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Pointwise Multipliers on $\mathbf{L}^1$ and Related Spaces

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Abstract. We consider completely continuous and weakly compact multiplication operators on certain classical function spaces, more precisely on Lebesgue spaces L^1, on spaces $\mathcal{C}(K)$ of continuous functions on a compact Hausdorff space K, and on the Hardy space H^1. We will describe such operators in terms of their defining symbols. Our characterizations extend corresponding results known from the literature. In any case, our results reveal the severe restrictions on the symbols of multiplication operators necessary to ensure complete continuity or weak compactness. The apparent simplicity of the obtained descriptions belie the deep and beautiful functional analytic principles that underlie them.

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Dedicated to the memory of Klaus Floret

Statement of Results

In this note we prove the following three results. In the first one (Ω, Σ, μ) is a (non-trivial) σ-finite measure space. It is well-known that Ω can be decomposed $\Omega = A \cup B$ where $A = \bigcup_n A_n$ is the countable union of pairwise disjoint μ-atoms $A_n \in \Sigma$ and $B \in \Sigma$ contains no μ-atoms. Given $C \in \Sigma$, we denote by μ_C the restriction of μ to C (so that $\mu_C(E) = \mu(C \cap E)$ for all $E \in \Sigma$). In a natural fashion, $L^1(\mu_C)$ appears as a (complemented) subspace of $L^1(\mu)$.

1 Theorem. *Let $M_F : L^1(\mu) \to L^1(\mu) : f \mapsto Ff$ be the multiplication operator induced by $F \in L^\infty(\mu)$. If M_F is completely continuous, then M_F*

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vanishes on $L^1(\mu_B)$*; if* M_F *is weakly compact then it induces a compact operator on* $L^1(\mu_A)$*, so that* $M_F : L^1(\mu) \to L^1(\mu)$ *is compact.*

Recall that a Banach space operator $u : X \to Y$ is completely continuous (or a Dunford-Pettis operator) if it maps weakly null sequences of X into norm null sequences of Y. To say that X has the Dunford-Pettis property means that every weakly compact operator with domain X is completely continuous. It is well-known that spaces $\mathcal{C}(K)$ and $L^1(\mu)$ enjoy the Dunford-Pettis property. Weakly compact operators with domain a space $\mathcal{C}(K)$ even coincide with the corresponding completely continuous operators. This applies in particular to spaces $L^\infty(\mu)$. For details see e.g. J. Diestel–J.J. Uhl [2].

We shall provide alternative proofs for the second case. Another one can be derived by duality from the following result. As usual, $\mathcal{C}(K)$ is the Banach space of continuous functions on a given compact Hausdorff space K.

2 Theorem. *Let* $F \in \mathcal{C}(K)$ *be such that* $M_F : \mathcal{C}(K) \to \mathcal{C}(K) : f \mapsto fF$ *is weakly compact. Then* M_F *is compact. In fact,* $A := K \setminus F^{-1}(0)$ *is discrete and countable, with all its cluster points in* $F^{-1}(0)$*. Moreover the closure of* $M_F(\mathcal{C}(K))$ *is an isomorphic copy of* $c_0(A)$*, and* M_F^* *takes its values in an isomorphic copy of* $\ell^1(A)$*.*

The third result is on Hardy spaces. Let $H^p(D)$, $0 < p < \infty$, be the classical Hardy space on D, the open unit disk in $\mathbb{C}$. Let $\mathbb{T} = \partial D$ be the unit circle $\{ \zeta \in \mathbb{C} \mid |\zeta| = 1 \}$, and let m be normalized Lebesgue measure on $\mathbb{T}$. By Fatou's Theorem, if f is in any $H^p(D)$, then $f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$ exists for m-almost all $\zeta \in \mathbb{T}$. Moreover, $f \mapsto f^*$ establishes an isometric embedding $H^p(D) \hookrightarrow L^p(\mathbb{T})$, so that $H^p(D)$ can be looked at as a subspace, denoted by $H^p(\mathbb{T})$, of $L^p(\mathbb{T})$; see P.L. Duren [3]. We follow the common habit of identifying $H^p(D)$ and $H^p(\mathbb{T})$ and using H^p as a common notation. Our third result is then

