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Higher order approximations at infinity to algebraic varieties

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Abstract. Let V be an algebraic variety in $\mathbb{C}^n$. For a curve γ in $\mathbb{C}^n$, going out to infinity, and $d \leq 1$, define $V_t := t^{-d}(V - \gamma(t))$. Then the currents defined by V_t converge to a limit current as t tends to infinity. This limit current is either zero or its support is an algebraic variety. Properties of such limit currents and examples are presented. These results are useful in the study of solvability questions for partial differential operators via Phragmén-Lindelöf conditions.

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Dedicated to the memory of Klaus Floret

Introduction

The basic work of Hörmander [12] and a number of results of other mathematicians (see, e.g., Andreotti and Nacinovich [1], Boiti and Nacinovich [3], Braun [4], Braun, Meise, and Vogt [10], Meise and Taylor [13], and Meise, Taylor, and Vogt [14,16]) showed that certain solvability properties of linear partial differential operators $P(D)$ with constant coefficients can be characterized in terms of conditions of Phragmén-Lindelöf type for plurisubharmonic functions on the complex zero variety $V(P)$ of the symbol P. For a long time these conditions could be understood in terms of the geometry of $V(P)$ only for very small dimensions n. Recently, a geometric characterization of Hörmander's condition was derived for polynomials in four variables in [6]. It is based on new necessary conditions for the local Phragmén-Lindelöf condition $\mathrm{PL}_{\mathrm{loc}}$ which reflect the fact that Phragmén-Lindelöf conditions are inherited by limit varieties. This effect was used already in [12], but in [6] more refined limit varieties are considered. The existence and properties of such higher order limit varieties at the origin were derived in [5].

To extend these results for the local Phragmén-Lindelöf condition to global conditions on algebraic varieties, one needs to know that such refined limit varieties also exist in a global sense, approximating the given algebraic variety V in certain areas near infinity. To formulate this in a more precise way, let V be an algebraic variety in $\mathbb{C}^n$ of pure dimension k, let $\gamma\colon \mathbb{C}\setminus(B(0,R)\cup]-\infty,0])\to\mathbb{C}$ be defined by $\gamma(t)=\sum_{j=-\infty}^{q} a_j t^{j/q}$ as a convergent Puiseux series, and fix $d\in]-\infty,1]$. Then, as t tends to infinity, the algebraic varieties $V_t := t^{-d}(V-\gamma(t))$ converge in the sense that the currents of integration over V_t converge to a limit current $T_{\gamma,d}[V]$. This limit current is either empty or a holomorphic k-chain, the support of which is an algebraic variety. An explicit description of $T_{\gamma,d}[V]$ in terms of algebraic equations can be derived using canonical defining functions. The behavior of the limit varieties is quite similar to the one of those defined in [5]. In fact, also the proofs are very similar to those in [5]. One might think that there should be an easy way to reduce everything to the results in the local case. However, it seems that technical problems do not make it simpler. Therefore, we have found it necessary to reuse the arguments given in [5]. However, once the existence of the limit varieties is proved one can obtain the limit varieties for an algebraic variety V by considering local limit varieties of a transformed variety $\tilde{V}$.

The results of the present paper are used in [8] and [7] as basic tools to characterize those $P\in\mathbb{C}[z_1,z_2,z_3]$ for which $P(D)$ admits a continuous linear right inverse on $\mathcal{D}'(\mathbb{R}^3)$, the space of all distributions on $\mathbb{R}^3$ or on $\mathcal{D}'_\omega(\mathbb{R}^3)$, the space of all ω-ultradistributions of Beurling type on $\mathbb{R}^3$. Also in [9], the results are applied to characterize the algebraic surfaces in $\mathbb{C}^n$ on which the condition (SPL), the analogue of the classical Pragmén-Lindelöf Theorem, holds.

1 Preliminaries about currents, k-chains, and convergence

In this section we introduce the basic notions and facts which are needed to introduce limit varieties and to investigate their properties. Most of the notions are taken from Chirka [11].

We denote by $\mathbb{N}$ the set of positive integers and by $B^n(z,r)$ the ball $\{w\in\mathbb{C}^n : |w-z|<r\}$, where $|\cdot|$ denotes the Euclidean norm. The exponent n may be omitted.

1 Definition. An *analytic variety* in $\mathbb{C}^n$ is defined as a closed analytic subset of some open set Ω in $\mathbb{C}^n$. If V is of pure dimension k, its *current of integration* $[V]$ is defined by

$$[V](\phi) := \int_V \phi,$$

where ϕ is any C^∞-form of bidegree (k,k) with compact support in Ω.

2 Definition. A *holomorphic k-chain* in an open set Ω in $\mathbb{C}^n$ is a locally finite sum

$$W = \sum n_i[V_i]$$

where $n_i \in \mathbb{Z}$ and $[V_i]$ is the current of integration over an irreducible analytic subvariety of Ω of dimension k. Recall that the *support* of W is equal to the union of those V_i for which $n_i \neq 0$.

3 Definition. The following definitions are taken from Chirka [11], 10.1, 11.1., 12.1, 12.2, and 11.3. Fix an analytic set $V \subset \mathbb{C}^n$ of pure dimension k, an affine plane $L \subset \mathbb{C}^n$ of dimension $n-k$, and an isolated point z of $V \cap L$. Then there is a neighborhood U of z such that the projection $\pi_L\colon U \cap V \to \pi_L(U \cap L) \subset L^\perp$ along L is an analytic cover. Its sheet number in z is denoted by $\mu_z(\pi_L|_V)$.

The minimum of the sheet numbers $\mu_z(\pi_L|_V)$ when L ranges over all $(n-k)$-dimensional affine subspaces for which z is an isolated point of $V \cap L$ is the *multiplicity* $\mu_z(V)$ of V at z.

If $D \subset \mathbb{C}^n$ is open and $D \cap V \cap L$ is finite, then the *intersection index* $i_D(V,L)$ is defined as

$$i_D(V,L) := \sum_{w \in D \cap L \cap V} \mu_w(\pi_L|_V).$$

If $W = \sum_{j=1}^m n_j[V_j]$ is a holomorphic k-chain and $D \cap L \cap \operatorname{Supp} W$ is finite, then

$$i_z(W,L) := \sum_{j=1}^m n_j \mu_z(\pi_L|_{V_j}) \quad \text{and} \quad i_D(W,L) := \sum_{w \in D \cap L \cap \operatorname{Supp} W} i_w(W,L).$$

$i_z(W,L)$ is called *sheet number* of the holomorphic chain W in z.

If V is a purely k-dimensional algebraic subset or a holomorphic k-chain in $\mathbb{C}^n$ with algebraic support and $L \subset \mathbb{C}^n$ is an affine $(n-k)$-dimensional subspace such that $V \cap L$ is finite and such that the projective closures of V and of L do not have points at infinity in common, then $i_{\mathbb{C}^n}(V,L)$ is the *degree* of V.

4 Remark. Note that, in the setting of Definition 3,

$$\mu_z(\pi_L|_V) = \mu_z(V)$$

whenever L is an affine $(n-k)$-dimensional subspace of $\mathbb{C}^n$ which is transversal to V at z (see [11], 11.2, Proposition 2).

5 Notation. Let V be an analytic variety of pure dimension k in some open set Ω in $\mathbb{C}^n$. For $a \in \Omega$ and $\rho > 0$ satisfying $B(a, \rho) \subset \Omega$ let $\operatorname{vol}(\rho, V, a)$ denote the $2k$-dimensional Hausdorff measure of $V \cap B(a, \rho)$. If $W = \sum_i n_i[V_i]$ is a holomorphic k-chain with nonnegative n_i, then $\operatorname{vol}(\rho, W, a) := \sum_i n_i \operatorname{vol}(\rho, V_i, a)$.

We say that a sequence $(W_j)_{j\in\mathbb{N}}$ of analytic varieties or holomorphic k-chains in some open set $\Omega \subset \mathbb{C}^n$ has locally uniformly bounded volume if for all $a \in \Omega$ there are $\rho, C > 0$ such that $B(a, \rho) \subset \Omega$ and $\operatorname{vol}(\rho, W_j, a) \leq C$ for all $j \in \mathbb{N}$.

In order to define convergence of holomorphic k-chains, we recall first the notion of convergence of a sequence of sets in a metric space (see Chirka [11], 15.5).

6 Definition. A sequence of sets $(V_j)_{j\in\mathbb{N}}$ in a metric space is said to *converge* to a set V if

(i) V coincides with the limit set of the sequence, i.e., consists of all points of the form $\lim_{\nu\to\infty} x_\nu$ where $x_\nu \in V_{j_\nu}$ for an arbitrary subsequence $(j_\nu)_{\nu\in\mathbb{N}}$ of $\mathbb{N}$, and

(ii) for any compact set $K \subset V$ and any $\epsilon > 0$, there is an index $j(\epsilon, K)$ such that K belongs to the ϵ neighborhood of V_j for all $j > j(\epsilon, K)$.

7 Definition. (a) A sequence $(T_j)_{j\in\mathbb{N}}$ of currents of bidegree $(n-k, n-k)$ on some open set Ω in $\mathbb{C}^n$ is said to *converge* to the current T if $T(\phi) = \lim_{j\to\infty} T_j(\phi)$ for each C^∞-form ϕ of bidegree (k, k) with compact support in Ω.

