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TOTAL SUBSPACES WITH LONG CHAINS OF NOWHERE NORMING WEAK* - SEQUENTIAL CLOSURES

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Abstract. *If a separable Banach space X is such that for some nonquasireflexive Banach space Y there exists a surjective strictly singular operator $T : X \to Y$ then for every countable ordinal α the dual of X contains a subspace whose weak* sequential closures of orders less than α are nowhere norming and whose weak* sequential closure of order $\alpha + 1$ coincides with X^*.*

Let X be a Banach space, X^* be its dual space. The closed unit ball and the unit sphere of X are denoted by $B(X)$ and $S(X)$ respectively. The term «operator» means a bounded linear operator.

Let us recall some definitions. A subspace M of X^* is said to be *total* if for every $0 \neq x \in X$ there is an $f \in M$ such that $f(x) \neq 0$. A subspace M of X^* is said to be *norming over a subspace* L of X if for some $c > 0$ we have

$$(\forall x \in L) \left(\sup_{f \in S(M)} |f(x)| \geq c\|x\| \right).$$

A subspace M of X^* is said to be *norming* if it is norming over X. If M is not norming over any infinite dimensional subspace of X then we shall say that M is *nowhere norming.*

The set of all limits of weak* convergent sequences in a subset M of X^* is called the *weak* sequential closure of* M and is denoted by $M_{(1)}$. If M is a subspace then $M_{(1)}$ is also a subspace.

This subspace need not be closed and all the more need not be weak* closed [6]. In this connection S. Banach introduced [3, p. 208, 213] the weak* sequential closures (S. Banach used the term «derivé faible») of other orders, including transfinite ones. For ordinal α the *weak* sequential closure of order* α of a subset M of X^* is the set $M_{(\alpha)} = \cup_{\beta<\alpha}(M_{(\beta)})_{(1)}$.

It should be noted that for separable X the notion of the weak* sequential closure of order α coincides with the notion of the derived set of order α considered in [1], [7], [8].

For the chain of the weak* sequential closures we have

$$M_{(1)} \subset M_{(2)} \subset \ldots \subset M_{(\alpha)} \subset M_{(\alpha+1)} \subset \ldots$$

If we have $M_{(\alpha)} = M_{(\alpha+1)}$ then all subsequent closures coincide with $M_{(\alpha)}$. The least ordinal α for which $M_{(\alpha)} = M_{(\alpha+1)}$ is called the *order* of M.

The present paper deals with one of the aspects of the following general prblem: how far from the norming subspaces can the total subspaces be and what is the structure of Banach spaces the duals of which contain such subspaces?

There are many works devoted to this problem (see [3, p. 208 - 215], papers [1, 4, 7, 8, 9, 10] and papers cited therein). We recall only the results which motivate us to carry out the present research.

1. There is a total subspace M of $l_\infty = (l_1)^*$ such that $M_{(n)}$ is nowhere norming for all $n \in \mathbf{N}$ [1].

2. Let X be a nonquasirelexive separable Banach space. Then, for every countable ordinal α, there is a total subspace of order $\alpha+1$ in X^* [9]. (V.B. Moscatelli [7], [8] obtained this result in the case when α is not greater than the first infinite ordinal. The explicit construction of [8] is useful in further investigation of such subspaces.)

3. Let X be a separable Banach space. Its dual contains a total nowhere norming subspace if and only if for some nonquasireflexive Banach space Y there exists a surjective strictly singular operator $T : X \to Y$ [10].

The main result of the present paper is the following.

Theorem. *Let X be a separable Banach space such that for some nonquasireflexive Banach space Y there exists a surjective strictly singular operator $T : X \to Y$. Then for every countable ordinal α there exists a subspace M of X^* such that $M_{(\alpha+1)} = X^*$ and for every ordinal $\beta < \alpha$ the subspace $M_{(\beta)} \subset X^*$ is nowhere norming.*

Let us introduce some notation. For a subset A of a Banach space X, lin(A) and cl(A) are, respectively, the linear span of A and the closure of A in the strong topology. By $w^* - \lim_{m\to\infty} x_m^*$ we denote the weak* limit of the sequence $\{x_m^*\}_{m=1}^\infty$ in the dual Banach space (if this limit exists). For a subset A of X^*, A^T is the set $\{x \in X : (\forall x^* \in A)\,(x^*(x) = 0)\}$.

