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Transfer principles and ergodic theory in Orlicz spaces

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Abstract. We extend to Orlicz spaces with weight a transfer principle of R. Coifman and G. Weiss, concerning L_p-inequalities on some convolution operators. We also extend to Orlicz spaces the Birkhoff's pointwise ergodic theorem, and L_p-inequalities on some maximal operators, using a transfer argument which follows Wiener and Calderon's ideas. We get a new characterization of the so called Dini-condition.

Keywords:

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Introduction

We are interested in the transfer principle of R. Coifman and G. Weiss [5] in the context of Orlicz spaces with weight. The idea of the transfer principle is to transfer some properties of convolution operators on classical groups G, such as locally compact amenable groups, to operators associated with more general measure spaces. The transfer principle was studied, developped by many authors, let us mention [1,5]. To be more precise, let k be in $L_1(G)$ and H_k be a convolution operator on G given by

$$H_k f = k \star f = \int_G k(y) f(\ \cdot y^{-1}) dy.$$

Let $R : u \to R_u$ be a representation of G on some Banach space. Then the transfered operator $H_k^{\#}$ is defined by letting

$$H_k^{\#} = \int_G k(y) R_{y^{-1}} dy.$$

Many properties of H_k are still valid for $H_k^{\#}$, in particular the preservation of L_p inequalities [1,5]. In Section 2 of this article we extend these results to Orlicz spaces with weight.

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Sections 3 and 4 are devoted to the extension of ergodic theorems to Orlicz spaces. More precisely, if $(\mathcal{M}, \mathcal{B}, \mu)$ is a measure space and $T : \mathcal{M} \to \mathcal{M}$ is a measure preserving measurable function, we define, for a measurable function f and x in $\mathcal{M}$:

$$A_n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x).$$

We prove that $\{A_n f(x)\}$ converges almost everywhere on $\mathcal{M}$, for each f in a reflexive Orlicz space $L_\phi(\mathcal{M}, \mathcal{B}, \mu)$. This result extends Birkhoff's pointwise ergodic theorem to Orlicz spaces.

We also prove that the operator A_∞ defined by

$$A_\infty f(x) = \sup_{n \geq 0} |A_n f(x)|$$

satisfies a strong type maximal inequality, i.e. there exists a constant $C > 0$ such that

$$\|A_\infty f\|_\phi \leq C \|f\|_\phi \qquad (\star)$$

for all f in $L_\phi(\mathcal{M}, \mathcal{B}, \mu)$ (where $\| \cdot \|_\phi$ is the norm defined in $L_\phi(\mathcal{M}, \mathcal{B}, \mu)$), and an extension of Wiener-Calderon's transfer principle [4,15] allows us to prove the result only for $L_\phi(\mathbb{N})$.

We also prove an inequality like $(\star)$ for other maximal functions which are defined with respect to integrals of functions in $L_\phi(\mathbb{R})$. This last result extends to Orlicz spaces a result of Hardy and Littlewood for $L_p(\mathbb{R})$ [14, 5.7.5]. Finally we get a new characterization of the Dini-condition.

1 Orlicz spaces

We start by recalling some well known facts about Orlicz spaces, for more details, see [12].

Orlicz spaces are defined with respect to an Orlicz function (or "N-function"). An Orlicz function ϕ is a function from $\mathbb{R}$ to $\mathbb{R}$ which can be defined by

$$\phi(u) = \int_0^{|u|} p(t)dt$$

where $p(t)$ is a right continuous function defined for $t \geq 0$, which is nondecreasing, positive for $t > 0$ and satisfies

$$p(0) = 0 \quad \text{and} \quad p(+\infty) = \lim_{t \to +\infty} p(t) = +\infty.$$

This definition implies that an Orlicz function is even, continuous, increases on $[0, +\infty[$ and $\phi(u) = 0$ if and only if $u = 0$.

The function ϕ is convex and has the following properties:

a) $\lim\limits_{u\to 0} \frac{\phi(u)}{u} = 0$ and $\lim\limits_{u\to +\infty} \frac{\phi(u)}{u} = +\infty$,

b) $\phi(u) \le up(u) \le \phi(2u)$ for $u > 0$,

c) $\phi(u) + \phi(v) \le \phi(u+v)$ for $u, v > 0$.

We will always denote by ψ the complementary function of an Orlicz function ϕ. It is defined by

$$\psi(u) = \sup_{v\ge 0} \{|u|v - \phi(v)\}$$

ψ is itself an Orlicz function and we have

$$uv \le \phi(u) + \varphi(v) \quad \text{for all } u, v > 0.$$

Let us recall two definitions.

An Orlicz function ϕ satisfies the Δ_2-*condition* if there exists a constant $K > 0$ and $u_0 \ge 0$ such that for all $u \ge u_0$, $\phi(2u) \le K\phi(u)$.

An Orlicz function ϕ satisfies the *Dini-condition* if there exists a constant $C > 0$ such that, for all $u > 0$,

$$\int_0^u \frac{p(t)}{t} dt \le Cp(u),$$

where $p(t)$ is the derivative of $\phi(t)$. Note that this condition is equivalent to the Δ_2-condition on ψ.

Let $(\mathcal{M}, \mathcal{B}, \mu)$ be a σ-finite measure space, where μ is a measure on $\mathcal{M}$ and $\mathcal{B}$ is the set of measurable subsets of $\mathcal{M}$. A *weight* w is a function on $\mathcal{M}$ which is positive and finite almost everywhere. The *Orlicz space* with weight w (denoted by $L_\phi(w)$) is the space of measurable functions $f : \mathcal{M} \to \mathbb{R}$ such that there exists $\lambda > 0$ with $\int\limits_{\mathcal{M}} \phi\left(\frac{f(x)}{\lambda}\right) w(x) d\mu(x) \le 1$.

The norm of f in $L_\phi(w)$ (denoted by $\|f\|_{\phi,w}$) is the infimum over all such λ.

We will simply denote by L_ϕ the Orlicz space with weight the constant function 1 and the norm is then denoted by $\|\cdot\|_\phi$.

For $f \in L_\phi(w)$, one can also define

$$\|f\|^\star_{\phi,w} = \sup\left\{\left|\int_{\mathcal{M}} f(x)g(x)w(x)d\mu(x)\right| ; \int_{\mathcal{M}} \psi(g(x))w(x)d\mu(x) \le 1\right\}.$$

Then $\|\cdot\|^\star_{\phi,w}$ is a norm which is equivalent to $\|\cdot\|_{\phi,w}$.

We also have Hölder inequality for Orlicz spaces, that is:

$$\left|\int_{\mathcal{M}} f(x)g(x)w(x)d\mu(x)\right| \le C\|f\|_{\phi,w}\|g\|_{\psi,w}.$$

Let us also remark that if ϕ and ψ satisfy the Δ_2-condition then the space $L_\phi(w)$ is reflexive and for all f in $L_\phi(w)$, $\int_{\mathcal{M}} \phi(f(x))w(x)d\mu(x)$ is finite.