(b) A sequence $(W_j)_{j\in\mathbb{N}}$ of holomorphic k-chains in Ω *converges* to a holomorphic k-chain W if

(i) the supports of W_j converge to $V := \operatorname{Supp} W$ as subsets of Ω in the sense of Definition 6, and

(ii) for each regular point $a \in V$ and each $(n-k)$-dimensional plane L through a, transversal to V at a, there is a neighborhood U of a such that $V \cap L \cap U = \{a\}$ and such that $i_U(W_j, L) = i_a(W, L)$ for all sufficiently large j.

2 Existence of limit varieties

In this section we show that for algebraic varieties V in $\mathbb{C}^n$ of pure dimension k, $d \leq 1$, and curves γ which tend to infinity and are given by certain Puiseux series, there exist limit currents. They are holomorphic k-chains for which the support is an algebraic variety in $\mathbb{C}^n$. To do this, we will use the following notions.

8 Definition. (a) A *simple curve* γ in $\mathbb{C}^n$ is a map $\gamma\colon \mathbb{C} \setminus (B(0,R) \cup]-\infty,0]) \to \mathbb{C}^n$ for some $R > 0$ which, for some $q \in \mathbb{N}$, admits a convergent Puiseux series expansion

$$\gamma(t) = \xi_0 t + \sum_{j=-\infty}^{q-1} \xi_j t^{j/q}, \quad |\xi_0| = 1,$$

where for a real number $d \leq 1$, t^d denotes the principal branch of the power function, i.e., $t^d = |t|^d \exp(id \arg(t))$, where $-\pi < \arg(t) < \pi$ for $t \in \mathbb{C} \setminus]-\infty, 0]$. The vector ξ_0 will be called the *limit* vector to γ at infinity.

(b) For a pure k-dimensional algebraic variety V in $\mathbb{C}^n$, a simple curve γ, and a real number $d \leq 1$ we let

$$V_t := V_{\gamma,d,t} := \{w \in \mathbb{C}^n : \gamma(t) + t^d w \in V\}, \quad t \in \mathbb{C} \setminus (B(0,\epsilon) \cup]-\infty, 0]) .$$

9 Remark. Note that V_t is a pure k-dimensional algebraic variety in $\mathbb{C}^n$. The following theorem shows that the currents $[V_t]$ have a limit.

10 Theorem. *Let V be a purely k-dimensional algebraic variety in $\mathbb{C}^n$, let γ be a simple curve, and let $d \leq 1$. For the varieties V_t defined in Definition 8(b), the currents $[V_t]$ converge to a limit current W as t tends to infinity in $\mathbb{C} \setminus]-\infty, 0]$. W is a holomorphic k-chain the support of which is an algebraic variety in $\mathbb{C}^n$.*

Most of this section will be devoted to the proof of Theorem 10.

11 Definition. Under the hypotheses of Theorem 10 we define

$$T_{\gamma,d}[V] := \lim_{t\to\infty} [V_t]$$

and call it the *limit current* of order d along the simple curve γ. Furthermore,

$$T_{\gamma,d}V := \operatorname{Supp} T_{\gamma,d}[V]$$

will be called the *limit variety* of V of order d along γ.

To prove the theorem, we will show that the varieties V_t have locally bounded volume, so that they form a relatively compact family of varieties. Therefore, the family of varieties will converge if we can prove that there is a unique limit variety in $\mathbb{C}^n$. This will be shown by studying the convergence of associated canonical defining functions for V_t. Lastly, it will be clear that the volume of the limit variety in a ball of radius r is $O(r^{2k})$, so that the limit is algebraic by Stoll's theorem. The first step is to find a bound for the volumes of the V_t. The bound is given in the following lemma, which can be proved in the same way as Lemma 3.4 in [5].

12 Lemma. *Let $V \subset \mathbb{C}^n$ be an algebraic variety of pure dimension k. Then there are constants $C, R > 0$ such that*

$$\operatorname{vol}(\rho, V, a) \leq C\rho^{2k}, \quad |a| - \rho > R,\ a \in \mathbb{C}^n.$$

Moreover, there are $R_0, r_0 > 0$ such that for each simple curve γ and each $d \leq 1$

$$\operatorname{vol}(r, V_t, 0) \leq Cr^{2k}$$

whenever $t \in \mathbb{C} \setminus]-\infty, 0]$ and $|t| > r_0$.

As in [5], Corollary 3.5, we get from Lemma 3.4, [5], Theorem 2.8, and the Bishop-Stoll Theorem (see [2], [17]) the following corollary.

13 Corollary. *For V, γ, and $d \leq 1$ as in Theorem 10 let $(t_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{C} \setminus]-\infty, 0]$ that tends to ∞.*

(1) There exists a subsequence $(t_{j_\nu})_{\nu\in\mathbb{N}}$ for which $[V_{t_{j_\nu}}]$ converges.

(2) If $\lim_{l\to\infty}[V_{t_l}] = W$ for some holomorphic k-chain W then $\operatorname{Supp} W$ is either empty or an algebraic variety of dimension k. Further, the degree of W is at most equal to the degree of V.

In the sequel we will complete the proof of Theorem 10 by showing that there is a unique limit for the convergent subsequences of $(V_{t_j})_{j\in\mathbb{N}}$. That is, there exists a holomorphic k-chain W_0 such that $\lim_{l\to\infty}[V_{t_l}] = W_0$ whenever $\lim_{l\to\infty}[V_{t_l}]$ exists for some sequence $(t_l)_{l\in\mathbb{N}}$ in $\mathbb{C} \setminus]-\infty, 0]$ tending to ∞. To prove this, fix V as in Theorem 10 and such a sequence $(t_l)_{l\in\mathbb{N}}$ and assume that $W = \lim_{l\to\infty}[V_{t_l}]$ exists. To describe W in a way which shows that it does not depend on the sequence $(t_l)_{l\in\mathbb{N}}$ we will study the canonical defining function of V as defined in Whitney [18], Appendix V, Section 7 (see also Chirka [11], 4.2). For that purpose we choose excellent coordinates for the varieties V and W in the sense of [18], 7.7. This means that we assume that for $\mathbb{C}^n = \mathbb{C}^{n-k} \times \mathbb{C}^k$ the projection $\pi\colon z = (z'', z') \mapsto (0, z')$ is proper when restricted to V and W and satisfies

$$\begin{aligned} |z| &\leq C(1 + |\pi(z)|), \quad z \in V, \\ |z| &\leq C(1 + |\pi(z)|), \quad z \in \operatorname{Supp} W. \end{aligned} \tag{1}$$

In the remainder of this section we will assume these hypotheses, even when they are not mentioned explicitly.

Note that in the above situation the $(n-k)$-dimensional subspace $\mathbb{C}^{n-k}\times\{0\}$ is transverse to V and $\operatorname{Supp} W$. The existence of such a subspace L follows from [11], 7.4 Theorem 2.

If B is the branch locus of $\pi\colon V\to\mathbb{C}^k$, then B and $\pi(B)$ are algebraic varieties of dimension at most $k-1$ and

$$\pi\colon V\setminus B\to\mathbb{C}^k\setminus\pi(B)$$

is a covering map. The number of points in a fiber over $z'\in\mathbb{C}^n\setminus\pi(B)$ is the degree of V. We denote it by m. Then we can write

$$\pi^{-1}(z')=\{(\alpha_i(z'),z'):1\le i\le m\}$$

where the $\alpha_i(z')=\alpha_i(z';V)$ are all distinct. We will also use the same notation for $z'\in\pi(B)$ by repeating each $\alpha_i(z')$ as many times as indicated by the sheet number $i_z(V,L)$, where $z:=(\alpha_i(z'),z')$ and $L:=\mathbb{C}^{n-k}\times\{z'\}$.

For $u,w\in\mathbb{C}^{n-k}$, let $\langle u,w\rangle=u_1w_1+\cdots+u_{n-k}w_{n-k}$ denote the standard bilinear form. Then the *canonical defining function* for V is defined as

$$P(z,\xi;V,\pi)=\prod_{i=1}^{m}\left\langle z''-\alpha_i(z'),\xi\right\rangle. \tag{2}$$

We will write

$$P(z,\xi)=P(z,\xi;V)=P(z,\xi;V,\pi)$$

when the missing data are clear from the context. A point z belongs to V if and only if

$$P(z,\xi)=0\quad\text{for all }\xi\in\mathbb{C}^{n-k}.$$

Equivalently, one can expand P as a homogeneous polynomial in ξ,

$$P(z,\xi)=\sum_{|\beta|=m}P_\beta(z)\xi^\beta$$

and then $z\in V$ if and only if $P_\beta(z)=0$ for all β.

Note that P is a polynomial of degree m in z'' and a homogeneous polynomial of degree m in $\xi\in\mathbb{C}^{n-k}$. It is defined at first for $z'\notin\pi(B)$ but extends, by the Riemann removable singularity theorem, to be analytic on all of $\mathbb{C}^{n-k}\times\mathbb{C}^k\times\mathbb{C}^{n-k}$. With the convention made about counting the points $\alpha_i(z')$ with multiplicity when $z'\in\pi(B)$, the formula (2) is still valid.

To express the canonical functions of V_t in a useful form, write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ where $\gamma_2(t) = \pi(\gamma(t))$ and then

$$P(\gamma(t) + t^d w, \xi) = \prod_{j=1}^{m} \left\langle \gamma_1(t) + t^d w'' - \alpha_j(\gamma_2(t) + t^d w'), \xi \right\rangle$$
$$= t^{md} \prod_{j=1}^{m} \left\langle w'' - \beta_j(w', t), \xi \right\rangle, \quad (3)$$

where

$$\beta_j(w', t) = \frac{\alpha_j(\gamma_2(t) + t^d w') - \gamma_1(t)}{t^d}. \quad (4)$$

The last formula gives the canonical functions for the varieties V_t with respect to the projection π onto the z' coordinates up to the scale factor t^{md}.

