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On the mathematical work of Klaus Floret

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Abstract. In section 1 of this article, a sketch of the life and the career of Klaus Floret is given. Then, in sections 2 to 5, the author reports on the part of the mathematical work of Klaus Floret which was devoted to four important topics in functional analysis in which Klaus worked and in which considerable progress was achieved during his lifetime. These topics are: locally convex inductive limits; bases and approximation properties; $\mathcal{L}_1$-, $\mathcal{L}_\infty$-spaces and the "problème des topologies"; tensor norms, operator ideals, spaces of polynomials. Some of the history of these topics is included, sometimes together with an account of some of the leading figures in the field. A prelude, an interlude and a coda present related remarks which do not fit in the main body of the article.

Keywords: curriculum vitae of Klaus Floret, list of publications of Klaus Floret; locally convex inductive limit, basis, approximation property, $\mathcal{L}_1$-space, $\mathcal{L}_\infty$-space, "problème des topologies", tensor norm, operator ideal, spaces of polynomials.

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In this article (as I did in the talk on which the article is based), I will first sketch Klaus Floret's curriculum vitae and career and then concentrate on four topics in functional analysis in which he made significant contributions. Both my report on his life and my report on his mathematical work involved making choices of which things to mention and which things to omit since there was just too much to be said and to be written. Thus, here is my personal choice. Somebody else might make a different choice, but I hope that my choice does justice to how Klaus lived and to how he contributed to mathematics.

Section 1 will be accessible to everybody; i.e., mathematicians and non-mathematicians. In the other sections, dealing with mathematics, I will try to include a bit of the history of the topics and will tell some stories about some of the mathematicians involved. After all, one tends to forget too easily that mathematics is made by mathematicians and that some of these mathematicians, apart from being very talented in mathematics, also have quite interesting personalities. Thus, parts of sections 2 to 5 should be accessible, to some extent, to people who do not know too much about the mathematics involved.

Note for the reader: In the article, references are given to items in four

[i]This is a revised and expanded version of my talk at the Memorial Colloquium held on the occasion of the first day of death of Klaus Floret at the University of Oldenburg, on July 4, 2003; there was a broad audience of mathematicians and non-mathematicians.

different lists. For $n \in I\!N$, clearly [n] refers to article n in the list of articles of Klaus Floret, while [An] and [Cn] refer to item n in the lists of his theses and books, respectively. On the other hand, abbreviations like [DS] always refer to items in the list of references to other people's work.

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1 Curriculum vitae

Klaus Floret was born at Mannheim, Germany on September 22, 1941. He went to elementary school and Gymnasium in Mannheim and got his Abitur in 1961. After the military service, he started studying mathematics at the University of Heidelberg in 1962. During his studies, he attended a sequence of lectures on functional analysis by Professor J. Wloka and heard a course on stochastics by Professor Krickeberg. At that time in Heidelberg there also were topologists like Dold and Seifert, plus analysts like Jörgens (who died too early) and Weidmann (now professor in Frankfurt).

The leading figure in functional analysis in Heidelberg, however, was *Professor Gottfried Köthe*, originally from Graz, Austria, then working with Emmy Noether and later with Otto Toeplitz. Köthe had done important work in locally convex spaces; Köthe echelon and co-echelon sequence spaces were named after him, and the duality of spaces of holomorphic functions in one variable and spaces of germs of holomorphic functions is sometimes called Köthe duality. Köthe had already published the first volume of his very influential book [K] on topological vector spaces. Volume 2 was to follow much later, after Köthe had left Heidelberg for Frankfurt.

Klaus actively participated in Köthe's seminar. Other participants of this seminar were H.G. Tillmann (later professor in Mainz, then in Münster) and D. Vogt (later Tillmann's assistant in Mainz, then professor in Wuppertal).

When Klaus participated in the seminar, people gave lectures on the work of Mityagin. In 1967, Klaus received the diploma degree (supervisor: Wloka); part of his diploma thesis [A1] was published in Manuscripta Math., see [1]. He stayed one more year as a scientific assistant in Heidelberg before he went, with Wloka, to the University of Kiel, where he was to remain until 1981.

In 1969, Klaus Floret became Dr. rer. nat. (supervisor: Wloka). His thesis [A2] treated inductive sequences of locally convex spaces with compact linking maps; it received the prize of the Faculty of Science of Kiel in 1970, and (part of) it was published in J. Reine Angew. Math. ("Crelle's journal") in 1971, see [5]. Already in 1971, Klaus got his Habilitation for mathematics. His Habilitationsschrift [A3] dealt with sequentially retractive locally convex inductive limits (see section 2 below), and the main part of it was published again in Crelle's journal, see [8]. Klaus was a scientific assistant of Wloka from 1968 to 1972, a Dozent from 1972 to 1976 and an apl. Professor from 1976 to 1981 at the University of Kiel. In Kiel, he had four PhD students: Volker Wrobel (1974), Carsten Schütt (1977), Jakob Harksen (1979), and Andreas Defant (1980).

During the winter term of 1975/76, Klaus held a visiting position at the Technical University of Berlin, and during the academic year 1977/78 he visited the S.U.N.Y. at Buffalo. While Klaus was in Kiel, he had become an enthusiastic yachtsman. During the spring break in Buffalo he went to Florida for some days of sailing (which nearly ended in disaster when a storm threatened to sink the ship). This pastime of sailing kept him from applying for positions at other German universities. It was not too easy to convince him that he had to apply for the open position of Full Professor for functional analysis at the University of Oldenburg, but finally he did and got the position.

From 1981 on, Klaus stayed in Oldenburg, and Andreas Defant had come there with him. They greatly enhanced mathematical research in Oldenburg. Klaus also brought some excellent analysts to Oldenburg: B. Carl (now professor in Jena), M. Langenbruch, and J. Voigt (now professor in Dresden). Klaus was Dean of the Faculty of Mathematics in Oldenburg from 1993 to 1995 and a member of the Senate of the University from 1995 to 1997.

In 1987, Klaus Floret became a corresponding member of the Société Royale des Sciences de Liège, Belgium. Jean Schmets from the University of Liège was a close friend of Klaus, mathematically and personally. When Jean was the Belgian Visiting Professor in Nordrhein-Westfalen, Germany for one semester in the 90s and came for this time to Paderborn, the Florets visited Paderborn for two days as well. Together we spent an unforgettable evening with an excellent dinner at the Castle of Bevern (in picturesque Weser-Renaissance style), about 70 km east of Paderborn.

In 2000, Klaus became a corresponding member of the Real Academia de Ciencias Exactas, Físicas y Naturales of Madrid, Spain, too. Besides his usual trips to Brazil (see below), Klaus also visited the University of Pretoria, South Africa, the University of Lecce, Italy, the universities of Valencia, Sevilla and Granada in Spain, and some places in South Korea.

