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Lifting the solutions of a Toeplitz type equation; the semigroup case $X = T_s X T_s^*$

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Abstract. A Toeplitz operator with respect to a contractive representation $\{T_s\}$ of an abelian semigroup Σ in a Hilbert space $\mathcal{H}$ is an operator $X \in \mathcal{B}(\mathcal{H})$ such that $X = T_s X T_s^*$ for all $s \in \Sigma$. We show that if $\{T_s\}$ has a minimal isometric dilation $\{U_s\} \subset \mathcal{B}(\mathcal{K})$, then Toeplitz operators can be obtained in a unique way as compressions of operators $Y \in \mathcal{B}(\mathcal{K})$, called Toeplitz symbols, such that $Y = U_s Y U_s^*$. This approach to lifting the Toeplitz equation $X = T_s X T_s^*$ is shown to be unitarily equivalent to the one proposed by Muhly in 1972. We use our approach to extend to this case a number of theorems about classical Toeplitz operators and, finally, we show that some classes of well-known operators —like Wiener-Hopf operators, Toeplitz operators in $H^2(\mathbb{T}^d)$ or Toeplitz operators in the sense of Murphy— fall within the class studied in this paper. The main objective of this paper is to provide a general framework that, we hope, will be useful in order to extend to wider classes of operators some of the more recent and deep advances in the theory of Toeplitz operators.

Keywords: Toeplitz operators, Wiener-Hopf operators, operator semigroups, minimal isometric dilation.

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To the memory of Klaus Floret, a fine mathematician and a good man

1 Introduction

Let $\{T_s\}$ be a semigroup of contractions having a minimal isometric dilation $\{U_s\}$. Can one lift simultaneously a collection of Toeplitz type equations $X = T_s X T_s^*$ to the corresponding collection $Y = U_s Y U_s^*$ and recover X as the compression of Y? It is widely known that the answer is affirmative for the semigroup $\{B^n : n = 0, 1, 2, \dots\}$, where B is the co-isometric unilateral backward shift in $H^2(\mathbb{T})$, and that the solutions of the simultaneous equations $X = B^n X B^{*n}$ are the classical Toeplitz operators.

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Every function $\phi \in L^\infty(\mathbb{T})$ defines a Toeplitz operator $T_\phi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ by means of $T_\phi f := P(\phi \cdot f)$ where P is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. In the language of dilation theory, T_ϕ is the compression of the multiplication operator M_ϕ induced by ϕ; this function ϕ is called the symbol of T_ϕ and it is unique. Toeplitz operators satisfy a simple characteristic relation given by Brown and Halmos in 1963 [6]; it tells us that a bounded linear map $X : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is a Toeplitz operator if, and only if, $X = BXB^*$, where B is the backward shift in $H^2(\mathbb{T})$. The development of the theory of Toeplitz operators in the last decades can be found in [2], [5], [7], [18] or [19].

It is no wonder that a number of possible generalizations of the notion of a Toeplitz operator have been proposed in the literature. We shall concentrate on generalizations made by exploiting the properties of the characteristic relation. Namely, take a contraction T defined on a Hilbert space $\mathcal{H}$ and study the properties of the operators $X : \mathcal{H} \to \mathcal{H}$, called generalized Toeplitz operators, that satisfy $X = TXT^*$. The purpose of this line of research —started by Douglas and Pearcy [13], Douglas [14] and [15], Rosenblum [41] and Sz.-Nagy and Foiaş [46] and [47], and continued by Muhly [26], Pták and Vrbová [39], Pták [38], two of the present authors [24] and [25], and Kérchy [22] and [23]— is two-fold: on one hand, to obtain new operators that share some of the properties that make the class of Toeplitz operators important and, on the other hand, to find out which of these interesting properties of the classical case depend only on the characteristic relation. Hyponormal operators and operators with one dimensional self-commutators are examples of generalized Toeplitz operators (see, respectively, [46] and [47], and [10]).

To study up to what point the properties of classical Toeplitz operators are valid within this approach, an essential problem is to find out what operators play the role of symbols. In the classical case, the symbols are multiplication operators induced by functions from $L^\infty(\mathbb{T})$ or, equivalently, operators Y such that $Y = UYU^*$ where U is the unitary bilateral backward shift in $L^2(\mathbb{T})$. From the point of view of dilation theory, U is both the minimal isometric dilation and the minimal unitary dilation of B, and U^* is the minimal unitary extension of the unilateral forward shift $S = B^*$, hence the key step is lifting the solutions of $BXB^* = X$ to solutions of $UYU^* = Y$, and these families of solutions are one-to-one related by the fact that each X is the compression of a Y to $H^2(\mathbb{T})$, that is $X = PY|H^2(\mathbb{T})$. This suggests that in the generalized setting the symbols should be solutions of a suitable lifting of the equation $X = TXT^*$ involving the minimal isometric-or-unitary dilation-or-extension of T.

If T is a co-isometry, then the approaches taken by Douglas, Sz.-Nagy and Foiaş, and Pták and Vrbová are formally the same, but there is a slight difference in their points of view. They proved that if T is a co-isometry then every solution

X of $X = TXT^*$ is the compression of a solution Y of $Y = UYU^*$ where, for Sz.-Nagy, Foiaş, Pták and Vrbová, U is the minimal isometric dilation of T [47, Thm. 2], [39, Thm. 2.11], [38, Thm. 2.5], whereas for Douglas, U^* is the minimal unitary extension of T^* [14, Thm. 2]. These different points of view produce, however, different approaches for the case when T is a general contraction, namely

(DM) Douglas proposed the following approach. Let $A := \sqrt{\lim_n T^n T^{*n}}$ be the asymptotic modulus of T^* and denote by $\mathcal{M}$ the closure of the range of A. Then there is an isometry V defined on $\mathcal{M}$ and such that $VA = AT^*$ and every solution X of the equation $X = TXT^*$ can be represented in the form $X = AX'A$ where X' is a solution of the equation $X' = V^*X'V$. Now, since V^* is a co-isometry, each of those X' can be represented as the compression of a solution Z of the equation $Z = WZW^*$ where W^* is the minimal unitary extension of V. This approach was later used by Muhly [26] to show how to lift simultaneously a family of equations of the form $X = T_s X T_s^*$ where $\{T_s : s \in \Sigma\}$ is a contractive representation of an abelian semigroup Σ on a Hilbert space $\mathcal{H}$.

(SF) Sz.-Nagy and Foiaş proposed the following approach. Let $U \in \mathcal{B}(\mathcal{K})$ be the minimal isometric dilation of T, let $\mathcal{R}$ be the residual subspace of U, that is, the largest reducing subspace where U is unitary, and denote by $U|\mathcal{R}$ the unitary part of the Wold decomposition of U. Then every solution X of the equation $X = TXT^*$ can be obtained from a solution Y of the equation $Y = (U|\mathcal{R})Y(U|\mathcal{R})^*$, by means of the equality $X = P(\mathcal{H})YP(\mathcal{R})|\mathcal{H}$, where $P(\mathcal{H})$ and $P(\mathcal{R})$ denote the respective orthogonal projections from $\mathcal{K}$.

(PV) Pták and Vrbová proposed the following approach. With the same notation as in (SF), every solution X of the equation $X = TXT^*$ is uniquely given as the compression $X = P(\mathcal{H})Y|\mathcal{H}$ of a solution Y of the equation $Y = UYU^*$. Moreover, these operators Y are essentially defined in $\mathcal{R}$, in the sense that they satisfy the equalities $Y = P(\mathcal{R})Y = YP(\mathcal{R})$, and they commute with U and U^*.

As it is described in the last sentence, for the case of an arbitrary contraction T, the approaches (SF) and (PV) are essentially the same, although we prefer the latter because the relation between X and Y is simpler in (PV) due to the fact that $\mathcal{H}$ is a subspace of $\mathcal{K}$ but not necessarily a subspace of $\mathcal{R}$. Within this approach a number of theorems have been extended from the classical to the generalized case in [24] and [25], complementing the results obtained in [14], [15], [46], [47], [39] and [38].

However, the approach (DM) is not, at least a priori, the same as the others if T is not a co-isometry, and the starting purpose of this paper was to clarify the situation. Then we realized, inspired essentially by the techniques used in [38], that a (SF)-(PV) approach to lifting a family of equations $X = T_s X T_s^*$, where $\{T_s\}$ is a semigroup of contractions, could be developed; this is done in Section 2 and we want to thank Prof. R. Douglas for calling our attention to Muhly's paper [26] and for suggesting us to extend the (SF)-(PV) approach to the case of semigroups. In Section 3 we prove that our approach is unitarily equivalent to the approach (DM) used by Muhly. In Section 4 we give extensions to the semigroup case of several theorems about classical Toeplitz operators. Finally, in Section 5 we show that some classes of well-known operators —like Wiener-Hopf operators, Toeplitz operators in $H^2(\mathbb{T}^d)$ or Toeplitz operators in the sense of Murphy— fall within the class studied in this paper.

The main objective of this paper is, thus, to provide a general framework that, we hope, will be useful in order to extend to wider classes of operators some of the more recent and deep advances in the theory of Toeplitz operators. We thank the referee for his helpful suggestions.

Notations. Our terminology and notations will be mostly standard, e.g., given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we shall denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all operators (bounded linear mappings) from $\mathcal{H}_1$ into $\mathcal{H}_2$ or simply $\mathcal{B}(\mathcal{H}_1)$ if $\mathcal{H}_1 = \mathcal{H}_2$. The closure $\overline{X(\mathcal{H}_1)}$ of the range $X(\mathcal{H})$ of an operator $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ will be denoted by ran(X) and its kernel by ker(X). The spectrum, right-spectrum and left-spectrum of X will be denoted, respectively, by $\sigma(X)$, $\sigma_r(X)$ and $\sigma_l(X)$. If $\mathcal{H}$ is a closed subspace of $\mathcal{K}$, the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$ will be denoted by $P(\mathcal{H})$. In general, we refer the reader to the excellent books by Böttcher and Silbermann [7], Douglas [16], Halmos [20] and [21], Nikolski [35], Sz.-Nagy and Foiaş [45] and Young [50].

2 Toeplitz operators with respect to a semigroup

Operator representations of semigroups. Let Σ be an abelian (additive) semigroup with identity e; we regard Σ as a directed set by saying that $r \leq s$ if $s = r + q$ for some $q \in \Sigma$. A family of contractions (respectively isometries, co-isometries, unitary operators) $\{T_s : s \in \Sigma\} \subset \mathcal{B}(\mathcal{H})$ is said to be a *contractive* (respectively *isometric, co-isometric, unitary*) *representation* of the semigroup Σ in the Hilbert space $\mathcal{H}$ if T_e is the identity operator id$(\mathcal{H})$ on $\mathcal{H}$ and $T_{r+s} = T_r T_s$ for all $r, s \in \Sigma$.