Our sources for Banach space theory are [3], [12].

At first we shall prove the theorem in the case when α is a nonlimit ordinal. In order to do this we need the following result.

Lemma 1. *Let Y be a separable nonquasireflexive Banach space. Then for every countable ordinal γ there exist a subspace N of Y^* and a bounded sequence $\{h_n\}_{n=1}^\infty$ in Y^{**} such that:*

A. *If a weak* convergent sequence $\{x_m^*\}_{m=1}^\infty$ is contained in $N_{(\beta)}$ for some $\beta < \gamma$ and $x^* = w^* - \lim_{m\to\infty} x_m^*$ then*

$$h_n(x^*) = \lim_{m\to\infty} h_n(x_m^*). \tag{1}$$

B. *There exists a collection $\{x_{n,m}^*\}_{n=1}^\infty,\ {}_{m=1}^\infty$ of vectors in $N_{(\gamma)}$ such that for every $k, n \in \mathbf{N}$ we have*

$$w^* - \lim_{m\to\infty} x_{n,m}^* = 0; \tag{2}$$

$$(\forall m \in \mathbf{N})\,(h_k(x_{n,m}^*) = \delta_{k,n}). \tag{3}$$

At first we shall finish the proof of the theorem in the case of a nonlimit α with the help of lemma 1.

We apply lemma 1 to $\gamma = \alpha - 1$ (this ordinal is correctly defined since α is nonlimit). Let $N, \{h_n\}$ and $\{x^*_{n,m}\}$ be as in lemma 1. Let $\{s^*_k\}_{k=1}^{\infty}$ be a normalized sequence in X^* such that $\operatorname{lin}(\{s^*_k\})$ is a norming subspace of X^*. Let $c_1 = \sup_n \|h_n\|$. Let $\nu_n > 0$ $(n \in \mathbf{N})$ be such that $\sum_{n=1}^{\infty} \nu_n \leq 1/(2c_1)$. We may assume without loss of generality that T is a quotient mapping. In this case $T^* : Y^* \to X^*$ is an isometric embedding. Therefore we may (and shall) identify Y^* with $T^*(Y^*) \subset X^*$.

Let $R : Y^* \to X^*$ be given by the equality

$$R(y^*) = y^* + \sum_{n=1}^{\infty} \nu_n h_n(y^*) s^*_n. \tag{4}$$

It is clear that

$$(\forall y^* \in Y^*)\,((1/2)\|y^*\| \leq \|Ry^*\| \leq (3/2)\|y^*\|). \tag{5}$$

Let $M = R(N)$. We prove that for every $\beta \leq \gamma$ we have

$$M_{(\beta)} = R(N_{(\beta)}). \tag{6}$$

We use transfinite induction. For $\beta = 0$ we have (6) by definition. Let us suppose that (6) is true for some $\beta < \gamma$ and prove that $M_{(\beta+1)} = R(N_{(\beta+1)})$. Let $x^* \in M_{(\beta+1)}$, i. e. $x^* = w^* - \lim_{m\to\infty} x^*_m$ for some sequence $\{x^*_m\}_{m=1}^{\infty}$ in $M_{(\beta)}$. Let $y^*_m \in N_{(\beta)}$ be such that $x^*_m = R(y^*_m)$. Denote $\sup_m \|x^*_m\|$ by c_2. By (5) we have $\|y^*_m\| \leq 2c_2$ for every $m \in \mathbf{N}$. Therefore by the separability of X we can select a weak* convergent subsequence $\{y^*_{m(i)}\}_{i=1}^{\infty}$ of $\{y^*_m\}_{m=1}^{\infty}$. Let $y^* = w^* - \lim_{i\to\infty} y^*_{m(i)}$. It is clear that $y^* \in N_{(\beta+1)}$. Since $\beta < \gamma$ then by the assertion A of lemma 1 we have

