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On not open linear continuous operators between Banach spaces

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Abstract. Let X and Y be infinite–dimensional Banach spaces. Let $T\colon X \to Y$ be a linear continuous operator with dense range and $T(X) \neq Y$. It is proved that, for each $\varepsilon > 0$, there exists a quotient map $q\colon Y \to Y_1$, such that Y_1 is an infinite–dimensional Banach space with a Schauder basis and $q \circ T$ is a nuclear operator of norm $\leq \varepsilon$. Thereby, we obtain with respect to the quotient spaces the proper analogue result of Kato concerning the existence of not trivial nuclear restrictions of not open linear continuous operators between Banach spaces.

As a consequence, it is derived a result of Ostrovskii concerning Banach spaces which are completions with respect to total nonnorming subspaces.

Keywords: not open linear continuous operators, nuclear operators, basic sequence, biorthogonal system, quotient spaces.

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Dedicated to the memory of Klaus Floret

Introduction

A well-known result of Kato is the following (see [6, Proposition 2.c.4]):

Let X and Y be infinite–dimensional Banach spaces. Let $T\colon X \to Y$ be a linear continuous operator with $T(X)$ not closed subspace of Y. Then, for every $\epsilon > 0$ there is an infinite–dimensional subspace Z of X so that Z has a Schauder basis and $T_{|Z}$ is a nuclear operator with norm $\leq \epsilon$.

This result played a central role in the study of the strictly singular operators and of the perturbation theory of Fredholm operators in the setting of Banach spaces (see [6, Section 2.c]).

The aim of this paper is to prove the proper analogue result of Kato with respect to the quotient spaces (see Theorems 1 and 2 of § 2). This type of result has been motivated by the study of some topological invariants in the context of Fréchet spaces (see [1,2]).

As a consequence, in §3 we derive a result of Ostrovskii [7] concerning Banach spaces which are completions with respect to total nonnorming subspaces.

Let us recall some basic definitions.

Let $(X, || \, ||)$ be a Banach space and $(X^*, || \, ||^*)$ be its dual topological space. We denote by B_X (B_{X^*} respectively) the closed unit ball of X (X^* respectively).

A closed subspace M of X^* is said to be *total* if for every $0 \neq x \in X$ there is an $f \in M$ such that $f(x) \neq 0$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be a *Schauder basis* if for every $x \in X$ there is a unique sequence of scalars $(\alpha_n)_{n\in\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$. A sequence $(x_n)_{n\in\mathbb{N}}$ of X is said to be a *basic sequence* if it is a Schauder basis for its closed linear span $[x_n]_{n\in\mathbb{N}}$. A pair of sequences $((x_n, x_n^*))_{n\in\mathbb{N}}$ in $X \times X^*$ is said to be a *biorthogonal system* if $x_n^*(x_n) = \delta_{nm}$ for every $n,\ m \in \mathbb{N}$.

Let $T\colon X \to Y$ be a linear continuous map with $(X, || \, ||)$ and $(Y, | \, |)$ Banach spaces. We denote by $||T||$ its operator norm and by $Rg(T)$ its range. The map T is said to be a *quotient map* if $\overline{TB_X} = B_Y$. The map T is said to be *nuclear* if there exist a bounded sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^*, a bounded sequence $(y_n)_{n\in\mathbb{N}}$ in Y, and an element $(\lambda_n)_{n\in\mathbb{N}} \in \ell^1$ such that $T(x) = \sum_{n=1}^{\infty} \lambda_n x_n^*(x) y_n$ for every $x \in X$. The notation $T_{|Z}$ means the restriction of T to the subspace Z of X.

Moreover, T is called *strictly cosingular* if there exists no closed subspace N of Y with $\operatorname{codim} N = \infty$ such that $q \circ T\colon X \to Y/N$ is onto where q denotes the canonical quotient map from Y onto Y/N (see [8]).

For a subset A of X, $\overline{A}$, $\overset{\circ}{A}$ and $A^{\perp}$ denote the closure of A in the strong topology, the set $\{x^* \in X^* :\ \forall x \in A\ |x^*(x)| \leq 1\}$ and the set $\{x^* \in X^* :\ \forall x \in A\ x^*(x) = 0\}$ respectively. For a subset A of X^*, $\overline{A}^{w^*}$, $\overset{\bullet}{A}$ and $A^{\top}$ denote the closure of A in the weak*–topology, the set $\{x \in X :\ \forall x^* \in A\ x^*(x) = 0\}$ and the set $\{x \in X :\ \forall x^* \in A\ x^*(x) = 0\}$ respectively.

Other notation for Banach spaces is standard and we refer the reader, for example, to [6].

1 Main Result

In this section we will prove the proper analogue result of KATO with respect to the quotient spaces. The proof of this result is inspired by the ones given in [3, Theorem 1.2, (2) implies (3)] and [9, Lemma 3].