An isometric representation $\{U_s : s \in \Sigma\}$ of Σ in $\mathcal{K}$ is said to be an *isometric dilation* of a contractive representation $\{T_s : s \in \Sigma\}$ in $\mathcal{H}$ if $\mathcal{H}$ is a subspace of $\mathcal{K}$ and $T_s = P(\mathcal{H}) U_s | \mathcal{H}$ for all $s \in \Sigma$, and is said to be *minimal* if $\mathcal{K}$ is the smallest subspace containing $\mathcal{H}$ and U_s-invariant for all $s \in \Sigma$ or, equivalently, if $\mathcal{K} = \bigvee_{s\in\Sigma} U_s\mathcal{H}$.

For instance, given any contraction $T \in \mathcal{B}(\mathcal{H})$, the family $\{T_n := T^n : n = 0, 1, 2, \dots\}$ is a contractive representation of the (additive) semigroup $\mathbb{Z}_+$ of non-negative integers in $\mathcal{H}$, and if $U \in \mathcal{B}(\mathcal{K})$ is the minimal isometric dilation of T, then $\{U_n := U^n : n = 0, 1, 2, \dots\}$ is a minimal isometric dilation of $\{T^n : n \in \mathbb{Z}_+\}$; in this case, all the minimal isometric dilations are the same up to unitary equivalence. However, it is not always the case that a contractive representation $\{T_s : s \in \Sigma\}$ of an arbitrary semigroup Σ has a minimal isometric dilation, and it may also happen that two minimal isometric dilations of the same representation are not isomorphic. Although we refer the reader to [45, I.6–I.9], where this topic is discused in detail, let us mention the following examples of contractive representations $\{T_s : s \in \Sigma\}$ of a semigroup Σ that have a minimal isometric dilation: **(a)** $\Sigma = \mathbb{Z}_+$, **(b)** $\Sigma = \big(\mathbb{Z}_+\big)^2$, **(c)** Σ is the additive semigroup $\mathbb{R}_+$ of all non-negative real numbers and $\{T_s : s \geq 0\}$ is continuous (that is, $\lim_{s\to 0^+} T_s = T_0$ in the strong operator topology), in this case $\{T_s : s \geq 0\}$ is usually called a *continuous one-parameter semigroup of contractions,* **(d)** $\{T_s : s \in \Sigma\}$ is a co-isometric representation (in which case the elements of a minimal isometric dilation $\{U_s\}$ are, in fact, unitary; namely, $\{U_s^*\}$ is the minimal unitary extension of the isometric representation $\{T_s^*\}$), and **(e)** $\{T_s : s \in \Sigma\}$ is doubly commuting (that is, $T_sT_r^* = T_r^*T_s$ for all $r, s \in \Sigma$). For more recent advances on the existence of minimal isometric or unitay dilations of a semigroup and related problems about lifting operator equalities and inequalities, we refer the interested reader to [3,4,8,9,17,34,48], and [49].

In what follows, Σ stands for a fixed abelian semigroup with identity e and $\{T_s\}$ for a contractive representation of Σ in a Hilbert space $\mathcal{H}$ having a minimal isometric dilation $\{U_s\} \subset \mathcal{B}(\mathcal{K})$.

1 Lemma. *For every $s \in \Sigma$ the following assertions hold true:*

(1) $P(\mathcal{H})U_sk = T_sP(\mathcal{H})k$ *for all* $k \in \mathcal{K}$.

(2) $U_s\mathcal{H}^\perp \subset \mathcal{H}^\perp$.

(3) $U_s^*\mathcal{H} \subset \mathcal{H}$.

(4) $U_s^*|\mathcal{H} = T_s^*$.

Proof. Since $\mathcal{K} = \bigvee_{r\in\Sigma} U_r\mathcal{H}$, to prove that (1) holds it will be enough to check that $P(\mathcal{H})U_sU_rh = T_sP(\mathcal{H})U_rh$ for all $r \in \Sigma$ and $h \in \mathcal{H}$. But

$$P(\mathcal{H})U_sU_rh = P(\mathcal{H})U_{s+r}h = T_{s+r}h = T_sT_rh = T_sP(\mathcal{H})U_rh,$$

as desired. Now, a straightforward computation shows that the four assertions are, indeed, equivalent (part (2) is essentially proved in [43] as a matter of fact). $\boxed{QED}$

Toeplitz operators and Toeplitz symbols. A bounded operator $X : \mathcal{H} \to \mathcal{H}$ is said to be a *Toeplitz operator with respect to* $\{T_s\}$ if

$$X = T_s X T_s^* \qquad \text{for all} \;\; s \in \Sigma.$$

In this equation one could consider $X : \mathcal{H}_1 \to \mathcal{H}_2$, a $\{T_{1s}\}$ representation of Σ in $\mathcal{H}_1$ on the right, and a $\{T_{2s}\}$ representation of the same semigroup in $\mathcal{H}_2$ on the left. Muhly [26] showed how to reduce this apparently more general situation to the case when both representations are the same, namely, X is a solution of $X = T_{2s} X T_{1s}^*$ if, and only if, $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ is a solution of

$$\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{2s} & 0 \\ 0 & T_{1s} \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_{2s} & 0 \\ 0 & T_{1s} \end{bmatrix}^* .$$

However, this way of reducing the equation $X = T_{2s} X T_{1s}^*$ to the equation $X = T_s X T_s^*$ is not suitable to study some properties, like invertibility. For the sake of avoiding a cumbersome notation, we shall stick to the equations $X = T_s X T_s^*$ but the reader can check that all of the results contained in this paper are true, with the obvious changes, for equations $X = T_{2s} X T_{1s}^*$ (and we shall make explicit use of this fact in the proof of assertion (2) of Theorem 3 below).

When T is a contraction and $\{T^n\}$ is the corresponding representation of $\mathbb{Z}_+$, we recover the notion of generalized Toeplitz operator described in the Introduction.

An operator $Y : \mathcal{K} \to \mathcal{K}$ is said to be a *Toeplitz symbol with respect to* $\{T_s\}$ if $Y = U_s Y U_s^*$ for all $s \in \Sigma$, that is, if Y is a Toeplitz operator with respect to the minimal isometric dilation $\{U_s\}$ of $\{T_s\}$. The key point here is that if Y is a Toeplitz symbol with respect to $\{T_s\}$, then the compression $X = P(\mathcal{H}) Y | \mathcal{H}$ is a Toeplitz operator with respect to $\{T_s\}$ because, by using Lemma 1 twice, for all $s \in \Sigma$ and $h \in \mathcal{H}$ we have

$$T_s X T_s^* h = T_s X U_s^* h = T_s P(\mathcal{H}) Y U_s^* h = P(\mathcal{H}) U_s Y U_s^* h = P(\mathcal{H}) Y h = X h.$$

Our first main result is that there is a one-to-one and isometric relation between Toeplitz operators and Toeplitz symbols.

2 Theorem. *If $X \in \mathcal{B}(\mathcal{H})$ is a Toeplitz operator with respect to $\{T_s\}$, then there exists a unique Toeplitz symbol $Y \in \mathcal{B}(\mathcal{K})$ such that $X = P(\mathcal{H}) Y | \mathcal{H}$. This symbol is given by*

$$Y k = \lim_{s \in \Sigma} U_s X P(\mathcal{H}) U_s^* k \qquad \textit{for all} \;\; k \in \mathcal{K}$$

and it satisfies that $\|Y\| = \|X\|$.

Proof. Since $\mathcal{K} = \bigvee_{r\in\Sigma} U_r\mathcal{H}$ and $\{U_s XP(\mathcal{H})U_s^* : s \in \Sigma\}$ is uniformly bounded, to prove that $\{U_s XP(\mathcal{H})U_s^* k : s \in \Sigma\}$ is a convergent net for each $k \in \mathcal{K}$, it will be enough to check that $\{U_s XP(\mathcal{H})U_s^*(U_r h) : s \in \Sigma\}$ converges for all $r \in \Sigma$ and $h \in \mathcal{H}$. Fix $r \in \Sigma$ and take $s \geq r$ so that $s = q + r$ for some $q \in \Sigma$. Now, define

$$\begin{aligned} k_q :=& U_s XP(\mathcal{H})U_s^* U_r h = U_{q+r} XP(\mathcal{H})U_{q+r}^* U_r h \\ =& U_{q+r} XP(\mathcal{H})U_q^* U_r^* U_r h = U_{q+r} XP(\mathcal{H})U_q^* h = U_{q+r} XT_q^* h. \end{aligned}$$

Let us see that $\langle k_{p+q}, k_q\rangle = \langle k_q, k_q\rangle$ for all $p \in \Sigma$. Indeed,

$$\begin{aligned} \langle k_{p+q}, k_q\rangle &= \left\langle U_{p+q+r} XT_{p+q}^* h, U_{q+r} XT_q^* h\right\rangle = \left\langle U_p XT_{p+q}^* h, XT_q^* h\right\rangle \\ &= \left\langle P(\mathcal{H})U_p XT_{p+q}^* h, XT_q^* h\right\rangle = \left\langle T_p XT_p^* T_q^* h, XT_q^* h\right\rangle \\ &= \left\langle XT_q^* h, XT_q^* h\right\rangle = \left\langle U_{q+r} XT_q^* h, U_{q+r} XT_q^* h\right\rangle = \langle k_q, k_q\rangle . \end{aligned}$$

Now, it follows that $\|k_{p+q} - k_q\|^2 = \|k_{p+q}\|^2 - \|k_q\|^2$ and this shows that $\{\|k_q\|^2 : q \in \Sigma\}$ is an increasing net of real numbers. Since this net is also bounded by $\|h\|^2$, it is convergent. The equality $\|k_{p+q} - k_q\|^2 = \|k_{p+q}\|^2 - \|k_q\|^2$ tells us now that $\{k_q : q \in \Sigma\}$ is a Cauchy net in $\mathcal{K}$, hence it converges and this finishes the proof that

$$Yk = \lim_{s\in\Sigma} U_s XP(\mathcal{H})U_s^* k \qquad \text{for all } k \in \mathcal{K}$$

defines a bounded operator in $\mathcal{K}$ such that $\|Y\| \leq \|X\|$.

Let us see now that X is the compression of Y to $\mathcal{H}$. Indeed, by Lemma 1, for all $h \in \mathcal{H}$ and $s \in \Sigma$ we have

$$P(\mathcal{H})U_s XP(\mathcal{H})U_s^* h = T_s XT_s^* h = Xh,$$

hence $P(\mathcal{H})Yh = \lim_{s\in\Sigma} P(\mathcal{H})U_s XP(\mathcal{H})U_s^* h = Xh$. It also follows $\|X\| \leq \|Y\|$ so that, as a matter of fact, $\|X\| = \|Y\|$.