The limit chain W of a sequence $([V_{t_j}])_{j\in\mathbb{N}}$ is not necessarily a current of integration over an analytic set, so its associated canonical defining function must take account of multiplicities. To fix the notation, let us suppose that

$$W = n_1[W_1] + \cdots + n_p[W_p] \quad (5)$$

where the W_j are the irreducible components of $\operatorname{Supp} W$, and

$$W_j = \{(\beta_{j,i}(w'), w') : w' \in \mathbb{C}^k, 1 \le i \le m_j\} \quad (6)$$

where m_j is the degree of W_j. Then

$$\nu := n_1 m_1 + \cdots + n_p m_p \quad (7)$$

is the degree of W, so $\nu \le m$ by Corollary 13. The canonical defining function of W is then

$$P(w, \xi; W) := P(w, \xi; W, \pi) := \prod_{j=1}^{p} \prod_{i=1}^{m_j} \left\langle w'' - \beta_{j,i}(w'), \xi \right\rangle^{n_j}. \quad (8)$$

14 Lemma. *The canonical defining functions $P(z, \xi; V, \pi)$ and $P(w, \xi; W, \pi)$ are polynomials. The total degrees of $P(z, \xi; V, \pi)$ and $P(w, \xi; W, \pi)$ are m and ν, respectively.*

Proof. The function $P(z, \xi; V, \pi)$ grows at the polynomial rate $O(|z, \xi|^m)$ since by (1) each of the factors grows linearly. Therefore, the claim follows from Liouville's theorem. The proof for $P(w, \xi; W, \pi)$ is completely analogous. $\boxed{QED}$

The degree of a limit chain will frequently be smaller than that of V_t. This occurs when some of the $\beta_j(w', t)$ go to infinity while the others converge to points in $\operatorname{Supp} W$.

The following lemma can be proved literally by the same arguments that we used in the proof of [5], Lemma 3.7. The main tool is Chirka [11], 12.2, Proposition 2, which establishes the continuous dependence of the intersection index of intersecting chains.

15 Lemma. *Suppose* $\lim_{l\to\infty}[V_{t_l}] = W$ *for some holomorphic k-chain W in* $\mathbb{C}^n$.

(i) For all $R, \epsilon > 0$ *there is* l_0 *such that for each* $l > l_0$ *and each* $w' \in \mathbb{C}^k$ *with* $|w'| \le R$ *there are exactly* ν *values of j for which the point* $(\beta_j(w', t_l), w')$ *lies in the* ϵ*-neighborhood of* $\operatorname{Supp} W$.

(ii) For each $R > 0$ *there is* $M(R) > 0$ *such that for each* $M > M(R)$ *there is* l_0 *such that for each* $l > l_0$ *and each* $w' \in \mathbb{C}^k$ *with* $|w'| \le R$ *there are exactly* $m - \nu$ *values of j for which the* $|\beta_j(w', t_l)| \ge M$.

(iii) For each $w = (w'', w') \in \mathbb{C}^n$ *there is* ϵ_0 *such that for each* $0 < \epsilon < \epsilon_0$ *there is* l_0 *such that for each* $l > l_0$ *the number of j with* $|\beta_j(w') - w''| < \epsilon$ *is exactly equal to the sheet number of W in w, i.e.,*

$$\left|\{j : |\beta_j(w', t_l) - w''| < \epsilon\}\right| = \sum_{j=1}^{p} n_j \left|\{i : \beta_{j,i}(w') = w''\}\right| \quad \text{for } l > l_0.$$

This implies that for fixed w' *the* β_j *can be numbered in such a way that* $\lim_{l\to\infty} \beta_j(w', t_l)$ *exists for* $1 \le j \le \nu$ *and* $|\lim_{l\to\infty} \beta_j(w', t_l)| = \infty$ *for* $j > \nu$. *The sequence of functions* $w' \mapsto \beta_j(w', t_l)$ *converges uniformly on compact sets* (*although they may have discontinuities*).

In order to derive a description of W from the canonical defining function $P(\cdot, \text{-}; V, \pi)$ of V we will use the following notation.

16 Definition. For $d \le 1$, $q \in \mathbb{N}$, and $l \in \mathbb{N}_0$ let p be a Laurent series in the variable $t^{1/q}$ with coefficients in $\mathbb{C}[w_1, \dots, w_n, \xi_1, \dots, \xi_l]$. Then p is called *d-quasihomogeneous* in w and t of *d-degree* ω if

$$p(\lambda^d w, \lambda t, \xi) = \lambda^\omega p(w, t, \xi), \quad \lambda > 0.$$

It is easy to check that p is d-quasihomogeneous of d-degree ω if and only if p has the form

$$p(w, t, \xi) = \sum_{j+d|\beta|=\omega} \sum_{\alpha\in\mathbb{N}_0^l} a_{j,\beta,\alpha} w^\beta t^j \xi^\alpha,$$

where β runs through $\mathbb{N}_0^n$ and j through a subset of $\frac{1}{q}\mathbb{Z}$ which is bounded from above.

17 Remark. For P as in (2), γ as in Theorem 10, and $d \le 1$, let

$$F(w,t,\xi) := P(\gamma(t)+w,\xi;V,\pi) = \sum_{j,\beta,\alpha} a_{j,\beta,\alpha} t^j w^\beta \xi^\alpha \tag{9}$$

where the sum is the Laurent series expansion of the holomorphic function $F(w,s^q,\xi)$ in $s=t^{1/q}$, w, ξ, where s runs through a neighborhood of ∞ and w through a neighborhood of the origin. Collecting all terms in (9) which have the same d-degree, we can regroup the series as

$$F(w,t,\xi) = F_{\omega_0}(w,t,\xi) + \sum_{\omega<\omega_0} F_\omega(w,t,\xi), \tag{10}$$

where F_ω is the d-quasihomogeneous part of d-degree ω of the series and

$$\omega_0 = \omega_0(d,V,\pi) = \max\{\omega : F_\omega \text{ does not vanish identically}\}. \tag{11}$$

Now note that for $t \in \mathbb{C} \setminus \big(B(0,R) \cup \,]-\infty,0]\big)$ the quasihomogeneity property implies $F(t^d w,t,\xi) = t^{\omega_0} F_{\omega_0}(w,1,\xi) + \sum_{\omega<\omega_0} t^\omega F_\omega(w,1,\xi)$ and hence

$$\lim_{t\to\infty} t^{-\omega_0} P(\gamma(t)+t^d w,\xi;V,\pi) = \lim_{t\to\infty} t^{-\omega_0} F(t^d w,t,\xi) = F_{\omega_0}(w,1,\xi), \tag{12}$$

where the convergence is uniform on compact subsets of $\mathbb{C}^n \times \mathbb{C}^{n-k}$.

The following two lemmas can be proved in the same way as Lemma 3.10 and 3.11 in [5].

18 Lemma. *Suppose that* $\lim_{l\to\infty}[V_{t_l}] = W$ *for a holomorphic* k*-chain* W*. Then there is a polynomial* Φ *on* $\mathbb{C}^k \times \mathbb{C}^{n-k}$ *such that*

$$F_{\omega_0}(\zeta'',\zeta',1,\xi) = P(\zeta'',\zeta',\xi;W)\Phi(\zeta',\xi)$$

for all $\zeta = (\zeta'',\zeta') \in \mathbb{C}^{n-k} \times \mathbb{C}^k$, $\xi \in \mathbb{C}^{n-k}$.

19 Lemma. *Suppose that* $\lim_{l\to\infty}[V_{t_l}] = W$ *for a holomorphic* k*-chain* W*. For each* $w' \in \mathbb{C}^k$ *the set* $\{\xi \in \mathbb{C}^{n-k} : \Phi(w',\xi) \ne 0\}$ *is open and dense in* $\mathbb{C}^{n-k}$.

20 Remark. We do not know any examples where the function Φ actually depends upon the variable w'.

The following proposition allows us to recover $P(\cdot,\text{-};W)$ from F_{ω_0}. Hence it shows the independence of $\lim_{l\to\infty}[V_{t_l}]$ from the sequence $(t_l)_{l\in\mathbb{N}}$ and thus completes the proof of the Main Theorem 10.

21 Proposition. *Suppose that* $\lim_{l\to\infty}[V_{t_l}] = W$ *for some holomorphic* k*-chain* W*. Let*

$$F_{\omega_0}(w,1,\xi) = \prod_{a=1}^{A} F_a(w,\xi)^{\lambda_a}$$

be the decomposition of F_{ω_0} *into powers of mutually nonproportional irreducible factors. Let* I *be the set of all* a *for which there is* $w \in \mathbb{C}^n$ *with* $F_a(w,\xi) = 0$ *for all* ξ*. Then there is* $c \neq 0$ *such that*

$$P(w,\xi;W) = c \prod_{a\in I} F_a(w,\xi)^{\lambda_a}, \quad w \in \mathbb{C}^n, \xi \in \mathbb{C}^{n-k}.$$

Proof. Recall that $F_{\omega_0}(w,1,\xi) = P(w,\xi;W)\Phi(w',\xi)$ by Lemma 18. If $a \in I$, then F_a must be a factor of P, since by Lemma 19 it cannot be a factor of Φ.