During the time in Oldenburg, Klaus organized seven so-called Wangerooge Meetings on Functional Analysis; most of them took place on the neighboring island of Spiekeroog in the North Sea. Many illustrious functional analysts attended these meetings and gave talks. There always was a special atmosphere at these island meetings. Sometimes the weather was cold, sometimes stormy. The participants took long walks along the beaches, talking about mathematics. The eighth (and possibly last) Wangerooge meeting was organized in November 2002 to honor the memory of Klaus Floret.

Another noteworthy series of events was the Northwest German Functional Analysis Colloquium, which Klaus founded, together with H.G. Tillmann (Münster). In the beginning these colloquia took place on one Saturday each semester, later once a year, at one of the participating universities. These were, first, Oldenburg, Münster, Osnabrück, and Bielefeld. Then Paderborn joined, and now also Dortmund, Essen, Wuppertal, and Düsseldorf are participating.

Klaus gave what, unfortunately, was to be his last scientific talk during the 23rd Northwest German Functional Analysis Colloquium in Paderborn early in 2001. After the talk he complained that he had had to look at his manuscript two or three times during the talk and that he had forgotten to mention his coauthor. (Klaus was not used to look at his manuscript during a talk, and even all his courses were given in this way.) Not too long after the talk in Paderborn it was discovered that Klaus had a brain tumor, of the worst sort. He underwent various operations, but in the end without success.

The last time Klaus attended a mathematical meeting was on the occasion of the 24th Northwest German Functional Meeting at Oldenburg, May 4, 2002. He told me that he knew all the participants of the colloquium, but that he did not remember a single name. He cordially greeted one of the speakers, Ioana Cioranescu, who was visiting Paderborn at that time and who had got her Habilitation at the University of Kiel when Klaus had also been there. What a pity that both Klaus and Ioana died in July 2002! Ioana died in Rio Piedras, Puerto Rico, where she was a professor. The date of Klaus' death in Oldenburg was July 2.

But let us go back to the year 1980. After I had visited Brazil for the second time, I told Klaus that he would certainly like this country and its inhabitants. He agreed when I offered to write a letter to Leopoldo Nachbin in Rio de Janeiro. Of course, Nachbin knew Floret's work and invited him to Brazil. The first visit was in 1982. Klaus immediately fell in love with Brazil and from then on often spent several months at Universidade Federal do Rio de Janeiro, IMECC, Unicamp at Campinas, S.P., or Univ. Fed. Fluminense at Niteroi. After a few years, Nachbin declared Klaus to be "an honorary Brazilian". In 1990, Klaus became scientific advisor of the German Federal Ministry for Education, Research and Technology for the cooperation in mathematics between Germany and Brazil (later including all of Latin America).

In 1987, Klaus was offered a full professorship at Unicamp. Since the situation in Brazil did not seem stable enough economically, I tried to convince him that he might take leaves from Oldenburg to go to Brazil, but that he should not go permanently, abandoning his professorship in Oldenburg. The offer of Unicamp was rejected in 1988, but he visited Unicamp for at least one month in the years 1989, 1993, 1994, 1995, 1996, 1997, 1998, and 2000.

At some point I got an e-mail of Klaus from Brazil in which he wrote that he had spent the most beautiful holiday of his life on the beaches in northeastern Brazil. At the end of this letter he had added: "Incidentally, she is named Andréa." Klaus Floret and Andréa Neves got married in Brazil in 1992, their daughter Sofia was born in 1998.

In the beginning of Andréa's time in Germany, Klaus always spoke Portuguese with her. But she learnt German at the Goethe Institute in Bremen and then was able to speak the German language very well. The time of Klaus' severe health problems was very hard for Andréa, but she soon learnt how to handle the situation, and she was a wonderful help to Klaus in the one and a half years which remained for him. And there were many friends of Klaus in Oldenburg (and in other places) who did their best to help Andréa and Klaus. It had not even been clear whether Klaus would live to see his 60th birthday, but mid-September to November 2001 was the best time for him between the various operations.

A week after Klaus came home after his second brain tumor operation, I visited the Florets. Klaus was quite fine, we had a nice conversation; he only needed some help with the stairs. Andréa went to the kindergarten, came home with Sofia, and put on the TV set: It was September 11, 2001, and Klaus and I watched with shock and disbelief what had happened in New York City. I will always keep Klaus Floret in mind as he looked on that ominous day.

Prelude

Before I will pass to the four mathematical topics which will be treated in more detail in sections 2 to 5, it should be mentioned that, during his visit to Buffalo, Klaus wrote the Springer Lecture Notes in Math. volume 801 (1980) “Weakly Compact Sets”, see [C2]. Although Klaus was not a specialist in this subject, his Lecture Notes volume has often been quoted as a standard reference, among other things because of the excellent treatment of angelic spaces. The reviewer of the book in Math. Reviews (P. Pérez Carreras) wrote: “The author has done a service in organizing the material carefully . . . The book fills a gap in the literature. In sum, an interesting monograph on a fascinating subject.” – B. Cascales and J. Orihuela from Murcia, Spain became well known to specialists when they solved a question of Klaus Floret in the Lecture Notes volume, proving that (LF)-spaces are angelic, see [CasOri1], [CasOri2].

Klaus also wrote a German textbook on measure and integration theory (an introduction) which appeared with Teubner in 1981, see [C3]. The two other books, the Springer Lecture Notes volume 56 (1968) on locally convex spaces [C1], together with Wloka, and the monograph in the series North-Holland Math. Studies (1993) [C4], together with Defant, will be treated in sections 2 and 5 below, respectively. – Incidentally, the list of publications of Klaus Floret consists of 58 articles (plus four books and three theses), which had a big impact on wide areas of functional analysis.

2 Locally convex inductive limits

The Springer Lecture Notes in Math. vol. 56 (1968) [C1] of K. Floret and J. Wloka gave an introduction to the theory of locally convex spaces which was easily readable and well focused. In many respects it was different from the by then “classical” books of Köthe [K] and Schaefer. Among other things, the authors used seminorms rather than 0-neighborhoods. And the treatment of countable locally convex inductive limits contained not only the theory of strict inductive limits, but also the theory of inductive limits of Banach spaces with compact linking maps. Such spaces were called (LS)-spaces in the book, while the term (DFS)-space has become standard in the literature. Unfortunately, the lecture notes of Floret-Wloka have not been translated to English. Many years after this book appeared, some authors still did not know that (DFS)-spaces have similarly good, even better properties than strict inductive limits. – Although the book had two authors, the style in which it was written already was typically “Klaus Floret”. Quite general, the style of Klaus’ papers was some kind of trademark, quite different from the style of most other mathematicians.