To prove that Y is a Toeplitz symbol, simply note that for each $r \in \Sigma$ and $k \in \mathcal{K}$ we have

$$U_r Y U_r^* k = \lim_{s\in\Sigma} U_r U_s XP(\mathcal{H})U_s^* U_r^* k = \lim_{s\in\Sigma} U_{r+s} XP(\mathcal{H})U_{r+s}^* k = Yk.$$

Finally, to prove the uniqueness of the symbol, asume that Z is a Toeplitz symbol such that $X = P(\mathcal{H})Z|\mathcal{H}$. Then, for all $s \in \Sigma$ and $k \in \mathcal{K}$ we have

$$\begin{aligned} Zk &= U_s Z U_s^* k = U_s\big[P(\mathcal{H}) + P(\mathcal{H}^\perp)\big] Z U_s^* k \\ &= U_s P(\mathcal{H}) Z\big[P(\mathcal{H}) + P(\mathcal{H}^\perp)\big] U_s^* k + U_s P(\mathcal{H}^\perp) Z U_s^* k \\ &= U_s P(\mathcal{H}) Z P(\mathcal{H}) U_s^* k + U_s P(\mathcal{H}) Z P(\mathcal{H}^\perp) U_s^* k + U_s P(\mathcal{H}^\perp) Z U_s^* k \\ &= U_s XP(\mathcal{H}) U_s^* k + U_s P(\mathcal{H}) Z P(\mathcal{H}^\perp) U_s^* k + U_s P(\mathcal{H}^\perp) U_s^* Z k. \end{aligned}$$

If we prove that $\lim_{s\in\Sigma} P(\mathcal{H}^\perp)U_s^* = 0$ in $\mathcal{K}$, this will imply that the last two summands of the last line of the chain of equalities displayed above are convergent to zero and, therefore, $Z = \lim_{s\in\Sigma} U_s X P(\mathcal{H}) U_s^* = Y$. But, by using once again that $\mathcal{K} = \bigvee_{r\in\Sigma} U_r\mathcal{H}$, in order to prove that $\lim_{s\in\Sigma} P(\mathcal{H}^\perp)U_s^* = 0$ in $\mathcal{K}$, it will be enough to check up on elements of the form $U_r h$ with $h \in \mathcal{H}$. Indeed, for $s \geq r$, we have $P(\mathcal{H}^\perp)U_s^* U_r h = P(\mathcal{H}^\perp)U_{s-r}^* h = 0$ because, according to Lemma 1, the space $\mathcal{H}$ is U_s^*-invariant for all $s \in \Sigma$. $\boxed{QED}$

As we mentioned above, when we have a contractive representation $\{T^n\}$ of $\Sigma = \mathbb{Z}_+$, the subspace $\mathcal{R}$, the residual part of the Wold decomposition of the minimal isometric dilation U of T, plays an essential role because, in that case, the Toeplitz symbols Y are essentially defined in $\mathcal{R}$ in the sense that $Y = P(\mathcal{R})Y = YP(\mathcal{R})$. In the general semigroup case, as we are about to see (Theorem 2 below), we have the same situation: the symbol is essentially defined in the unitary part $\mathcal{R}$ of the Wold decomposition of a semigroup of isometries introduced by Suciu [44] as follows. Since U_s is an isometry, we have that $U_sU_s^*$ is a projection for each $s \in \Sigma$. Therefore, $\{U_sU_s^* : s \in \Sigma\}$ is a decreasing net of commuting projections that converges strongly to the orthogonal projection $P(\mathcal{R})$ onto the subspace defined by $\mathcal{R} := \bigcap_{s\in\Sigma} U_s\mathcal{K}$. Moreover, $\mathcal{R}$ is U_s-reducing and $U_s|\mathcal{R}$ is unitary for all $s \in \Sigma$ [44, Thm. 1]. (The Wold decomposition of a semigroup of isometries has three parts called unitary, totally non-unitary or shift, and strange or evanescent; we refer the interested reader to [44] and [27]). Let us note at this point that if $\{T_s\}$ is a co-isometric representation of Σ, then each U_s is unitary, hence $\mathcal{R} = \mathcal{K}$. This shows that the importance of the role played by $\mathcal{R}$ is hidden in the classical case because if T is the backward shift B in $H^2(\mathbb{T})$, then $\mathcal{R} = \mathcal{K} = L^2(\mathbb{T})$. In our case, each single U_s might not be the minimal isometric dilation of the corresponding T_s and $\mathcal{R}$ might not be the residual subspace of the Wold decomposition of U_s but, nevertheless, this subspace $\mathcal{R}$ has similar properties, so we shall call it *the residual subspace* of the minimal isometric dilation $\{U_s\}$ of $\{T_s\}$. We record now for later use some of the properties of the representations $\{T_s\}$ and $\{U_s\}$ related to $\mathcal{R}$ (see, e.g., [38, Lemmas 2.3 and 2.4] and [45, II.3] for the case of a single contraction).

3 Lemma. *Let $\mathcal{R} := \bigcap_{t\in\Sigma} U_t\mathcal{K}$ be the residual subspace of the minimal isometric dilation $\{U_s\}$ of $\{T_s\}$. Let $\mathcal{P}$ be the closure of $P(\mathcal{R})\mathcal{H}$. Then the following assertions hold:*

(1) *For each $s \in \Sigma$, the residual subspace $\mathcal{R}$ is a U_s-reducing subspace of $\mathcal{K}$ contained in the residual subspace of the Wold decomposition of U_s.*

(2) *$U_s|\mathcal{R}$ is a unitary operator.*

(3) *$\mathcal{R} = \mathcal{P} \oplus (\mathcal{R} \cap \mathcal{H}^\perp)$, where $\mathcal{H}^\perp$ is the orthocomplement of $\mathcal{H}$ in $\mathcal{K}$.*

(4) *$P(\mathcal{P})h = P(\mathcal{R})h$ for every $h \in \mathcal{H}$.*

(5) *$P(\mathcal{H})P(\mathcal{P}) = P(\mathcal{H})P(\mathcal{R})$.*

(6) *For all $h \in \mathcal{H}$ and $s \in \Sigma$ the following chain of equalities holds*

$$U_s^* P(\mathcal{P})h = P(\mathcal{P})U_s^* h = P(\mathcal{P})T_s^* h = U_s^* P(\mathcal{R})h = P(\mathcal{R})U_s^* h = P(\mathcal{R})T_s^* h.$$

(7) *$\mathcal{P}$ is U_s^*-invariant and $U_s^*|\mathcal{P}$ is an isometry.*

(8) *For each $s \in \Sigma$, define the co-isometry $R_s := (U_s^*|\mathcal{P})^* \in \mathcal{B}(\mathcal{P})$. Then $\{R_s\}$ is a co-isometric representation of Σ in $\mathcal{P}$ and $\{U_s|\mathcal{R}\}$ is a minimal isometric (unitary, in fact) dilation of $\{R_s\}$.*

Proof. (1) Since $\mathcal{R} = \bigcap_{t\in\Sigma} U_t\mathcal{K}$, it is clear that $\mathcal{R}$ is U_s-invariant. Now, given an element $k = \lim_t U_t U_t^* k \in \mathcal{R}$, we have

$$U_s^* k = \lim_t U_s^* U_t U_t^* k = \lim_r U_s^* U_{s+r} U_{s+r}^* k = \lim_r U_r U_r^* U_s^* k = P(\mathcal{R})U_s^* k,$$

and it follows that $\mathcal{R}$ is U_s^*-invariant. Finally, if $\mathcal{R}_s$ is the residual subspace of the isometry U_s then, as it is well-known,

$$\mathcal{R}_s = \bigcap_{n\in\mathbb{N}} U_s^n \mathcal{K} = \bigcap_{n\in\mathbb{N}} U_{ns}\mathcal{K} \supset \bigcap_{t\in\Sigma} U_t\mathcal{K} = \mathcal{R}.$$

(2) follows from (1).

(3) For every $x \in \mathcal{R}$ and $h \in \mathcal{H}$ we have

$$\langle x, h\rangle = \langle P(\mathcal{R})x, h\rangle = \langle x, P(\mathcal{R})h\rangle$$

so it follows, by using that $P(\mathcal{R})\mathcal{H}$ is dense in $\mathcal{P}$, that $\langle x, \mathcal{P}\rangle = 0$ if, and only if, $x \in \mathcal{H}^\perp$.

Noting that $P(\mathcal{R} \cap \mathcal{H}^\perp)h = 0$ for all $h \in \mathcal{H}$, we have that (4) follows from (3).

(5) also follows from (3) because

$$P(\mathcal{H})P(\mathcal{R}) = P(\mathcal{H})\big[P(\mathcal{P}) + P(\mathcal{R} \cap \mathcal{H}^\perp)\big] = P(\mathcal{H})P(\mathcal{P}).$$

(6) By Lemma 1, $U_s^* h = T_s^* h \in \mathcal{H}$. By using now assertions (1) and (4), we have

$$U_s^* P(\mathcal{P})h = U_s^* P(\mathcal{R})h = P(\mathcal{R})U_s^* h = P(\mathcal{P})U_s^* h = P(\mathcal{P})T_s^* h.$$

By using (5) we obtain the whole chain of equalities.

(7) follows from (6) and (2).

(8) Since $\mathcal{P}$ is U_s^*-invariant for each $s \in \Sigma$ we have that

$$R_{s+t} = \left(U_{s+t}^*|\mathcal{P}\right)^* = \left(U_t^* U_s^*|\mathcal{P}\right)^* = \\ = \left((U_t^*|\mathcal{P})(U_s^*|\mathcal{P})\right)^* = (U_s^*|\mathcal{P})^*(U_t^*|\mathcal{P})^* = R_s R_t.$$

Trivially, R_e is the identity on $\mathcal{P}$, hence $\{R_s\}$ is a co-isometric representation of Σ in $\mathcal{P}$.