For the proof of the other direction fix a such that F_a is a factor of P. Choose $w = (w'', w') \in \mathbb{C}^n$ and $\xi_0 \in \mathbb{C}^{n-k}$ with $F_a(w,\xi_0) \neq 0$. Consider $F_a(\zeta'', w', \xi)$ as a polynomial in $\mathbb{C}[\zeta'', \xi]$. Then it is a factor of

$$P(\zeta'', w', \xi) = \prod_{j=1}^{p} \prod_{i=1}^{m_j} \langle \zeta'' - \beta_{j,i}(w'), \xi\rangle^{n_j}.$$

In particular, there is a pair (j,i) such that $\langle \zeta'' - \beta_{j,i}(w'), \xi\rangle$ divides $F_a(\zeta'', w', \xi)$ in $\mathbb{C}[\zeta'', \xi]$. Then $F_a(\beta_{j,i}(w'), w', \xi) = 0$ for all $\xi \in \mathbb{C}^{n-k}$ and hence $a \in I$. $\boxed{QED}$

Proof of Theorem 10. For each sequence $(t_l)_{l\in\mathbb{N}}$ in $\mathbb{C}\setminus\big(B(0,R)\cup]-\infty,0]\big)$ with $\lim_{l\to\infty} t_l = \infty$ the sequence $([V_{t_l}])_{l\in\mathbb{N}}$ has an accumulation point W by Corollary 13(a). This accumulation point is unique and does not depend on the sequence $(t_l)_{l\in\mathbb{N}}$ by Proposition 21. Hence W is the limit. Its support is either empty or algebraic of pure dimension k by Corollary 13(b). $\boxed{QED}$

In [8] we will apply the following corollary of Theorem 10, which is obvious from the proof of this theorem.

22 Corollary. *Let* $V \subset \mathbb{C}^n$ *be an algebraic variety of pure dimension* k*, let* γ *be a simple curve, and let* $d \le 1$*,* $R > 0$*, and a sequence* $(t_j)_{j\in\mathbb{N}}$ *in* $\mathbb{C} \setminus \big(B(0,R) \cup]-\infty, 0]\big)$ *satisfying* $\lim_{j\to\infty} t_j = \infty$ *be given. Then the varieties* V_{t_j} *converge to* $T_{\gamma,d}V$ *in the sense of Meise, Taylor, and Vogt [15], 4.3, as* j *tends to infinity.*

It is possible to determine the sheet number of $T_{\gamma,d}[V]$ at each point. This provides a purely geometric description of $T_{\gamma,d}[V]$. If p is a polynomial in one variable, we denote by $\operatorname{ord}_0 p$ the vanishing order of p at the origin.

23 Proposition. *Let $w = (w'', w')$ be excellent coordinates for V and for $T_{\gamma,d}V$ as in (1). For $w' \in \mathbb{C}^k$ set $L_{w'} := \mathbb{C}^{n-k} \times \{w'\}$. For $w \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^{n-k}$ consider the polynomial $p_{w,\xi} : \tau \mapsto F_{\omega_0}(w'' + \tau\xi, w', 1, \xi)$, $\tau \in \mathbb{C}$. Then*

$$i_w(T_{\gamma,d}[V], L_{w'}) = \min\{\operatorname{ord}_0 p_{w,\xi} : \xi \in \mathbb{C}^{n-k}\}.$$

PROOF. We start with the proof of "$\leq$".

Choose $\xi_0 \in \mathbb{C}^{n-k}$ with $\min\{\operatorname{ord}_0 p_{w,\xi}\} = \operatorname{ord}_0 p_{w,\xi_0}$. Assume for convenience $\xi_0 = (1, 0, \ldots, 0)$. By Proposition 21, p_{w,ξ_0} is a multiple of

$$P(w_1 + \tau, w_2, \ldots, w_n, \xi_0; W) = \prod_{j=1}^{p} \prod_{i=1}^{m_j} (w_1 + \tau - \beta_{j,i}^{(1)}(w'))^{n_j},$$

where $\beta_{j,i}^{(1)}$ denotes the first coordinate of $\beta_{j,i}$ as in (8). The definition of $\beta_{j,i}$ implies that the order of this polynomial is not smaller than $i_w(T_{\gamma,d}[V], L_{w'})$.

To prove the converse inequality, use Lemma 19 to choose $\xi_0 \in \mathbb{C}^{n-k}$ such that $\Phi(w', \xi_0) \neq 0$ and such that $\langle \zeta'', \xi_0 \rangle \neq \langle w'', \xi_0 \rangle$ whenever $(\zeta'', w') \in W$ and $\zeta'' \neq w''$. Again, we may assume $\xi_0 = (1, 0, \ldots, 0)$. Then

$$p_{w,\xi_0}(\tau) = P(w_1 + \tau, w_2, \ldots, w_n, \xi_0; W) = \prod_{j=1}^{p} \prod_{i=1}^{m_j} (w_1 + \tau - \beta_{j,i}^{(1)}(w'))^{n_j} \Phi(w', \xi_0).$$

This shows that $\operatorname{ord}_0 p_{w,\xi_0} = i_w(T_{\gamma,d}[V], L_{w'})$ for the special choice of ξ_0. The proposition is proved. $\boxed{QED}$

Proposition 23 holds under the general hypothesis that the coordinates are excellent for V and for $T_{\gamma,d}V$ (see (1)). When investigating examples, one wants to be able to see from F_{ω_0} that a given system of coordinates is excellent. So let us assume that the standard coordinate system is excellent for V, i.e., the first inequality in (1) is valid. Then it is possible to define the canonical defining function $P(z, \xi; V, \pi)$ as in (2) and, for a given simple curve γ and some $d \leq 1$, the expansion of $P(\gamma(t) + z, \xi; V, \pi)$ into d-quasihomogeneous terms as in (10) exists. Hence

$$Z := \{w \in \mathbb{C}^n : F_{\omega_0}(w, 1, \xi) = 0 \text{ for all } \xi \in \mathbb{C}^{n-k}\}$$

is defined, and the following holds:

24 Proposition. *Assume that $\dim Z = k$ and that the standard coordinate system is excellent for V and for Z. Then it is excellent for $T_{\gamma,d}V$. In particular, $Z = T_{\gamma,d}V$ and Proposition 23 holds in these coordinates.*

Proof. Since(1) is inherited by subvarieties of the same dimension, it suffices to show $T_{\gamma,d}V \subset Z$. So fix $w \in T_{\gamma,d}V$ and an arbitrary sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim_{l\to\infty} t_l = \infty$. By Definition 6 there is a sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_n \in V_{\gamma,d,t_n}$ and $\lim_{n\to\infty} z_n = w$. Fix $\xi \in \mathbb{C}^{n-k}$. Then

$$\begin{aligned} 0 &= t_n^{-\omega_0} P(\gamma(t_n) + t_n^d z_n, \xi; V, \pi) \\ &\qquad = F_{\omega_0}(z_n, 1, \xi) + \sum_{\omega<\omega_0} t_n^{\omega-\omega_0} F_\omega(z_n, 1, \xi) \xrightarrow{n\to\infty} F_{\omega_0}(w, 1, \xi). \end{aligned}$$

Hence $w \in Z$, and the claim is shown. $\boxed{QED}$

25 Definition. Let h be a Laurent series in $t^{1/q}$ with coefficients in the polynomial ring $\mathbb{C}[w_1, \ldots, w_n]$. Fix $d \le 1$ and expand h into a convergent series

$$h(w,t) = \sum_{\omega\in\mathbb{Z}/q+d\mathbb{Z}} h_\omega(w,t)$$

such that h_ω is zero or d-quasihomogeneous of d-degree ω in w and t. Then for $\omega_0 := \max\{\omega : h_\omega \neq 0\}$ the term h_{ω_0} is called the *d-quasihomogeneous principal part* of h.

If h does not depend on t then the d-quasihomogeneous principal part of h coincides with the principal part in the classical sense.

In the case of a hypersurface the vanishing ideal is principal, and its generator replaces the canonical defining function as indicated in the next statement. Since its proof is the same as the one of [5], Corollary 3.16, we omit it.

26 Corollary. *Let $p \in \mathbb{C}[z_1, \ldots, z_n]$ and let $V := \{z \in \mathbb{C}^n : p(z) = 0\}$. Furthermore, assume that there is a dense open subset A of V with* $\operatorname{grad} p(z) \neq 0$ *for all $z \in A$. Let γ be a simple curve and set $f(w,t) := p(\gamma(t) + w)$. For $d \le 1$ let f_{ω_0} be the d-quasihomogeneous principal part of f. Then $T_{\gamma,d}V = \{w \in \mathbb{C}^n : f_{\omega_0}(w,1) = 0\}$. Let $W_1, \ldots, W_N$ be the irreducible components of $T_{\gamma,d}V$, and let n_j denote the multiplicity of f_{ω_0} at an arbitrary regular point of W_j. Then*

$$T_{\gamma,d}[V] = \sum_{j=1}^N n_j [W_j].$$

If f defines the hypersurface V geometrically without generating the corresponding ideal (i.e., if f is not square-free), then it is still possible to determine the limit variety $T_{\gamma,d}V$.