Let us recall that one cannot say too much about arbitrary locally convex inductive limits; they do not have any nice permanence properties. If E is an *ultrabornological* space; i.e., an inductive limit of Banach spaces, and F is an arbitrary separable infinite dimensional Banach space, then M. Valdivia (see [Val1]), improving a theorem of Raikov, showed that E can be represented as $\mathrm{ind}_\alpha E_\alpha$, where all the spaces E_α are topologically isomorphic to F and where all the linking maps are nuclear. That is, even for "good" spaces and "excellent" linking maps the uncountable inductive limit is nothing more than just ultrabornological. Klaus reported on Valdivia's theorem in his talk at the Second Paderborn Conference on Functional Analysis, see [18].

The situation for countable locally convex inductive limits is much better. For the purposes of our report here we will always assume that a countable locally convex inductive limit is Hausdorff and that it is the union of an increasing sequence of linear subspaces with continuous linking maps; we will sometimes also omit "locally convex" in connection with inductive limits. Historically, the need to treat inductive limits arose from distribution theory: The space $\mathcal{D}$ of all test functions, the predual of the space $\mathcal{D}'$ of all distributions, is a countable locally convex inductive limit of Fréchet spaces; that is, an *(LF)-space.* This limit is strict, and strict (LF)-spaces were first treated in 1949 in an article by J. Dieudonné and L. Schwartz, see [DS]. (Note that, in the beginning, (LF)-spaces were always supposed to be strict inductive limits. In today's literature, however, the requirement that the inductive limit is strict is omitted.)

Jean Dieudonné was a very active member of the N. Bourbaki group. He wrote a well known Treatise on Analysis with four volumes. In the spring semester of 1972, I was a Research Associate at the University of Maryland at College Park, where both Dieudonné and Köthe were visiting. Dieudonné impressed the secretaries because he was typing the text of one of his books when they arrived in the morning, he was typing all the day, except for one hour at lunch time, and still typed in the evening when they left. Dieudonné could be very excited very fast: "This country, this people – there are no such things as Baire measures or Borel measures, there are only Radon measures." (Remember Bourbaki's point of view of measure and integration theory.) But then Dieudonné usually also calmed down very fast.

In 1972, the functional analysis seminar at the University of Maryland was held in the evening in the basement of the house of John Brace. One night the speaker was a member of the Romanian academy who had published a book in French. Dieudonné and Köthe were sitting right next to each other on a settee just in front of me. The speaker announced that he would talk about the book that he was just writing, on fixed point theory. Several minutes he kept turning pages of a manuscript and then wrote a theorem on the blackboard. Dieudonné: "That's a simple exercise in my book!" At some point during the talk, Horváth told Dieudonné in French that this was the worst lecture he had ever heard in the seminar. Dieudonné was very agitated. Köthe put his arm around Dieudonné's shoulder and said: "Please, Professor Dieudonné, calm down. This will not do you any bodily harm." When the speaker had finished, Dieudonné asked him whether his results had any applications, and when he answered something like "not yet", Dieudonné exclaimed: "That's what I had supposed."

Laurent Schwartz discovered (he chose not to say "invented"!) distributions and received a Fields Medal, the highest mathematical prize, for this work in 1950. His article [SDVV] on vector valued distributions introduced the ε-product, a construction related to the complete ε-tensor product which was investigated in the "thèse" [Groth] of Schwartz's student Alexander Grothendieck. When the name "espace de Schwartz" appeared for the first time in Schwartz's "Théorie des distributions à valeurs vectorielles", there was a footnote: "Cette dénomination est due à M. GROTHENDIECK!".

In 1980, while I was spending some time at Universidade Federal do Rio de Janeiro, Schwartz visited Rio de Janeiro, after not having been in Brazil for a long time because of the military government. Schwartz gave a talk in Nachbin's seminar and wanted to give it in Portuguese language. Nachbin was asked to correct the language very strictly. Nachbin did so, very often, since Schwartz sometimes used Spanish expressions instead of Portuguese ones, and at one point Schwartz got so confused that he uttered a perfect German sentence thinking that it was Portuguese!

When Pepe Bonet and I completed our characterization of the distinguished Köthe echelon spaces, we sent a note, "à langue dominante anglaise", to Laurent Schwartz, and he presented it to the French academy, see [BiBoCR].

Schwartz was against the French war in Algeria and against the American war in Vietnam, he actively pleaded for human rights. During his visit to Brazil in 1980, he went to Uruguay, where the mathematician (and communist) Massera was in jail and had been mistreated, and spoke with the responsible minister. Some time later, Massera was let out. Schwartz also was a hunter and collector of butterflies. After visiting Rio de Janeiro in 1980, he went to Panama because of butterflies. Schwartz was as proud of his Fields medal as of the fact that he had discovered a new butterfly, which was then named after him. His collection of butterflies exceeded 20,000 pieces and was exhibited once more during the "Hommage à la mémoire de Laurent Schwartz: Mathématiques - Politique - Entomologie" at the École polytechnique at Palaiseau, France from July 1 to

4, 2003. Two days after Klaus Floret had passed away, Laurent Schwartz died on July 4, 2002, aged 87.

Countable inductive limits of Banach spaces with compact linking maps were investigated by Sebastião e Silva in 1953 and by Raikov in 1958/59. Examples of spaces of this type are spaces of germs of holomorphic functions, and they also appear in the context of co-echelon spaces. If one has a countable inductive spectrum of locally convex spaces, the inductive limit in the category of topological vector spaces coincides with the one in the category of locally convex spaces. In (DFS)-spaces, however, it also coincides with the inductive limit in the category of all topological spaces. (DFS)-spaces are complete, their strong duals are Fréchet-Schwartz spaces, abbreviated (FS)-spaces. Since (DFS)-spaces and (FS)-spaces are reflexive, a (DFS)-space is the strong dual of an (FS)-space, hence the term which is now standard.

In the literature, (DFS)-spaces $E = \operatorname{ind}_n E_n$ are always represented as inductive limits of increasing sequences of Banach spaces; i.e., the compact linking maps are injective. In his article [5], based on his doctoral dissertation, however, Klaus investigated, among other things, what happens when one omits the requirement that the linking maps are injective. But [5] contained much more material. E.g., as a corollary to a more general theorem, it was deduced that for an (FS)-space E and a (DFS)-space F the space $\mathcal{L}_b(F, E)$ of all continuous linear operators from E to F, equipped with the topology of uniform convergence on the bounded subsets of E, is again an (FS)-space. A larger class of countable inductive limits of Banach spaces, namely those with weakly compact linking maps, was also treated, and results of Komatsu, Makarov and de Wilde were recovered. In [5], *totally reflexive* Fréchet spaces (i.e., Fréchet spaces all quotients of which are reflexive) appeared, and it was proved that (FS)-spaces are totally reflexive. Later M. Valdivia [Val2] characterized the totally reflexive Fréchet spaces as those in which the canonical countable projective spectrum of Banach spaces generating the Fréchet space has weakly compact linking maps.