2 Transfer principles

2.1 R. Coifman – G. Weiss transfer principle

Let us recall this principle. In what follows G will be a locally compact amenable group, that is: given a compact set $K \subset G$ and $\epsilon > 0$ there exists an open neighborhood V of the identity having finite left (or right) Haar measure $\nu(V)$ such that:

$$\nu(VK^{-1})/\nu(V) \leq 1 + \epsilon. \tag{1}$$

Any compact group and any locally compact abelian group is amenable (for more details about topological amenable groups, see [8,10]).

Let $(\mathcal{M}, \mathcal{B}, \mu)$ be a σ-finite measure space and R be a representation of G acting on $L_p(\mathcal{M})$ ($1 \leq p < \infty$). That is $R : u \to R_u$ is continuous as a mapping from G into the space of bounded operators on $L_p(\mathcal{M})$ and $R_{uv} = R_u R_v$, for all $u, v \in G$.

We suppose R_e, e the identity of G, is the identity operator and the family $\{R_u\}$ is uniformly bounded, that is: there is a constant $C > 0$ such that for all $u \in G$, for all $F \in L_p(\mathcal{M})$,

$$\|R_u F\|_p \leq C \|F\|_p.$$

Let $k \in L_1(G)$ have compact support and $N_p(k)$ be the operator norm of the convolution operator $H_k : f \to k \star f$ on $L_p(G)$. R. Coifman and G. Weiss [5] proved that the transfered operator

$$H_k^{\#} F = \int_G k(u) R_{u^{-1}} F du$$

is defined on $L_p(\mathcal{M})$, maps into $L_p(\mathcal{M})$ and is bounded with an operator norm not exceding $C^2 N_p(k)$.

We will extend this result to Orlicz spaces $L_\phi(w)$ with weight w. We suppose that the family $\{R_u\}$ is uniformly bounded in the following sense: there exists a constant $C > 0$ such that for all $u \in G$, for all $F \in L_\phi(w)$,

$$\int_{\mathcal{M}} \phi(R_u F(x)) w(x) d\mu(x) \leq \int_{\mathcal{M}} \phi(CF(x)) w(x) d\mu(x). \tag{2}$$

Let us remark that this integral inequality is not equivalent to the norm inequality

$$\|R_u F\|_{\phi,w} \leq C \|F\|_{\phi,w}.$$

The norm inequality is weaker than the integral one. The integral inequality is equivalent to uniform boundedness of a family of norm inequalities. This result is proved in Orlicz spaces (see [2,3]) and it is still true in Orlicz spaces with weight. More precisely, we have:

1 Proposition. *Let $(\mathcal{M}, \mathcal{B}, \mu)$ be a σ-finite measure space, $L(\mathcal{M})$ be the measurable functions on $\mathcal{M}$ and T be a quasi-linear operator on $L(\mathcal{M})$. Then,*

$$\int_{\mathcal{M}} \phi(Tf(x))w(x)d\mu(x) \leq \int_{\mathcal{M}} \phi(Cf(x))w(x)d\mu(x) \tag{1'}$$

if and only if, for all $\epsilon > 0$,

$$\|Tf\|_{\phi, w\epsilon d\mu} \leq C\|f\|_{\phi, w\epsilon d\mu},$$

where

$$\|f\|_{\phi, w\epsilon d\mu} = \inf\left\{\lambda > 0, \int_{\mathcal{M}} \phi\left(\frac{f(x)}{\lambda}\right) w(x)\epsilon d\mu(x) \leq 1\right\}.$$

Let $k \in L_1(G)$, we denote by $N_\phi(k)$ the infimum of $C > 0$ such that for all $F \in L_\phi(G)$

$$\int_G \phi(k \star F(g))dg \leq \int_G \phi(CF(g))dg.$$

And more generally we denote, for a sequence $\{k_j\}$ in $L_1(G)$, by $N_\phi(\{k_j\})$ the smallest constant $C > 0$ such that for all $F \in L_\phi(G)$,

$$\int_G \phi(\sup_j |k_j \star F|(g))dg \leq \int_G \phi(CF(g))dg.$$

We will prove the following transfered theorem:

2 Theorem. *The transfered operator $H_k^\#$ is defined on $L_\phi(w)$, maps into $L_\phi(w)$ and satisfies the following inequality:*

$$\int_{\mathcal{M}} \phi(H_k^\# F(x))w(x)d\mu(x) \leq \int_{\mathcal{M}} \phi(C^2 N_\phi(k)F(x))w(x)d\mu(x) \tag{3}$$

In particular,

$$\|H_k^\#\|_{\phi,w} \leq C^2 N_\phi(k).$$

Before starting the proof, let us mention that we do not assume any more that k has compact support. Since following [1] we get a general lemma (for a sequence $\{k_j\}$ in $L_1(G)$) that will be also useful later.

3 Lemma. *Let $\{k_j\}_{j=1}^n$ be a finite sequence in $L_1(G)$ and $(k_{j,p})_{p=1}^\infty$ be a sequence in $L_1(G)$ for each j such that $\|k_{j,p} - k_j\|_1 \overset{p\to\infty}{\longrightarrow} 0$ then,*

$$N_\phi(\{k_{j,p}\}) \overset{p\to\infty}{\longrightarrow} N_\phi(\{k_j\}).$$

Suppose we have proved (3) for $k \in L_1(G)$ with compact support. Take an arbitrary $k \in L_1(G)$. Then there is a sequence $\{k_p\}$ with compact support that tends to k in $L_1(G)$. By Lemma 3, we also have $N_\phi(k_p) \overset{p\to\infty}{\longrightarrow} N_\phi(k)$. For each p we have the corresponding inequality (3). Or equivalently, for all $\epsilon > 0$,

$$\begin{aligned}\|H^{\#}_{k_p}F\|_{\phi,w\epsilon d\mu} &\le C^2 N_\phi(k_p)\|F\|_{\phi,w\epsilon d\mu} \\ &\le C^2 N_\phi(\{k_p\})\|F\|_{\phi,w\epsilon d\mu}\end{aligned}$$

Then, if we prove that the norm $\|H^{\#}_{k_p}F - H^{\#}_k F\|_{\phi,w\epsilon d\mu}$ tends to zero, we have done. We will show that the equivalent norm $\|H^{\#}_{k_p}F - H^{\#}_k\|^{\star}_{\phi,w\epsilon d\mu}$ tends to zero. By definition of this norm, we have

$$\begin{aligned}&\|H^{\#}_{k_p}F - H^{\#}_k F\|^{\star}_{\phi,w\epsilon d\mu} \\ &\quad = \sup\left\{\left|\int_{\mathcal{M}}\left(H^{\#}_{k_p} - H^{\#}_k\right)(F) g w\epsilon d\mu\right|, \int_{\mathcal{M}}\psi(g)w\epsilon d\mu \le 1\right\} \\ &\quad = \sup\left\{\left|\int_{\mathcal{M}}\left[\int_G (k_p - k)(u) R_{u^{-1}}F(x)du\right] g(x)w(x)\epsilon d\mu(x)\right|,\right. \\ &\qquad\qquad \left.\int_{\mathcal{M}}\psi(g)w\epsilon d\mu \le 1\right\} \\ &\quad \sup\left\{\int_G |(k_p - k)(u)|\left[\int_{\mathcal{M}}|R_{u^{-1}}F(x)|\,|g(x)|\,w(x)\epsilon d\mu(x)\right]du,\right. \\ &\qquad\qquad \left.\int_{\mathcal{M}}\psi(g)w\epsilon d\mu \le 1\right\} \\ &\quad \le \int_G |(k_p - k)(u)|\cdot\|R_{u^{-1}}F\|^{\star}_{\phi,w\epsilon d\mu}du \\ &\quad \le A\|k_p - k\|_1\|F\|^{\star}_{\phi,w\epsilon d\mu}\end{aligned}$$

And we then get the inequality (3) for $k \in L_1(G)$.