3 Theorem. *Let* $M_F : H^1 \to L^1(\mathbb{T})$ *be the multiplication operator induced by* $F \in L^\infty(\mathbb{T})$*. If* $M_F : H^1 \to L^1(\mathbb{T})$ *is completely continuous, or weakly compact, then* $F = 0$ *m-a.e.*

Weakly compact operators and completely continuous operators with domain H^1 do not contain each other. In fact, it is well-known (see [7] and [11]) that ℓ^1 as well as ℓ^2 are isomorphic to complemented subspaces of H^1.

We are indebted to S. Goldstein for several stimulating discussions.

Proofs

We shall freely make use of standard notation and terminology of measure and integration theory; see e.g. K.Floret's book [4].

Let (Ω, Σ, μ) be a (non-trivial) σ-finite measure space. We continue to write

$\Omega = A \cup B$ where $A = \bigcup_n A_n$ is the (finite or infinite, but countable) union of pairwise disjoint μ-atoms $A_n \in \Sigma$ and $B \in \Sigma$ contains no μ-atoms. It is well-known (W. Rudin [8], Lemma 6.9) that there is a function $h \in L^1(\mu)$ such that $0 < h(\omega) < 1$ for all $\omega \in \Omega$. Note that the measure μ and the finite measure $\mu_h := h\, d\mu$ have the same null sets and the same atoms. For every $0 < p < \infty$, $f \mapsto h^{1/p} f$ defines an isometric isomorphism of $L^p(\mu_h)$ onto $L^p(\mu)$, whereas $L^\infty(\mu_h)$ and $L^\infty(\mu)$ coincide as Banach spaces. Every multiplication operator $M_F : L^p(\mu) \to L^q(\mu)$ gives rise to a multiplication operator $M_G : L^p(\mu_h) \to L^q(\mu_h)$ where $G = h^{(1/p)-(1/q)} F$. Clearly, M_F is weakly compact (completely continuous, compact,...) if and only if M_G is. But for (essential) boundedness of F to imply boundedness of G we need $p \le q$.

As for our multiplication operators, it suffices to look at the case $F \ge 0$. In fact, write $F = g \cdot |F|$ where $g \in L^\infty(\mu)$ satisfies $|g(\omega)| = 1$ for all $\omega \in \Omega$. Since $M_g : L^q(\mu) \to L^q(\mu)$ is an isometric isomorphism, $M_F : L^p(\mu) \to L^q(\mu)$ is weakly compact (completely continuous, compact,...) if and only if $M_{|F|} : L^p(\mu) \to L^q(\mu)$ has this property.

We need a simple lemma which we prove for the sake of completeness:

4 Lemma. *Let (Ω, Σ, μ) be a finite measure space, $A \in \Sigma$ a μ-atom and $f \in L^1(\mu)$. Then*

$$f(\omega) \;=\; \frac{1}{\mu(A)} \cdot \int_A f\, d\mu \quad \textit{for } \mu\textit{-almost all } \omega \in A.$$

Proof. Write $a_f := \mu(A)^{-1} \cdot \int_A f\, d\mu$. First look at a real-valued f. Since A is a μ-atom, precisely one of the sets $A^+ := \{\, \omega \in A \mid f(\omega) \ge a_f \,\}$ and $A^- := \{\, \omega \in A \mid f(\omega) < a_f \,\}$ is a μ-null set. Accordingly, $\int_A |f - a_f|\, d\mu \;=\; \big(\int_{A^+} - \int_{A^-}\big) f d\mu - a_f\big(\mu(A^+) - \mu(A^-)\big) \;=\; 0$.