In many respects, Klaus' article [5] was an inspiration for us when Reinhold Meise and I started studying weighted inductive limits of spaces of continuous or holomorphic functions, see [BM]. We used theorems and methods of [5] and applied them to the function spaces we were dealing with. When in [BM] we had proved the abstract result that for two (DFS)-spaces E and F the ε-product $E\varepsilon F$ is again a (DFS)-space, Klaus first believed that this result had already been shown in [5], but then thought about it for a moment and agreed that this had not been the case. – Incidentally, the research started with [BM] culminated in the joint paper [BMS] of Meise, Summers and myself; for a recent survey on the state of the art in this direction, see [BiLaubin].

It is known that, for an (LF)-space, complete $\Rightarrow$ quasicomplete $\Rightarrow$ sequentially complete $\Rightarrow$ *regular*; that is, for every bounded set B in the inductive limit $E = \operatorname{ind}_n E_n$ there is n such that B is contained and bounded in E_n. Grothendieck's problem whether regularity must imply completeness is still unsolved, even for *(LB)-spaces*; i.e., countable inductive limits of Banach spaces. – Somebody from the state of Washington in the U.S.A. has claimed at least three times to have solved this problem, but his proofs contained severe gaps or were simply wrong. Because of this, Klaus wrote a letter to the editor of the journal in which these articles had been published in order to tell him that he should find better referees for his journal.

In the article [18], which also contained a survey on some interesting aspects of locally convex inductive limits like well-locatedness of subspaces (cf. [15]), Klaus at some point considered co-echelon spaces $k_0(V)$ of order 0. Köthe and Grothendieck had given examples of such spaces which were incomplete. In Klaus' article it was mentioned that E. Dubinsky had now even constructed a non-regular space of this type. Later on, in [BMS] and [BMSK], Meise, Summers and I characterized the situation as follows: $k_0(V)$ is regular $\Leftrightarrow$ $k_0(V)$ is complete $\Leftrightarrow$ V has a property which we call "regularly decreasing" $\Leftrightarrow$ $(k_0(V))'_b = \lambda_1(A)$ is quasinormable. (Part of this characterization is also due to Valdivia, independently.)

Klaus introduced *strong regularity conditions* for inductive limits into the literature. An inductive limit $E = \operatorname{ind}_n E_n$ is *sequentially retractive* if for each convergent sequence $(x_k)_k$ in E there is an index n such that $(x_k)_k$ is contained and convergent in E_n. With this notion, Klaus was able in [8], based on his Habilitationsschrift, to prove a variant of Grothendieck's famous factorization theorem. This theorem asserts that for a Fréchet space F and an (LF)-space $E = \operatorname{ind}_n E_n$, any continuous linear map from F into E factorizes continuously through some E_n. In Klaus' result, F can be taken to be a metrizable locally convex space and $E = \operatorname{ind}_n E_n$ is a sequentially retractive limit of arbitrary locally convex spaces.

Meise and I [BM] also considered variants of "sequentially retractive"; e.g., compactly regular and boundedly retractive. $E = \operatorname{ind}_n E_n$ is *boundedly retractive* if for each bounded subset B of E there is n such that B is contained in E_n and such that the topologies of E and E_n coincide on B. After some work of H. Neus and C. Fernández, J. Wengenroth [Wen] finally proved that in (LF)-spaces all the strong regularity conditions just mentioned coincide and that they are equivalent to Retakh's condition (M) (which is important for the topological subspace problem in inductive limits).

In [18] the new, and as it turned out, very important and useful class of locally convex spaces with the *countable neighborhood property* was introduced, and it was proved (the proof was attributed to J. Harksen) that

$$E \otimes_\alpha (\oplus_{n=1}^\infty F_n) = \oplus_{n=1}^\infty (E \otimes_\alpha F_n)$$

holds if E has the countable neighborhood property, F_n are arbitrary locally convex spaces, and α is an arbitrary tensornorm topology. Bonet [Bocnp] later showed that a locally convex space E has the countable neighborhood property if and only if

$$E \otimes_\pi (\oplus_{\mathbb{N}} \mathbb{K}) = \oplus_{\mathbb{N}} E.$$

In her thorough treatment of locally convex properties of spaces of continuous linear operators between locally convex spaces, S. Dierolf [Die] made extensive use of the countable neighborhood property (which will reappear in sections 3 and 4 below).

A problem which interested Klaus at that time was the interchange of inductive limits and tensor products equipped with some tensornorm topology. A variant of this problem for the ε-topology replaces the complete tensor product by Schwartz's ε-product. The following question was asked in [BM]: *If $E = \mathrm{ind}_n E_n$ is a (DFS)-space, does one have*

$$(\mathrm{ind}_n E_n)\, \varepsilon F = \mathrm{ind}_n (E_n \varepsilon F)$$

topologically for each Banach space F? One could also ask if this actually holds for each locally convex space F with the countable neighborhood property. For the complete π-tensor product instead of the ε-product such an interchange is true by Grothendieck's work, hence it holds in the above context if $E = \mathrm{ind}_n E_n$ is nuclear. Our philosophy at that time was that it might hold without the full assumption of nuclearity, only assuming that $E = \mathrm{ind}_n E_n$ is a (DFS)-space. However, we had forgotten that nuclearity does not only imply Schwartz, it also implies the approximation property.

The situation was cleared up completely many years later by Alfredo Peris [PerStu], [PerMA] (also cf. my survey [BiTrier]): In general the question has a negative answer, but if one assumes that the linking maps are not only compact, but even *approximable* (i.e., limits of continuous linear mappings of finite rank in the operator norm), which is the case if the inductive limit has the bounded approximation property, then the interchange of the inductive limit with the ε-product is indeed true. This leads us directly to the topic which we will treat in the next section.

Interlude

In the 1970s and early 1980s, Klaus was the master of the abstract theory of locally convex inductive limits worldwide; one could learn a lot from his papers on the subject. Then the main part of his research switched to other topics.

After two series of talks on locally convex inductive limits, weighted inductive limits and topological tensor products, which I gave at the University of Maryland at College Park during the spring semester of 1978 and at Universidade Federal do Rio de Janeiro in August-September 1980, I presented an introduction to inductive limits and their applications at the Autumn School on "Functional Analysis and its Applications" at the International Centre for Pure and Applied Mathematics (ICPAM) in Nice, France in September 1986. For an expanded version see [BiIL]. In the acknowledgments of this article, I wrote: "I am especially indebted to Klaus Floret for several helpful conversations relating to the selection of the material for these notes and other details."

