Note di Matematica **25,** n. 1, 2005/2006, 63–67.

The Pullback for bornological and ultrabornological spaces

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Abstract. Counter examples show that the notion of a pullback cannot be transferred from the category of locally convex spaces to the category of bornological or ultrabornological locally convex spaces. This answers in the negative a question asked to the authors by W. Rump.

Keywords: pullback, bornological spaces, ultrabornological spaces.

MSC 2000 classification: 46A05, 46A09.

To the memory of our dear friend Klaus Floret

Introduction

The purpose of this paper is to construct the counter examples mentioned in the abstract. Unexplained notation about locally convex spaces can be seen in [8] and [10].

Let X, Y be locally convex spaces, $L \subset X$ a linear subspace, $q : X \longrightarrow X/L$ the corresponding quotient map and let $j : Y \longrightarrow X/L$ be a linear continuous map. Then the space $Z := \{(x, y) \in X \times Y : q(x) = j(y)\}$ provided with the relative topology induced by the product $X \times Y$ together with the restricted projections $p_X : Z \longrightarrow X$ and $p_Y : Z \longrightarrow Y$ (the latter of which is a quotient map) is called the corresponding pullback in the category of locally convex spaces LCS. In fact, Z has the following universal property. Whenever a triple $(E,\ f,\ g)$ consisting of a locally convex space E and linear continuous maps $f : E \longrightarrow X$, $g : E \longrightarrow Y$ such that $q \circ f = j \circ g$, then there is a linear continuous map $h : E \longrightarrow Z$ satisfying $f = p_X \circ h$ and $g = p_Y \circ h$.

If j is injective, then Z is (via p_X) topologically isomorphic to the subspace $q^{-1}(j(Y))$ provided with the initial topology w.r.t. to the inclusion $q^{-1}(j(Y)) \hookrightarrow X$ and the restricted quotient map $q|q^{-1}(j(Y)) : q^{-1}(j(Y)) \longrightarrow Y$. In this shape, the pullback had several applications to three-space-problems (providing counter examples, cf. [2], [3], [4] and [6]).

Trivially, Z inherits from X and Y all properties that are stable under initial topologies in LCS. Concerning other properties, the behaviour of Z will be rather bad in general.

In fact, let L be an arbitrary Hausdorff locally convex space, X a product of Banach spaces containing L as a topological subspace and let Y be an arbitrary subspace of X/L provided with the strongest locally convex topology; let $q : X \longrightarrow X/L$ denote the quotient map. Then $Z := q^{-1}(Y)$, endowed with the initial topology mentioned above, contains the given locally convex space L as a topologically complemented subspace. Thus Z can be obtained as bad as possible, whereas X and Y are nice spaces. In the following construction, the map $j : Y \longrightarrow X/L$ will be even bijective. By the example of Köthe and Grothendieck (see [9]) of a Montel echelon space of type 1, having ℓ^1 as a quotient, the topological direct sum $X := \bigoplus_{\mathbb{N}} \ell^\infty$ contains a closed linear subspace L which is not a DF-space, hence not countably quasibarrelled. Let $Y := X/L$ be endowed with the strongest locally convex topology. Again the pullback $Z := q^{-1}(Y)$ contains L as a complemented subspace. Since in the above two examples, the restricted quotient map $q|Z : Z \longrightarrow Y$ leads into a space with the strongest locally convex topology, it will remain an open map, if Z is given its associated bornological topology. On the other hand, a continuous and open linear map $f : X \longrightarrow Y$ will not remain open in general as a map $f : X^{\mathrm{bor}} \longrightarrow Y^{\mathrm{bor}}$ between the associated bornological spaces. In order to provide an example, we recall that every locally convex space E is a quotient of a suitable complete locally convex space F, in which all bounded sets have finite dimensional linear span (see [5]). Putting E to be any bornological space, which does not carry the strongest locally convex topology, we are done.

Returning to the pullback, W. Rump asked, whether there is a pullback in the category of bornological spaces, which amounts to the problem, whether in the above setting with X and Y bornological, the restricted projection $p_Y : Z \longrightarrow Y$, which is easily shown to be open, remains open as a map $p_Y : Z^{\mathrm{bor}} \longrightarrow Y$, where Z^{bor} denotes the associated bornological space.