27 Corollary. *Let $A \in \mathbb{C}[z_1, \ldots, z_n]$ and let $V := \{z \in \mathbb{C}^n : A(z) = 0\}$. Let γ be a simple curve and define $f(w,t) = A(\gamma(t) + w)$. For $d \le 1$ let f_{ω_0} be the d-quasihomogeneous principal part of f. Then $T_{\gamma,d}V = \{w \in \mathbb{C}^n : f_{\omega_0}(w,1) = 0\}$.*

PROOF. Decompose A in $\mathbb{C}[z_1, \ldots, z_n]$. Thus $A = A_1^{m_1} \cdots A_l^{m_l}$ with mutually nonproportional irreducible polynomials A_j. Then $r := A_1 \cdots A_l$ satisfies the hypotheses of Corollary 26, hence $T_{\gamma,d}V = \{w \in \mathbb{C}^n : g_{\sigma_0}(w) = 0\}$ where g_{σ_0} is the d-homogeneous principal part of $g(w,t) := r(\gamma(t) + w)$. Let f_{j,ω_j} be the d-homogeneous principal part of $f_j(w,t) := A_j(\gamma(t) + w)$. It is easy to see that d-homogeneous principal parts are multiplicative, hence $g_{\sigma_0} = f_{1,\omega_1} \cdots f_{l,\omega_l}$ and $f_{\omega_0} = f_{1,\omega_1}^{m_1} \cdots f_{l,\omega_l}^{m_l}$. Thus the zero sets of g_{σ_0} and of f_{ω_0} coincide. $\boxed{QED}$

3 Properties of the limit varieties

It is convenient to record some simple properties of the limit varieties before studying specific examples in Section 4. Invariance properties of $T_{\gamma,d}[V]$ are studied in Proposition 29, while in Proposition 31 the influence of d is discussed. The main tool in the proof of the latter is the Newton polygon.

In Proposition 35 we show how limit varieties can be interpreted as approximations in conoids.

28 Definition. For an algebraic variety $V \subset \mathbb{C}^n$ let V_h denote the limit cone at infinity, i.e., if $\overline{V}$ denotes the closure of V in $\mathbb{P}^n$, then

$$V_h = \left\{ z \in \mathbb{C}^n : (0 : z_1 : \cdots : z_n) \in \overline{V} \right\}.$$

Here, homogeneous coordinates on $\mathbb{P}^n$ are written in the form $(z_0 : z_1 : \cdots : z_n)$ with the understanding that $z \in \mathbb{C}^n$ corresponds to $(1 : z_1 : \cdots : z_n)$.

29 Proposition. *Let V be an algebraic variety in $\mathbb{C}^n$. Let γ be a simple curve as in Definition 8 with ξ_0 as a limit vector at infinity, $d \leq 1$, and $T_{\gamma,d}[V]$ the limit current defined in Definition 11.*

(i) If $\tilde{\gamma}(t)$ is another simple curve and if $\tilde{\gamma}(t) = \gamma(t) + o(|t|^d)$, then $T_{\gamma,d}[V] = T_{\tilde{\gamma},d}[V]$.

(ii) If $d = 1$, then $T_{\gamma,1}V = V_h - \xi_0$; more precisely, if $j(w) = \xi_0 + w$, then $j_*(T_{\gamma,1}[V]) = [V_h]$.

(iii) If $d < 1$ and $\lambda \in \mathbb{C}$, then $w \in T_{\gamma,d}V$ if and only if $w + \lambda\xi_0 \in T_{\gamma,d}V$; or in terms of the currents, if $j(w) = w + \lambda\xi_0$, then $j_*(T_{\gamma,d}[V]) = T_{\gamma,d}[V]$.

(iv) $T_{\gamma,d}V$ is empty if and only if for every relatively compact open set $\Omega \Subset \mathbb{C}^n$ there exists $r_0 > 0$ so small that the conoid with core γ, opening exponent d, and profile Ω, with tip truncated at r_0, then

$$\Gamma(\gamma, d, \Omega, r_0) = \bigcup_{t > r_0} (\gamma(t) + t^d \Omega) \tag{13}$$

has empty intersection with V.

(v) If $\tilde{\gamma}$ is another simple curve such that $\gamma(]R_1, \infty[) = \tilde{\gamma}(]R_2, \infty[)$ for some constants $R_1, R_2 > 0$, then $T_{\gamma,d}[V] = T_{\tilde{\gamma},d}[V]$ for each $d \leq 1$.

PROOF. Choose coordinates as in Section 2 and recall the canonical defining function $P(w, \xi) = P(w, \xi; V, \pi)$ associated to this choice of coordinates along with the functions F and F_{ω_0} as in (9) and (11). It follows from the hypotheses about γ and $\tilde{\gamma}$, (10), and the quasihomogeneity property of the F_ω that

$$\lim_{t\to\infty} t^{-\omega_0} P(\tilde{\gamma}(t) + t^d w, \xi) = \lim_{t\to\infty} t^{-\omega_0} P(\gamma(t) + t^d (w + o(1)), \xi)$$
$$= \lim_{t\to\infty} F_{\omega_0}((w+o(1)), 1, \xi) + \lim_{t\to\infty} \sum_{\omega<\omega_0} t^{\omega-\omega_0} F_\omega((w+o(1)), 1, \xi) = F_{\omega_0}(w, 1, \xi).$$

Therefore, under the hypothesis of (i), ω_0 and the function F_{ω_0} are unchanged if γ is replaced by $\tilde{\gamma}$. By Proposition 21 this yields (i).

To prove (ii), we can assume that $\gamma(t) = \xi_0 t$ because of part (i). Then $V_t = \{w \in \mathbb{C}^n : \xi_0 + w \in \frac{1}{t} V\}$, so $[V_t]$ is the translate of the current $[\frac{1}{t} V]$ by $-\xi_0$. Consequently, the same is true of the limit varieties.

To prove (iii), Proposition 21 implies that it suffices to show that $F_{\omega_0}(w + \lambda\xi_0, 1, \xi) = F_{\omega_0}(w, 1, \xi)$. By analytic continuation, it is enough to prove this equation for $\lambda > 0$. Set $\tilde{t} = t + \lambda t^d$ so that $t = \tilde{t} - \lambda \tilde{t}^d + o(\tilde{t}^d)$. Then since $d < 1$,

$$\begin{aligned}
F_{\omega_0}(w, 1, \xi) &= \lim_{\tilde{t}\to\infty} \tilde{t}^{-\omega_0} P(\gamma(\tilde{t}) + \tilde{t}^d w, \xi) \\
&= \lim_{t\to\infty} (t + o(t))^{-\omega_0} P(\gamma(t) + t^d \lambda \xi_0 + o(t^d) + (t + o(t))^d w, \xi) \\
&= \lim_{t\to\infty} t^{-\omega_0} P(\gamma(t) + t^d (w + \lambda\xi_0 + o(1)), \xi) + o(1) \\
&= \lim_{t\to\infty} F_{\omega_0}(w + \lambda\xi_0 + o(1), 1, \xi) + o(1) \\
&= F_{\omega_0}(w + \lambda\xi_0, 1, \xi)
\end{aligned}$$

so part (iii) is proved.

Part (iv) is a consequence of Lemma 15. The same Lemma and Proposition 21 show that $T_{\gamma,d}V$ is nonempty except when all the points $(\beta_j(w', t), w') \in V_t$ diverge to ∞ as $t \to \infty$. From the definition of the β's in (4) we see that this means exactly that V_t has no points in any conoid $\Gamma(\gamma, d, \Omega, r_0)$ with relatively compact profile when r_0 is sufficiently large.

For the proof of part (v) let ξ_0 denote the limit vector to γ at infinity. Since

$$\xi_0 = \lim_{t\to\infty} \frac{\gamma(t)}{|\gamma(t)|},$$

the limit vectors of γ and $\tilde{\gamma}$ coincide. We may assume $\xi_0 = (0, \dots, 0, 1)$. Let γ_n and $\tilde{\gamma}_n$ denote the last component of γ and $\tilde{\gamma}$, respectively. Since both are injective, we can define $\rho := \gamma_n^{-1} \circ \tilde{\gamma}_n$. Then $\tilde{\gamma} = \gamma \circ \rho$. Both have the same limit vector, so it is immediate that $\lim_{t\to\infty} \rho(t)/t = 1$. Set $\tilde{F}(w,t,\xi) := P(\tilde{\gamma}(t)+w,\xi)$, and let $\tilde{\omega}_0$ be defined by (11), but with F replaced by $\tilde{F}$. Then

$$\begin{aligned}\tilde{F}_{\omega_0}(w,1,\xi) &= \lim_{t\to\infty} t^{-\omega_0} P(\tilde{\gamma}(t),\xi) \\ &= \lim_{t\to\infty} \left(\frac{t}{\rho(t)}\right)^{-\omega_0} \rho(t)^{-\omega_0} P(\gamma(\rho(t),\xi) = F_{\omega_0}(w,1,\xi). \quad (14)\end{aligned}$$

Since the right hand side does not vanish, this shows that $\tilde{\omega}_0 \geq \omega_0$. Interchanging γ and $\tilde{\gamma}$ in the preceding argument, we conclude that $\omega_0 = \tilde{\omega}_0$. Now (14) completes the proof. $\boxed{QED}$

30 Remark. Part (iv) of Proposition 29 implies that $T_{\gamma,1}V$ is never empty and that $T_{\gamma,d}V$ is empty whenever $d < 1$ and the limit vector of γ at infinity is not in V_h.

31 Proposition. *Let V be an algebraic variety in $\mathbb{C}^n$. Let m be its degree, let $\gamma(t)$ be a simple curve as in 8, $d \leq 1$, and $T_{\gamma,d}[V]$ the limit current defined in Definition 11.*

(i) There are rational numbers $1 = d_1 > d_2 > \cdots > d_p$, where $1 \leq p \leq m+1$, such that $T_{\gamma,d}[V] = T_{\gamma,d'}[V]$ whenever $d_i > d \geq d' > d_{i+1}$ for $1 \leq i < p$ or $d_p > d \geq d'$.