3 Bases and approximation properties

The history of this topic started with the group around one of the "founding fathers" of functional analysis, *Stefan Banach*. His book "Théorie des opérations linéaires" [Ban] contained the fundamental theorems of functional analysis: the Open Mapping Theorem, the Closed Graph Theorem, the Uniform Boundedness Theorem, the Banach-Steinhaus Theorem and the Hahn-Banach Theorem. Banach had proved what is now called the Hahn-Banach Theorem and had deduced many interesting consequences and applications when he became aware that an analog had already been published before by H. Hahn in Crelle's journal. He immediately published a "reconnaissance du droit de l'auteur" in Studia Math.

Banach lived in Lvov or Lemberg in Poland; the city is now part of Ukraine and named Lviv. Banach and some of his colleagues used to meet in the Scottish Café for long discussions of mathematics. S. Ulam reported: "It was difficult to outlast or outdrink Banach during these sessions." The legendary Scottish Book was created in which all their mathematical problems were noted. Sometimes a prize was offered for the solution of a problem. Mazur's *"basis problem"* (number 153, dated November 6, 1936) asked if every separable Banach space had a (Schauder) basis; the prize to be given was the highest of all problems in the Scottish Book: a live goose.

Recall that a sequence $(x_n)_n$ in a locally convex space E is a *basis* if each $x \in E$ has a unique expansion $x = \sum_{n=1}^{\infty} a_n x_n$ with scalars a_n. The sequence is said to be a *Schauder basis* if all the linear coefficient functionals l_n, $l_n(x) := a_n$ in the above expansion, are continuous. It is well known that each basis in a Fréchet space is a Schauder basis.

In [4], Klaus Floret proved that also each basis in a sequentially retractive (LF)-space must be a Schauder basis; this result was improved by de Wilde. In [Ori], J. Orihuela showed a link between the Closed Graph Theorem and the Weak Basis Theorem ("every basis in the weak topology is a Schauder basis in the original topology of the space") and thus established some sort of "ultimate" theorem in this direction, generalizing all previous results.

In the first part of the 1950s, *Alexander Grothendieck* worked very hard on the famous *"approximation problem"* which is related to the basis problem: Does every locally convex space E have the approximation property? Remember that E is said to have the *approximation property (AP)* if the identity of E can be approximated by continuous linear operators of finite rank, uniformly on the compact subsets of E. A Banach space with a basis clearly has the approximation property.

Grothendieck's "thèse", see [Groth], treated the above approximation problem, investigated topological tensor products of locally convex spaces, posed a question called "problème des topologies" (see section 4 below) and introduced and thoroughly studied the important class of nuclear spaces, named after Schwartz's famous kernel theorem. Grothendieck was born in Berlin, his "thèse" was dedicated to his mother, in German: "Meiner Mutter, Hanka Grothendieck, in Verehrung und Dankbarkeit gewidmet." It appeared in the Memoirs of the Amer. Math. Soc. and brought the theory of locally convex spaces to a much higher level. But the approximation problem remained unsolved at that time.

Quite clearly, Alexander Grothendieck is the mathematician whose work influenced Klaus Floret most. Around 1955, Grothendieck left functional analysis and turned to algebraic geometry; for his work in that field he received a Fields medal.

I saw Grothendieck for the first time during the International Congress of Mathematicians in Nice, France in 1970. He wore shorts and sandals, but no socks, and preached for his "Action Survivre". He told the audience that, while he had been doing mathematical research, for many years he did not have time enough to read newspapers. Now he had studied the newspapers and found that something was wrong with the present world. Around this time, Grothendieck, together with a mathematical talk, often also gave a lecture along the lines of the "Action Survivre" concept. Much later, he refused to accept a prestigious prize of the Swedish Academy and in an accompanying letter accused today's mathematicians of stealing other people's results and of not giving proper credit. It is said that Grothendieck nowadays lives in a small village in the Pyrenees to which no roads or railroad tracks lead.

While he was still in Kiel, Klaus asked Defant and Gomoletz to compile all articles of Grothendieck on functional analysis (except the "thèse"), plus the corresponding reviews in Math. Reviews and Zbl. MATH, in a book; a few copies of this were distributed in 1982 to colleagues. Klaus also sent his joint paper [37] with Defant, which was some sort of preliminary version of the later monograph, to Grothendieck, and the master did send an answer. But later on, the copy of the monograph itself, posted to Grothendieck's address, was returned as undeliverable.

In 1972, Per Enflo [Enf] solved both Grothendieck's approximation problem and Mazur's basis problem by constructing a Banach space without the approximation property. During a major media event in Poland he received a live goose for his solution. The later development showed that for each sequence space l_p, $1 \leq p \leq \infty$, but $p \neq 2$, there exists a closed linear subspace without (AP). It still came as a surprise when A. Szankowski [Sza] was able to show that the space $\mathcal{L}(l_2, l_2)$ of all continuous linear operators on the separable Hilbert space l_2 does not have (AP), but also this "shocking" result was based essentially on the methods developed by Enflo.

The only different construction of Banach spaces without the approximation property is due to G. Pisier [PiP]. He solved another problem of Grothendieck by finding an infinite dimensional Banach space P for which the ε- and π-tensor topologies coincide on $P \otimes P$; such a space P, now called *Pisier space*, cannot have (AP). Recall that a locally convex space E is *nuclear* if for any locally convex space F the ε- and π-tensor topologies coincide on $E \otimes F$ and that the intersection of the class of nuclear spaces with the normed spaces is just the class of all finite dimensional spaces. – The Pisier space would become very important for the work of Klaus Floret, see section 5 below. For a recent survey on approximation properties in Banach spaces, see [CaAP].

A locally convex space E is said to have the *bounded approximation property (BAP)* if there exists an equicontinuous net of continuous linear operators of finite rank which converges pointwise (or, equivalently, uniformly on each compact subset) to the identity of E. Clearly, in a large class of locally convex spaces the existence of a Schauder basis implies (BAP), and (BAP) always implies (AP). Each nuclear space does have the approximation property, but Mityagin and Zobin [MZ] proved that there exist nuclear Fréchet spaces (for short: (FN)-spaces) without a basis. E. Dubinsky [Dub] exhibited an example of an (FN)-space without (BAP); the construction of such spaces was greatly simplified by D. Vogt [Vogt].