Suppose now that $f = g + i\, h$ where $g, h \in L^1(\mu)$ are real-valued. Then, by the first step, $\int_A |f - a_f|\, d\mu = \int_A [(g - a_g)^2 + (h - a_h)^2]^{1/2} d\mu \le \int_A (|g - a_g| + |h - a_h|)\, d\mu = 0$.

In either case, we can conclude that $f \chi_A = a_f$ μ-a.e. *QED*

Proof of Theorem 1. We first look at a weakly compact $M_F : L^1(\mu) \to L^1(\mu)$. By the preceding discussion, we may assume that $F \ge 0$ and that μ is a probability measure. For notational convenience, let us also assume that there are infinitely many atoms A_n. By the lemma, $f \chi_{A_n} = \mu(A_n)^{-1} \int_{A_n} f d\mu$ μ-a.e. for each n. It follows that $\ell^1 \to L^1(\mu_A) : (\xi_n) \mapsto \sum_n \xi_n \mu(A_n)^{-1} \chi_{A_n}$ is an onto isometric isomorphism; the inverse is $L^1(\mu_A) \to \ell^1 : f \mapsto \big(\int_{A_n} f d\mu\big)_{n \in \mathbb{N}}$. Now M_F maps $L^1(\mu_A)$ into itself in a weakly compact manner. Compactness of M_F on $L^1(\mu_A)$ follows from the fact ℓ^1 has the Schur property (i.e., the identity is completely continuous); see e.g. [2].

Now for the statement on μ_B when $M_F : L^1(\mu) \to L^1(\mu)$ is just completely continuous. It suffices to consider the case $\mu = \mu_B$. Since μ has no atoms, we can construct a sequence (r_n) of "Rademacher functions" as follows:

Write $\Omega = \Omega_{1,0} \cup \Omega_{1,1}$ where $\Omega_{1,0}$ and $\Omega_{1,1}$ are disjoint sets of measure 1/2 and put $r_1(\omega) = 1$ for $\omega \in \Omega_{1,0}$ and $r_1(\omega) := -1$ for $\omega \in \Omega_{1,1}$. Next split $\Omega_{1,0} = \Omega_{2,0} \cup \Omega_{2,1}$ and $\Omega_{1,1} = \Omega_{2,2} \cup \Omega_{2,3}$ into disjoint sets from Σ of measure 1/4, and we define $r_2 = \sum_{k=0}^{3} (-1)^k \chi_{\Omega_{2,k}}$.

Continue inductively. The result is an orthonormal sequence of unimodular functions $r_n \in L^\infty(\mu)$, which certainly converge weakly to zero in $L^1(\mu)$. By hypothesis, $\lim_n \|M_F r_n\|_{L^1(\mu)} = 0$. But $\|M_F r_n\|_{L^1(\mu)} = \int_\Omega |F r_n| \; d\mu = \int_\Omega F d\mu$ for each n, which proves what we wanted. QED

It follows from these considerations that if $M_F : L^1(\mu) \to L^1(\mu)$ is completely continuous then, using the above notation, it can be represented as

$$M_F f \;=\; \sum_n F f \; \chi_{A_n} \;=\; \sum_n \frac{1}{\mu(A_n)} \, \Big(\int_{A_n} F f \; d\mu \Big) \, \chi_{A_n}$$

(μ–a.e.) for all $f \in L^1(\mu)$; the series being absolutely convergent.

We pass to the announced alternate proof of Theorem 1 in the case of weak compactness.

($\bullet_1$) Suppose that $M_F : L^1(\mu) \to L^1(\mu)$, $F \in L^\infty(\mu)$, is weakly compact. Again we can assume that (Ω, Σ, μ) is finite and that $F \geq 0$. It suffices to show that if μ has no atoms then $M_F = 0$.

