of $\pi(p)$. Remark that from $[T, U_i] = 0$, we obtain

$$D_{U_i} D_T X = D_T D_{U_i} X = D_T 0 = 0,$$
$$D_{U_i} D_T Y = D_T D_{U_i} Y = D_T 0 = 0.$$

Thus $D_T X$ and $D_T Y$ are vector fields which are constant along $M_\mu(p)$ and which are orthogonal to $P^n(p)$. This means that they are in the direction $\pi(p)$, such that the spaces $\pi(p)$ and hence $P^n(p)$ are parallel.

Now if we move P^n along $M_\lambda(p)$, the intersection with $\mathbb{M}^n \times \mathbb{R}$ gives a rotation hypersurface with axis π. QED

5 Some applications

In this last section, we will first classify the rotation hypersurfaces of $\mathbb{M}^n \times \mathbb{R}$ which are intrinsically flat. Then we will prove that all rotation hypersurfaces of $\mathbb{M}^n \times \mathbb{R}$ are normally flat in $\mathbb{E}^{n+2}$ resp. $\mathbb{L}^{n+2}$, and to conclude we give a classification of minimal rotation hypersurfaces of $\mathbb{M}^n \times \mathbb{R}$.

5.1 Rotation hypersurfaces which are intrinsically flat

Rotation hypersurfaces of $\mathbb{E}^{n+1}$ are flat if and only if $n = 2$ and the profile curve is an open part of a line. We will now classify flat rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$.

3 Theorem. *Let M^n be a rotation hypersurface of $\mathbb{S}^n \times \mathbb{R}$, with axis P^2 as above, which is intrinsically flat. Then $n = 2$ and the profile curve is either a vertical line on $\mathbb{S}^1 \times \mathbb{R}$ or it is parametrized as follows:*

$$\alpha(s) = \left(\cos(s),\ 0,\ \sin(s),\ \pm \int_{s_0}^{s} \sqrt{C \cos(\sigma)^2 - 1}\, d\sigma\right) \tag{8}$$

with $C \in \mathbb{R}$.

Proof. Let M^n be a flat rotation hypersurface of $\mathbb{S}^n \times \mathbb{R}$.

If $n \geq 3$, the equation of Gauss (1) yields

$$\begin{cases} \lambda\mu + 1 - \|T\|^2 = 0 \\ \mu^2 + 1 = 0. \end{cases} \tag{9}$$

It is clear that this system has no solutions for λ and μ.

For $n = 2$, the equation of Gauss only yields the first equation of (9), which is equivalent to

$$\lambda\mu = -\cos^2\theta. \tag{10}$$

Using the results from section 3, we see that this equation is satisfied if α is a vertical line. If α is not a vertical line, and it is parametrized as before, one sees that the left-hand side of this equation can be rewritten as

$$\lambda\mu = \frac{a'(s)a''(s)\cot(s)}{(1 + a'(s)^2)^2},$$

and the right-hand side as

$$\begin{aligned} -\cos^2\theta = \sin^2\theta - 1 = \langle \frac{\partial}{\partial x_{n+2}}, \frac{T}{\|T\|}\rangle^2 - 1 &= \langle \frac{\partial}{\partial x_{n+2}}, \frac{\alpha'}{\|\alpha'\|}\rangle^2 - 1 \\ &= \frac{a'(s)^2}{1 + a'(s)^2} - 1 = -\frac{1}{1 + a'(s)^2}. \end{aligned}$$

This means that equation (10) is equivalent to

$$(a'(s)^2)' + 2\tan(s)a'(s)^2 = -2\tan(s),$$

for which the general solution is given by

$$a'(s)^2 = C\cos(s)^2 - 1, \qquad C \in \mathbb{R}.$$

Thus an explicit parametrisation of α is given by (8). QED

Next, we look at the flat rotation hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$.