1 Theorem. *Let $(X, || \, ||)$ and $(Y, | \, |)$ be infinite–dimensional Banach spaces. Let $T\colon X \to Y$ be a linear continuous map with dense range and $T(X) \neq Y$. Then for each $\varepsilon > 0$ there exists a quotient map $q\colon Y \to Y_1$, such that Y_1 is an infinite–dimensional Banach space with a Schauder basis and the map $q \circ T$ is nuclear with norm $\leq \varepsilon$.*

PROOF. We first notice that, by Closed Range Theorem, if T has no closed range then T^* hasn't, and also T^* restricted to any finite–codimensional subspace of Y^* hasn't.

Let $\varepsilon > 0$ be fixed. We choose a sequence $(A_n)_{n\in\mathbb{N}}$ of positive real numbers with $A_1 > 1$ and $A_{n+1} > 1 + \sum_{k=1}^{n} A_k$ for all $n \in \mathbb{N}$, and we construct by induction a biorthogonal system $((y_n, y_n^*))_{n\in\mathbb{N}}$ in $Y \times Y^*$ such that

$$|y_n| \leq A_n,\ |y_n^*|^* = 1 \text{ and } ||T^* y_n^*||^* \leq \frac{\varepsilon}{A_n} 2^{-n} \tag{1}$$

for all $n \in \mathbb{N}$.

Clearly, there exist $y_1 \in Y$ and $y_1^* \in Y^*$ such that $y_1^*(y_1) = 1$ and (1) holds with $n = 1$. Turning to our induction step, we assume that $(y_k)_{k=1}^n \subset Y$ and $(y_k^*)_{k=1}^n \subset Y^*$ have been chosen in such a way that $((y_k, y_k^*))_{k=1}^n$ is a biorthogonal system and (1) is satisfied for $k \in \{1, \ldots, n\}$. Let define $P\colon Y \to Y$ by $Py = \sum_{k=1}^n y_k^*(y) y_k$. Then we set $Q = I - P$ and choose $y_{n+1}^* \in Rg(Q^*) = \operatorname{span}\{y_1, \ldots, y_n\}^\perp \subset Y^*$ so that

$$|y_{n+1}^*|^* = 1 \text{ and } ||T^* y_{n+1}^*||^* \leq \frac{\varepsilon}{A_{n+1}} 2^{-(n+1)}.$$

Since $y_{n+1}^* = Q^*(y_{n+1}^*)$, we have

$$1 = |y_{n+1}^*|^* = \sup_{|y|\leq 1} |y_{n+1}^*(Qy)| \leq \sup_{z\in Rg(Q),\ |z|\leq ||Q||} |y_{n+1}^*(z)|.$$

By construction $A_{n+1} > 1 + ||P|| \geq ||Q||$ so that we can find y_{n+1} in Y with the desired properties.

Next, let $S\colon X \to Y$ be the map so defined

$$Sx := \sum_{n=1}^{\infty} (T^* y_n^*)(x) y_n, \quad x \in X. \tag{2}$$

Then $S \in L(X, Y)$ is nuclear with norm $\leq \varepsilon$. Since

$$S^* y^* = \sum_{n=1}^{\infty} y^*(y_n) T^* y_n^*$$

for all $y^* \in Y^*$, we obtain that $T^* = S^*$ on $\operatorname{span}\{y_1^*, y_2^*, \ldots\}$. We set $Z = \{y \in Y \ :\ \forall n \in \mathbb{N}\ y_n^*(y) = 0\}$. Clearly Y/Z is infinite–dimensional. Since $y_n^*(Tx - Sx) = ((T^* - S^*)y_n^*)(x) = 0$ for all $n \in \mathbb{N}$ and $x \in X$, we have that $q \circ T = q \circ S$ where $q\colon Y \to Y/Z$ is the canonical quotient map. Consequently, $q \circ T$ is a nuclear map with norm $\leq \varepsilon$.

In order to obtain a quotient with a Schauder basis, we notice that $Rg(q \circ T)$ is dense in Y/Z and $Rg(q \circ S)$ is separable. Therefore the Banach space Y/Z is separable and then we can apply [6, Theorem 1.b.7] (see [5]) to get an infinite–dimensional quotient of Y/Z with a Schauder basis. Finally, the composition of the quotient maps fulfills all required properties and the proof is complete. $\boxed{QED}$

As an immediate consequence, we obtain the following:

2 Theorem. *Let $(X, \|\ \|)$ and $(Y, |\ |)$ be infinite–dimensional Banach spaces. Let $T\colon X \to Y$ be a not open linear continuous operator. Then for each $\varepsilon > 0$ there exists a quotient map $q\colon Y \to Y_1$, such that Y_1 is an infinite–dimensional Banach space and $q \circ T$ is a nuclear operator of norm $\leq \varepsilon$.*

Proof. Let $Z = \overline{T(X)}$. By assumption, T is a linear continuous map from X into Z with dense range and $T(X) \neq Z$. Taking any $\varepsilon > 0$, by Theorem 1, there exists a quotient map $q_Z\colon Z \to Z_1$, such that Z_1 is an infinite dimensional space with a Schauder basis and $q_Z \circ T$ is a nuclear operator of norm $\leq \varepsilon$.