Suppose that M_F doesn't vanish. Then there exists a $\lambda > 0$ such that $E_0 = \{ \, \omega \in B \mid F(\omega) \geq \lambda \, \}$ has positive measure. Since μ has no atoms, we can find $E_n \in \Sigma$ of positive measure satisfying $E_{n+1} \subseteq E_n \subseteq E_0$ for all n and $\lim_n \mu(E_n) = 0$; so $E = \bigcap_k E_k$ is a μ-null set. The functions $f_n := \chi_{E_n}/\mu(E_n)$ constitute norm one vectors in $L_1(\mu_B) \subseteq L^1(\mu)$, with $\lim_n f_n(\omega) = 0$ for all $\omega \in \Omega \setminus E$. It follows that $\lim_n F f_n = 0$ pointwise μ-a.e. By weak compactness of M_F, $(F f_n)$ is uniformly integrable in $L^1(\mu)$. In particular, there is a $\delta > 0$ such that $\sup_n \int_C F f_n \; d\mu \leq \lambda/2$ for all $C \in \Sigma$ with $\mu(C) < \delta$. But $\int_{E_N} F f_N d\mu < \lambda/2$ for some large $N \in \mathbb{N}$ since $\lim_n \mu(E_n) = 0$, and we reach a contradiction:

$$\lambda \;\leq\; \frac{1}{\mu(E_N)} \int_{E_N} F \; d\mu \;=\; \|M_F f_N\|_{L^1(\mu)} \;=\; \int_{E_N} F f_N \; d\mu \;<\; \lambda \, . \qquad \text{QED}$$

($\bullet_2$) Still another proof exploits the fact that $L^1(\mu)$ has the Dunford-Pettis property:

Let $M_F : L^1(\mu) \to L^1(\mu)$ be weakly compact, where $0 \le F \in L^\infty(\mu)$. Look at the bounded functions $\varphi := \sqrt{F}$ and $\varphi_\varepsilon = F/\sqrt{F+\varepsilon}$ ($\varepsilon > 0$). Note that $\lim_{\varepsilon\to 0} \|M_\varphi - M_{\varphi_\varepsilon}\| = 0$ because of

$$\begin{aligned} \|\varphi - \varphi_\varepsilon\| &= \|\sqrt{F} - \sqrt{F+\varepsilon} + \varepsilon/\sqrt{F+\varepsilon}\| \\ &\le \|\sqrt{F} - \sqrt{F+\varepsilon}\| + \|\varepsilon/\sqrt{F+\varepsilon}\| \le 2\sqrt{\varepsilon}\ . \end{aligned}$$

Together with M_F, the multiplication operators $M_{\varphi_\varepsilon} = M_F \circ M_{1/\sqrt{F+\varepsilon}}$ are weakly compact, so that the limit M_φ is weakly compact as well. By the Dunford-Pettis property of $L^1(\mu)$, M_φ is completely continuous, so that $M_F = M_\varphi \circ M_\varphi$ is compact. QED

– Notice that this argument applies equally well for $\mathcal{C}(K)$ spaces, K any compact Hausdorff space, and so in particular for $L^\infty(\mu)$ spaces. Of course a more measure theoretic description of the state of affairs as regards $L^\infty(\mu)$ may be obtained by duality from Theorem 1.

Let further K be any compact Hausdorff space, let $\mathcal{B}(K)$ be K's Borel algebra and let X be a Banach space. Given any operator $T : \mathcal{C}(K) \to X$, we can define the weak* countably additive (vector) measure $G : \mathcal{B}(K) \to X^{**} : E \mapsto T^{**}(\chi_E)$. It is well-known that T is weakly compact iff the $G(E)$'s even belong to X, for all $E \in \mathcal{B}(K)$. In such a case, G is (norm) countably additive, and there is a measure λ in $\mathcal{C}(K)^*$ which 'controls' T^* in the sense that $T^*x^* \ll \lambda$ for all $x^* \in X^*$, equivalently, which satisfies $G \ll \lambda$ (Theorem of Bartle-Dunford-Schwartz). We may even take $\lambda = |\langle x_0^*, G(\cdot)\rangle|$ for some $x_0^* \in X^*$ (Rybakov's Theorem). We refer to [2] (p.267 ff) for details.