$$\lim_{i\to\infty} h_n(y^*_{m(i)}) = h_n(y^*)$$

for every $n \in N$. By the definition of R it follows that

$$\lim_{i\to\infty} \sum_{n=1}^{\infty} \nu_n h_n(y^*_{m(i)}) s^*_n = \sum_{n=1}^{\infty} \nu_n h_n(y^*) s^*_n,$$

where the limit is taken in the strong topology. Therefore we have:

$$w^* - \lim_{i\to\infty} R(y^*_{m(i)}) = R(y^*).$$

Hence, $x^* = R(y^*)$ and $M_{(\beta+1)} \subset R(N_{(\beta+1)})$. The inclusion $R(N_{(\beta+1)}) \subset M_{(\beta+1)}$ follows immediately from (1), (4) and (6).

The case of a limit ordinal $\beta \leq \gamma$ is more simple:

$$M_{(\beta)} = \cup_{\tau<\beta} M_{(\tau)} = \cup_{\tau<\beta} R(N_{(\tau)}) = R(\cup_{\tau<\beta} N_{(\tau)}) = R(N_{(\beta)}).$$

Therefore formula (6) is proved. In particular, we have $M_{(\gamma)} = R(N_{(\gamma)})$. Let us show that this equality implies that $M_{(\gamma)}$ is nowhere norming.

Suppose that this is not the case. Let an infinite dimensional subspace L of X be such that $M_{(\gamma)}$ is norming over L.

Recall that if U and V are subspaces of a Banach space X then the number

$$\delta(U, V) = \inf\{||u - v|| : u \in S(U),\ \ v \in V\}$$

is called the *inclination* of U to V.

Since T is a strictly singular quotient mapping then X does not contain an infinite dimensional subspace with non-zero inclination to $\ker(T)$. Using well-known arguments (see [2], [5] or [11]) we can find a normalized sequence $\{z_i\}_{i=1}^{\infty}$ in L such that for some sequence $\{t_i\}_{i=1}^{\infty}$ in $\ker(T)$ we have $||z_i - t_i|| < 2^{-i}$ and, furthermore,

$$(\forall n \in \mathbb{N})\ (\lim_{i\to\infty} s^*_n(t_i) = 0). \tag{7}$$

Let $c > 0$ be such that

$$(\forall x \in L)\ (\exists f \in S(M_{(\gamma)}))\ (|f(x)| \geq c||x||).$$

In particular,

$$(\forall i \in \mathbb{N})\ (\exists f_i \in S(M_{(\gamma)}))\ (|f_i(z_i)| \geq c).$$

Since $M_{(\gamma)} = R(N_{(\gamma)})$ then we can find $y^*_i \in N_{(\gamma)}$ such that $f_i = R(y^*_i)$, i. e. $f_i = y^*_i + \sum_{n=1}^{\infty} \nu_n h_n(y^*_i)\, s^*_n$. By (5) we have $||y^*_i|| \leq 2$. Furthemore, we have

$$c \leq |f_i(z_i)| \leq |f_i(z_i - t_i)| + |f_i(t_i)| < 2^{-i} +$$
$$+ |\sum_{n=1}^{\infty} \nu_n h_n(y^*_i)\, s^*_n(t_i)| \leq 2^{-i} + c_1 ||y^*_i|| \sum_{n=1}^{\infty} \nu_n |s^*_n(t_i)|.$$

Using (7) and the boundedness of the sequences $\{s_n^*\}$ and $\{t_i\}$ we obtain a contradiction. Hence, the subspace $M_{(\gamma)}$ is nowhere norming. Thus we proved that M satisfies the second assertion of the theorem.

It remains to prove that $M_{(\gamma+2)} = X^*$.