Now, to prove that $R_s = P(\mathcal{P})U_s|\mathcal{P}$, it will be enough to prove that the equality holds for elements of the form $P(\mathcal{R})h$ with $h \in \mathcal{H}$. But

$$R_s P(\mathcal{R})h = (U_s^*|\mathcal{P})^* P(\mathcal{R})h = P(\mathcal{P})U_s P(\mathcal{P})P(\mathcal{R})h = P(\mathcal{P})U_s P(\mathcal{R})h.$$

Finally, since $\mathcal{K} = \bigvee_{s\in\Sigma} U_s\mathcal{H}$, we have, by using that $\mathcal{R}$ is U_s-reducing for all $s \in \Sigma$, that

$$\mathcal{R} = P(\mathcal{R})\mathcal{K} = \bigvee_{s\in\Sigma} P(\mathcal{R})U_s\mathcal{H} = \bigvee_{s\in\Sigma} U_s P(\mathcal{R})\mathcal{H} = \bigvee_{s\in\Sigma} U_s\mathcal{P}.$$

This finishes the proof that $\{U_s|\mathcal{R}\}$ is a minimal isometric dilation of $\{R_s\}$. $\boxed{QED}$

4 Theorem. (1) *If Y is a Toeplitz symbol with respect to $\{T_s : s \in \Sigma\}$, then Y is an operator defined essentially in $\mathcal{R} = \bigcap_{s\in\Sigma} U_s\mathcal{K}$ in the sense that $Y = YP(\mathcal{R}) = P(\mathcal{R})Y$ and Y commutes with both U_s and U_s^* for all $s \in \Sigma$.*

(2) *The compression $Y' = P(\mathcal{P})Y|\mathcal{P}$ is a Toeplitz operator with respect to the co-isometric representation $\{R_s\}$ of Σ in $\mathcal{P}$. Conversely, if Y' is a Toeplitz operator with respect to $\{R_s\}$ and $Y \in \mathcal{B}(\mathcal{R})$ is its Toeplitz symbol with respect to $\{R_s\}$, then the operator $P(\mathcal{R})YP(\mathcal{R}) \in \mathcal{B}(\mathcal{K})$ is a Toeplitz symbol with respect to $\{T_s\}$.*

Proof. (1) Since every U_s is an isometry and we have that $Y = U_s Y U_s^*$, it follows that $YU_s = U_s Y$ and that $U_s^* Y = YU_s^*$. Therefore,

$$Y = U_s Y U_s^* = Y U_s U_s^* \qquad \text{for all } s \in \Sigma.$$

Taking limits as $s \in \Sigma$, we obtain $Y = YP(\mathcal{R})$. The equality $P(\mathcal{R})Y = Y$ can be proved analogously.

(2) Since $Y = P(\mathcal{R})YP(\mathcal{R})$ and, according to Lemma 2 above, $\{U_s|\mathcal{R}\}$ is a minimal isometric dilation of $\{R_s\}$ and $\mathcal{R}$ is U_s-reducing for each $s \in \Sigma$, the equality $Y = U_s Y U_s^*$ implies that

$$Y|\mathcal{R} = (U_s|\mathcal{R})Y(U_s|\mathcal{R})^* = (U_s|\mathcal{R})(Y|\mathcal{R})(U_s|\mathcal{R})^*,$$

hence $Y|\mathcal{R} = P(\mathcal{R})(Y|\mathcal{R})P(\mathcal{R})$ is a Toeplitz symbol with respect to $\{R_s\}$. It follows now from Theorem 1 that $Y' = P(\mathcal{P})Y|\mathcal{P}$ is a Toeplitz operator with respect to the co-isometric representation $\{R_s\}$ of Σ in $\mathcal{P}$.

Conversely, if Y' is a Toeplitz operator with respect to $\{R_s\}$ and $Y \in \mathcal{B}(\mathcal{R})$ is its Toeplitz symbol with respect to $\{R_s\}$, then $Y = (U_s|\mathcal{R})Y(U_s|\mathcal{R})^*$. Now, by using again that $\mathcal{R}$ is U_s-reducing, we have

$$P(\mathcal{R})YP(\mathcal{R}) = U_sP(\mathcal{R})YP(\mathcal{R})U_s^*$$

hence $P(\mathcal{R})YP(\mathcal{R}) \in \mathcal{B}(\mathcal{K})$ is a Toeplitz symbol with respect to $\{T_s\}$. $\boxed{QED}$

5 Corollary. *There is a one-to-one and isometric relation between the set of all Toeplitz operators with respect to a contractive representation $\{T_s\}$ and the set of all Toeplitz operators with respect to the co-isometric representation $\{R_s\}$ defined by $R_s := (U_s^*|\mathcal{P})^*$ for each $s \in \Sigma$, and related Toeplitz operators share, essentially, the same symbol.*

Analytic Toeplitz operators. Since the set of classical analytic symbols is $H^\infty(\mathbb{T})$ and this space can be seen as the set of symbols such that the associated Toeplitz operator commutes with the forward shift S or, also, as the set of symbols that leave $H^2(\mathbb{T})$ invariant, we may keep the analogy with the classical and say that a Toeplitz symbol Y is *analytic* if $Y(\mathcal{H}) \subset \mathcal{H}$. Thus a Toeplitz symbol Y is analytic when $P(\mathcal{H}^\perp)Y|\mathcal{H} = 0$. A Toeplitz operator X is said to be *analytic* if its symbol Y is analytic and, in that case, X is simply the restriction of Y, i.e. $X = Y|\mathcal{H}$. For the case when $\{T_s\}$ is a co-isometric representation of Σ, Douglas [15] proved the existence of (what we call here) analytic Toeplitz symbols and also that a Toeplitz operator X is analytic if, and only if, $XT_s^* = T_s^*X$ for each $s \in \Sigma$ as in the classical case. For the general case, we shall need the following

6 Lemma. *Let $X : \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then X is an analytic Toeplitz operator with respect to $\{T_s\}$ if, and only if, $XT_s^* = T_s^*X$ for all $s \in \Sigma$ and* $\operatorname{ran}(X) \subset \mathcal{H} \cap \mathcal{R}$.

Proof. Assume that X is an analytic Toeplitz operator with respect to $\{T_s\}$ and let Y be the symbol of X. Since, by Theorem 2, $Y = YP(\mathcal{R}) = P(\mathcal{R})Y$, if Y is analytic, then $X = Y|\mathcal{H} \subset \mathcal{R}$, hence $X(\mathcal{H}) \subset \mathcal{H} \cap \mathcal{R}$. Moreover, by using Lemma 1 and Theorem 2, for all $h \in \mathcal{H}$ and $s \in \Sigma$ we have

$$XT_s^*h = YU_s^*h = U_s^*Yh = U_s^*Xh = T_s^*Xh.$$

Conversely, assume that $XT_s^* = T_s^*X$ for all $s \in \Sigma$ and that $\operatorname{ran}(X) \subset \mathcal{H}\cap\mathcal{R}$. Let us check first that X is a Toeplitz operator with respect to $\{T_s\}$. By using

Lemma 1 (4), we have that

$$T_s X T_s^* = T_s T_s^* X = T_s U_s^* X.$$

Now, by using that $\mathrm{ran}(X) \subset \mathcal{H} \cap \mathcal{R} \subset \mathcal{P} \subset \mathcal{R}$ and Lemma 2 (6), the chain of equalities continues

$$T_s X T_s^* = T_s U_s^* X = T_s U_s^* P(\mathcal{P}) X = T_s P(\mathcal{R}) T_s^* X = P(\mathcal{H}) U_s P(\mathcal{R}) U_s^* X.$$

Finally, by using that $\mathcal{R}$ is reducing for U_s, that U_s is unitary on $\mathcal{R}$ and, again, that $\mathrm{ran}(X) \subset \mathcal{H} \cap \mathcal{R}$, we conclude

$$T_s X T_s^* = P(\mathcal{H}) U_s P(\mathcal{R}) U_s^* X = P(\mathcal{H}) P(\mathcal{R}) U_s U_s^* X = P(\mathcal{H}) P(\mathcal{R}) X = X.$$

This shows that X is a Toeplitz operator such that $X T_s^* = T_s^* X$ for all $s \in \Sigma$. To prove that X is analytic proceed as follows. For each $h \in \mathcal{H}$ we have, by using Lemma 1 and Theorem 1, that

$$Yh = \lim_{s \in \Sigma} U_s X P(\mathcal{H}) U_s^* h = \lim_{s \in \Sigma} U_s X T_s^* h = \lim_{s \in \Sigma} U_s T_s^* X h.$$

Since $\mathrm{ran}(X) \subset \mathcal{H} \cap \mathcal{R}$, we have $X T_s^* h = T_s^* X h = U_s^* X h \in \mathcal{R}$ and, by using that $U|\mathcal{R}$ is unitary in the U_s-reducing subspace $\mathcal{R}$, the computation above yields

$$Yh = \lim_{s \in \Sigma} U_s U_s^* X h = Xh \in \mathcal{H}.$$

Therefore, $Y\mathcal{H} \subset \mathcal{H}$ and this proves that Y is analytic. $\boxed{QED}$

Note that if every T_s is a co-isometry, then the equality $X T_s^* = T_s^* X$ implies immediately that $X = T_s X T_s^*$, hence any operator commuting with each T_s^* is a Toeplitz operator. Since, in this case we also have that $\mathcal{R} = \mathcal{K}$ so that $\mathcal{R} \cap \mathcal{H} = \mathcal{H}$, Lemma 3 tells us, in particular, that if every T_s is a co-isometry, then X is an analytic Toeplitz operator if, and only if, $X T_s^* = T_s^* X$, so we recover Douglas's result [15, Thm. 2] quoted above.

3 Comparison to Muhly's approach

Description of Muhly's approach. As we mentioned above, Muhly [26] extended to the semigroup case the lifting of a Toeplitz type equation proposed by Douglas [14]. We start by describing this approach, and the first thing we must mention is that Muhly studies the equation $T_s^* X T_s = X$ rather than $X = T_s X T_s^*$, so our description of his results from that paper will be adapted to the convention we are using here.

The (DM) lifting is a two-step procedure. First step [26, Thm. I]: Let A be the positive square root of the strong limit of the net $\{T_s T_s^* : s \in \Sigma\}$ and denote by $\mathcal{M}$ the closure of the range of A, then there is an isometric representation $\{V_s\}$ of Σ in $\mathcal{M}$ such that $V_s A = AT_s^*$ for each $s \in \Sigma$. This isometric representation has the following property: an operator $X \in \mathcal{B}(\mathcal{H})$ is a Toeplitz operator with respect to $\{T_s\}$ if, and only if, there is an operator $X' \in \mathcal{B}(\mathcal{M})$ such that (i) $\|X'\| = \|X\|$, (ii) X' is a Toeplitz operator with respect to $\{V_s^*\}$, and (iii) $X = AX'A$. Although this is not explicitely mentioned in [26], it is easy to check that the relationship between X and X' is one-to-one. Simply note that since A is an operator from $\mathcal{H}$ into $\mathcal{M}$, the equality $X = AX'A$ must be read as $X = (A|\mathcal{M})X'A$, so that if $X = 0$ then $(A|\mathcal{M})X'(A\mathcal{H}) = 0$ hence $(A|\mathcal{M})X' = 0$ and, taking adjoints, $0 = (X')^* P(\mathcal{M})A|\mathcal{H}$, so that $X' = 0$.