Proof of Theorem 2. Weakly compact operators on $\mathcal{C}(K)$ are completely continuous, and conversely. Therefore $M_F : \mathcal{C}(K) \to \mathcal{C}(K)$ is weakly compact iff $M_{|F|} : \mathcal{C}(K) \to \mathcal{C}(K)$ has this property.

By weak compactness, M_F^{**} maps $\mathcal{C}(K)^{**}$ into $\mathcal{C}(K)$ so that each $M_F^{**}(\chi_E) = F \cdot \chi_E$ ($E \in \mathcal{B}(K)$) belongs to $\mathcal{C}(K)$. In particular, $F \cdot \chi_{\{x\}} \in \mathcal{C}(K)$ for all $x \in K$, which can only happen if A is discrete. But then $\chi_{\{x\}} \in \mathcal{C}(K)$ for all $x \in A$ so that the singleton $\{x\}$ is a clopen subset of K. Therefore no element of A can be a cluster point of A.

The above vector measure G is now $E \mapsto F \cdot \chi_E$. The operator M_F^* assigns to each $\mu \in \mathcal{C}(K)^*$ the measure $\mu^F : \mathcal{B}(K) \to \mathbb{C} : E \mapsto \int_E F d\mu$. By Rybakov's Theorem, there is a measure $0 \le \lambda \in \mathcal{C}(K)^*$ such that λ^F controls M_F^*. So if $E \in \mathcal{B}(K)$ and $\lambda^F(E) = 0$ then $\mu^F(E) = 0$ for all $\mu \in \mathcal{C}(K)^*$. In particular, $(\delta_x)^F \ll \lambda^F$ for all $x \in K$. So if $E \in \mathcal{B}(K)$ is a null set for λ^F then $E \subseteq F^{-1}(0)$. The converse is also true: if $E \subseteq F^{-1}(0) = |F|^{-1}(0)$, then $\lambda^{|F|}(E) = 0$ and so $\lambda^F(E) = 0$ since $|\lambda^F| \le \lambda^{|F|}$.

Again let $K = B \cup \bigcup_{n\in J} A_n$ be a decomposition of K into pairwise disjoint λ-atoms A_n and a Borel set B without λ-atoms ($J \subseteq \mathbb{N}$). On $L^1(\lambda)$, M_F^* coincides with M_F, and this operator multiplies $L^1(\lambda_A)$, resp. $L^1(\lambda_B)$, into itself in a weakly compact manner. By the argument used in the proof of Theorem 1, M_F vanishes on $L^1(\lambda_B)$ so that $B \subseteq F^{-1}(0)$ up to a λ-null set. Put $A := K \setminus F^{-1}(0)$. If $x \in A$, then $\{x\}$ is an atom for λ^F and also for λ so that A must be countable. As before, $L^1(\lambda_A)$ and $\ell^1(A)$ are isomorphic. Since M_F^* maps $\mathcal{C}(K)^*$ into $L^1(\lambda_A)$, weak compactness and the Schur property of $\ell^1(A)$ implies compactness.

It remains to show that the range of M_F is dense in a copy of $c_0(A)$. To this end assume that A is infinite and let $(x_n)_{n\in\mathbb{N}}$ be any enumeration of A. By compactness and what we showed earlier, each subsequence of (x_n) has a subsequence which converges to a point in $F^{-1}(0)$. In terms of F this means that each subsequence of $(F(x_n))$ has a subsequence which converges to zero, or equivalently that $(F(x_n)) \in c_0$. It is now an exercise to show that the representation $M_F f = \sum_n Ff\, \delta_{x_n}$ ensures that $Ff \mapsto (F(x_n)f(x_n))_n$ is an isometric embedding of $M_F(\mathcal{C}(K)$ into $c_0(A)$. Finally, given any finite scalar sequence $\zeta = (\zeta_1, \ldots, \zeta_N, 0, \ldots)$, the function f_ζ defined on K by $f_\zeta(x) = \zeta_n / F(x_n)$ if $x = x_n$ ($1 \leq n \leq N$) and $f_\zeta(x) = 0$ otherwise, clearly belongs to $\mathcal{C}(K)$. Since now ζ corresponds to the element Ff_ζ of $M_F(\mathcal{C}(K))$, it follows that $M_F(\mathcal{C}(K))$ is isometrically isomorphic to a dense subspace of $c_0(A)$. *QED*

In particular, the measure λ associated with our weakly compact operator $M_F : \mathcal{C}(K) \to \mathcal{C}(K)$ is purely atomic iff $F \neq 0$ λ-a.e.