Recently, P. Domański and D. Vogt [DV] showed that the space of all real analytic functions on $\mathbb{R}^d$, $d \geq 1$, does not have a basis. This nuclear space is rather complicated topologically, but it is the only known concrete function space without basis (which was not constructed for this very purpose, but already existed in the literature for a long time). It is an open problem, however, whether the space of real analytic functions has (BAP). And it is still open whether the non-separable Banach space $H^\infty(D)$ of all bounded holomorphic functions on the unit disk under the sup-norm has (AP).

Let us now turn to the work of Klaus Floret in this direction. Motivated by applications in infinite dimensional holomorphy, L.A. de Moraes had asked the following question: *Let $E = \operatorname{ind}_n E_n$ be a strict inductive limit of locally convex spaces E_n with continuous norms. Does then E admit a continuous norm as well?*

Klaus had already written the joint paper [19] with S. Dierolf about the extendibility of continuous norms. With his work on the above question Klaus now struck some sort of "gold mine". He first showed in the article [25] that the question has a positive answer for countable direct sums and for inductive sequences with a partition of unity (in the sense of de Wilde). The answer is also positive if all E_n have the countable neighborhood property, or (essentially a result of S. Dineen) if $E = \operatorname{ind} E_n$ is a strict inductive limit of Fréchet spaces which has an unconditional basis. However, the main result of [19] was a counterexample: There is an (FN)-space G with an increasing sequence $(E_n)_n$ of closed linear subspaces, each having a continuous norm, but such that the strict inductive limit $E = \operatorname{ind}_n E_n$ does not admit any continuous norm. Then the (FN)-spaces E_n are not countably normed in the sense of I.M. Gelfand and G.E. Shilov and do not have (BAP); moreover, E does not have a basis.

Concordance properties of norms like in the work of Gelfand and Shilov were the key to the counterexample. A. Pełczyński had observed that (BAP) of a separable metrizable space implies that the space is countably normed. Dubinsky had then given an example of an (FN)-space with continuous norms which is not countably normed, and hence does not have (BAP), see [Dub]. Vogt [Vogt] had also used concordance of norms in his much simpler construction of such a space.

On the other hand, V.B. Moscatelli [Mos] had constructed Fréchet spaces without continuous norms and without bases in a different way; constructions of this type are nowadays called "of Moscatelli type", and spaces constructed in this way can often be used as counterexamples. (Unfortunately, one cannot get counterexamples to Grothendieck's question whether a regular (LB)-space must be complete by use of constructions of Moscatelli type, as shown by J. Bonet and S. Dierolf [BoDi].)

In [29], Floret and Moscatelli combined their efforts. Using a suitably more general version of a lemma due to Dineen, they reproved that the "twisted" nuclear Fréchet spaces of [Mos] do not admit a basis. They also showed that there is a Fréchet space without an unconditional basis which is a strict projective limit of reflexive Banach spaces with unconditional bases. In [30], the authors proved that a twisted Fréchet space does not have an unconditional basis.

The joint article [39] of Defant and Floret is devoted to another variant of (AP). In his "thèse" Grothendieck had shown that a Banach space E has (AP) if and only if the canonical map $E\hat{\otimes}_\pi F \longrightarrow E\check{\otimes}_\varepsilon F$ is injective for all locally convex spaces F. In [39] it is proved that the same holds if E is a Fréchet space or a semi-Montel (gDF)-space or if $E = \operatorname{ind}_n E_n$ is an (LF)-space such that $\operatorname{ind}_n(E_n\hat{\otimes}_\pi F)$ is complete for each Banach space F. In these three cases, E also has (AP) if and only if the nuclear operators on E have a trace.

In the same article [39] the authors answered a question of R. Hollstein by proving that Hollstein's ε-spaces have (AP). The main tool was a representation of the topological dual of $\mathcal{L}(E,F)$ with the compact-open topology, by Radon-Nikodym techniques. There was an interesting connection with the duality of ε- and π-tensor products: If E is a Fréchet-Montel space with (AP) and F is a Fréchet space, then

$$E'_b\hat{\otimes}_\pi F'_b = (E\check{\otimes}_\varepsilon F)'$$

holds algebraically.

One of Buchwalter's dualities for tensor products had been reinterpreted by Reinhold Meise and myself [BM] as a duality between Schwartz's ε-product and the complete π-tensor product; in his second volume, Köthe [K] did the same for the other duality: For Fréchet spaces E and F one has

$$(E\varepsilon F)'_c = E'_c\hat{\otimes}_\pi F'_c, \quad (E\hat{\otimes}_\pi F)'_c = E'_c\varepsilon F'_c.$$

Here all the dual spaces are equipped with the topology of uniform convergence on the compact subsets. (Incidentally, the second of the dualities above was used in [BM] to give a simple proof of the fact that the ε-product of two (DFS)-spaces is again a (DFS)-space.) For more general results along these lines and the connection with the localization of compact sets in the ε-product, see the article [26] of Defant and Floret.

4 $\mathcal{L}_1$-, $\mathcal{L}_\infty$-spaces and the "problème des topologies"

For $1 \leq p \leq \infty$, the class of $\mathcal{L}_p$-spaces was introduced and studied by J. Lindenstrauss and A. Pełczyński [LiPe]. Such spaces may have a quite complicated global structure, but by definition their *local structure*; i.e., the structure of their finite dimensional subspaces, is (uniformly) close to the structure of finite dimensional l_p-spaces.

In [11], Klaus Floret refined a result of Stegall and Retherford by proving that a Banach space Z is a $\mathcal{L}_1$-space if and only if $Z \otimes_\pi F$ is a topological subspace of $Z \otimes_\pi E$ for each topological linear subspace F of an arbitrary locally convex space E. The main result of [11], however, was a theorem on liftings: A Banach space Z is a $\mathcal{L}_1$-space if and only if, for each Banach space E and each closed linear subspace F of E, any continuous linear operator from Z into the quotient E/F has a lifting from Z into the bidual E'' of E. The article [11] also contained some interesting remarks on liftings in locally convex spaces.

While, in the last section, we have reported on the solution of the basis and approximation problems and on Klaus' work related to bases and approximation properties, the rest of the present section is devoted to the solution of another problem of Grothendieck and to Klaus' contributions in this direction.

Grothendieck's equally famous *"problème des topologies"* asked: Let E and F be Fréchet spaces and let B be a bounded subset of $E \hat{\otimes}_\pi F$. Must then exist bounded sets $C \subset E$ and $D \subset F$ such that $B \subset \overline{\Gamma}(C \otimes D)$, where $\overline{\Gamma}$ denotes the closed absolutely convex hull? If this does hold for a fixed pair (E, F) of Fréchet spaces, it is now customary to say that (E, F) satisfies property (BB). In this terminology, Grothendieck [Groth] proved that (E, F) satisfies (BB) if one of the spaces is nuclear or a Köthe echelon space $\lambda_1(A)$.