We assume in the sequel that the set $\{1 = d_1, \ldots, d_p\}$ is minimal, i.e., that (i) *holds for no proper subset.*

(ii) If $d_i > d > d_{i+1}$, $1 \leq i < p$, then $T_{\gamma,d}V$ is homogeneous and nonvoid.

(iii) If $d < d_p$, then $T_{\gamma,d}V$ is homogeneous or empty.

(iv) If $p = m+1$, then $T_{\gamma,1}[V] = T_{\gamma,d}[V]$ for $1 \geq d > d_2$.

Proof. The proof relies on the Newton polygon for the function F defined in (9), i.e., of the series $F(w,t,\xi) = \sum_{j,\beta,\alpha} a_{j,\beta,\alpha} w^\beta t^j \xi^\alpha$. Let M be the *support* of that series, i.e.,

$$M := \{(j,l) : qj \in \mathbb{Z},\ l \in \mathbb{N}_0,\ a_{j,\beta,\alpha} \neq 0 \text{ for some } \beta \text{ with } |\beta| = l \text{ and } |\alpha| = m\}.$$

For $\theta \in \mathbb{R}^2 \setminus \{0\}$ and $b \in \mathbb{R}$ define the closed half plane

$$H_{\theta,b} := \{x \in \mathbb{R}^2 : \langle x,\theta\rangle \leq b\}.$$

We call it admissible if $\theta \in [0,\infty[\times \mathbb{R}$ and $M \subset H_{\theta,b}$. The Newton polygon N is the intersection of all admissible half planes. Note that all vertices of N are elements of M. In particular, if (j, l) is a vertex of N, then $l \in \mathbb{N}_0$ and $l \le m$ by Lemma 14. Hence N has at most $m+1$ vertices and at most m edges between them (plus two unbounded edges).

If we use the convention that the slope of a vertical edge is $-\infty$, then we claim that $s \in [-\infty, -1] \cup [0, \infty[$ whenever s is the slope of an edge of N. To see this, note first that $(0, m) \in \partial N$. On the other hand, no point of N can be strictly above the line through $(0, m)$ and $(m, 0)$, since the Puiseux series expansion of γ admits no exponent strictly exceeding 1. Hence the slope of the edge through $(0, m)$ cannot exceed -1. Since among the edges with negative slope the nonhorizontal edge through $(0, m)$ admits the largest slope, the intermediate claim is shown.

Note that the slope 0 is obtained at the unbounded edges only, and let $1 = d_1 > d_2 > \cdots > d_p$ be an enumeration of

$$\{1\} \cup \left\{ -\frac{1}{s} : \ s \text{ is the slope of a bounded edge of } N \right\}.$$

Then $p \le m+1$ is obvious, and if $p = m+1$, then there is no edge with slope -1.

For $d \le 1$ let $\omega_0(d) := \omega_0(d, V, \pi)$ be as in (11). Then the line

$$\partial H_{(1,d),\,\omega_0(d)} = \{(j, l) : j + dl = \omega_0(d)\}$$

has nonempty intersection with M. Fix i with $1 \le i < p$. Then there is a pair $(j(i), l(i))$ (a vertex of the Newton polygon) such that $M \cap \partial H_{(1,d),\,\omega_0(d)} = \{(j(i), l(i))\}$ for each $d \in \,]d_i, d_{i+1}[$. Hence

$$F_{\omega_0}(w, 1, \xi) = \sum_{|\beta|=l(i)} \sum_{|\alpha|=m} a_{j(d),\beta,\alpha} w^\beta \xi^\alpha. \tag{15}$$

By Proposition 21 this shows the part of (i) dealing with $d, d' > d_p$. The identity (15) also implies that $F_{\omega_0}(w, 1, \xi)$ is homogeneous and thus so is $T_{\gamma,d}V$. If $T_{\gamma,d}V$ were empty, then $l(i) = 0$, since otherwise $0 \in T_{\gamma,d}V$. However, the construction of the Newton polygon shows that then $T_{\gamma,d'}V = \emptyset$ for each $d' \le d$, thus contradicting the minimality of the set $\{d_1, \ldots, d_p\}$. This completes the proof of (ii).

To show (iii) and finish the proof of (i), fix $d < d_p$. Then again there is a vertex $(j(p), l(p))$ of the Newton polygon such that $M \cap \partial H_{(1,d),\,\omega_0(d)} = \{(j(p), l(p)\}$ for each $d < d_p$. This shows the independence of $T_{\gamma,d}[V]$ on $d < d_p$ and thus completes the proof of (i). The homogeneity of $T_{\gamma,d}V$ follows as before.

If $p = m+1$ there is no edge with slope -1, hence the proof of (ii) applies also in this case. $\boxed{QED}$

32 Definition. For V and γ as in 31, we call the elements of the minimal set $\{d_1, \ldots, d_p\}$ satisfying 31 (i) the *critical values* for γ and V.

The following result is a partial converse to 31(iii). We do not know whether it also holds in the case of arbitrary codimension.

33 Corollary. *For $A \in \mathbb{C}[z_1, \ldots, z_n]$ let $V := \{z \in \mathbb{C}^n : A(z) = 0\}$ and let γ be a simple curve. Let $d_1, \ldots, d_p$ be as in Proposition 31. If $2 \leq i \leq p$, then $T_{\gamma,d_i}V$ is not homogeneous.*

Proof. Set $f(w,t) := A(\gamma(t) + w)$ and let $f_{\omega_0}(w,t) = \sum_{j+d\beta=\omega_0} a_{j,\beta} t^j w^\beta$ be the d-quasihomogeneous principal part of f. The proof of Proposition 31 shows that it suffices to show that $T_{\gamma,d}V$ is inhomogeneous if $-1/d$ is the slope of an edge of the Newton polygon. In that case, there are at least two pairs $(j_1, l_1) \neq (j_2, l_2)$ such that, for $i = 1, 2$, $j_i + dl_i = \omega_0$ and $a_{j_i,\beta_i} \neq 0$ for some β_i satisfying $|\beta_i| = l_i$. Then $f_{\omega_0}(w, 1)$ contains at least the terms $a_{j_i,\beta_i} w^{\beta_i}$, $i = 1, 2$. Since they have different degrees, Corollary 27 yields the claim. $\boxed{QED}$

Finally, we show that limit varieties $T_{\gamma,d}V$ approach V like $o(|z|^d)$ in conoids around γ that open like $|z|^d$, $d < 1$. This is analogous to the well known result that V approaches V_h like $o(|z|)$ when $|z| \to \infty$.

For $z \in \mathbb{C}^n$ denote the n-th coordinate by z_n and the n-th coordinate of a simple curve γ by γ_n.

34 Definition. Let $\gamma \colon \mathbb{C} \setminus \big(B(0,R) \cup {]-\infty, 0]}\big) \to \mathbb{C}^n$ be a simple curve satisfying $\gamma_n(t) = t$ for all t, and let $d < 1$. Define

$$W_{\gamma,d} := \left\{\gamma(t) + t^d a : t \in \mathbb{C} \setminus (B(0,R) \cup {]-\infty, 0]}),\ a \in T_{\gamma,d}V,\ a_n = 0\right\}.$$

The next result, whose proof is completely analogous to the proof of Proposition 5.4 in [5], shows that $W_{\gamma,d}$ approximates V of order d in conoids $\Gamma(\gamma, d, \Omega, r)$ as in (13).

35 Proposition. *Let V be an algebraic set in $\mathbb{C}^n$, let γ be a simple curve satisfying $\gamma_n(t) = t$ for all t, let $d < 1$, and let Ω be a relatively compact open subset of $\mathbb{C}^n$.*

(1) For each $\epsilon > 0$ there is $R > 0$ such that for each $z \in V \cap \Gamma(\gamma, d, \Omega, R)$ there is $w \in W_{\gamma,d}$ with $|z - w| < \epsilon |z|^d$.

(2) For each $\epsilon > 0$ there is $R > 0$ such that for each $w \in W_{\gamma,d}$ with $|w| > R$ there is $z \in V$ with $|w - z| < \epsilon |w|^d$.

The next result describes $W_{\gamma,d}$ using the map F_{ω_0} defined in (11). Its proof is analogous to the one of [5], Proposition 5.3.

36 Proposition. *For γ as in Definition 34 and each relatively compact open subset Ω of $\mathbb{C}^n$ there is $R > 0$ such that $W_{\gamma,d} \cap \Gamma(\gamma, d, \Omega, R)$ is a closed analytic subset of $\Gamma(\gamma, d, \Omega, R)$.*

More precisely, if F_{ω_0} *is as in* (11) *then*

$$W_{\gamma,d} \cap \Gamma(\gamma, d, \Omega, R)$$
$$= \left\{ w \in \Gamma(\gamma, d, \Omega, R) : F_{\omega_0}(w - \gamma(w_n), w_n, \xi) = 0 \text{ for all } \xi \in \mathbb{C}^{n-k} \right\}. \quad (16)$$

4 Examples

In this section we provide examples to illustrate the results of the preceding sections. To do this, we first indicate that the limit currents for an algebraic variety $V(P)$ with respect to a given simple curve γ tending to infinity can be obtained also as limit currents of an algebraic variety $V(Q)$ with respect to a curve σ tending to zero, investigated in [5]. Hence we can derive examples from [5], Section 6.