1 Remark. A partial positive result can be obtained easily:

Let X, Y be bornological spaces, $L \subset X$ a linear subspace, $q : X \longrightarrow X/L$ the quotient map, $j : Y \hookrightarrow X/L$ linear and continuous, and let Z, p_X, p_Y be as before. If for each bounded set A in Y there is a bounded set B in X such that $q(B) \supset j(A)$, then $p_Y : Z^{\mathrm{bor}} \longrightarrow Y$ is open.

In fact, given $A \subset Y$ bounded choose $B \subset X$ bounded such that $q(B) \supset j(A)$; the set $C := (B \times A) \cap Z$ is bounded in Z and for all $a \in A$ there is $b \in B$

with $j(a) = q(b)$ which implies $(b, a) \in C$. Thus $p_Y(C) \supset A$, and we are done.

On the other hand, the following example shows that the answer to Rump's question is negative in general.

2 Example. Let E, F be Banach spaces with unit balls B_E and B_F, respectively, and continuous inclusion $F \hookrightarrow E$ and such that $B_F \subset B_E$ and $C := F \cap \overline{B_F}^E$ is not absorbed by B_F. Let $X := \bigoplus_{\mathbb{N}} E \times c_0(F)$, let $G := \bigoplus_{\mathbb{N}} E + c_0(F) = \operatorname{ind}_{n\to} G_n$, where $G_n := E^{n-1} \times c_0((F)_{k\geq n})$, be the corresponding LB-space of Moscatelli type (which is not regular in this case, see [1]), and let $q : X \longrightarrow G$, $((x_n)_n, (y_n)_n) \mapsto (x_n + y_n)_n$ denote the natural quotient map. $L := \{(\frac{1}{n}y)_n : y \in F\}$ is a linear subspace of $G_1 = c_0(F)$. We first show that for each $m \in \mathbb{N}$, G_m and G_1 induce the same topology on L. In fact, let $(y^{(k)})_k$ be a sequence in F such that $((\frac{1}{n}y^{(k)})_n)_k$ converges to $(0)_n$ in G_m; then $(\frac{1}{m}y^{(k)})_k$ converges to 0 in F, hence $(y^{(k)})_k$ converges to 0 in F, from which one easily obtains that $((\frac{1}{n}y^{(k)})_n)_k$ converges to $(0)_n$ in G_1.

Next we define $A := \{(\frac{1}{n}y)_n : y \in C\} \subset L$, and show that A is a bounded subset of G. In fact, $B_F^{\mathbb{N}} \cap G$ is clearly bounded in G, and it suffices to prove that $A \subset \overline{B_F^{\mathbb{N}} \cap G}^G$. For that purpose let $\varepsilon > 0$ and $(\varepsilon_n)_n \in (0, \infty)^{\mathbb{N}}$ be given; moreover, let $y \in C$. Choose $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{n}y \in \varepsilon B_F$ for all $n \geq n_\varepsilon$; furthermore, for all $n < n_\varepsilon$, $\frac{1}{n}y \in \frac{1}{n}C = \overline{\frac{1}{n}B_F}^E \subset \frac{1}{n}B_F + \varepsilon_n B_E$. Thus $(\frac{1}{n}y)_n \in G \cap B_F^{\mathbb{N}} + \varepsilon B_F^{\mathbb{N}} + \bigoplus_{\mathbb{N}} \varepsilon_n B_E$.

Obviously, A is absorbing in L; consequently the Minkowski functional p_A is a norm on L, and the inclusion $j : Y := (L, p_A) \hookrightarrow G$ is continuous.

Let, as above, $Z := \{(x, y) \in X \times Y : q(x) = j(y)\}$ and let $p_Y : Z \longrightarrow Y$ denote the restricted projection. We claim that $p_Y : Z^{\text{bor}} \longrightarrow Y$ is not open. For that purpose we want to show that there is a bornivorous absolutely convex set U in Z such that $p_Y(U)$ does not absorb A.