We now give the proof of Theorem2. This proof follows the one of [5].

Proof. Let $k \in L_1(G)$ with compact support K. Let us mention as, for $F \in L_\phi(w)$,

$$\int_G \|k(u)R_{u^{-1}}F\|_{\phi,w}du \le C\|k\|_1\|F\|_{\phi,w},$$

the function $u \to k(u)R_{u^{-1}}F$ is Bochner integrable and

$$\|H^{\#}_k F\|_{\phi,w} \le C\|k\|_1\|F\|_{\phi,w}.$$

Let V be an open neighborhood satisfying (1). If we take in (2), $R_u H^{\#}_k F$ instead of F and u^{-1} instead of u, we get

$$\int_{\mathcal{M}}\phi(H^{\#}_k F(x))w(x)d\mu(x) \le \int_{\mathcal{M}}\phi(CR_u H^{\#}_k F(x))w(x)d\mu(x).$$

Integration on V gives :

$$\nu(V)\int_{\mathcal{M}}\phi(H_k^{\#}F(x))w(x)d\mu(x)\leq\int_V\left[\int_{\mathcal{M}}\phi(CR_uH_k^{\#}F(x))w(x)d\mu(x)\right]du.$$

In fact, $R_uH_k^{\#}F(x)=\int_G k(g)R_{ug^{-1}}F(x)dg$. Put χ the characteristic function of the set VC^{-1}, then the second member of the preceding inequality is by permuting integrations,

$$\begin{aligned}&\int_{\mathcal{M}}\left[\int_G\phi\left(C\int_G k(g)R_{ug^{-1}}F(x)\chi(ug^{-1})dg\right)du\right]w(x)d\mu(x)\\&\leq\int_{\mathcal{M}}\left[\int_G\phi\left(CN_\phi(k)R_gF(x)\chi(g)dg\right)\right]w(x)d\mu(x)\\&\leq\nu(VC^{-1})\int_{\mathcal{M}}\phi\left(C^2N_\phi(k)F(x)\right)w(x)d\mu(x)\end{aligned}$$

And we then get the inequality (3). QED

2.2 N. Asmar, E. Berkson, T.A. Gillepsie transfer of strong type maximal inequalities

N. Asmar, E. Berkson, P.A. Gillepsie [1] do not consider only one single operator but a sequence of operators in L_p, see [1]. We will recall the situation they consider. Let us consider $(\mathcal{M},\mathcal{B},\mu)$ an arbitrary measure space.

Let us recall two definitions. An operator T in $L_p(\mathcal{M})$ is *separation-preserving* (respectively, *positivity-preserving*) provided that whenever $f,g\in L_p(\mathcal{M})$ and $f\cdot g=0$ μ a.e. (respectively, $f\in L_p(\mathcal{M})$ and $f\geq 0$ μ a.e.) we have $(Tf)(Tg)=0$ μ a.e. (respectively, $Tf\geq 0$ μ a.e.).

Let G be a locally compact abelian group and $R:u\to R_u$ be a representation of G in $L_p(\mathcal{M})$ $(1\leq p<\infty)$ such that the family $\{R_u\}$ is uniformly bounded and let us put $C=\sup\{\|R_u\|,u\in G\}$. Suppose that, for each $u\in G$, R_u is separation-preserving.

Let $\{k_j\}$ be any (finite or infinite) sequence in $L_1(G)$ such that $N_p(\{k_j\})$ is finite, where $N_p(\{k_j\})$ is the smallest constant $a>0$ such that, for all $f\in L_p(G)$,

$$\|\sup_j|k_j\star f|\,\|_p\leq a\|f\|_p.$$

Let us denote by $H^{\#}$ the maximal operator corresponding to the operators $\{H_{k_j}^{\#}\}$. That is $H^{\#}=\sup_j\left|H_{k_j}^{\#}\right|$.

With these notations and conditions, N. Asmar, E. Berkson, T.A. Gillepsie got the following result:

4 Theorem ([1]). *The maximal operator $H^{\#}$ has a norm satisfying:*

$$\left\|H^{\#}\right\|_{L_p(\mathcal{M})} \leq C^2 N_p(\{k_j\}).$$

Let us mention that this theorem ceases to be true without separation-preserving assumption, see [1] for an example.

Let us consider the L_ϕ-extension. Our conditions will be the following: G is a locally compact abelian group, $\{k_j\}$ is a sequence in $L_1(G)$, $(\mathcal{M}, \mathcal{B}, \mu)$ is a σ-finite measure space and for each $u \in G$, R_u is positivity-preserving in $L_\phi(w)$ and the family $\{R_u\}$ satisfies the condition of uniform boundedness (2). We get:

5 Theorem. *The maximal operator $H^{\#}$ satisfies the following integral inequality:*

$$\int_{\mathcal{M}} \phi\left(H^{\#}F(x)\right) w(x)d\mu(x) \leq \int_{\mathcal{M}} \phi\left(C^2 N_\phi(\{k_j\})F(x)\right) w(x)d\mu(x). \tag{4}$$

And in particular,

$$\|H^{\#}\|_{\phi,w} \leq C^2 N_\phi(\{k_j\}).$$

Proof. We only give the beginning of the proof, the rest follows the proof of Theorem 2 and [1]. By monotone convergence we need only to consider a finite sequence $\{k_j\}_{j=1}^n$. We can also suppose that k_j has compact support: let $k_j \in L^1(G)$, then there exists $k_{j,p} \in L^1(G)$ with compact support tending to k_j in $L^1(G)$. By Lemma 3, we also have:

$$N_\phi(\{k_{j,p}\}) \overset{p\to\infty}{\longrightarrow} N_\phi(\{k_j\}).$$

Suppose we have for each p the inequality (4).

We have, for all $\epsilon > 0$:

$$\left\|\max_{1\leq j\leq n} H^{\#}_{k_j,p}F - \max_{1\leq j\leq n} H^{\#}_{k_j}F\right\|_{\phi,w\epsilon d\mu} \leq \sum_{j=1}^{n} \left\|H^{\#}_{k_j,p}F - H^{\#}_{k_j}F\right\|_{\phi,w\epsilon d\mu}$$

And as we did previously, we get :

$$\left\|H^{\#}_{k_j,p}F - H^{\#}_{k_j}F\right\|_{\phi,w\epsilon d\mu} \overset{p\to+\infty}{\longrightarrow} 0.$$

And we get the inequality (4) for k.