5 Remark. It is clear that, in the situation of Theorem 2, the compact operator $M_F^* : \mathcal{C}(K)^* \to \mathcal{C}(K)^*$ satisfies $M_F^*\mu = \sum_{x\in A} F(x)\mu(\{x\})\delta_x$ for each $\mu \in \mathcal{C}(K)^*$, the series being absolutely convergent. If we identify M_F^* with the operator $\mathcal{C}(K)^* \to \ell^1(A) : \mu \mapsto (F(x)\mu(\{x\}))_{x\in A}$, then its compactness is reflected by the statement $(F(x))_{x\in A} \in c_0(A)$. This leads to a reduction to compact operators among sequence spaces. In a similar spirit, properties stronger than compactness (e.g. nuclearity) can be characterized. We do not enter such topics.

Proof of Theorem 3. As for complete continuity, we can use the following very simple observation. Let μ be a σ-finite, non-atomic measure, and let X be a subspace of $L^1(\mu)$ which contains a weak null sequence (ψ_n) consisting of unimodular functions. Then a multiplication operator $M_F : X \to L^1(\mu)$, $F \in L^\infty(\mu)$, can only be completely continuous if $F = 0$ μ-a.e. In fact, $\lim_{n\to\infty} \|M_F\psi_n\| = 0$, and for each n, $\|M_F\psi_n\| = \|F\|$.

The monomials $\psi_n(\zeta) = \zeta^n$ ($n \geq 0$) form a weak null sequence in $H^1 \subset L^1(\mathbb{T})$ consisting of unimodular functions.

The case of weak compactness can be settled by modifying the argument used for ($\bullet_1$). Again there is no loss in assuming $F \geq 0$. We argue contrapositively and suppose that $F \neq 0$. Then there is a $\lambda > 0$ such that $E_0 = \{F \geq \lambda\}$ satisfies $m(E_0) > 0$. As before there are Borel sets $E_n \subseteq E_0$ such that $E_{n+1} \subseteq E_n$ and $m(E_n) > 0$ for each n and $\lim_n m(E_n) = 0$, i.e., $E = \bigcap_n E_n$ is an m-null set.

Let (ε_n) be a decreasing sequence of positive numbers such that $\lim_n \varepsilon_n = 0$. The functions $f_n := (\chi_{E_n}/\mu(E_n)) + \varepsilon_n$ form a bounded sequence in $L^1(\mathbb{T})$ which converges to zero pointwise on $\mathbb{T} \setminus E$. It follows that $Ff_n \to 0$ m-a.e.

Each function $\log f_n$ is integrable, so that $|\varphi_n^*| = f_n$ m-a.e for some $\varphi_n \in H^1(D)$ (P.L. Duren [3], p.24). Since M_F is weakly compact, $(F\varphi_n^*)_n$ is uniformly integrable in $L^1(\mathbb{T})$. In particular, there is a $\delta > 0$ such that $\int_B |F\varphi_n^*|\, dm = \int_B Ff_n\, dm \leq \lambda/2$ for every Borel set $B \subseteq \mathbb{T}$ with $m(B) < \delta$. As before this leads to a contradiction. QED

We refer to [1] for further results related to the weak compactness part of Theorem 3.

Some Consequences

We may, for example, reprove a result from H. Takagi–K. Yokouchi [9] (p.326).