The "problème des topologies" is closely related to two questions of Grothendieck on (DF)-spaces. Recall that the class of *(DF)-spaces* had been introduced by Grothendieck [GroDF]; it contains all strong duals of Fréchet spaces and all (LB)-spaces. Each (DF)-space (more generally, each (gDF)-space) has the countable neighborhood property. On the other hand, the strong dual of each (DF)-space is a Fréchet space.

Grothendieck asked: *(1) If E is a Fréchet space and G is a (DF)-space, must then the space $\mathcal{L}_b(E, G)$ of all continuous linear operators from E into G, equipped with the topology of uniform convergence on the bounded subsets of E, be a (DF)-space? (2) Suppose that G and H are (DF)-spaces. Must then $G \otimes_\varepsilon H$ also be a (DF)-space?*

In a certain sense these two questions are dual to the "problème des topologies": If (E, F) satisfies (BB), then $\mathcal{L}_b(E, F'_b) = (E \hat{\otimes}_\pi F)'_b$ holds; i.e., it is the strong dual of the Fréchet space $E\hat{\otimes}_\pi F$, hence a (DF)-space. On the other hand, one can show that if, for two separable Fréchet spaces E and F (or for two Fréchet spaces E and F one of which is separable and has (BAP)), $\mathcal{L}_b(E, F_b')$ is a (DF)-space, then (E, F) must satisfy (BB). For more information on the "problème des topologies", related problems etc., from a modern point of view, see section 6 of the recent survey [BiBoFS] on some aspects of the modern theory of Fréchet spaces. This survey is dedicated "to the memory of Professor Klaus Floret, our good friend".

The first counterexamples to the "problème des topologies" were given in 1985 by Jari Taskinen, see [T1], [T2], [T3] for more details. It may suffice to note here the following: There is a Fréchet-Montel space E such that both (E, E) and (E, l_2) do not satisfy (BB); none of the spaces $\mathcal{L}_b(E, E_b')$, $\mathcal{L}_b(E, l_2)$, $E_b' \otimes_\varepsilon E_b'$, and $E_b' \otimes_\varepsilon l_2$ is a (DF)-space.

Klaus found the work of Taskinen so interesting that he asked me if we could combine some funds of our two universities to invite Taskinen to both Oldenburg and Paderborn, and of course I agreed. However, such a visit turned out to be non-trivial; at that time Taskinen was not so easily persuaded to leave Finland. Only after he had had a positive experience when he visited Paris in order to give a talk in the seminar of Pisier, he agreed to come to Oldenburg and Paderborn.

The outcome of his visit to Oldenburg was the beautiful joint article [38] of A. Defant, K. Floret and J. Taskinen. Its main results were: (1) A Banach space F is a $\mathcal{L}_1$-space if and only if (F, E) satisfies (BB) for each Fréchet space E, and this holds if and only if $\mathcal{L}_b(F, G)$ is a (DF)-space for each reflexive (DF)-space G. (2) A Banach space F is a $\mathcal{L}_\infty$-space if and only if $E \otimes_\varepsilon F$ is a (DF)-space for each (DF)-space E, and this holds if and only if $E\check{\otimes}_\varepsilon F$ is a (DF)-space for each inductive limit E of reflexive Banach spaces with the approximation property. – The article also contained some permanence properties for "quasibarrelled" and "bornological" under (complete) ε-tensor products.

Bonet, Schmets and I in [BBS], an article dedicated to the memory of G. Köthe, used a variant of (1) above to show that there is a reflexive Fréchet space E such that the space $H^\infty(D, E_b') = \mathcal{L}_b(E, H^\infty(D))$ of all bounded holomorphic functions on the unit disk D with values in E_b' is not a (DF)-space.

New positive results on the permanence of the (DF)-property under tensor products equipped with tensor norms can also be found in the joint article [40] of Defant and Floret, as well dedicated to the memory of G. Köthe. – Incidentally, by now Jari Taskinen has visited Paderborn various times.

5 Tensor norms, operator ideals, spaces of polynomials

Contrary to Grothendieck's well known "thèse", his article "Résumé de la théorie métrique des produits tensoriels topologiques" [GroR] remained widely unnoticed for many years. One reason probably was that this article was published in a journal to which not many libraries would subscribe. Another, more mathematical reason maybe was that the article did not contain proofs – with the exception of the main theorem, which is nowadays called "Grothendieck's inequality".

The "Résumé" did not deal with locally convex spaces, but was restricted to the framework of Banach spaces. And apparently specialists in the geometry of Banach spaces (in contrast to those doing research in locally convex spaces) were rather reluctant to work with tensor products; they preferred the setting of spaces of continuous linear operators. Only when in 1968 Lindenstrauss and Pełczyński [LiPe] presented Grothendieck's "théorème fondamental de la théorie métrique des produits tensoriels" in form of an inequality, many people in Banach space theory became aware of Grothendieck's work in the "Résumé". Now ideas of Grothendieck also brought the geometry of Banach spaces to a higher level. In retrospect, one noted that Grothendieck had already developed much of a "local theory" of Banach spaces.

Around the time when [LiPe] was published, Albrecht Pietsch in Jena and his school started to develop the theory of operator ideals. The development culminated in Pietsch's book [PiOI], which was some sort of encyclopedia on the theory of operator ideals. Most researchers in Banach space theory now worked with operator ideals rather than with the still quite unpopular tensor products.

Only after Pisier's work [PiP] with the Pisier space P and after his research explained in the book [PiFac], it became clear that both aspects together, the one of tensor products and the one of operator ideals, would yield more insight and would enhance research. Andreas Defant and Klaus Floret set out to fill a gap in the literature and (after the preliminary version [37]) finally published their monograph "Tensor Norms and Operator Ideals" [C4]. It really was a masterpiece of enormous influence, widely used and quoted. Of course, the theory of tensor products of Banach spaces and bilinear forms were treated from scratch. All of the fourteen natural tensor norms of Grothendieck were investigated, all relevant results on minimal and maximal Banach operator ideals were given, the concept of type and cotype of a Banach space was considered. Grothendieck's inequality took a central place. The reviewer in Math. Reviews (A. Pietsch) wrote: "It is impossible to describe the full content of this well-written book, which will be a rich source for all researchers in the field."

One merit of the monograph also was that the authors treated the accessibility of tensor norms in such a transparent fashion that it was finally possible for Pisier to solve the last of Grothendieck's open questions from the "Résumé": He came up with examples of non-accessible tensor norms (or, equivalently, with examples of non-accessible maximal normed operator ideals); in these examples the Pisier space P played the fundamental role. This solution arrived at Oldenburg shortly before the monograph was finished, and, with it, it was possible to present a very satisfactory theory.