Let $\{x_{n,m}^*\}_{n=1}^{\infty}{}_{,}{}_{m=1}^{\infty}$ be the collection whose existence is asserted in lemma 1. By (6) we have $R(x_{n,m}^*) \in M_{(\gamma)}$ for every $m, n \in \mathbf{N}$. By (3) we have $R(x_{n,m}^*) = x_{n,m}^* + \nu_n s_n^*$. Since the sequence $\{x_{n,m}^*\}_{m=1}^{\infty}$ is weak* null and $\nu_n \neq 0$, then we have $s_n^* \in M_{(\gamma+1)}$ for every $n \in \mathbf{N}$, therefore $\text{lin}(\{s_n^*\}) \subset M_{(\gamma+1)}$. Since the subspace $\text{lin}(\{s_n^*\})$ is norming then by [3, p. 213] we have $M_{(\gamma+2)} = X^*$. The theorem is proved in the case when α is nonlimit.

Proof of lemma 1. By [4, theorem 2] Y contains a bounded away from 0 basic sequence $\{z_n\}_{n=0}^{\infty}$ such that the set

$$\left\{\sum_{i=j}^{k} z_{i(i+1)/2+j}\right\}_{j=0,\,k=j}^{\infty\quad\infty}$$

is bounded. We may assume without loss of generality that $||z_i|| \leq 1$ for every $i \in \mathbf{N}$. Let $Z = \text{cl}(\text{lin}(\{z_n\}_{n=0}^{\infty}))$. It is easy to see that the following claims are true.

1. The space Z^{**} may be identified with the weak* closure of Z in Y^{**}.
2. Every weak* null sequence in Z^* has a weak* null sequence of extensions to Y.
3. If we denote the canonical embedding of Z into Y by ξ then for every ordinal α and every subspace N of Z^* we shall have

$$(\xi^*)^{-1}(N_{(\alpha)}) = ((\xi^*)^{-1}N)_{(\alpha)}.$$

(In this connection see lemma 1 in [9].)

4. If $z^{**} \in Z^{**}$ and $y^* \in Y^*$ then the value $z^{**}(y^*)$ depends only on the restriction of y^* to Z.

These claims imply that it is sufficient to prove lemma 1 with Y^* and Y^{**} replaced by Z^* and Z^{**} respectively.

Let us introduce some notation. We shall write z_i^j for $z_{(j+i-1)(j+i)/2+j} (j = 0, 1, 2, \ldots, i \in \mathbf{N})$, while biorthogonal functionals to the system will be denoted by $\tilde{z}_n$ or $\tilde{z}_i^j$. By the result of [4] cited above we have

$$\sup_{j,m} ||\sum_{i=1}^{m} z_i^j|| = c_1 < \infty.$$

Therefore for every $j = 0, 1, 2, \ldots$ the sequence $\left\{\sum_{i=1}^{m} z_i^j\right\}_{m=1}^{\infty}$ has at least one weak* limit point in Z^{**}. Let us choose one of these limit points and denote it by f_j. It is clear that $||f_j|| \leq c_1$.

We need the following result from [9].

Lemma 2. *For every vector $g_0 \in Z^{**}$ of the form $af_j + z_s^r (a > 0, r \neq j)$, every countable ordinal α and every infinite subset $A \subset \mathbf{N}$ such that $j, r \notin A$ there exists a countable subset $\Omega(g_0, \alpha, A)$ of Z^{**} such that:*

1. *For a subspace $K(g_0, \alpha, A)$ of Z^* defined by $K(g_0, \alpha, A) = (\Omega(g_0, \alpha, A))^T$ we have $(K(g_0, \alpha, A))_{(\alpha)} \subset \ker(g_0)$.*

2. *All vectors $h \in \Omega(g_0, \alpha, A)$ are of the form $h = a(h) f_{j(h)} + z_{s(h)}^{r(h)}$ with $j(h), r(h) \in A \cup \{j, r\}, a(h) > 0$ and for every $h \neq g_0$ from $\Omega(g_0, \alpha, A)$ we have $j(h) \neq r, r(h) \neq r$.*