Second step [26, Thm. II]: Let $\{W_s\} \subset \mathcal{B}(\mathcal{N})$ be the minimal unitary dilation of the co-isometric representation $\{V_s^*\}$ (in other words, $\{W_s^*\}$ is the minimal unitary extension of the isometric representation $\{V_s\}$). Then an operator $X' \in \mathcal{B}(\mathcal{M})$ is a Toeplitz operator with respect to $\{V_s^*\}$ if, and only if, there is a unique operator $Z \in \mathcal{B}(\mathcal{N})$ such that (i) $\|Z\| = \|X'\|$, (ii) Z is a Toeplitz operator with respect to $\{W_s\}$, and (iii) X' is the compression of Z to $\mathcal{M}$, i.e., $X' = P(\mathcal{M})Z|\mathcal{M}$. Plainly, this second step can be seen as our Theorem 1 above applied to the particular case when every T_s is a co-isometry.

Equivalence of both approaches. These two steps yield a representation of a Toeplitz operator X as $X = AP(\mathcal{M})ZA$, where Z is a Toeplitz operator with respect to a unitary representation of Σ such that $\|Z\| = \|X\|$. Our Theorem 1 provides a representation of a Toeplitz operator X as $X = P(\mathcal{H})Y|\mathcal{H}$, where Y is a Toeplitz operator with respect to the minimal isometric dilation of $\{T_s\}$ such that $\|Y\| = \|X\|$. The relationship between Y and Z is linear, one-to-one and isometric, so both approaches are formally equivalent under the hypothesis, common to Lemma 4 and Theorem 3, of the existence of an isometric dilation for $\{T_s\}$. In Theorem 3 below we go further by showing an explicit unitary link between the Toeplitz operator $Y' \in \mathcal{B}(\mathcal{P})$ defined in Theorem 2 and $X' = P(\mathcal{M})Z|\mathcal{M}$.

Which approach should one use? Of course, unitary operators are simpler than isometries and (DM) approach works even if $\{T_s\}$ has no minimal isometric dilation. However, on the other hand, the relationship between X and Y is simpler than the relationship between X and Z and, as we shall see in Section 4, it enables us to obtain a number of results about the linking between properties of X and those of Y.

7 Lemma. *Let $\{T_s\}$ be a contractive representation of a semigroup Σ in a Hilbert space $\mathcal{H}$ having a minimal isometric dilation $\{U_s\} \subset \mathcal{B}(\mathcal{K})$, with residual subspace $\mathcal{R}$, and let $\mathcal{P}$ be the closure of $P(\mathcal{R})\mathcal{H}$. Then the following hold:*

(1) $A^2 = P(\mathcal{H})P(\mathcal{R})|\mathcal{H}$.

(2) *There exists a unitary operator* $E : \mathcal{M} \to \mathcal{P}$ *such that* $EAh = P(\mathcal{R})h$ *for all* $h \in \mathcal{H}$.

(3) *The unitary operator* E *intertwines the isometries* V_s *and* $U_s^*|\mathcal{P} = R_s^*$ *for all* $s \in \Sigma$.

Proof. To prove assertion (1), simply compute. For every $h \in \mathcal{H}$ we have, by using Lemma 1, that

$$A^2h = \lim_s T_sT_s^*h = \lim_s P(\mathcal{H})U_sU_s^*h = P(\mathcal{H})\lim_s U_sU_s^*h = P(\mathcal{H})P(\mathcal{R})h.$$

(2) Given an arbitrary element $h \in \mathcal{H}$, take $x = Ah \in A(\mathcal{H})$ and $y = P(\mathcal{R})h \in P(\mathcal{R})\mathcal{H}$, and define $E : A(\mathcal{H}) \to P(\mathcal{R})\mathcal{H}$ by $Ex := y$ or, in other terms, define E by the relation $EA = P(\mathcal{R})|\mathcal{H}$. First of all, let us note that E is well-defined because if $Ah_1 = Ah_2$ then $A^2(h_1 - h_2) = 0$ so that, by assertion (1), $P(\mathcal{H})P(\mathcal{R})(h_1 - h_2) = 0$. This proves that $P(\mathcal{R})(h_1 - h_2) \in \mathcal{H}^\perp \cap \mathcal{R}$, where $\mathcal{H}^\perp$ is the orthocomplement of $\mathcal{H}$ in $\mathcal{K}$. But, as we have seen in Lemma 2 above, $\mathcal{R} \cap \mathcal{H}^\perp$ is the orthocomplement of $\mathcal{P}$ in $\mathcal{R}$, hence $P(\mathcal{R})(h_1 - h_2) \in \mathcal{R} \ominus \mathcal{P}$. On the other hand, $P(\mathcal{R})(h_1 - h_2) \in \mathcal{P}$ by definition, thus $P(\mathcal{R})h_1 = P(\mathcal{R})h_2$ and this shows that E is a well-defined linear operator from $A(\mathcal{H})$ into $P(\mathcal{R})\mathcal{H}$. Let us see now that E is an isometry. By using assertion (1) again, we have

$$\begin{aligned} \|x\|^2 = \|Ah\|^2 &= \left\langle A^2h, h \right\rangle = \langle P(\mathcal{H})P(\mathcal{R})h, h\rangle \\ &= \langle P(\mathcal{R})h, h\rangle = \left\langle P(\mathcal{R})^2h, h\right\rangle = \|P(\mathcal{R})h\|^2 = \|y\|^2. \end{aligned}$$

Therefore, E can be extended to a unitary operator $E : \mathcal{M} \to \mathcal{P}$ such that $EAh = P(\mathcal{R})h$ for all $h \in \mathcal{H}$.

(3) We have to prove that $EV_s = (U_s^*|\mathcal{P})E$ for all $s \in \Sigma$. By density, it will be enough to check that $EV_sAh = (U_s^*|\mathcal{P})EAh$ for all $h \in \mathcal{H}$. Since, as we mentioned above, $U_s^*P(\mathcal{R})h = P(\mathcal{R})T_s^*h$, we have

$$EV_sAh = EAT_s^*h = P(\mathcal{R})T_s^*h = U_s^*P(\mathcal{R})h = (U_s^*|\mathcal{P})EAh.$$

This finishes the proof. $\boxed{QED}$

8 Theorem. *Let* $\{T_s\}$ *be a contractive representation of a semigroup* Σ *in a Hilbert space* $\mathcal{H}$ *having a minimal isometric dilation* $\{U_s\} \subset \mathcal{B}(\mathcal{K})$. *Then* (1) *The unitary operator* $E : \mathcal{M} \to \mathcal{P}$ *defined above provides a unitary relationship between the Toeplitz operators with respect to the co-isometric representation* $\{V_s^*\}$ *of* Σ *in* $\mathcal{M}$ *and the Toeplitz operators with respect to the co-isometric representation* $\{R_s\}$ *of* Σ *in* $\mathcal{P}$. *Namely, an operator* $X' \in \mathcal{B}(\mathcal{M})$ *satisfies* $X' =$

$V_s^* X' V_s$ for all $s \in \Sigma$ if, and only if, the operator $Y' = EX'E^* \in \mathcal{B}(\mathcal{P})$ satisfies $Y' = R_s Y' R_s^*$ for all $s \in \Sigma$.

(2) *The unitary operator E can be extended to a unitary operator $J : \mathcal{N} \to \mathcal{R}$ that intertwines the symbol Y, considered as an operator in $\mathcal{R}$, and the operator $Z \in \mathcal{B}(\mathcal{N})$ constructed in Muhly's second step described above.*

Proof. (1) By Lemma 4, we have that $EV_s^* = (U_s^*|\mathcal{P})^* E$. Hence if $X' = V_s^* X' V_s$, then

$$EX'E^* = EV_s^* X' V_s E^* = (U_s^*|\mathcal{P})^* EX'E^* (U_s^*|\mathcal{P}) = R_s EX'E^* R_s^*.$$

The converse is analogous.

(2) By Lemma 4, $EV_s = (U_s^*|\mathcal{P})E = R_s^* E$ for each $s \in \Sigma$. As in the description of Muhly's lifting second step, let $\{W_s\} \subset \mathcal{B}(\mathcal{N})$ be the minimal isometric dilation of $\{V_s^*\}$. First, we shall extend the unitary operator $E : \mathcal{M} \to \mathcal{P}$ to a unitary operator $J : \mathcal{N} \to \mathcal{R}$ such that $JW_s^* = U_s^* J$ for each $s \in \Sigma$. Since $\{V_s^*\}$ and $\{R_s\}$ are co-isometric, the equality $EV_s = R_s^* E$ tells us (Lemma 3 above or [15]) that the $E : \mathcal{M} \to \mathcal{P}$ is an analytic Toeplitz operator with respect to the semigroup representations $\{V_s^*\}$ and $\{R_s\}$, hence (Theorem 1 and Lemma 3) $E : \mathcal{M} \to \mathcal{P}$ can be extended to an analytic Toeplitz symbol $J : \mathcal{N} \to \mathcal{R}$, that is, $E = J|\mathcal{M}$ and $JW_s^* = U_s^* J$ for each $s \in \Sigma$. Obviously, J is defined on finite sums of the form $\sum W_s m_s$ by $J(\sum W_s m_s) := \sum U_s E m_s$ and extends to all of $\mathcal{N}$ by density. We claim that J is also unitary, and we shall prove this fact in the Corollary to Theorem 4 below. Let us then, continue, and prove that $YJ = JZ$.