The monograph was published in the series North-Holland Math. Studies (continuation of the "Notas de Matemática"), edited by *Leopoldo Nachbin*. Nachbin was the leading figure of Brazilian functional analysis who in 1982 received the Houssay Prize in the Exact Sciences by the Organization of American States. He had worked, among other things, on order and topology, topological vector spaces, and ordered topological vector spaces. In an article on L^1-spaces, Grothendieck introduced a "propriété de Nachbin" for Banach spaces. According to the introduction of Grothendieck's "Résumé", he and Nachbin intended to write a monograph on locally convex spaces, including many aspects of the metric theory of tensor products, but, unfortunately, this never happened. For an extensive survey of Nachbin's life and work until the mid-1980s, see [Hor].

In the early 1960s Nachbin and his students had worked in approximation theory, on the so-called *"weighted approximation problem"*; i.e., on deriving generalizations of the Stone-Weierstrass Theorem in the framework of Nachbin's weighted spaces $CV_0(X)$ of continuous functions, where the system V of weights, satisfying certain assumptions which can be made without loss of generality, was usually called a "Nachbin family". S. Machado, J.B. Prolla and G. Zapata worked on in this direction after Nachbin had left the field (see below). – Let us note in passing that Silvio Machado (who, unfortunately, died rather young) proved an important quantitative version of Bishop's theorem of which T.J. Ransford later gave a simple proof.

In the second half of the 1960s, Nachbin's research switched from approximation to infinite dimensional holomorphy, see [Nach], where he became very influential internationally as one of the "founding fathers". He was the leader of a large school of people in holomorphy, among them in Brazil: Barroso, Matos, L.A. de Moraes, and J. Mujica. Since Klaus Floret went to Brazil so often, his research was more and more influenced by the mathematical atmosphere around Nachbin, and he started working in holomorphy. Following the trend in holomorphy at that time (and since then), he mainly dealt with multilinear mappings,

n-fold (symmetric) tensor products and polynomials on Banach spaces.

In the article [47] (which is dedicated to the memory of Nachbin), Floret and Matos proved a Khintchine inequality for the n-Rademacher functions of Aron and Globevnik, with constants which, for the first time, were independent of n. Then some applications to the theory of polynomials and holomorphic mappings were given.

There are two joint articles of Klaus with Raymundo Alencar, a cousin of his wife Andréa. In [48], they proved multilinear generalizations of results of Pełczyński and Pitt. From a general theorem with applications to polynomials, they derived that for $1 < p_i < \infty$, $i = 1, \ldots, N$, the following assertions are equivalent: (a) $\sum_{i=1}^{N} 1/p_i < 1/q$, (b) every continuous N-linear mapping from $l_{p_1} \times \cdots \times l_{p_N}$ into l_q is compact, (c) the Banach space of all continuous N-linear mappings from $l_{p_1} \times \cdots \times l_{p_N}$ into l_q is reflexive. Moreover, some results on type and cotype, rank and Banach-Saks type of Banach spaces were proved. In [49], the authors used the concepts and results of [48] to give several new results, and to simplify some known facts, involving Tsirelson's space T.

The last papers of Klaus dealt with two topics: ideals of polynomials and multilinear mappings between Banach spaces ([54], [56], [57], and [58]), and the theory of symmetric tensor products of normed spaces ([51], [53], and [55]). On [57], the reviewer in Zbl. MATH (I. Patyi) said: "The paper under review in Banach space theory studies composition ideals of homogeneous polynomials on Banach and quasi-Banach spaces, with minimal ideals in focus ... The paper is well readable and may serve as a building block in the emerging theory of polynomial composition ideals."

In [51], Ansemil and Floret presented an explicit formula for the n-fold symmetric tensor product of a finite direct sum of locally convex spaces. The reviewer of the article in Math. Reviews (J. Bonet) wrote: "The article also gives a very clear and compact introduction to symmetric tensor products and symmetric tensor topologies." On [53], the reviewer in Math. Reviews (R.M. Aron) said: "This extensive survey has two aims: (1) to introduce the basic algebraic theory of symmetric tensor products and the two extremal norms π_s and ε_s, and (2) to prepare a theory of s-tensor norms."

The main result of [55] showed that every s-tensor norm on an n-symmetric tensor product of normed spaces (for fixed n) is equivalent to the restriction to the symmetric tensor product of a tensor norm in the sense of Grothendieck on a "full" n-fold tensor product of normed spaces. As a consequence, a large part of the isomorphic theory of norms on symmetric tensor products can be deduced from the theory of "full" tensor norms. Among the references of [55], there is the following one: "K. Floret, The metric theory of symmetric tensor products of normed spaces, in preparation". As far as I know, this article was never completed.

Coda

Certain articles of Klaus Floret have not been mentioned here since they were outside of the four topics which I had selected. Among them is the "gem" [13] in which Klaus treats some aspects of the inductive topology on the dual of a locally convex space. The style of [13] is typical: He begins with an illustrative introduction/summary of the main results of the article. There are five paragraphs in clear logical order, usually starting with a definition or a short remark. Then a statement of the result is given, followed by a concise proof. – Also, Klaus' work on Grothendieck's precompactness lemma, see [20], [22], should at least be mentioned here. The reviewer of [22] in Math. Reviews (T. Terzioğlu) wrote: "Some elegant examples are given to demonstrate the applicability of this lemma." Finally, we quote what the reviewer in Math. Reviews (M.S. Ramanujan) said on [23]: "The author achieves a very simple construction of Fréchet-Montel spaces which are not Schwartz spaces."

Let me conclude with some general remarks. I hope that my account of part of the mathematical work of Klaus Floret has served to show, first, that Klaus contributed important notions, methods and results to the topics in functional analysis in which he worked. It should also have served to demonstrate, secondly, that in the parts of functional analysis in which Klaus worked, fine results were obtained in the last 30 years, and questions which had been open for quite a while were solved. In each case a question (like the approximation problem, the "problème des topologies") was solved, this was not the end of the story, but led to new, stronger research with further, even more interesting results. Still, there are very important open problems left, and it is a pity that Klaus can no longer discuss these problems with us. We all miss Klaus' view of mathematics, his clear and illuminating way of analyzing the situation. And I, personally, deeply miss Klaus' always benevolent and charming personality and his persistent friendship.

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[A2] Lokalkonvexe Sequenzen mit kompakten Abbildungen, Doctoral dissertation, University of Kiel, 88 p. (1969) (published in [3]–[5]).

[A3] Folgenretraktive lokalkonvexe Sequenzen, "Habilitationsschrift", University of Kiel, 59 p. (1971) (published in [7] and [8]).

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