3. *If we denote by $Q(b, g_0, \alpha, A)$ the intersection of the set*

$$(8) \qquad b\tilde{z}_s^r + u, \qquad \text{where} \qquad u \in \operatorname{lin}\left(\{\tilde{z}_k^t\}_{k=1, t \in A\cup\{j\}}^{\infty}\right)$$

with $K(g_0, \alpha, A)$ then the set $(Q(b, g_0, \alpha, A))_{(\alpha)}$ contains all vectors of the form (8) which are in $\ker(g_0)$.

Let us introduce the functionals $h_n \in Z^{**}$ by the equalities $h_n = f_{2n-1}$ $(n \in \mathbf{N})$ and the vectors $x_{n,m}^*$ by the equalities $x_{n,m}^* = \tilde{z}_m^{2n-1}$. It is clear that the vectors h_n and $x_{n,m}^*$ satisfy equalities (2) and (3).

Let $\{A_n\}_{n=0}^{\infty}$ be a partition of the set of even natural numbers into pairwise disjoint infinite sets. Let $\varepsilon_{n,k} > 0$ $(n, k \in \mathbf{N})$ be such that $\sum_{n,k=1}^{\infty} \varepsilon_{n,k} < \infty$. Define the family $\{g_{n,k}\}_{n,k=1}^{\infty}$ in the following way:

$$g_{n,k} = z_k^{2n-1} + \varepsilon_{n,k} f_{j(n,k)},$$

where the mapping $j : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ is such that $j(n,k) \in A_n$ and $j(n,k) \neq j(n,l)$ for $k \neq l$.

Let $\{D_{n,k}\}_{n,k=1}^{\infty}$ be a partition of A_0 into pairwise disjoint infinite sets.

The cases of limit and nonlimit γ will be treated separately.

Let γ be a nonlimit ordinal and let $\Omega(g_{n,k}, \gamma - 1, D_{n,k})$ be the sets whose existence is asserted in lemma 2. Let us define $N \subset Z^*$ by $N = (\cup_{n,k=1}^{\infty} \Omega(g_{n,k}, \gamma - 1, D_{n,k}))^T$. Let us show that N satisfies the conditions of lemma 1.

Let $\{x_m^*\}_{m=1}^{\infty}$ be a weak* convergent sequence in $N_{(\beta)} (\beta \leq \gamma - 1)$ and let $x^* = w^* - \lim_{m \to \infty} x_m^*$. By lemma 2 we have

$$(9) \qquad (\forall n, k \in \mathbf{N})\, (x_m^* \in \ker(g_{n,k}))$$

Since the sequence $\{z_n\}_{n=0}^{\infty}$ is a basis of Z, then its biorthogonal sequence $\{\tilde{z}_n\}_{n=0}^{\infty}$ is a w^*-Schauder basis of Z^* [12, p. 155]. (It means that every vector $z^* \in Z^*$ can be represented as $z^* = w^* - \lim_{k\to\infty} \sum_{n=0}^{k} a_n \tilde{z}_n$, where $a_n = z^*(z_n)$).

Using (9) we can estimate some of the coefficients of the weak* decompositions of the vectors $x_m^*(m \in \mathbb{N})$. Precisely, if we denote $\sup_m \|x_m^*\|$ by c_2, then we obtain

$$|x_m^*(z_k^{2n-1})| \leq \varepsilon_{n,k} c_1 c_2 .$$

Therefore x_m^* can be represented as $u_m^* + v_m^*$, where $u_m^* = \sum_{n,k}^{\infty} x_m^*(z_k^{2n-1}) \tilde{z}_k^{2n-1}$ (we note that this series converges unconditionally, therefore we need not indicate the order of summability), and

$$(10) \qquad (\forall n, k \in \mathbb{N})\ (v_m^*(z_k^{2n-1}) = 0)$$

Since weak* convergence implies coordinatewise convergence for w^*-Schauder bases, we can represent x^* in an analogous way, $x^* = u^* + v^*$.