Let us assume that the contrary is true. Let $(\varepsilon_n)_n \in (0, \infty)^{\mathbb{N}}$ be arbitrary. Then

$$U := \sum_{n\in\mathbb{N}} \frac{\varepsilon_n}{2} \left(\left(\left(\bigoplus_{k<n} B_E \times \prod_{k\geq n} \{0\} \right) \times \left(B_F^{\mathbb{N}} \cap c_0(F) \right) \times A \right) \cap Z \right)$$

is clearly bornivorous in Z. By assumption, A is absorbed by

$$p_Y(U) \subset \sum_{n\in\mathbb{N}} \frac{\varepsilon_n}{2} \left(\left(\left(\bigoplus_{k<n} B_E \times \prod_{k\geq n} \{0\} \right) + \left(B_F^{\mathbb{N}} \cap c_0(F) \right) \right) \cap A \right) \subset$$

$$\subset \sum_{n\in\mathbb{N}} \varepsilon_n \left(\left(\bigoplus_{k<n} B_E \times \prod_{k\geq n} B_F \right) \cap L \right).$$

This latter set is a typical O-nbhd in $\text{ind}_{n\to}(L, \mathcal{S}_n \cap L)$ where $\mathcal{S}_n$ denotes the topology of G_n. Thus we obtain that A is bounded in $\text{ind}_{n\to}(L, \mathcal{S}_n \cap L)$. Since $\mathcal{S}_n \cap L = \mathcal{S}_1 \cap L$ for all $n \in \mathbb{N}$, A is bounded in $(G_1, \mathcal{S}_1) = c_0(F)$. Therefore $pr_1(A) = C$ is a bounded subset of F, a contradiction. (L is in fact a subspace of $G = \text{ind}_{n\longrightarrow} G_n$ which is not a limit subspace).

A suitable modification of the construction in the above example yields a negative answer to a pullback in the category of ultrabornological spaces.

3 Example. By [7] there exist Banach spaces containing dense ultrabornological hyperplanes H. Comparing H with a closed hyperplane in the same Banach space, one obtains a Banach space $(E, ||\cdot||)$ admitting a strictly finer ultrabornological normed topology $\mathcal{S}$. Let us put $F := (E, \mathcal{S})$. Then the identity map $id : (E, \mathcal{S}) \longrightarrow (E, ||\cdot||)$ is continuous, and we may clearly assume that the unit ball B_F in $F = (E, \mathcal{S})$ is dense in the unit ball B_E of $(E, ||\cdot||)$. Clearly, B_E is not absorbed by B_F. Repeating the construction of Example 1 verbatim (we never utilized the completeness of F in Example 1), we obtain:

There is a bornivorous absolutely convex set U in

$$Z \subset \left(\bigoplus_{\mathbb{N}} (E, ||\cdot||) \times c_0((E, \mathcal{S})) \right) \times (L, p_A)$$

such that $p_Y(U)$ does not absorb the set $A = \left\{ \left(\frac{1}{n}y\right)_n : y \in C := B_E \right\}$. As A is closed in the unit ball of the Banach space $\{x = (x_n) \in E^{\mathbb{N}} : |||x||| := \sup_n ||\frac{1}{n}x_n|| < \infty\}$ (diagonal transform of $\ell^\infty(E, ||\cdot||)$), A is a Banach disc and $Y = (L, p_A)$ a Banach space. Consequently, $p_Y : Z \longrightarrow Y$ is not even open w. r. to the associated bornological topology on Z.

It remains to prove that $X = \bigoplus_{\mathbb{N}} (E, ||\cdot||) \times c_0(E, \mathcal{S})$ is ultrabornological, or - equivalently - that $c_0(E, \mathcal{S})$ is ultrabornological.

Since $(E, \mathcal{S})$ is ultrabornological, the proof of 1.8.9 in [10] yields that in $c_0(E, \mathcal{S})$ every absolutely convex set that absorbs all bounded Banach discs is bornivorous, hence a 0-nbhd in the normed space $c_0(E, \mathcal{S})$.

Acknowledgements. The research of J. Bonet was partially supported by MCYT and FEDER Proyecto no. BFM 2001 - 2670.

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