As we mentioned in the description of Muhly's lifting second step, Z is obtained from X' as described in Theorem 1 (this can also be directly checked in the proof of [26, Thm. II]), namely

$$Zx = \lim_s W_s X' P(\mathcal{M}) W_s^* x \qquad \text{for all } x \in \mathcal{N}.$$

We have proved above that $Y' = EX'E^*$ is a Toeplitz operator with respect to $\{R_s\}$ that, by the one-to-one relation between X, X' and Y' and Theorem 2, has the same Toeplitz symbol as X, that is

$$YP(\mathcal{R})k = \lim_s U_s Y' P(\mathcal{P}) U_s^* P(\mathcal{R}) k \qquad \text{for all } k \in \mathcal{K}.$$

Hence, by using that $JW_s = U_s J$ for all $s \in \Sigma$ and that $J^* P(\mathcal{P}) J = P(\mathcal{M})$ because $J : \mathcal{N} \to \mathcal{R}$ is unitary and extends $E : \mathcal{M} \to \mathcal{P}$, it follows that for all $x \in \mathcal{N}$ we have

$$\begin{aligned} YJx &= \lim_s U_s Y' P(\mathcal{P}) U_s^* Jx = \lim_s U_s Y' P(\mathcal{P}) J W_s^* x \\ &= \lim_s U_s J X' J^* P(\mathcal{P}) J W_s^* x = \lim_s J W_s X' P(\mathcal{M}) W_s^* x = JZx. \end{aligned}$$

This finishes the proof. $\boxed{QED}$

9 Corollary. *For the case of a single contraction, the approaches* (DM) *and* (PV) *described above are unitarily equivalent.*

4 Properties of Toeplitz operators

Within the framework of Toeplitz operators with respect to a single contraction, we gave in [24] and [25] appropriate extensions, as well as clarifying examples, of a number of results about classical Toeplitz operators; namely, Wintner's theorem of invertibility of analytic Toeplitz operators, Widom and Devinatz's invertibility criteria for Toeplitz operators with unitary symbols, Hartman and Wintner's theorem about Toeplitz operators having Fredholm symbols, Hartman and Wintner's estimate of the norm of a compactly perturbed Toeplitz operator, the non-existence of compact classical Toeplitz operators due to Brown and Halmos, and some spectral properties that complemented the work done by Sz.-Nagy and Foiaş.

The tools that we used in [24] and [25] (the existence and uniqueness of symbols, the residual subspace, the associated Toeplitz operator Y', etc.) work similarly in the semigroup case, so the proofs of our results there can be carried out almost word by word to the present situation. To prevent from making this paper unnecessarily long, we shall only state and prove, under the assumption that $\{T_s\}$ is a co-isometric representation of Σ, the extensions of Wintner's theorem of invertibility of analytic Toeplitz operator, the spectral inclusions between the left and right spectra of a Toeplitz symbol and the corresponding spectra of its associated Toeplitz operator, and Hartman and Wintner's theorem about Toeplitz operators having Fredholm symbols.

The hypothesis that $\{T_s\}$ is a co-isometric representation of Σ in $\mathcal{H}$, implies that every $U_s = W_s$ is a unitary operator on $\mathcal{K} = \mathcal{R} = \mathcal{N}$, that $\mathcal{H} = \mathcal{M} = \mathcal{P}$, that $X = X' = Y'$, that $Z = Y$, and that X is an analytic Toeplitz operator if, and only if, $XT_s^* = T_s^*X$. These consequences help us to avoid technicalities —otherwise necessary in the most general case as the examples in [24] and [25] show— in the arguments. Nevertheless, the results that we offer give a flavor of the situation and can be still applied to some examples, as we shall see in Section 5. Let us remark at this point that these results will sound undoubtly familiar to the reader of the papers we quote here; please bear in mind that our main purpose here is to show that they fit into a common framework.

10 Theorem. *Let X be an analytic Toeplitz operator with respect to the co-isometric representation $\{T_s\}$. Then X is invertible if, and only if, its symbol Y is invertible and Y^{-1} is also an analytic Toeplitz symbol, in which case the Toeplitz operator associated to Y^{-1} is X^{-1}.*

Proof. Assume that X is invertible. Since X is analytic, it commutes with each T_s^*. According to Douglas's characterization [15, Thm. 2] quoted after Lemma 3 above to prove that X^{-1} is also an analytic Toeplitz operator it will be enough to prove that X^{-1} commutes with each T_s^*. Indeed,

$$T_s^* X^{-1} = X^{-1} X T_s^* X^{-1} = X^{-1} T_s^* X X^{-1} = X^{-1} T_s^*.$$

Let Y_0 be the analytic symbol of X^{-1}. We have to prove now that $YY_0 = Y_0 Y = \mathrm{id}(\mathcal{K})$. Since $\mathcal{K} = \bigvee_{s\in\Sigma} U_s\mathcal{H}$, we only need to check up elements of the form $U_r h$ with $r \in \Sigma$ and $h \in \mathcal{H}$. But, by using Theorem 1, we have

$$\begin{aligned} YY_0U_rh &= \lim_{s\in\Sigma} YU_sX^{-1}P(\mathcal{H})U_s^*U_rh \\ &= \lim_{s>r} YU_sX^{-1}P(\mathcal{H})U_s^*U_rh = \lim_{s>r} YU_sX^{-1}P(\mathcal{H})U_{s-r}^*h. \end{aligned}$$

Now, by using that Y commutes with each U_s (by Theorem 2), that $\mathcal{H}$ is U_{s-r}^*-invariant (by Lemma 1) and that $X = Y|\mathcal{H}$ due to the analyticity, we have

$$\begin{aligned} YY_0U_rh &= \lim_{s>r} YU_sX^{-1}P(\mathcal{H})U_{s-r}^*h = \lim_{s>r} U_sYX^{-1}U_{s-r}^*h \\ &= \lim_{s>r} U_sXX^{-1}U_{s-r}^*h = \lim_{s>r} U_sU_{s-r}^*h = U_rh. \end{aligned}$$

The equality $Y_0YU_rh = U_rh$ follows analogously.

Conversely, if Y is invertible and Y^{-1} is also an analytic symbol with Toeplitz operator X_0, then $\mathcal{H}$ is Y-invariant and Y^{-1}-invariant so that

$$XX_0 = Y|\mathcal{H}Y^{-1}|\mathcal{H} = YY^{-1}|\mathcal{H} = \mathrm{id}(\mathcal{H}).$$

Analogously, one can prove that $X_0X = \mathrm{id}(\mathcal{H})$. Therefore X is invertible. $\boxed{QED}$

11 Corollary. *Let $\{T_s\}$ be a co-isometric representation of a semigroup Σ and let $\{U_s\}$ be its minimal unitary dilation. Let X be a unitary operator that commutes with every T_s. Then X is an analytic Toeplitz operator with respect to $\{T_s\}$ that can be uniquely extended to a unitary operator $Y \in \mathcal{B}(\mathcal{K})$, its analytic symbol, that commutes with every U_s.*

Proof. By taking adjoints in the equality $T_sX = XT_s$ and using that X is unitary, it follows that $X^*T_s^* = T_s^*X^*$ and that $XT_s^* = T_s^*X$ for every $s \in \Sigma$. Therefore, X and X^* are both analytic Toeplitz operators with respect to $\{T_s\}$. Since X is invertible and its inverse is X^*, it follows from Theorem 4 that its symbol Y is invertible and that its inverse is the symbol Y^* of X^*. Hence Y is unitary, extends X because it is analytic, and $YU_s = U_sY$ by Theorem 2. $\boxed{QED}$

When every T_s is a co-isometry, it is clear that $\lambda\,\mathrm{id}(\mathcal{H})$ is an analytic Toeplitz operator with respect to $\{T_s\}$ for each complex number λ and it follows from the theorem that for analytic Toeplitz operators we have $\sigma(Y) \subset \sigma(X)$. This is true for arbitrary Toeplitz operators with respect to a co-isometric representation, as was proved by Muhly [26, Thm. III]. We can go a bit further.

12 Theorem. *Let X be a Toeplitz operator with respect to a co-isometric representation $\{T_s\}$ and let Y be the symbol of X. Then*

$$\sigma_l(Y) \subset \sigma_l(X) \qquad \textit{and} \qquad \sigma_r(Y) \subset \sigma_r(X).$$

Moreover, if X is analytic then $\sigma_l(Y) = \sigma_l(X)$.

Proof. Since $X - \lambda\,\mathrm{id}(\mathcal{H})$ is a Toeplitz operator with symbol $Y - \lambda\,\mathrm{id}(\mathcal{K})$, to prove that $\sigma_l(Y) \subset \sigma_l(X)$ it will be enough to prove that if X is left-invertible then so is Y. If X is left-invertible, then there exists $\varepsilon > 0$ such that $\|Xh\| \geq \varepsilon \|h\|$ for each $h \in \mathcal{H}$. Now, for each $r \in \Sigma$ and $h \in \mathcal{H}$ we have

$$\|YU_r h\| = \|U_r Y h\| = \|Yh\| \geq \|P(\mathcal{H})Yh\| = \|Xh\| \geq \varepsilon \|h\| = \varepsilon \|U_r h\|.$$

Now take a finite linear combination $k = \sum_{s\in\Phi} U_s h_s$ and consider $r = \sum_{s\in\Phi} s$. Using again that $\mathcal{H}$ is U_s^*-invariant for all $s \in \Sigma$, we have that $h = \sum_{s\in\Phi} U_{r-s}^* h_s$ is in $\mathcal{H}$. Hence, by the inequality we have just established, it follows that

$$\|Yk\| = \left\| Y(\sum_{s\in\Phi} U_s h_s) \right\| = \left\| YU_r(\sum_{s\in\Phi} U_{r-s}^* h_s) \right\| = \|YU_r h\| \geq \varepsilon \|U_r h\| = \varepsilon \|k\|.$$

Since $\mathcal{K} = \bigvee_{s\in\Sigma} U_s\mathcal{H}$, we have $\|Yk\| \geq \varepsilon \|k\|$ for all $k \in \mathcal{K}$ so that Y is left-invertible.

To prove that $\sigma_r(Y) \subset \sigma_r(X)$, simply note that X^* is a Toeplitz operator with Toeplitz symbol Y^* and take complex conjugates in the inclusion $\sigma_l(Y^*) \subset \sigma_l(X^*)$.

Now assume that X is analytic and that X is not left-invertible. Then there exists a sequence of unit vectors $(h_n) \subset \mathcal{H}$ such that $\lim_n \|Xh_n\| = 0$. This implies, by using that $X = Y|\mathcal{H}$, that $\lim_n \|Yh_n\| = 0$, so that Y is not left-invertible. This shows that if X is analytic then the inclusion $\sigma_l(X) \subset \sigma_l(Y)$ also holds. $\boxed{QED}$

It is well-known that for $\phi \in L^\infty$ the spectrum of M_ϕ is the essential range of ϕ. On the other hand, Wintner's Theorem (see, e.g., [16, 7.21], [21, Prob. 247] or [35, p. 320]) says that the spectrum of a classical analytic Toeplitz operator T_ϕ equals $\overline{\tilde{\phi}(\mathbb{D})}$ where $\tilde{\phi}$ is the analytic extension of ϕ to the open unit disc $\mathbb{D}$.

So even under the most favourable (but non-trivial) conditions there is no hope of obtaining that $\sigma(Y) = \sigma(X)$ or, for that matter, $\sigma_r(Y) = \sigma_r(X)$.