We have

$$u^* = w^* - \lim_{m\to\infty} u_m^* .$$

By (10) and by the definition of h_n it follows that

$$(11) \qquad (\forall m, n \in \mathbb{N})\ (h_n(v_m^*) = 0);$$

$$(12) \qquad (\forall n \in \mathbb{N})\ (h_n(v^*) = 0).$$

Since the vectors $\{u_m^*\}_{m=1}^{\infty}$ and u^* are contained in the strongly compact set

$$C = \left\{ \sum_{n,k=1}^{\infty} a_{n,k} \tilde{z}_k^{2n-1} : |a_{n,k}| \leq \varepsilon_{n,k} c_1 c_2 \right\},$$

the weak* convergence of $\{u_m^*\}$ to u^* implies the weak convergence of $\{u_m^*\}$ to u^*. Therefore for every $n \in \mathbb{N}$ we have $\lim_{m\to\infty} h_n(u_m^*) = h_n(u^*)$. From here by (11) and (12) we obtain $\lim_{m\to\infty} h_n(x_m^*) = h_n(x^*)$. Thus the assertion **A** of lemma 1 is proved.

In order to prove the assertion **B** it is sufficient to check that $\tilde{z}_k^{2n-1} \in N_{(\gamma)}$. Let $\tilde{z}(t) = \tilde{z}_k^{2n-1} - \tilde{z}_t^{j(n,k)}/\varepsilon_{n,k} (t \in \mathbb{N})$. It is clear that $\tilde{z}(t) \in \ker(g_{n,k})$ and that $\tilde{z}_k^{2n-1} = w^* -$

$\lim_{t\to\infty}(\tilde{z}(t))$. Furthermore, the vectors $\tilde{z}(t)$ are of the form (8) with $b = 1, r = 2n - 1, s = k$ and $j = j(n,k)$. Therefore $\tilde{z}(t) \in (Q(1, g_{n,k}, \gamma - 1, D_{n,k}))_{(\gamma-1)}$ and $\tilde{z}_k^{2n-1} \in (Q(1, g_{n,k}, \gamma - 1, D_{n,k}))_{(\gamma)}$.

It remains to show that

$$(13) \qquad (\forall r, s \in \mathbb{N})\,(Q(1, g_{n,k}, \gamma - 1, D_{n,k}) \subset (\Omega(g_{r,s}, \gamma - 1, D_{r,s}))^T)$$

For $r = n, s = k$ this follows immediately from the definition of Q. Let $(r,s) \neq (n,k)$. Recall that every element of $\Omega(g_{r,s}, \gamma - 1, D_{r,s})$ is a weak* limit point of linear combinations of $z_s^{2r-1}, z_t^{j(r,s)} (t \in \mathbb{N})$ and $z_q^p (p \in D_{r,s}, q \in \mathbb{N})$ and that $Q(1, g_{n,k}, \gamma - 1, D_{n,k})$ consists of linear combinations of $\tilde{z}_k^{2n-1}, \tilde{z}_t^{j(n,k)}$ $(t \in \mathbb{N})$ and $\tilde{z}_q^p (p \in D_{n,k}, q \in \mathbb{N})$. Our construction is such that the sets $\{2n-1, j(n,k)\} \cup D_{n,k}$ and $\{2r-1, j(r,s)\} \cup D_{r,s}$ intersect if and only if $2n - 1 = 2r - 1$. Since in this case we have $s \neq k$, we obtain (13).