Let now k_j be arbitrary in $L^1(G)$.

$$\begin{aligned} H^{\#}_{k_j} R_{-u}F &= \int_G k_j(v) R_{-v-u}F \; dv \\ &= R_{-u}\left[\int_G k_j(v) R_{-v}F \; dv\right]. \end{aligned}$$

As by positivity-preserving of R_u,

$$|R_u F| = \left|R_u(F^+ - F^-)\right| \leq R_u F^+ + R_u F^- = R_u|F|.$$

$\left|H^{\#}_{k_j} R_{-u} F\right| \leq R_{-u} \left|H^{\#}_{k_j} F\right|$. And again by positivity-preserving:

$$\max_{1\leq j\leq n} \left|H^{\#}_{k_j} R_{-u} F\right| \leq R_{-u} \left(\max_{1\leq j\leq n} \left|H^{\#}_{k_j} F\right|\right).$$

Take $R_u F$ instead of F to get :

$$\begin{aligned}
&\int_{\mathcal{M}} \phi\left(\max_{1\leq j\leq n} \left|H^{\#}_{k_j} F(x)\right|\right) w(x) d\mu(x) \\
&\leq \int_{\mathcal{M}} \phi\left(R_{-u}\left(\max_{1\leq j\leq n} \left|H^{\#}_{k_j} R_u F(x)\right|\right)\right) w(x) d\mu(x) \\
&\leq \int_{\mathcal{M}} \phi\left(C \max_{1\leq j\leq n} \left(H^{\#}_{k_j} R_u F(x)\right)\right) w(x) d\mu(x).
\end{aligned}$$

And then follow the proof of [1] to get the required inequality. QED

6 Remark. We suppose here that R_u is positivity-preserving. Then following [1], R_u is also separation-preserving. If we only suppose R_u separation-preserving, we then need the Δ_2-condition for the Orlicz function ϕ. Under this condition we get the density of the simple functions in $L_\phi(w)$ and we then get (as in [11]) that if T is a separation-preserving operator in $L_\phi(w)$ then there is a positivity-preserving operator $|T|$ in $L_\phi(w)$ such that

$$\forall f \in L_\phi, \quad |Tf| = |T|(|f|)\mu\text{a.e.}$$

It suffices to follow the proof of [1] to get (4).

2.3 Applications

2.3.1 Hilbert transforms

For a function $f \in L_p(\mathbb{T})$, $1 < p < \infty$, let us consider the following Hilbert transforms:

(a) the truncated Hilbert transform

$$\left(H_{(N)} f\right)(x) = \int_{\frac{1}{N}\leq |t|\leq N} \frac{f(x-t)}{t} dt = (k \star f)(x),$$

with the kernel $k(t) = \frac{1}{t}\chi_{\{\frac{1}{N}\leq |s|\leq N\}}(t)$,

(b) the maximal Hilbert transform

$$(H^{\#}f)(x) = \sup_{N>0} \left| \left(H_{(N)}f\right)(x) \right|,$$

(c) the conjugate function operator

$$\tilde{f}(x) = P.V. \int_0^1 f(x-s)\,\mathrm{cotg}(\pi s)ds.$$

All these operators are bounded in $L_p(\mathbb{T})$, for all p $(1 < p < \infty)$ and of weak type $(1,1)$ [5]. Using interpolation, when $L_\phi(\mathbb{T})$ is reflexive we can deduce that all these operators are then bounded in $L_\phi(\mathbb{T})$. We transfer them and using the different transfer principles we get the expected inequalities for the corresponding transfered operators. To be more precise: let ϕ be an Orlicz function. We denote by R_ϕ an Orlicz function generating an Orlicz space L_{R_ϕ} such that

$$R_\phi(u) = u \int_1^u t^{-2}\phi(t)dt, u \geq 1.$$

(In what follows the definition of R_ϕ, $0 \leq u < 1$ does not play any role). Suppose the function ϕ satisfies the Δ_2-condition, then R_ϕ also satisfies the Δ_2-condition and $L_{R_\phi} \subset L_\phi$ ([7,16]). Let us recall the following Marcinkiewicz's interpolation type theorem

7 Theorem ([16]). *Let G be of finite measure and T be a quasi-linear operator which is bounded in $L_p(G)$, for all $1 < p < \infty$ and of weak-type (1,1). Let ϕ satisfy the Δ_2-condition. Then T is defined on L_{R_ϕ} and T is bounded as an operator from L_{R_ϕ} to L_ϕ.*

It can be showed that in fact $\left(L_\phi(G) = L_{R_\phi}(G)\right)$ if and only if ($L_\phi(G)$ is reflexive) [7]. Then if $L_\phi(G)$ is reflexive the operator T is bounded in $L_\phi(G)$.

2.3.2 Integral transform with radial kernels

A function is said to be radial if it is invariant under the action of rotations of $\mathbb{R}^n$. Let k be a continuous radial function in $\mathbb{R}^n$ $(n \geq 3)$ with compact support. In particular k has the form $k(y) = k^0(|y|)$, where k^0 is a function defined on the non negative reals. Let H_k be the operator mapping $f \in L_\phi(\mathbb{R}^n)$ into $k \star f$. It can be shown [5] that

$$H_k f(x) = \frac{\omega_{n-1}}{\omega_{n-2}} \int_{SO(n)} du \left[\int_{\mathbb{R}^{n-1}} |y| k(y) f(x - uy) dy \right]$$

where $\mathcal{SO}(n)$ is the group of rotations of $\mathbb{R}^n$, ω_{n-1} is the "surface area" of the unit sphere $\Sigma_{n-1} \subset \mathbb{R}^n$.

The locally compact abelian group G is $\mathbb{R}^{n-1}$, the representation of G depends on $u \in \mathcal{SO}(n)$ and is given by $(R_y^u f)(x) = f(x+uy)$, when f is a function defined on $\mathcal{M} = \mathbb{R}^n$. The inequality (2) is satisfied. If the kernel $h(y) = |y|k(y)$ induces an operator $f \to h \star f$ on $L_\phi(G)$ satisfying: $\exists C > 0 \quad \forall f \in L_\phi(\mathbb{R}^{n-1})$,

$$\int_{\mathbb{R}^{n-1}} \phi(h \star f(g))dg \leq \int_{\mathbb{R}^{n-1}} \phi(Cf(g))dg.$$

Then

$$\int_{\mathbb{R}^n} \phi(H_k f(x))dx = \frac{\omega_{n-1}}{\omega_{n-2}} \int_{\mathbb{R}^n} \phi\left[\int_{\mathcal{SO}(n)} du \left[\int_{\mathbb{R}^{n-1}} h(y)f(x-uy)dy\right]\right] dx$$

As $\int_{\mathcal{SO}(n)} du = 1$, by Jensen's inequality and Fubini's theorem,

$$\int_{\mathbb{R}^n} \phi\left(H_k f(x)\right) dx \leq \frac{\omega_{n-1}}{\omega_{n-2}} \int_{\mathcal{SO}(n)} \left[\int_{\mathbb{R}^n} \phi\left[\int_{\mathbb{R}^{n-1}} h(y)f(x-uy)dy\right] dx\right] du.$$

Applying Theorem 2, we get

$$\int_{\mathbb{R}^n} \phi\left(H_k f(x)\right) dx \leq \frac{\omega_{n-1}}{\omega_{n-2}} \int_{\mathbb{R}^n} \phi(Cf(x))dx.$$