Now, denote by $\mathcal{F}(\mathcal{H})$ the set of all Fredholm operators. Atkinson's theorem [16, 5.17] tells us that $\mathcal{F}(\mathcal{H})$ can be written as $\mathcal{F}(\mathcal{H}) = \mathcal{F}_+(\mathcal{H}) \cap \mathcal{F}_-(\mathcal{H})$ where

$$\mathcal{F}_+(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : A\mathcal{H} \text{ is closed and } \ker A \text{ is finite-dimensional}\}$$
$$\mathcal{F}_-(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : A\mathcal{H} \text{ is closed and } \ker A^* \text{ is finite-dimensional}\}.$$

13 Theorem. *Let X be a Toeplitz operator with respect to a co-isometric representation $\{T_s\}$ and let Y be the symbol of X. Then the following hold:*

(1) *If $X \in \mathcal{F}_+(\mathcal{H})$ then $Y \in \mathcal{F}_+(\mathcal{K})$.*

(2) *If $X \in \mathcal{F}_-(\mathcal{H})$ then $Y \in \mathcal{F}_-(\mathcal{K})$.*

(3) *If $X \in \mathcal{F}(\mathcal{H})$ then $Y \in \mathcal{F}(\mathcal{K})$.*

Proof. Since an operator is in $\mathcal{F}_+$ if, and only if, its adjoint is in $\mathcal{F}_-$, we only need to prove (1). Now, if $\mathcal{K}$ is finite-dimensional then it is obvious that $Y \in \mathcal{F}_+(\mathcal{K})$ and there is nothing to prove. So we may, and do, assume that $\mathcal{K}$ is infinite-dimensional. We shall use the following general characterization for operators in $\mathcal{F}_+(\mathcal{H})$ [7, 1.11(g)]:

Let $\mathcal{H}$ be a Hilbert space and take $A \in \mathcal{B}(\mathcal{H})$. If $A \in \mathcal{F}_+(\mathcal{H})$ and P_0 is the orthogonal projection from $\mathcal{H}$ onto $\ker(A)$ *then there exists $\delta > 0$ such that $\|Ax\| + \|P_0x\| \geq \delta \|x\|$ for all $x \in \mathcal{H}$. Conversely, if there is a finite number of compact operators $K_1, K_2, \ldots, K_n \in \mathcal{B}(\mathcal{H})$ and $\delta > 0$ such that $\|Ax\| + \sum_{j=1}^{n} \|K_jx\| \geq \delta \|x\|$ for all $x \in \mathcal{H}$, then $A \in \mathcal{F}_+(\mathcal{H})$.*

So assume $X \in \mathcal{F}_+(\mathcal{H})$ and let P_0 be the orthogonal projection from $\mathcal{H}$ onto $\ker(X)$. Then, according to the characterization written above, there exists $\delta > 0$ such that $\|Xh\| + \|P_0h\| \geq \delta \|h\|$ for all $h \in \mathcal{H}$. Therefore,

$$\|P(\mathcal{H})YP(\mathcal{H})k\| + \|P_0P(\mathcal{H})k\| \geq \delta \|P(\mathcal{H})k\| \qquad \text{for all } k \in \mathcal{K}.$$

Add $\delta \left\|P(\mathcal{H}^\perp)k\right\|$ to both sides of this inequality to obtain

$$\|P(\mathcal{H})YP(\mathcal{H})k\| + \delta \left\|P(\mathcal{H}^\perp)k\right\| + \|P_0P(\mathcal{H})k\| \geq \delta \|k\| \quad \text{for all } k \in \mathcal{K}.$$

In particular, fix $k \in \mathcal{K}$ such that $k \neq 0$, since U_s^* is unitary for all $s \in \Sigma$, we have

$$\|P(\mathcal{H})YP(\mathcal{H})U_s^*k\| + \delta \left\|P(\mathcal{H}^\perp)U_s^*k\right\| + \|P_0P(\mathcal{H})U_s^*k\| \geq \delta \|U_s^*k\| = \delta \|k\|.$$

Now use that each U_s is an isometry to write

$$\|U_s P(\mathcal{H}) Y P(\mathcal{H}) U_s^* k\| + \delta \left\| P(\mathcal{H}^\perp) U_s^* k \right\| + \|P_0 P(\mathcal{H}) U_s^* k\| \geq \delta \|k\| . \tag{i}$$

We shall analyze the behaviour of each one of the three summands in the left hand side of (i). For the first one we know, by Theorem 1 and the equality $XP(\mathcal{H}) = P(\mathcal{H})YP(\mathcal{H})$, that $\lim_{s\in\Sigma} U_s P(\mathcal{H}) Y P(\mathcal{H}) U_s^* k = Yk$. Therefore, there exists $s_0 \in \Sigma$, depending on k, such that if $s \geq s_0$ then

$$\|U_s P(\mathcal{H}) Y P(\mathcal{H}) U_s^* k\| - \|Yk\| \leq \|U_s P(\mathcal{H}) Y P(\mathcal{H}) U_s^* k - Yk\| < \frac{\delta}{4} \|k\| . \tag{ii}$$

Concerning the second summand, use that $\mathcal{K} = \bigvee_{s\in\Sigma} U_s \mathcal{H}$ to find $h_1, h_2, \ldots, h_n \in \mathcal{H}$ and $r_1, r_2, \ldots, r_n \in \Sigma$ such that $\|k - \sum_{i=1}^n U_{r_i} h_i\| \leq 1/4 \|k\|$. Then, by using that $\mathcal{H}$ is U_s^*-invariant, for all $s \geq s_1 := r_1 + r_2 + \cdots + r_n$, we have

$$\begin{aligned} \left\| P(\mathcal{H}^\perp) U_s^* k \right\| &\leq \left\| P(\mathcal{H}^\perp) U_s^* \big(k - \sum_{i=1}^n U_{r_i} h_i\big) \right\| + \left\| P(\mathcal{H}^\perp) U_s^* \sum_{i=1}^n U_{r_i} h_i \right\| \\ &\leq 1/4 \|k\| + \left\| P(\mathcal{H}^\perp) \sum_{i=1}^n U_{s-r_i}^* h_i \right\| = 1/4 \|k\| . \end{aligned} \tag{iii}$$

Now, for the third summand note that given $x \in \mathcal{K}$ all the terms of the net

$$\mathfrak{C}(x) := \{P_0 P(\mathcal{H}) U_s^* : s \in \Sigma\}$$

belong to the finite-dimensional subspace $\ker(X)$. Since it is clear that $\mathfrak{C}(x)$ is bounded, it follows that $\mathfrak{C}(x)$ is relatively compact and Tikhonov's theorem ensures that the product set $\prod_{x\in\mathcal{K}} \mathfrak{C}(x)$ is relatively compact for the product topology. Therefore, the net

$$\{(P_0 P(\mathcal{H}) U_s^* x)_{x\in\mathcal{K}} : s \in \Sigma\}$$

has an adherent point $(Cx)_{x\in\mathcal{K}}$ in the product space $\prod_{x\in\mathcal{K}} \mathfrak{C}(x)$. This gives us a function $C : \mathcal{K} \to \ker(X)$ and a standard proof shows that C is a bounded linear mapping with finite rank, hence C is a compact operator. Note that C does not depend on our previously fixed $k \in \mathcal{K}$. Going back to this $k \in \mathcal{K}$, there exists an cofinal set $\Sigma_k \subset \Sigma$ such that for all $s \in \Sigma_k$ the following holds

$$\|P_0 P(\mathcal{H}) U_s^* k\| - \|Ck\| < \frac{\delta}{4} \|k\| . \tag{iv}$$

Now, since Σ_k is cofinal, we may take $s \in \Sigma_k$ such that $s \geq s_0$ and $s \geq s_1$. With this s plug (ii), (iii) and (iv) in (i) to obtain

$$\begin{aligned} \|Yk\| + \|Ck\| > \|U_s P(\mathcal{H}) Y P(\mathcal{H}) U_s^* k\| + \delta \left\| P(\mathcal{H}^\perp) U_s^* k \right\| \\ + \|P_0 P(\mathcal{H}) U_s^* k\| - \frac{3\delta}{4} \|k\| \geq \frac{\delta}{4} \|k\| . \end{aligned} \tag{v}$$

Since k was arbitrary in $\mathcal{K}$, the characterization quoted at the beginning of the proof tells us that $Y \in \mathcal{F}_+(\mathcal{K})$. $\boxed{QED}$

5 Examples

The purpose of this final section is to show that a number of widely studied classes of operators can be defined as the families of all Toeplitz operators with respect to suitable semigroups.

Toeplitz operators in the sense of Murphy. In the series of papers [29–32] and [33], Murphy has extended the notion, and many of the properties, of classical Toeplitz operator to Hardy spaces generated by function algebras (a similar extension was introduced by Cowen and Douglas [11]); we shall describe his framework as explained in [32]. Let Ω be a function algebra on a compact Hausdorff space G having a unique representing measure μ for a character of Ω. Then a great deal of the theory of Hardy spaces on $\mathbb{T}$ extends to this setting. Let $H^2(\mu)$ be the closure of Ω in $L^2(\mu)$. The functions in $H^2(\mu)$ are called analytic; in particular, the analytic and unimodular functions are called μ-inner, as in the classical case. Every function $\phi \in L^\infty(\mu)$ defines a Toeplitz operator $T_\phi : H^2(\mu) \to H^2(\mu)$ by $T_\phi f := P(\phi \cdot f)$ where P is the orthogonal projection from $L^2(\mu)$ onto $H^2(\mu)$. Let Σ be the semigroup of all μ-inner functions, then Murphy proved [32, Thm. 2.1] that $X : H^2(\mu) \to H^2(\mu)$ is a Toeplitz operator if, and only if, $X = T_\phi^* X T_\phi$ for every μ-inner ϕ. Whence, in our language, $X : H^2(\mu) \to H^2(\mu)$ is a Toeplitz operator in the sense of Murphy if, and only if, X is a Toeplitz operator with respect to the semigroup $\{T_\phi^* : \phi \in \Sigma\}$. Our results in Section 4 yield some of the results obtained by Murphy for individual Toeplitz operators. He also produced a number of interesting contributions to the study of the Toeplitz algebra generated by Toeplitz operators; we refer the interested reader to his papers.

Toeplitz operators on $H^2(\mathbb{T}^d)$. Let $L^2(\mathbb{T}^d)$ and $H^2(\mathbb{T}^d)$ be the corresponding Lebesgue and Hardy spaces of functions of d variables $f : \mathbb{T}^d \to \mathbb{C}$. As in the one dimensional case, every function $\phi \in L^\infty(\mathbb{T}^d)$ defines a Toeplitz operator on $H^2(\mathbb{T}^d)$ by $T_\phi f := P(\phi \cdot f)$, where P is the orthoprojection from $L^2(\mathbb{T}^d)$ onto $H^2(\mathbb{T}^d)$. The operator T_ϕ is said to be analytic if the symbol $\phi \in H^\infty(\mathbb{T}^d)$. This class of operators has been studied by many authors; we refer the reader to the book by Böttcher and Silbermann [7, Ch. 8] and references therein (see also [1]).