Thus we have finished the proof of lemma 1 in the case when γ is a nonlimit ordinal. Let us pass to the case when γ is a limit ordinal. Let $\{\gamma_n\}_{n=1}^{\infty}$ be an increasing sequence of ordinals for which $\gamma = \lim_{n\to\infty} \gamma_n$. Let us introduce the subspace $N \subset Z^*$ by the equality

$$N = (\cup_{n,k=1}^{\infty} \Omega(g_{n,k}, \gamma_{n+k}, D_{n,k}))^T.$$

We shall show that N satisfies all the conditions of lemma 1. Let $\{x_m^*\}_{m=1}^{\infty}$ be a weak* convergent sequence in $N_{(\beta)}$ with $\beta < \gamma$ and let $x^* = w^* - \lim_{m\to\infty} x_m^*$. Let $i \in \mathbb{N}$ be such that $\gamma_{i-1} < \beta \leq \gamma_i$ (we let $\gamma_0 = 0$). The definition of the sets $\Omega(g_{n,k}, \gamma_{n+k}, D_{n,k})$ implies that for those pairs (n,k) for which $n + k \geq i$ we have $x_m^* \in \ker(g_{n,k})$, and, consequently, we have

$$|x_m^*(z_k^{2n-1})| \leq \varepsilon_{n,k} c_1 c_2.$$

At the same time, since $\|z_k^{2n-1}\| \leq 1$, then

$$(\forall n, k \in \mathbb{N})\,(|x_m^*(z_k^{2n-1})| \leq c_2).$$

Therefore we may argue in the same way as in the first part of lemma 1 if we define the set C in the following way:

$$C = \left\{ \sum_{k=1}^{\infty} a_{n,k} \tilde{z}_k^{2n-1} : |a_{n,k}| \leq \varepsilon_{n,k} c_1 c_2 \text{ if } n + k \geq i \text{ and } |a_{n,k}| \leq c_2 \text{ if } n + k < i \right\}.$$

The proof of the lemma 1 is complete.

Thus in the case when α is a nonlimit ordinal the proof of the theorem is finished.

Let us describe the changes which should be made in the proof of the theorem in the case when α is a limit ordinal.

Let $\{\alpha_n\}_{n=1}^{\infty}$ be an increasing sequence of nonlimit ordinals such that $\alpha = \lim_{n\to\infty} \alpha_n$. Instead of lemma 1 we shall use the following result.

Lemma 3. *There exists a subspace N of Y^* and a bounded sequence $\{h_n\}_{n=1}^{\infty}$ in Y^{**} such that:*

A$^{\text{new}}$. *If a weak* convergent sequence $\{x_m^*\}_{m=1}^{\infty}$ is contained in $N_{(\beta)}$ for some $\beta < \alpha$ and $x^* = w^* - \lim_{m\to\infty} x_m^*$, then we have*

$$h_n(x^*) = \lim_{m\to\infty} h_n(x_m^*)$$

for those n for which $\beta < \alpha_n$.

B$^{\text{new}}$. *For every $n \in \mathbb{N}$ there exists a sequence $\{x_{n,m}^*\}_{m=1}^{\infty}$ in $N_{(\alpha_n)}$ such that the conditions (2) and (3) are satisfied.*

At first we finish the proof of the theorem with the help of lemma 3. Let $\{s_k^*\}_{k=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ be the same as in the first part of the theorem. Let $R : Y^* \to X^*$ be defined by equality (4). For every ordinal $\beta < \alpha$ we denote by $W(\beta)$ the set of natural numbers n for which $\alpha_n < \beta$. It is clear that $W(\beta)$ is a finite set.

Let us show that for every ordinal $\beta < \alpha$ we have

$$M_{(\beta)} = \text{lin}\,(R(N_{(\beta)}) \cup \{s_k^*\}_{k\in W(\beta)}). \tag{14}$$