We have therefore established

8 Theorem. *Suppose k^0 is a continuous function on $\mathbb{R}^+$ having compact support. If $h(y) = |y|k^0(|y|)$ satisfies, for all $f \in L_\phi(\mathbb{R}^{n-1})$:*

$$\int_{\mathbb{R}^{n-1}} \phi\left[\int_{\mathbb{R}^{n-1}} h(y)f(z-y)dy\right] dz \leq \int_{\mathbb{R}^{n-1}} \phi(Cf(y))dy.$$

Then, for all $f \in L_\phi(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \phi\left[\int_{\mathbb{R}^n} k^0(|y|)f(x-y)dy\right] dx \leq \int_{\mathbb{R}^n} \phi(Cf(y))dy. \tag{5}$$

By iteration we can reduce the question of L_ϕ-boundedness of radial convolutions on $\mathbb{R}^n$ to one-dimensional problem: let k be the kernel then the operator $f \to k \star f$ satisfies (5) in $L_\phi(\mathbb{R}^n)$ provided $h(t) = |t|^{n-1}k^0(|t|)$ gives us a convolution on $L_\phi(\mathbb{R})$ satisfying the corresponding integral inequality.

3 Maximal ergodic theorem and transference from the integers

Let $(\mathcal{M}, \mathcal{B}, \mu)$ be a σ-finite measure space, and let $T : \mathcal{M} \to \mathcal{M}$ be a measure preserving measurable function on $\mathcal{M}$. Then we can define a linear operator on the Orlicz space $L_\phi = L_\phi(\mathcal{M})$, which we also denote by T, by setting for all f in L_ϕ

$$(Tf)(x) = f(Tx).$$

We have, as T is measure preserving, for all f in L_ϕ

$$\|Tf\|_\phi = \|f\|_\phi$$

We use the notations $A_n f$ (for n a natural number) and $A_\infty f$ respectively for the operators defined by:

$$\begin{aligned} A_n f(x) =& \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x) \\ A_\infty f(x) =& \sup_{n \geq 0} |A_n f(x)| \end{aligned}$$

We will prove that the maximal operator A_∞ satisfies a strong type inequality, i.e. there exists a constant $C > 0$ such that

$$\|A_\infty f\|_\phi \leq C \|f\|_\phi.$$

First we will prove such an inequality in the particular case of $L_\phi(\mathbb{N})$. Let τ be the translation on $\mathbb{N}$ i.e. $\tau(n) = n + 1$.

We put, for $F \in L_\phi(\mathbb{N})$ and $n \in \mathbb{N}$:

$$\begin{cases} \alpha_n F(i) = \frac{1}{n+1} \sum_{k=0}^{n} F(i+k) \\ \alpha_\infty F(i) = \sup_{n \geq 0} |\alpha_n F(i)| \end{cases}$$

We get

9 Proposition. *Suppose that ϕ satisfies the Dini-condition. There exists a constant $C > 0$ such that, for all F in $L_\phi(\mathbb{N})$:*

$$\sum_{i=0}^{+\infty} \phi(\alpha_\infty F(i)) \leq \sum_{i=0}^{\infty} \phi(CF(i)).$$

In particular,

$$\|\alpha_\infty F\|_\phi \leq C \|F\|_\phi.$$

Proposition 9 extends to Orlicz spaces the L_p-inequality

$$\sum_{i=0}^{+\infty} (\alpha_\infty F(i))^p \leq \sum_{i=0}^{+\infty} (CF(i))^p$$

which can be proved using a result of Hardy and Littlewood [9] and the following well-known inequality for $p > 1$:

$$\sum_{i=0}^{n} \left(\frac{a_0 + a_1 + \cdots + a_i}{i+1} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{i=0}^{n} a_i^p \qquad (6)$$

Before starting the proof, let us mention a few facts.

Let $a_0, \ldots, a_n$ be a finite sequence of positive numbers, let s be a non decreasing function defined on $\mathbb{R}$, and let $k = k(i)$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that $k(i) \leq i$ for all i in $\mathbb{N}$. We denote

$$A(i, k, a) = \frac{a_k + a_{k+1} + \cdots + a_i}{i - k + 1},$$

and let $a_0^\star, a_1^\star, \ldots, a_n^\star$ be the sequence $a_0, a_1, \ldots, a_n$ arranged in decreasing order. Then following [9]

$$\sum_{i=0}^{n} s(A(i, k, a)) \leq \sum_{i=0}^{n} s(A(i, 0, a^\star)).$$

We will prove the following result, which extend inequality (6) to Orlicz spaces.

10 Lemma. *Suppose that ϕ satisfies the Dini-condition. There exists a constant $C > 0$ such that for all finite sequences $a_0, a_1, \ldots, a_n$ of positive numbers:*

$$\sum_{i=0}^{n} \phi \left(\frac{a_0 + a_1 + \cdots + a_i}{i+1} \right) \leq \sum_{i=0}^{n} \phi(C a_i).$$

Proof. To simplify the notations, we denote

$$f : \{0, \ldots, n\} \to \mathbb{R}^+ : i \to a_i$$

and

$$Mf(i) = \frac{a_0 + a_1 + \cdots + a_i}{i+1}.$$

Then we have

$$\begin{aligned}\sum_{i=0}^{n}\phi(Mf(i)) =& \sum_{i=0}^{n}\int_0^{Mf(i)} p(t)dt \qquad (p(t) \text{ is the derivative of } \phi)\\ =& \sum_{i=0}^{n}\int_0^{+\infty} p(t)\cdot\chi_{[0,Mf(i)]}(t)dt\\ =& \sum_{i=0}^{n}\int_0^{\infty} p(t)\chi_{[t,+\infty[}(Mf(i))dt\\ =& \int_0^{\infty} p(t)(\#\{0\le i\le n : Mf(i)\ge t\})dt\end{aligned}$$

But we have

$$\# \{0 \le i \le n : Mf(i) \ge t\} = \\ \# \left\{0\le i\le n : i+1\le \frac{1}{t}\sum_{k=0}^{i} f(k)\right\} \le \frac{1}{t}\sum_{k=0}^{n} f(k). \quad (7)$$

For $a > 0$, $a \in \mathbb{R}$, we denote

$$\begin{aligned}\lambda_f(a) =& \{0\le i\le n : f(i)\ge a\}\\ \lambda_f^c(a) =& \{0,1,\dots,n\}\setminus\lambda_f(a)\\ (f)_a =& f\chi_{\lambda_f^c(a)} + a\chi_{\lambda_f(a)}\\ (f)^a =& (f-a)\chi_{\lambda_f(a)}\end{aligned}$$

and we have $f = (f)^a + (f)_a$, $(f)_a \le a$ and then $M(f)_a \le a$.