Given $\vec{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}_+^d$ consider the shift $S_{\vec{n}}$ defined on $H^2(\mathbb{T}^d)$ by

$$(S_{\vec{n}} f)(\zeta_1, \zeta_2, \ldots, \zeta_d) := \zeta_1^{n_1} \zeta_2^{n_2} \cdots \zeta_d^{n_d} f(\zeta_1, \zeta_2, \ldots, \zeta_d).$$

If we now define $T_{\vec{n}} := S_{\vec{n}}^*$, then it is clear that $\{T_{\vec{n}} : \vec{n} \in \mathbb{Z}_+^d\}$ is a co-isometric representation of $\mathbb{Z}_+^d$ in $H^2(\mathbb{T}^d)$ that has a minimal isometric dilation $\{U_{\vec{n}} : \vec{n} \in$

$\mathbb{Z}_+^d\}$ defined on $L^2(\mathbb{T}^d)$ by

$$(U_{\vec{n}}f)(\zeta_1,\zeta_2,\ldots,\zeta_d) := \zeta_1^{-n_1}\zeta_2^{-n_2}\cdots\zeta_d^{-n_d} f(\zeta_1,\zeta_2,\ldots,\zeta_d).$$

(Recall that an arbitrary contractive representation of $\mathbb{Z}_+^d$ does not necessarily have a minimal isometric dilation if $d > 2$ [45, I.6.3].)

Following the lines of the proof given by Brown and Halmos for the one dimensional case, it is easy to see, and surely well-known, that $X \in \mathcal{B}(H^2(\mathbb{T}^d))$ is a Toeplitz operator if, and only if, it is a Toeplitz operator with respect to $\{T_{\vec{n}} : \vec{n} \in \mathbb{Z}_+^d\}$ and that analytic Toeplitz operators T_ϕ with $\phi \in H^\infty(\mathbb{T}^d)$ correspond to analytic Toeplitz operators with respect to $\{T_{\vec{n}} : \vec{n} \in \mathbb{Z}_+^d\}$. Theorems 4 and 6 above do not provide new information, only alternative proofs, about analytic Toeplitz operators in $H^2(\mathbb{T}^d)$. Nevertheless, one of the consequences of Theorem 5, the fact that if $\phi \in H^\infty(\mathbb{T}^d)$, then $\sigma_l(\phi) = \sigma_l(T_\phi)$, is probably well-known at least, as we mentioned above, for Toeplitz operators in the sense of Murphy [31, 5.6] so, in particular, for the one dimensional case; however, we have been unable to locate a reference for the d-dimensional case. Recall that if $\phi \in H^\infty(\mathbb{T}^d)$ then $\sigma(\phi) = \sigma_l(\phi)$ hence, in fact, the part covered by the spectrum of ϕ within the spectrum of its Toeplitz operator T_ϕ (which is greater in general) is exactly the left spectrum of T_ϕ.

Toeplitz operators with respect to a continuous semigroup. Let $\{T_s : s \geq 0\}$ be a continuous one-parameter semigroup of contractions and let

$$A := \lim_{s\to 0^+} \frac{T_s - \mathrm{id}(\mathcal{H})}{s} \qquad \text{and} \qquad T := \big(A + \mathrm{id}(\mathcal{H})\big)\big(A - \mathrm{id}(\mathcal{H})\big)^{-1}$$

be, respectively, the generator and the co-generator of the semigroup. As it is well-known, A is a closed linear mapping densely defined in $\mathcal{H}$ but generally unbounded, and T is a contraction in $\mathcal{H}$ that determines $\{T_s : s \geq 0\}$ uniquely (see [45, III. 8] for details). As a matter of fact, the following hold

$$T = \lim_{s\to 0^+} \phi_s(T_s) \qquad \text{where} \quad \phi_s(\zeta) = \frac{\zeta - 1 + s}{\zeta - 1 - s} \text{ for } \zeta \in \mathbb{D}, \text{ and}$$
$$T_s = e_s(T) \qquad \text{where} \quad e_s(\zeta) = \exp\big(s(\zeta+1)/(\zeta-1)\big) \text{ for } \zeta \in \mathbb{D}.$$

It is well-known that $\{T_s : s \geq 0\}$ is, respectively, isometric, co-isometric or unitary if, and only if, so is the co-generator T. On the other hand, we also know that if U is the minimal isometric dilation of T, then U is the co-generator of a continuous semigroup $\{U_s : s \geq 0\}$ which is the minimal isometric dilation of $\{T_s : s \geq 0\}$; moreover, in this case, the reducing subspace of $\{U_s : s \geq 0\}$ equals the reducing susbpace of U. By using these properties and the fact that the symbols commute with the dilations on the residual subspace, it is easy to prove the following result.

14 Theorem. *An operator $X \in \mathcal{B}(\mathcal{H})$ is a Toeplitz operator with respect to a continuous one-parameter semigroup of contractions $\{T_s : s \geq 0\}$ if, and only if, it is a generalized Toeplitz operator with respect to the co-generator T of the semigroup.*

Wiener-Hopf operators. Wiener-Hopf operators fall within the class of Toeplitz operators with respect to a continuous semigroup described above. To see this, let us consider the translation co-isometries T_s defined on $L^2(\mathbb{R}_+)$ for each $s \geq 0$ by

$$(T_s f)(x) = \chi_{\mathbb{R}_+}(x) f(x+s) \qquad \text{for } f \in L^2(\mathbb{R}_+).$$

Then $\{T_s : s \geq 0\}$ is a continuous co-isometric semigroup and its minimal isometric, in fact unitary, dilation is the semigroup $\{U_s : s \geq 0\}$ consisting of the translation operators U_s defined on $L^2(\mathbb{R})$ by $(U_s f)(x) = f(x+s)$. It is also known [45, III. 9] that the co-generator of $\{T_s : s \geq 0\}$ is the co-isometry T defined on $L^2(\mathbb{R}_+)$ by

$$(Tf)(x) = f(x) - 2e^{x} \int_x^\infty f(\xi) e^{-\xi}\, d\xi \qquad \text{for } f \in L^2(\mathbb{R}_+).$$

According to our Theorem 2, the Toeplitz symbols with respect to $\{T_s : s \geq 0\}$ are the operators $Y \in \mathcal{B}(L^2(\mathbb{R}))$ that commute with the translations U_s for all $s \in \mathbb{R}$ and, as it is well-known [7, 9.2], these are precisely the operators of the form $Y = F^{-1} M_\phi F$ where F is the Fourier transform on $L^2(\mathbb{R})$ and M_ϕ is the operator of multiplication by a function $\phi \in L^\infty(\mathbb{R})$. Consequently, $X \in \mathcal{B}(L^2(\mathbb{R}_+))$ is a Toeplitz operator with respect to $\{T_s : s \geq 0\}$ if, and only if, it is of the form $X = \chi_{\mathbb{R}_+} F^{-1} M_\phi F | L^2(\mathbb{R}_+)$. The operators of this form are called Wiener-Hopf integral operators and have been widely studied in the literature (we refer the reader to [7, Ch. 9], see also [11] and [28] for the study of algebras generated by Wiener-Hopf operators). It was proved by Rosenblum and Devinatz that Wiener-Hopf operators are unitarily equivalent to classical Toeplitz operators on $H^2(\mathbb{T})$ (see [18] for a different approach), and we now see that both classes are particular cases of a more abstract situation. By using Theorem 7 above we obtain that $X \in \mathcal{B}(L^2(\mathbb{R}_+))$ is a Wiener-Hopf operator if, and only if, $X = S^* X S$ where S is the adjoint of the co-generator T or, in other words, S the isometry defined by

$$(Sf)(x) = f(x) - 2e^{-x} \int_0^x f(\xi) e^{\xi}\, d\xi \qquad \text{for } f \in L^2(\mathbb{R}_+).$$

(This isometry S is a disguised form of the Laguerre shift R defined on $L^2(\mathbb{R}_+)$ by $(Rf)(x) = f(x) - 2e^{-x/2} \int_0^x f(\xi) e^{\xi/2}\, d\xi$ which can also be used to define Wiener-Hopf operator as the solutions of $X = R^* X R$; see [42, Ch. 3]).

When the function ϕ in the symbol $Y = F^{-1}M_\phi F$ is in $H^\infty(\mathbb{R})$ the Wiener-Hopf operator $X = \chi_{\mathbb{R}_+}Y|L^2(\mathbb{R}_+)$ is said to be analytic; this corresponds to the analyticity of the classical Toeplitz operator which is unitarily equivalent to X. Let us see that this is also consistent with our definition of analytic Toeplitz operator with respect to $\{T_s : s \geq 0\}$. Indeed, since the semigroup $\{T_s : s \geq 0\}$ is co-isometric, a symbol $Y = F^{-1}M_\phi F$ is analytic (in the sense introduced in Section 2 above) if $YL^2(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ or, equivalently,

$$F^{-1}M_\phi FL^2(\mathbb{R}_+) \subset L^2(\mathbb{R}_+).$$

But, since by the Paley-Wiener representation theorem F maps $L^2(\mathbb{R}_+)$ unitarily onto $H^2(\mathbb{R})$, this is the same as saying that M_ϕ maps $H^2(\mathbb{R})$ into itself and this happens if, and only if, $\phi \in H^\infty(\mathbb{R})$ as desired.

Bearing in mind that the multiplication operator M_ϕ by a function $\phi \in L^\infty(\mathbb{R})$ is Fredholm if, and only if, it is invertible, our results in Section 4 yield the well-known characterizations of invertible analytic Wiener-Hopf operators and Fredholm Wiener-Hopf operators [7, Ch. 9].

Again, one of the consequences of Theorem 5 is the fact that if $\phi \in H^\infty(\mathbb{R})$, then the left spectrum of the analytic Wiener-Hopf operator defined by ϕ coincides with the spectrum of corresponding symbol that, as in the classical case, is the essential range of ϕ.

Let us finally mention that, along a series of papers, Devinatz, Pellegrini (also for unbounded operators), Reeder and Shinbrot considered invertibility properties of so-called generalized Wiener-Hopf operators, defined as the class of operators $X \in \mathcal{B}(\mathcal{H})$ which are compressions $X = P(\mathcal{H})Y|\mathcal{H}$ of operators Y defined on a superspace $\mathcal{K}$ of $\mathcal{H}$ (see, e.g., [12], [36], [37] [40], and references therein). The class of generalized Wiener-Hopf operators is obviously larger than the class of Toeplitz operators with respect to semigroups considered in this paper and share some of the properties in a weaker sense (no Toeplitz type equations or commutativity involved).

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