Put $N = \ker q_Z$. Clearly, $N \subset Z$ is also a closed subspace of Y and Z_1 is isometrically isomorph to a closed subspace of $Y_1 = Y/N$ via the canonical map $j\colon Z_1 \to Y_1$ defined by $j(x + N) = x + N$ for all $x \in Z$. Consequently, denoting by q the canonical quotient map from Y onto Y_1, $j \circ q_Z = q \circ i$ (where $i\colon Z \to Y$ is the canonical inclusion) and hence $q \circ T = j \circ q_Z T$ is also a nuclear operator of norm $\leq \varepsilon$. This completes the proof. $\boxed{QED}$

2 A Consequence

Let $(X, \|\ \|)$ be an infinite–dimensional Banach space. Let M be a total subspace of X^*. Define the completion of X with respect to M as the completion of X under the norm

$$\|x\|_M = \sup\{|x^*(x)| : \ x^* \in M, \ \|x^*\|^* \leq 1\}.$$

Denote by X_M this completion.

If the norm $\|\ \|_M$ is equivalent to the initial norm of X, then the subspace M is said to be *norming*. It is clear that if M is norming, then $X_M = X$. If M is a total nonnorming subspace of X, then $X_M^* = \operatorname{span} \overline{B}_M^{\sigma(X^*,X)} \supset M$.

Total nonnorming subspaces were studied by many authors. In particular, in [4] Davis and Lindenstrauss proved that a Banach space X has a total nonnorming subspace in its dual if and only if X has infinite codimension in its second dual.

In [7] Ostrovskii considered the problem to characterize what kind of Banach spaces are completions of some other Banach spaces with respect to a total nonnorming subspace. In particular, he showed that:

3 Theorem. *If a Banach space Z is the completion of some other Banach space with respect to a total nonnorming subspace, then Z^* contains a norming* *subspace M and a $\sigma(Z^*, Z)$–closed infinite–dimensional subspace N such that $\delta(M, N) > 0$ and the quotient $Z/N^\top$ is separable.*

Recall that if U and V are subspaces of a Banach space $(X, || \, ||)$ the number $\delta(U, V) = \inf\{||u - v|| : \ u \in U \text{ and } ||u|| = 1, \ v \in V\}$ is called *the inclination of U to V*.

Now, this result follows from Theorem 1 as a consequence. Indeed:

PROOF. Let $Z = X_M$ for some Banach space $(X, || \, ||)$ and a total nonnorming subspace M of X^*. Every element of M is a linear functional on X_M with the same norm. Further, M considered as a subspace of Z^* is clearly norming.

Since the inclusion $i_M : (X, || \, ||) \hookrightarrow (X_M, || \, ||_M)$ is a linear continuous map with dense range and not open, taking e.g. $\varepsilon = 1/2$, as it was proved in Theorem 1, there is a biorthogonal system $((x_n, x_n^*))_{n \in \mathbb{N}}$ in $X_M \times X_M^*$ satisfying the conditions in (1) such that, if we set $M_1 = \{x \in X_M : \ x_n^*(x) = 0 \text{ for all } n \in \mathbb{N}\}$, the quotient space X_M/M_1 is infinite–dimensional and separable.

Let $N = M_1^\perp \subset X_M^*$. Clearly, N is $\sigma(X_M^*, X_M)$–closed. Since X_M/M_1 is separable as already observed, it remains only to show that $\delta(M, N) > 0$.

Let $z^* \in M$ with $||z^*||_M^* = 1 = ||z^*||^*$. Then, taking any $\epsilon > 0$, there is $x \in X$ with $||x|| = 1$ such that $z^*(x) \geq 1 - \epsilon$. Put $w = x - \sum_{n=1}^\infty x_n^*(x) x_n = x - Sx$ (cf. (2)). Then $w \in M_1 = N^\top$ and $||w - x|| \leq 1/2$. It follows that

$$z^*(w) \geq z^*(x) + z'(w - x) \geq 1 - \epsilon - \frac{1}{2} = \frac{1}{2} - \epsilon;$$

hence, for each $y^* \in N$,

$$||z^* - y^*||_M^* \geq \frac{2}{3} |(z^* - y^*)(w)| = \frac{2}{3} |z^*(w)| \geq \frac{2}{3} (\frac{1}{2} - \epsilon) = \frac{1}{3} - \frac{2}{3} \epsilon.$$

By the arbitrarity of ϵ, it follows that $||z^* - y^*||_M^* \geq 1/3$ for all $y^* \in N$, thereby implying that $\delta(M, N) \geq 1/3 > 0$. QED

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