Fix $0 < r < 1$, we have

$$\begin{aligned}Mf(i) =& M(f)^{rt}(i) + M(f)_{rt}(i)\\ \le& M(f)^{rt}(i) + rt \qquad \text{for all } t > 0.\end{aligned}$$

We obtain:

$$\begin{aligned}\#\{0\le i\le n : Mf(i)\ge t\} =& \#\{0\le i\le n : M(f)^{rt}(i)\ge (1-r)t\}\\ \le& \frac{1}{(1-r)t}\sum_{k=0}^{n}(f)^{rt}(k) \quad \text{by (7)}\\ =& \frac{1}{(1-r)t}\sum_{\substack{k=0\\ f(k)\ge rt}}^{n}(f(k)-rt)\end{aligned}$$

$$\leq \frac{1}{(1-r)t} \sum_{\substack{k=0 \\ f(k)\geq rt}}^{n} f(k).$$

We use this last inequality to obtain

$$\begin{aligned}
\sum_{i=0}^{n} \phi(Mf(i)) &\leq \frac{1}{1-r} \int_0^{+\infty} \frac{p(t)}{t} \left(\sum_{\substack{k=0 \\ f(k)\geq rt}}^{n} f(k) \right) dt \\
&= \frac{1}{1-r} \int_0^{+\infty} \frac{p(t)}{t} \left(\sum_{k=0}^{n} f(k) \chi_{\lambda_f(rt)}(k) \right) dt \\
&= \frac{1}{1-r} \sum_{k=0}^{n} \left[f(k) \int_0^{+\infty} \frac{p(t)}{t} \chi_{\lambda_f(rt)}(k) dt \right] \\
&= \frac{1}{1-r} \sum_{k=0}^{n} \left[f(k) \int_0^{\frac{f(k)}{r}} \frac{p(t)}{t} dt \right]
\end{aligned}$$

By the Dini-condition on ϕ, we get there exists $C \geq 1$ such that

$$\begin{aligned}
\sum_{i=0}^{n} \phi(Mf(i)) &\leq \frac{1}{1-r} \sum_{k=0}^{n} \left[f(k) C p\left(\frac{f(k)}{r} \right) \right] \\
&\leq \frac{Cr}{1-r} \sum_{k=0}^{n} \phi\left(\frac{2f(k)}{r} \right)
\end{aligned}$$

as $up(u) \leq \phi(2u)$ for all $u > 0$.

In particular, this last inequality is true for $r = \frac{1}{2}$ and we obtain

$$\sum_{i=0}^{n} \phi(Mf(i)) \leq C \sum_{k=0}^{n} \phi(4f(k)) \leq \sum_{k=0}^{n} \phi(4Cf(k)),$$

by convexity of ϕ. *QED*

Proof of Proposition 9. Let $F \in L_\phi(\mathbb{N})$. As $\alpha_\infty F \leq \alpha_\infty |F|$, we can suppose $F \geq 0$. Without lost of generality, we can also suppose that there exists $N \in \mathbb{N}$ such that $F(n) = 0$ for $n > N$.

First note that for $n \geq N - i$, we have

$$\alpha_n F(i) \leq \alpha_{N-i} F(i),$$

so we have

$$\alpha_\infty F(i) = \sup_{0\leq n\leq N-i} \alpha_n F(i).$$

Now, we have to prove

$$\sum_{i=0}^{N} \phi\left(\sup_{0\le n\le N-i} \alpha_n F(i)\right) \le \sum_{i=0}^{N} \phi(CF(i))$$

for a constant $C > 0$ independant of the choice of F.

Following our notations, let us take $a_i = F(N-i)$ for $i = 0, 1, \ldots, N$, $s = \phi$ and $k = k(i)$ the least value of the indice k such that

$$\sup_{0\le n\le N-i} A_n F(i) = A_k F(i).$$

We then obtain

$$\begin{aligned}\sum_{i=0}^{N} \phi\left(\sup_{0\le n\le N-i} A_n F(i)\right) &= \sum_{i=0}^{N} \phi\left(A\left(N-i, k, a\right)\right)\\ &\le \sum_{i=0}^{N} \phi\left(A\left(N-i, 0, a^\star\right)\right)\\ &= \sum_{i=0}^{N} \phi\left(\frac{a_0^\star + a_1^\star + \cdots + a_i^\star}{i+1}\right)\end{aligned}$$

We then use Lemma 10 to obtain

$$\sum_{i=0}^{N} \phi\left(\sup_{0\le n\le N-i} A_n F(i)\right) \le \sum_{i=0}^{N} \phi(Ca_i^\star) = \sum_{i=0}^{N} \phi(Ca_i) = \sum_{i=0}^{N} \phi(CF(i))$$

where C is a constant independant of F. QED

Now we get the general result by transfering from the integers. Note that our Transfer Principle extends to Orlicz spaces the transfer principle of Wiener and Calderon ([4], [15]).

11 Proposition. *(Transfer Principle). Suppose that there exists $C > 0$ such that for all F in $L_\phi(\mathbb{N})$*

$$\sum_{i=0}^{+\infty} \phi\left(\alpha_\infty F(i)\right) \le \sum_{i=0}^{+\infty} \phi(CF(i)).$$

Then we have for all f in $L_\phi(\mathcal{M})$

$$\int_{\mathcal{M}} \phi\left(A_\infty f(x)\right) d\mu(x) \le \int_{\mathcal{M}} \phi(Cf(x)) d\mu(x).$$

Proof. Let $f \in L_\phi$. As $A_\infty f \leq A_\infty |f|$, we can suppose $f \geq 0$.
Fix $J \in \mathbb{N}$, we note

$$M_J f(x) = \sup_{0 \leq n \leq J} A_n f(x).$$

For $N \in \mathbb{N}$, $N \gg J$, and $x \in X$, we set

$$F(n) = \begin{cases} f(T^n x) & \text{if} \quad n \leq N \\ 0 & \text{if} \quad n > N \end{cases}$$

If $i \leq N - J$, we then have

$$\sup_{0 \leq n \leq J} \alpha_n F(i) = \sup_{0 \leq n \leq J} A_n f(T^i x) = M_J f(T^i x)$$

and

$$\begin{aligned} \sum_{i=0}^{n-J} \phi\left(M_J f(T^i x)\right) &= \sum_{i=0}^{n-J} \phi\left(\sup_{0 \leq n \leq J} \alpha_n F(i)\right) \\ &\leq \sum_{i=0}^{+\infty} \phi(CF(i)) = \sum_{i=0}^{N} \phi(CF(i)) = \sum_{i=0}^{N} \phi(Cf(T^i x)) \end{aligned}$$

We integrate over $\mathcal{M}$ and use the T-invariance of the measure μ to obtain:

$$(N - J + 1) \int_{\mathcal{M}} \phi(M_J f(x)) d\mu(x) \leq (N + 1) \int_{\mathcal{M}} \phi(Cf(x)) d\mu(x)$$

or

$$\int_{\mathcal{M}} \phi(M_J f(x)) d\mu(x) \leq \frac{N+1}{N-J+1} \int_{\mathcal{M}} \phi(Cf(x)) d\mu(x).$$

Let $N \to +\infty$, we obtain:

$$\int_{\mathcal{M}} \phi(M_J f(x)) d\mu(x) \leq \int_{\mathcal{M}} \phi(Cf(x)) d\mu(x).$$

Then we let $J \to +\infty$ and apply Fatou's lemma to obtain

$$\int_{\mathcal{M}} \phi(A_\infty f(x)) d\mu(x) \leq \int_{\mathcal{M}} \phi(Cf(x)) d\mu(x).$$

QED

We then get

12 Theorem. *Suppose that ϕ satisfies the Dini-condition. Then there exists a constant $C > 0$ such that, for all f in L_ϕ*

$$\int_{\mathcal{M}} \phi(A_\infty f(x))d\mu(x) \leq \int_{\mathcal{M}} \phi(Cf(x))d\mu(x).$$

In particular,

$$\|A_\infty f\|_\phi \leq C\|f\|_\phi.$$

Let us mention that this result is more general than the one obtained by interpolation. Indeed, D. Gallardo [8] proved that if ϕ and ψ satisfy the Δ_2-condition the every operator of weak-type (1,1) and of type p $(1 < p < \infty)$ is defined on L_ϕ, it has values in L_ϕ and it satisfies the integral inequality (1').