If σ is a simple curve in the sense of [5], Definition 3.1, i.e., a simple curve tending to zero, if $\delta \geq 1$, and if W is an analytic variety in a neighborhood of the origin, then the limit current in the sense of [5], Definition 3.3, will be denoted by $T^0_{\sigma,\delta}[W]$ and its support by $T_{\sigma,\delta}W$. With this notation, we recall from [5], Proposition 4.7 and Corollary 4.4, the following result.

37 Proposition. *Let σ be a simple curve in the sense of [5], Definition 3.1, let W be an analytic variety in some neighborhood of the origin, and let $\delta \in [1, \infty[$. Then there are $p \in \mathbb{N}$ and rational numbers $1 = \delta_1 < \cdots < \delta_p$ such that the following holds*:

(i) $T^0_{\sigma,\delta}[W] = T^0_{\sigma,\delta'}[W]$ *whenever* $\delta_i < \delta \leq \delta' < \delta_{i+1}$ *for* $1 \leq i < p$ *or* $\delta_p < \delta \leq \delta'$.

(ii) If $\delta_i < \delta < \delta_{i+1}$, $1 \leq i < p$, *then* $T^0_{\sigma,\delta}W$ *is homogeneous and nonvoid.*

(iii) If $\delta > \delta_p$ *then* $T^0_{\sigma,\delta}W$ *is either empty or homogeneous.*

(iv) If W *is a hypersurface, then* $T^0_{\sigma,\delta_i}W$ *is not homogeneous for* $2 \leq i \leq p$.

38 Definition. (a) The numbers $1 = \delta_1 < \delta_2 < \ldots < \delta_p$ are called the *critical values* for σ and W, provided that they are minimal with respect to condition (i). Statements (ii)–(iv) hold if $\delta_1, \ldots, \delta_p$ are the critical values.

(b) For $P \in \mathbb{C}[z_1, \ldots, z_n]$ of degree $m > 0$ expand $P = \sum_{j=\nu}^m P_j$, with P_j either homogeneous of degree j or identically zero and $P_\nu \not\equiv 0$. Then P_ν is called the *localization* of P at the origin.

39 Proposition. *Let* $P \in \mathbb{C}[z_1, \ldots, z_n]$ *be of degree* $m \geq 1$, *denote by* P_m *its principal part, and let* $\gamma\colon \mathbb{C} \setminus (B(0, \alpha) \cup]-\infty, 0]) \to \mathbb{C}^n$ *be a simple curve of the form* $(\gamma_1(t), \ldots, \gamma_{n-1}(t), t)$, *where* $\lim_{t\to\infty} \gamma_j(t)/t = 0$ *for* $1 \leq j < n$.

Define $G := \{z \in \mathbb{C}^n : z_n \neq 0\}$ *and* $\Phi\colon G \to G$, $\Phi(s) := s/s_n^2$, *as well as* $\tilde{Q}(s) := s_n^{2m} P(\Phi(s))$ *and* $\sigma(\tau) := \Phi(\gamma(1/t))$, $\tau \in B(0, 1/\alpha) \setminus\,]-\infty, 0]$. *Then the following assertions hold:*

(a) Φ *is a biholomorphic map.*

(b) $\tilde{Q}$ *extends to a polynomial* $Q \in \mathbb{C}[s_1, \ldots, s_n]$ *which has* P_m *as localization at the origin.*

(c) σ *is a simple curve in the sense of [5], 3.1, satisfying* $\lim_{\tau \to 0} \sigma(\tau) = 0$.

(d) $T_{\gamma,d}[V(P)] = T^0_{\sigma, 2-\delta}[V(Q)]$ *for* $d \in\,]-\infty, 1]$.

(e) If $1 = d_1 > \cdots > d_p$ *(resp.* $1 = \delta_1 < \delta_2 < \ldots < \delta_l$*) denote the critical values for* γ *and* $V(P)$ *(resp.* σ *and* $V(Q)$*) then* $p = l$ *and* $d_j + \delta_j = 2$ *for* $1 \leq j \leq p = l$.

Proof. (a) This follows from $\Phi \circ \Phi = \mathrm{id}_G$.

(b) We expand $P = \sum_{j=0}^m P_j$, where P_j is either homogeneous of degree j or identically zero. Then

$$\tilde{Q}(s) = s_n^{2m} P(s/s_n^2) = s_n^{2m} \sum_{j=0}^m s_n^{-2j} P_j(s) = \sum_{j=0}^m s_n^{2(m-j)} P_j(s),\ s \in G. \tag{17}$$

Therefore, $\tilde{Q}$ is the restriction to G of the polynomial Q defined by $Q(s) := \sum_{j=0}^m s^{2(m-j)} P_j(s)$. If $P_j \not\equiv 0$ then $s \mapsto s_n^{2(m-j)} P_j(s)$ has degree $2m - j$. Therefore, the localization of Q at the origin is P_m.

(c) This is easy to check.

(d) Note first that by (a), for each $w \in G$ also $z := \Phi(w)$ is in G. This implies

$$Q(z) = P(w)/w_n^{2m}.$$

Next fix $w \in G$, $t \in \mathbb{C}$, satisfying $|t| > \max(\alpha, |w_n|)$, and $d < 1$. Then

$$\begin{aligned}
&\Phi(\gamma(t) + t^d w) \\
&\quad = (t + t^d w_n)^{-2} (\gamma(t) + t^d w) \\
&\quad = t^{-2} \left((1 + t^{d-1} w_n)^{-2} - 1 + 1 \right) \left(\gamma(t) + t^d w \right) \\
&\quad = t^{-2} \gamma(t) + t^{d-2} w + t^{-2} \left((1 + t^{d-1} w_n)^{-2} - 1 \right) (\gamma(t) + t^d w) \\
&\quad = \sigma(1/t) + (1/t)^{2-d} \left(w + \big((1 + t^{d-1} w_n)^{-2} - 1 \big) \big((1/t)^{2-d} \sigma(1/t) + w \big) \right) \\
&\quad = \sigma(1/t) + (1/t)^{2-d} \phi(t, w),
\end{aligned} \tag{18}$$

where

$$\phi(t, w) = w + \big((1 + t^{d-1}w_n)^{-2} - 1\big)\big((1/t)^{2-d}\sigma(1/t) + w\big).$$

Since d is smaller than 1, it is easy to see that

$$\lim_{t \to \infty} \phi(t, w) = w. \tag{19}$$

Next assume that the curve γ admits the Puiseux series expansion $\gamma(t) = \sum_{j=0}^{\infty} a_j t^{(q-j)/q}$ for some $q \in \mathbb{N}$. Then, for $s \in B(0, 1/\alpha) \setminus]-\infty, 0]$, we have

$$\sigma(s) = \frac{1}{s^2}\gamma(\frac{1}{s}) = \frac{1}{s^2}\sum_{j=0}^{\infty} a_j (\frac{1}{s})^{(q-j)/q} = \sum_{j=0}^{\infty} a_j s^{(q+j)/q}.$$

Now let $\delta := 2 - d$ and expand, according to [5], Corollary 3.17,

$$Q(\sigma(s) + w) = \sum_{j \in \mathbb{Z}, \alpha \in \mathbb{N}_0^3} b_{j,\alpha} s^{j/q} w^{\alpha} = \sum_{\omega \geq \omega_1} G_\omega(s, w), \tag{20}$$

where

$$\omega_1 = \omega_1(\delta) = \min\{\frac{j}{q} + \delta|a| : b_{j,\alpha} \neq 0\}.$$

Similarly, we obtain from Corollary 27 that

$$P(\gamma(t) + w) = \sum_{j \in \mathbb{Z}, \alpha \in \mathbb{N}_0^3} c_{j,\alpha} t^{j/q} w^{\alpha} = \sum_{\omega \leq \omega_0} F_\omega(t, w), \tag{21}$$

where

$$\omega_0 = \omega_0(d) = \max\{\frac{j}{q} + d|\alpha| : \ c_{j,\alpha} \neq 0\}.$$

Next we use (17), (18), (19), and the quasihomogeneity of the functions G_ω to get

$$\begin{aligned} P(\gamma(t) + t^d w) &= (t + w_n)^{2m} Q(\Phi(\gamma(t) + t^d w)) \\ &= (t + w_n)^{2m} Q(\sigma(t^{-1}) + t^{-\delta}\phi(t, w)) \\ &= (t + w_n)^{2m} \sum_{\omega \geq \omega_1} G_\omega(t^{-1}, t^{-\delta}\phi(t, w)) \\ &= t^{-\omega_1}(t + w_n)^{2m} \sum_{\omega \geq \omega_1} (1/t)^{\omega - \omega_1} G_\omega(1, \phi(t, w)). \end{aligned} \tag{22}$$

Using the expansion (21), we get

$$P(\gamma(t) + t^d w) = t^{\omega_0}\Big(\sum_{\omega \leq \omega_0} t^{\omega - \omega_0} F_\omega(1, w)\Big).$$

Combining it with (22) and using (19), we get

$$F_{\omega_0}(1,w) = \lim_{t\to\infty} t^{-\omega_0-\omega_1+2m}\left(1+\frac{w_n}{t}\right)^{2m} G_{\omega_1}(1,w).$$

Since $G_{\omega_1}(1,\cdot) \not\equiv 0$, this implies $\omega_0 + \omega_1 = 2m$ and $F_{\omega_0}(1,\cdot) \equiv G_{\omega_1}(1,\cdot)$. By Proposition 23 and [5], Proposition 3.14, this implies for $d < 1$:

$$T_{\gamma,d}[V(P)] = T^0_{\sigma,2-d}[V(Q)]. \tag{23}$$

For $d = 1$ we get from Proposition 29 (ii) that

$$T_{\gamma,1}V(P) = V_h - \xi_0 = V(P_m) - \xi_0 \quad \text{for } \xi_0 := (0,\dots,0,1).$$

By [5], Proposition 4.2 (ii) and (b), we have

$$T^0_{\sigma,1}V(Q) = T_0V(Q) - \xi_0 = V(P_m) - \xi_0.$$

Since it is not difficult to interpret the previous equations in the sense of currents, (23) holds for all $d \le 1$, and the proof of (d) is complete.