6 Corollary. *Let μ be a non-atomic σ-finite measure and $0 < p < q < \infty$. If $F \in L^\infty(\mu)$ is such that $M_F(L^p(\mu)) \subseteq L^q(\mu)$ then $F = 0$ μ-a.e.*

Proof. We may assume that μ is a probability measure and that $F \geq 0$. M_F maps $L^p(\mu)$ into $L^q(\mu)$ if and only if M_{F^p} maps $L^1(\mu)$ into $L^{q/p}(\mu)$. The space $L^{q/p}(\mu)$ is reflexive so that M_{F^p}, as an operator $L^1(\mu) \to L^1(\mu)$, is weakly compact. Now apply Theorem 1. QED

We have already alluded to the fact that duality leads immediately to the following

7 Corollary. *Suppose that μ is non-atomic and σ-finite. If the multiplication operator $M_F : L^\infty(\mu) \to L^\infty(\mu)$ given by $F \in L^\infty(\mu)$ is weakly compact then $F = 0$ μ-a.e.*

There is another proof which is based on the following result of J.J. Uhl [10]:

($\diamond_1$) *Let X be a Banach space and ν a finite measure. Let $j_p : L^p(\nu) \hookrightarrow L^1(\nu)$ be the formal identity, $1 < p \leq \infty$. An operator $u : L^1(\nu) \to X$ is completely continuous iff $u \circ j_p : L^p(\nu) \to X$ is compact, for some (and then every) $1 < p \leq \infty$.*

Hence, in the situation of Corollary 2, if $M_F : L^\infty(\mu) \to L^\infty(\mu)$ is weakly compact, then $j_\infty \circ M_F$ is compact (since j_∞ is completely continuous), so that

$(\diamond_1)$ yields complete continuity of $M_F : L^1(\mu) \to L^1(\mu)$. The result then follows from Theorem 1.

We have already noted that weak compactness and complete continuity for operators with domain $L^\infty(\mu)$ are the same.

As mentioned before, the reduction from σ-finite measures to finite ones may present problems for multiplication operators $M_F : L^p(\mu) \to L^q(\mu)$ if $p > q$. Therefore we now restrict to probability measures.

8 Corollary. *Let μ be a non-atomic probability measure and $F \in L^\infty(\mu)$. If $M_F : L^p(\mu) \to L^q(\mu)$ exists as a compact operator for some $p > 1$ and $q \geq 1$, then $F = 0$ μ-a.e.*

Proof. Let $j_p : L^p(\mu) \hookrightarrow L^1(\mu)$ and $j_q : L^q(\mu) \hookrightarrow L^1(\mu)$ be the formal identities. Note that $L^1(\mu) \to L^1(\mu) : f \mapsto Ff$ exists. To distinguish this operator from $M_F : L^p(\mu) \to L^q(\mu)$, we will denote it by $\widetilde{M}_F$. Then $j_q\widetilde{M}_F = M_F j_p$. The latter is compact so that $\widetilde{M}_F : L^1(\mu) \to L^1(\mu)$ is completely continuous by $(\diamond_1)$. Now apply Theorem 1. QED

There is a perfect analogue of Uhl's result for Hardy spaces:

$(\diamond_2)$ *Let X be a Banach space and $i_p : H^p \hookrightarrow H^1$ the formal identity, $1 < p \leq \infty$. An operator $u : H^1 \to X$ is completely continuous if and only if $u \circ i_p : H^p \to X$ is compact, for some (and then every) $1 < p \leq \infty$.*

See [6, Theorem 3] (the proof there suffers from a 'self-correcting error').

9 Corollary. *Let $p > 1$, $q \geq 1$ and $F \in L^\infty(\mathbb{T})$. If $M_F : H^p \to L^q(\mathbb{T})$ is compact then $F = 0$ m-a.e.*

To see this, repeat the proof of Corollary 3, replacing $(\diamond_1)$ by $(\diamond_2)$ and appealing to Theorem 2 instead of Theorem 1.

A generalization of this to a non-commutative setting is intended to appear in [5].

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