The operator A_∞ is of weak-type (1,1) and of type p $(1 < p < \infty)$ [11]. Thus, when ϕ and ψ satisfy the Δ_2-condition then we directly get the result by interpolation.

4 Pointwise ergodic theorem

In this section, we will extend Birkhoff's ergodic theorem to Orlicz spaces, in the case where these spaces are reflexive. We use the same notations as in the previous section.

If the Orlicz space L_ϕ is reflexive, then we can deduce, from the mean ergodic theorem of Von Neumann, that the sequence of operators $\{A_n\}_{n\geq 0}$ converges in the strong topology of operators, as $n \to +\infty$ [6, chapter VIII.5].

Further, we have the decomposition

$$L_\phi = F \oplus \overline{(I-T)(L_\phi)}$$

where $F = \{f \in L_\phi : Tf = f\}$ and $\overline{(I-T)(L_\phi)}$ is the closure of the range of $I - T$ (I is the identity operator) [6, chapter VIII.5].

We will then prove the following result, which extend Birkhoff's ergodic theorem to L_ϕ.

13 Theorem. *Suppose that ϕ and ψ satisfy the Δ_2-condition, and let $f \in L_\phi$. Then the sequence $\{A_n f(x)\}_{n\geq 0}$ converges almost everywhere on $\mathcal{M}$.*

Proof. As L_ϕ is reflexive, we have

$$L_\phi = F \oplus \overline{(I-T)(L_\phi)}.$$

If $f \in F$, then $A_n f = f$ for all $n \geq 0$, and the sequence $\{A_n f(x)\}_{n\geq 0}$ converges trivially for all x in $\mathcal{M}$.

If $f \in (I-T)(L_\phi)$, we can then write

$$f = g - Tg \quad \text{where } g \in L_\phi.$$

Then, we have

$$A_n f = \frac{1}{n+1}\left(g - T^{n+1}g\right)$$

and so

$$|A_n f(x)| \leq \frac{1}{n+1}\left(|g(x)| + \left|g(T^{n+1}x)\right|\right).$$

As ϕ is even, convex and increases on $[0,+\infty[$, we have (if $n \geq 1$):

$$\phi\left(A_n f(x)\right) \leq \frac{2}{n+1}\phi\left(\frac{1}{2}|g(x)| + \frac{1}{2}\left|g(T^{n+1}x)\right|\right)$$

and again by convexity of ϕ and the definition of T:

$$\phi\left(A_n f(x)\right) \leq \frac{1}{n+1}\left((\phi \circ g)(x) + \left(T^{n+1}(\phi \circ g)\right)(x)\right).$$

But, if ϕ satisfies the Δ_2-condition, then we have $\phi \circ f \in L_1$ for all f in L_ϕ.

We can then use Birkhoff's ergodic theorem for L_1 to deduce

$$\frac{1}{n+1}\left((\phi \circ g)(x) + \left(T^{n+1}(\phi \circ g)\right)(x)\right) \stackrel{n\to+\infty}{\longrightarrow} 0$$

almost everywhere on $\mathcal{M}$, and in particular

$$\phi\left(A_n f(x)\right) \stackrel{n\to+\infty}{\longrightarrow} 0$$

almost everywhere on $\mathcal{M}$.

As ϕ is continuous, increases on $[0,+\infty[$ and satisfies $\phi(u) = 0$ if and only if $u = 0$, we obtain

$$A_n f(x) \stackrel{n\to+\infty}{\longrightarrow} 0$$

almost everywhere on $\mathcal{M}$.

So we have now proved that $\{A_n f(x)\}_{n\geq 0}$ converges almost everywhere on $\mathcal{M}$, for f in a dense subset of L_ϕ.

On the other hand, we also have for all $n \geq 0$, and for all x in $\mathcal{M}$

$$\phi(A_n f(x)) \leq \frac{1}{n+1}\sum_{k=0}^{n} \phi\left(|f(x)|\right).$$

So we have for all x in $\mathcal{M}$,

$$\sup_{n\geq 0} \phi(A_n f(x)) \leq \sup_{n\geq 0}(A_n(\phi \circ f))(x)$$

As $\phi \circ f \geq 0$ and $\phi \circ f \in L_1$, we have

$$\sup_{n\geq 0}(A_n(\phi \circ f))(x) < +\infty$$

almost everywhere on $\mathcal{M}$ [6, chapter VIII.5].

As ϕ is continuous and increases on $[0, +\infty[$, we can deduce that

$$\sup_{n\geq 0}|A_n f(x)| < +\infty$$

almost everywhere on $\mathcal{M}$.

We can then use a theorem of Banach (see [6, IV.11.2]) to deduce that $\{A_n f(x)\}_{n\geq 0}$ converges almost everywhere on $\mathcal{M}$, for all f in L_ϕ. QED

5 Other maximal functions

The maximal functions we will consider here involve the integrals of functions defined on the real line, and were introduced by Hardy and Littlewood [9]. We denote by $(\mathbb{R}, \mathcal{B}, m)$ the measure space consisting of the real line $\mathbb{R}$, with the Lebesgue measure m and the measurable sets $\mathcal{B}$.

Given a Lebesgue measurable function f on $\mathbb{R}$, the maximal functions M^+f and M^-f are defined respectively by

$$M^+f(x) = \sup\left\{(u-x)^{-1}\left|\int_x^u f(t)dt\right| : u > x\right\}$$
$$M^-f(x) = \sup\left\{(x-u)^{-1}\left|\int_u^x f(t)dt\right| : x > u\right\}$$

We also set

$$\begin{aligned} Mf(x) &= \max\left(M^+f(x), M^-f(x)\right) \\ &= \sup\left\{\left|(u-x)^{-1}\int_u^x f(t)dt\right| : u \neq x\right\} \end{aligned}$$

Our aim is to extend Hardy-Littlewood's result to the Orlicz space $L_\phi(\mathbb{R}, \mathcal{B}, m)$.