4 Theorem. *Let M^n be a rotation hypersurface of $\mathbb{H}^n \times \mathbb{R}$, with axis P^2 as above, which is intrinsically flat. If $n \geq 3$, then P^2 is either Lorentzian or degenerate and the profile curve α satisfies the following:*

(i) $\alpha(s) = (\cosh(s),\ 0,\ \ldots,\ 0,\ \sinh(s),\ \pm\cosh(s) + C)$, *with* $C \in \mathbb{R}$, *if* P^2 *is Lorentzian,*

(ii) α *is a vertical line on* $\mathbb{H}^1 \times \mathbb{R}$ *if* P^2 *is degenerate.*

If $n = 2$, then the profile curve α is either a vertical line on $\mathbb{H}^1 \times \mathbb{R}$ or it is parametrized as follows, with $C \in \mathbb{R}$:

(i) $\alpha(s) = \left(\cosh(s),\ 0,\ \sinh(s),\ \pm\displaystyle\int_{s_0}^{s} \sqrt{C\cosh(\sigma)^2 - 1}\, d\sigma\right)$ *if* P^2 *is Lorentzian,*

(ii) $\alpha(s) = \left(\cosh(s),\ 0,\ \sinh(s),\ \pm\displaystyle\int_{s_0}^{s} \sqrt{C\sinh(\sigma)^2 - 1}\, d\sigma\right)$ *if* P^2 *is Riemannian,*

(iii) $\alpha(s) = \left(s,\ 0,\ -\dfrac{1}{2s},\ \pm\displaystyle\int_{s_0}^{s} \sqrt{C - \frac{1}{\sigma^2}}\, d\sigma\right)$, *with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *defined above, if* P^2 *is degenerate.*

Proof. Let M^n be a flat rotation hypersurface of $\mathbb{H}^n \times \mathbb{R}$.
If $n \geq 3$, the equation of Gauss (1) yields

$$\begin{cases} \lambda\mu - 1 + \|T\|^2 = 0 \\ \mu^2 - 1 = 0. \end{cases} \tag{11}$$

Remark that the first equation of this system is equivalent to

$$\lambda\mu = \cos^2\theta. \tag{12}$$

If α is not a vertical line, it follows immediately from our results in section 3 that the equation $\mu^2 = 1$ only has a solution if P^2 is Lorentzian, namely $a(s) = \pm\cosh(s) + C$. The formula for $\cos^2\theta$, deduced in the proof of Theorem 3, is still valid and hence it is easy to check that the resulting hypersurface also satisfies (12) and hence is indeed flat. If α is a vertical line, the equation $\mu^2 = 1$ has no solutions if P^2 is non-degenerate. If P^2 is degenerate, both equations of (11) are satisfied for every vertical line.

If $n = 2$, the equation of Gauss only yields equation (12). We can then proceed as in the second part of the proof of Theorem 3. QED

5.2 Rotation hypersurfaces which are normally flat in $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$

Let M^n be a hypersurface of $\mathbb{M}^n \times \mathbb{R}$ and denote, as above, by N a unit normal on M^n, tangent to $\mathbb{M}^n \times \mathbb{R}$, and by ξ a unit normal on $\mathbb{M}^n \times \mathbb{R}$. Let S_N and S_ξ be the corresponding shape operators.

The normal connection of M^n as a submanifold of $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$ is given by

$$\begin{aligned}\nabla^\perp_X \xi = \langle \nabla^\perp_X \xi, N\rangle N = \langle D_X \xi, N\rangle N \\ = -\langle X, \frac{\partial}{\partial x_{n+2}}\rangle \langle N, \frac{\partial}{\partial x_{n+2}}\rangle N = -\langle X, T\rangle \cos\theta \; N \end{aligned} \quad (13)$$

and

$$\nabla^\perp_X N = \varepsilon \langle \nabla^\perp_X N, \xi\rangle \xi = -\varepsilon \langle N, \nabla^\perp_X \xi\rangle \xi = \varepsilon \langle X, T\rangle \cos\theta \, \xi. \quad (14)$$

We can now prove the following:

5 Theorem. *Let M^n be a rotation hypersurface of $\mathbb{M}^n \times \mathbb{R}$. Then M^n is normally flat in $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$.*

Proof. Denote by $R^\perp$ the normal curvature tensor of M^n as a submanifold of $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$ and let X and Y be tangent vector fields to M^n, which are orthogonal to T.

From (13) and (14), we see that $R^\perp(X, Y) = 0$.