(e) This follows from (d) by the Propositions 37 and 31, together with Corollary 33. $\boxed{QED}$

40 Example. Define $P \in \mathbb{C}[x,y,z]$ by

$$P(x,y,z) = y(x^2-y^2) - yz + z$$

and let $V = V(P)$. Then

$$V_h = \{(x,y,z) \in \mathbb{C}^3 : y(x^2-y^2) = 0\}$$

and $\Theta := (0,0,1)$ is a singular point of V_h. Define $\gamma(t) := t\Theta$. Then we have

$$d_1 = 1, \quad d_2 = \frac{1}{2}, \quad d_3 = 0,$$

and the following limit varieties:

$$\begin{aligned}
&T_{\gamma,d}V = V_h, && \tfrac{1}{2} < d \le 1,\\
&T_{\gamma,d}V = \{(x,y,z)\in\mathbb{C}^3 : y(x^2-y^2-1)=0\}, && d = \tfrac{1}{2},\\
&T_{\gamma,d}V = \{(x,y,z)\in\mathbb{C}^3 : y=0\}, && 0 < d < \tfrac{1}{2},\\
&T_{\gamma,d}V = \{(x,y,z)\in\mathbb{C}^3 : -y+1=0\}, && d = 0,\\
&T_{\gamma,d}V = \emptyset, && d < 0.
\end{aligned}$$

These statements can be obtained by constructing the corresponding Newton polygon as described in the proof of Proposition 31. However, it is also possible to apply Proposition 39 together with examples that we treated in [5]. Using the notation introduced in Proposition 39, we have

$$Q(s,t,u) = t(s^2 - t^2) - tu^3 + u^5$$

and $\sigma(\tau) = (0,0,\tau)$. Therefore, the assertions above follow from Proposition 39 and [5], Example 6.6. Since $T_{\gamma,1/2}V$ has $(1,0,\lambda)$ and $(-1,0,\lambda)$, $\lambda \in \mathbb{C}$, as singular points, it is reasonable to define $\kappa(t) := (\sqrt{t}, 0, t)$, $t \in \mathbb{C}\backslash]-\infty, 0]$, and to consider $T_{\kappa,d}V$ for $d \le \frac{1}{2}$. Using Proposition 39 and [5], Example 6.6 again, we get

$$\begin{aligned}
T_{\kappa,d}V &= \{(x,y,z) \in \mathbb{C}^3 : y((x+1)^2 - y^2 - 1) = 0\}, && d = \frac{1}{2},\\
T_{\kappa,d}V &= \{(x,y,z) \in \mathbb{C}^3 : 2xy = 0\}, && \frac{1}{4} < d < \frac{1}{2},\\
T_{\kappa,d}V &= \{(x,y,z) \in \mathbb{C}^3 : 2xy + 1 = 0\}, && d = \frac{1}{4},\\
T_{\kappa,d}V &= \emptyset, && d < \frac{1}{4}.
\end{aligned}$$

We could perturb the curve again, but this would not lead to new insights as $T_{\kappa,1/4}V$ does not have any singularities left.

Applying Proposition 39, a number of further examples can be derived from the examples in [5], Section 6. We conclude this section with two more examples which show how the results of the present paper can be used directly to compute limit varieties.

41 Example. Define the polynomial P by

$$P(x,y,z) := x(x^2 - y^2)(x^2 - 4y^2) - y(y^2 - 4x^2)z - z^2 + 1.$$

Then for $V := V(P)$ we have

$$V_h = \{(x,y,z) \in \mathbb{C}^3 : \ x(x^2 - y^2)(x^2 - 4y^2) = 0\}$$

and consequently

$$(V_h)_{\text{sing}} \cap S^2 = \{(0,0,1), (0,0,-1)\}.$$

If we define $\gamma_\pm(t) := (0,0,\pm t)$ for $t > 0$, then we get the following limit varieties in $\mathbb{C}^3$:

$$\begin{aligned}
T_{\gamma\pm,d}V &= \{(x,y,z) : x(x^2-y^2)(x^2-4y^2)=0\}, && \frac{1}{2} < d \le 1,\\
T_{\gamma\pm,d}V &= \{(x,y,z) : x(x^2-y^2)(x^2-4y^2) \pm y(y^2-4x^2)=0\}, && d = \frac{1}{2},\\
T_{\gamma\pm,d}V &= \{(x,y,z) : y(y^2-4x^2)=0\}, && \frac{1}{3} < d < \frac{1}{2},\\
T_{\gamma\pm,d}V &= \{(x,y,z) : y(y^2-4x^2) \pm 1=0\}, && d = \frac{1}{3},\\
T_{\gamma\pm,d}V &= \emptyset, && d < \frac{1}{3}.
\end{aligned}$$

To prove these statements, note first that those on V_h are either obvious or follow from a standard computation. In order to derive those on the limit varieties, we use Corollary 27 in connection with the proof of Proposition 31. To do so we expand $P(\gamma_\pm(t)+w)$ and get as set M in the proof of Proposition 31:

$$M = \{(0,5),(0,4),(0,2),(0,0),(1,3),(1,1),(2,0)\}.$$

Hence the bounded edges of the Newton polygon N of M are the segments $[(0,5),(1,3)]$ and $[(1,3),(2,0)]$, which have slope -2 and -3. By Proposition 31 this implies

$$d_1 = 1,\ d_2 = \frac{1}{2},\ d_3 = \frac{1}{3}.$$

The equations for the limit varieties are obtained from Corollary 27 by grouping the terms in the expansion according to their d-degree.

42 Example. As an example in higher codimensions we consider the *rational normal curve*

$$V := \{(t,t^2,t^3) \in \mathbb{C}^3 : t \in \mathbb{C}\}.$$

The projection $\pi\colon (x,y,z) \mapsto (0,0,z)$ is excellent for V. Once all calculations are completed, it will be obvious that π is also excellent for all limit varieties. Let λ be a primitive third root of unity and note that $1+\lambda+\lambda^2=0$. Some computation shows that the canonical defining function of V with respect to π is

$$\begin{aligned}
&P(x,y,z,\xi,\eta;V)\\
&\quad = \Big\langle (x,y)-(z^{1/3},z^{2/3}),(\xi,\eta)\Big\rangle \Big\langle (x,y)-(\lambda z^{1/3},\lambda^2 z^{2/3}),(\xi,\eta)\Big\rangle \times\\
&\qquad \times \Big\langle (x,y)-(\lambda^2 z^{1/3},\lambda z^{2/3}),(\xi,\eta)\Big\rangle\\
&\quad = (x^3-z)\xi^3 + 3x(xy-z)\xi^2\eta + 3y(xy-z)\xi\eta^2 + (y^3-z^2)\eta^3.
\end{aligned}$$

We choose $\gamma(t) := (0, 0, t)$ and determine F as defined in (9).

$$F(x, y, z, t, \xi, \eta) = P(\gamma(t) + (x, y, z), (\xi, \eta); V)$$
$$= (x^3 - z - t)\xi^3 + 3x(xy - z - t)\xi^2\eta + 3y(xy - z - t)\xi\eta^2 + (y^3 - z^2 - 2zt - t^2)\eta^3.$$

For M as in the proof of Proposition 31 we find

$$M = \big\{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 0)\big\}.$$

Hence there are only two critical values, namely $d_1 = 1$ and $d_2 = 2/3$. In the case $d = 2/3$ we find

$$F_{\omega_0}(x, y, z, t, \xi, \eta) = x^3\xi^3 + 3x^2y\xi^2\eta + 3xy^2\xi\eta^2 + (y^3 - t^2)\eta^3.$$

For all other values of d the calculations are even simpler. We list all limit varieties:

$$\begin{aligned} T_{\gamma,d}V &= \{(0, 0)\} \times \mathbb{C}, && \frac{2}{3} < d \le 1,\\ T_{\gamma,d}V &= (\{(0, 1)\} \times \mathbb{C}) \cup (\{(0, \lambda)\} \times \mathbb{C}) \cup (\{(0, \lambda^2)\} \times \mathbb{C}), && d = \frac{2}{3},\\ T_{\gamma,d}V &= \emptyset, && d < \frac{2}{3}. \end{aligned}$$

For $d = 2/3$, the three components of $T_{\gamma,d}V$ are all simple. In the case $2/3 < d \le 1$, the only component has multiplicity 3.

We could interpret this result as a resolution of the singularity of V at infinity. There are certainly some aspects where our work is connected to the theory of resolution of singularities. Our emphasis, however, is on analytic limit processes as in Theorem 10 and Corollary 22.

Note added in proof: The problems stated in Remark 20 and right after Definition 32 are both solved in R. W. Braun, R. Meise, and B. A. Taylor: Higher order tangents to analytic varieties along curves II, to appear in Canad. J. Math..

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