Our main result of this section is

14 Theorem. *If ϕ satisfies the Dini-condition, then there exists a constant $A > 0$ such that for all $f \in L_\phi$*

i) $\int_{\mathbb{R}} \phi(M^+f(x))dx \leq \int_{\mathbb{R}} \phi(Af(x))dx$

ii) $\int_{\mathbb{R}} \phi(M^-f(x))dx \leq \int_{\mathbb{R}} \phi(Af(x))dx$

iii) $\int_{\mathbb{R}} \phi(Mf(x))dx \leq \int_{\mathbb{R}} \phi(2Af(x))dx$

In particular,

1) $\|M^+f\|_\phi \leq A\|f\|_\phi$

2) $\|M^-f\|_\phi \leq A\|f\|_\phi$

3) $\|Mf\|_\phi \leq 2A\|f\|_\phi$

Our proof uses ideas of the proof of theorem (5.7.5) in [14], and needs the following two lemmas.

For a Lebesgue measurable function f, we denote, for $a > 0$, $a \in \mathbb{R}$:

$$\lambda_f(a) = \{x \in \mathbb{R} : f(x) > a\}.$$

15 Lemma ([14], 5.2.2). *Let f be a Lebesgue measurable function, and let $E \subset \mathbb{R}$, $E \in \mathcal{B}$. Also, let θ be a real valued function, which is absolutely continuous on each finite interval $[0, \alpha[$, $\alpha > 0$, of the real line. If $\theta(x) = 0$ if and only if $x = 0$, and if either $\theta' \in L^1(\mathbb{R}^+)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, or $\theta' \geq 0$, or $\theta' \leq 0$ on $\mathbb{R}^+$, then*

i) $m(\lambda_f(a)) \leq \frac{1}{\theta(a)} \int_{\lambda_f(a)} \theta(|f(x)|)dx$ *for all* $a > 0$*;*

ii) $\int_E \theta(|f(x)|)dx = \int_0^\infty m(E \cap \lambda_f(t))\theta'(t)dt.$

16 Lemma ([14], 5.7.4). *Let f be a non-negative Lebesgue measurable function on $\mathbb{R}$. Then for $0 < k < 1$ and $t > 0$*

i) $m\lambda_{M^+f}(t) \leq t^{-1}(1-k)^{-1} \int_{\lambda_f(kt)} f(y)dy$
and similarly with M^- in place of M^+, and

ii) $m\lambda_{Mf}(t) \leq 2t^{-1}(1-k)^{-1} \int_{\lambda_f(kt)} f(y)dy$

Proof of Theorem 14. Let $f \in L_\phi$, without lost of generality, we assume $f \geq 0$. We apply Lemma 15 ii) to M^+f with $E = \mathbb{R}$ and $\theta = \phi$ (note that an Orlicz function ϕ satisfies the conditions of Lemma 15). We obtain:

$$\int_{\mathbb{R}} \phi(M^+f(x))dx = \int_0^\infty m(\lambda_{M^+f}(t))p(t)dt$$

(where $p(t) = \phi'(t)$).

For $0 < k < 1$, we have, by Lemma 16:

$$\begin{aligned}\int_{\mathbb{R}} \phi(M^+ f(x))dx \leq & \int_0^\infty t^{-1}(1-k)^{-1} p(t) \int_{\lambda_f(kt)} f(y)dy\ dt \\ =&(1-k)^{-1} \int_0^\infty \frac{p(t)}{t} \int_{\mathbb{R}} f(y)\chi_{\lambda_f(kt)}(y)dy\ dt \\ =&(1-k)^{-1} \int_{\mathbb{R}} f(y) \int_0^\infty \frac{p(t)}{t} \chi_{\lambda_f(kt)}(y)dt\ dy \\ =&(1-k)^{-1} \int_{\mathbb{R}} f(y) \int_0^{\frac{f(y)}{k}} \frac{p(t)}{t} dt\ dy\end{aligned}$$

By the Dini-condition, there exists a constant $C \geq 1$ such that

$$\begin{aligned}\int_{\mathbb{R}} \phi(M^+ f(x))dx \leq & C(1-k)^{-1} \int_{\mathbb{R}} f(y) p\left(\frac{f(y)}{k}\right) dy \\ \leq & Ck(1-k)^{-1} \int_{\mathbb{R}} \phi\left(\frac{2f(y)}{k}\right) dy\end{aligned}$$

(as $up(u) \leq \phi(2u)$ for all $u > 0$).

We take $k = \frac{1}{2}$ and we use convexity of ϕ to obtain finally:

$$\int_{\mathbb{R}} \phi(M^+ f(x))dx \leq \int_{\mathbb{R}} \phi(4Cf(y))dy.$$

The results involving M^- and M are proved similarly. *QED*

17 Remark. Q. Lai proved that the inequality (i) (of Theorem 13) implies that ψ satisfies the Δ_2-condition [13]. We thus get a new characterization of the Dini-condition:

18 Corollary. *ϕ satisfies the Dini-condition if and only if there exists a constant $A > 0$ such that for every f in L_ϕ,*

$$\int_{\mathbb{R}} \phi(M^+ f(x))dx \leq \int_{\mathbb{R}} \phi(Af(x))dx.$$

References

[1] N. Asmar, E. Berkson, T.A. Gillepsie: *Transference of strong type maximal inequalities by separation-preserving representations*, Amer. J. Math. **113** (1991), 47–74.

[2] S. Bloom, R. Kerman: *Weighted L_ϕ-integral inequalities for operators of Hardy type*, Studia Math. **110** (1) (1994), 35–52.

[3] S. Bloom, R. Kerman: *Weighted Orlicz space integral inequalities for the Hardy-Littlewood maximal operator*, Studia Math. **110** (2) (1994), 105–114.

[4] A.P. Calderon: *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. U.S.A. **59** (1968), 349–353.

[5] R.R. Coifman, G. Weiss: Transference methods in analysis, C.B.M.S. Regional Conf. Series in Math. **31**, Amer. Math. Soc. Providence, R.I., 1977 (reprinted 1986).

[6] N. Dunford, J.T. Schwartz: Linear Operators – Part I. Interscience Publishers. New-York, 1967.

[7] C. Finet, G.E. Tkebuchava: *Some classical function systems in separable Orlicz spaces*, Studia Math. **121** (2) (1996), 193–205.

[8] F.P. Greenleaf: Invariant means on topological groups and their applications, Van Nostrand Reinhold, New-York, Toronto and London, 1969.

[9] G.H. Hardy, J.E. Littlewood: *A maximal theorem with function-theoretic applications*, Acta Mathematica **54**, 1930.

[10] E. Hewitt, K.A. Ross: Abstract harmonic analysis II, Grundlehren, Band. **152**, Springer-Verlag, Berlin and New-York, 1970.

[11] C. Kan *Ergodic properties of Lamperti operators*, Can. J. Math. **30** (1978), 1206–1214.

[12] M.A. Krasnosel'skii, Ya.B. Rutickii: Convex functions and Orlicz spaces, P. Noordhoff Ltd. The Netherlands, 1961.

[13] Q. Lai: *Two-weight mixed* Φ*-inequalities for the one-sided maximal function*, Studia Math. **115** (1) (1995), 1–22.

[14] G.O. Okikiolu: Aspects of the Theory of Bounded Integral Operators in L^p-spaces, Academic Press London, 1971.

[15] N. Wiener: *The ergodic theorem*, Duke Math. J. **5**, 1939, 1–18.

[16] A. Zygmund: Trigonometric Series, Vol.II, Cambridge Univ. Press, 1968.