Remark that $[T, X]$ has no component in the direction of T. Indeed, using (3) we obtain

$$\begin{aligned} \langle [T,X], T\rangle &= \langle \nabla_T X - \nabla_X T, T\rangle \\ &= -\langle X, \nabla_T T\rangle - \frac{1}{2} X[\langle T, T\rangle] \\ &= -\langle X, \cos\theta \, \lambda T\rangle - \frac{1}{2} X[1 - \cos^2\theta] \\ &= \cos\theta \, X[\cos\theta] \\ &= -\cos\theta \langle \mu X, T\rangle \\ &= 0. \end{aligned}$$

This implies that

$$\begin{aligned} R^\perp(T,X)N &= \nabla^\perp_T \nabla^\perp_X N - \nabla^\perp_X \nabla^\perp_T N - \nabla^\perp_{[T,X]} N \\ &= -\nabla^\perp_X (\varepsilon \cos\theta \, \sin^2\theta \, N) \\ &= -\varepsilon X[\cos\theta \sin^2\theta] N, \end{aligned}$$

which is zero because θ is constant in all directions orthogonal to T, due to (3). We conclude that $R^\perp = 0$. QED

5.3 Rotation hypersurfaces which are minimal

Minimal rotation surfaces in $\mathbb{E}^3$ are catenoids, whereas minimal rotation hypersurfaces of $\mathbb{E}^{n+1}$ are generalized catenoids in the sense of Blair, see for example [1]. The following theorem gives all minimal rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$.

6 Theorem. *Let M^n be a minimal rotation hypersurface of $\mathbb{S}^n \times \mathbb{R}$, with axis P^2 as above. Then the profile curve is either the vertical line*

$$\alpha(s) = (0, \ldots, 0, 1, s)$$

or it is given by

$$\alpha(s) = \left(\cos(s),\ 0,\ \ldots,\ 0,\ \sin(s),\ \int_{s_0}^{s} \frac{C}{\sqrt{\sin(\sigma)^{2(n-1)} - C^2}}\, d\sigma \right),$$

with $C \in \mathbb{R}$.

Proof. In order to find minimal rotation hypersurfaces, we have to solve the equation

$$\lambda + (n-1)\mu = 0. \tag{15}$$

If α is a vertical line, the equation reduces to $\cot(c) = 0$, which gives the first profile curve in the theorem. Remark that the resulting rotation hypersurface is totally geodesic.

If α is not a vertical line, equation (15) becomes

$$\frac{a''(s)}{(1 + a'(s)^2)^{3/2}} + (n-1)\frac{a'(s)\cot(s)}{(1 + a'(s)^2)^{1/2}} = 0,$$

which is equivalent to

$$\frac{a''(s)}{a'(s)(1 + a'(s)^2)} = -(n-1)\cot(s).$$

Integrating both sides of the equation yields

$$a'(s) = \frac{C}{\sqrt{\sin(s)^{2(n-1)} - C^2}}, \qquad C \in \mathbb{R}.$$

QED

In an analogous way, we can prove the following result.

7 Theorem. *Let M^n be a minimal rotation hypersurface of $\mathbb{H}^n \times \mathbb{R}$, with axis P^2 as above. Then the profile curve is described as follows, with $C \in \mathbb{R}$:*

(i) $\alpha(s) = \left(\cosh(s),\ 0,\ \ldots,\ 0,\ \sinh(s),\ \pm\displaystyle\int_{s_0}^{s} \frac{C}{\sqrt{\sinh(\sigma)^{2(n-1)} - C^2}}\, d\sigma\right)$ *if* P^2 *is Lorentzian,*

(ii) $\alpha(s) = \left(\cosh(s),\ 0,\ \ldots,\ 0,\ \sinh(s),\ \pm\displaystyle\int_{s_0}^{s} \frac{C}{\sqrt{\cosh(\sigma)^{2(n-1)} - C^2}}\, d\sigma\right)$ *or* $\alpha(s) = (1, 0, \ldots, 0, s)$ *if* P^2 *is Riemannian,*

(iii) $\alpha(s) = \left(s,\ 0,\ \ldots,\ 0,\ -\dfrac{1}{2s},\ \pm\displaystyle\int_{s_0}^{s} \frac{C}{\sigma\sqrt{\sigma^{2(n-1)} - C^2}}\, d\sigma\right)$*, with respect to the basis* $\{e_1, \ldots, e_{n+2}\}$ *defined above, if* P^2 *is degenerate.*

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