We shall prove this with the aid of transfinite induction. For $\beta = 0$ (14) follows immediately from the definition. Let us suppose that (14) is valid for some $\beta < \alpha$ and prove the analogous inclusion for $\beta + 1$. Let $x^* \in M_{(\beta+1)}$, i. e. $x^* = w^* - \lim_{m\to\infty} x_m^*$, where $x_m^* \in M_{(\beta)} \subset \text{lin}\,(R(N_{(\beta)}) \cup \{s_k^*\}_{k\in W(\beta)})$. It is clear that the last space is a subspace of $\text{cl}\,(R(N_{(\beta)})) \oplus F$, where F is some subspace of $\text{lin}\,(\{s_k^*\}_{k\in W(\beta)})$. Therefore, we may write $x_m^* = u_m^* + v_m^*$, where $u_m^* \in \text{cl}\,(R(N_{(\beta)}))$ and $v_m^* \in F$. It is clear that the sequences $\{u_m^*\}_{m=1}^{\infty}$ and $\{v_m^*\}_{m=1}^{\infty}$ are bounded and that we may suppose without loss of generality that $u_m^* \in R(N_{(\beta)})$. Let $u_m^* = Ry_m^*$. By (5) the sequence $\{y_m^*\}$ is also bounded. So we can find a sequence $\{m(i)\}_{i=1}^{\infty}$ of natural numbers such that the sequences $\{v_{m(i)}^*\}_{i-1}^{\infty}$ and $\{y_{m(i)}^*\}_{i=1}^{\infty}$ are weak* convergent. Let v^* and y^* be the corresponding weak* limits. By lemma 3 we have $h_n(y^*) = \lim_{i\to\infty} h_n(y_{m(i)}^*)$ for all n for which $\alpha_n > \beta$. Therefore,

$$\lim_{i\to\infty} \sum_{n\notin W(\beta+1)} \nu_n h_n(y_{m(i)}^*) s_n^* = \sum_{n\notin W(\beta+1)} \nu_n h_n(y^*) s_n^*,$$

where the limit is taken in the strong topology.

Since the set $W(\beta+1)$ is finite, without loss of generality we may assume that the sequence $\left\{ \sum_{n \in W(\beta+1)} \nu_n h_n(y^*_{m(i)}) s^*_n \right\}_{i=1}^{\infty}$ is strongly convergent. Let $\sum_{n \in W(\beta+1)} a_n s^*_n$ be its limit. It follows that

$$x^* = R(y^*) - \sum_{n \in W(\beta+1)} \nu_n h_n(y^*) s^*_n + \sum_{n \in W(\beta+1)} a_n s^*_n + v^* \in \text{lin}\,(R(N_{(\beta+1)}) \cup \{s^*_n\}_{n \in W(\beta+1)}).$$

Therefore $M_{(\beta+1)} \subset \text{lin}\,(R(N_{(\beta+1)}) \cup \{s^*_n\}_{n \in W(\beta+1)})$. The converse inclusion follows immediately from $\mathbf{B}^{\text{new}}$ and the induction hypothesis.

If β is a limit ordinal and (14) is proved for all ordinals less than β, then (14) follows immediately from the following fact: $W(\beta) = W(\tau)$ for some $\tau < \beta$. So (14) is proved for all ordinals which are less than α.

It is not hard to check that the linear span of the union of a finite-dimensional and a nowhere norming subspaces is nowhere norming. Therefore, by (14) and the arguments of the first part of the theorem, $M_{(\beta)}$ is a nowhere norming subspace for every $\beta < \alpha$.

The proof of the equality $M_{(\alpha+1)} = X^*$ is the same as in the first part of the theorem.

Proof of Lemma 3. We repeat the arguments of the proof of lemma 1 up to the passage where A_0 was presented in the form $A_0 = \cup_{n,k=1}^{\infty} D_{n,k}$. Now we continue in the following way. Let $\{\alpha_n\}_{n=1}^{\infty}$ be the increasing sequence of nonlimit ordinals introduced above. Let $\Omega(g_{n,k}, \alpha_n - 1, D_{n,k})$ be the sets whose existence is guaranteed by lemma 2. Let us introduce the subspace $N \subset Z^*$ by the equality

$$N = (\cup_{n,k=1}^{\infty} \Omega(g_{n,k}, \alpha_n - 1, D_{n,k}))^T.$$

The assertion $\mathbf{A}^{\text{new}}$ of lemma 3 is proved in the same way as the assertion $\mathbf{A}$ of lemma 1. The only distinction is that we have relation (9) not for all natural n but only for $n \in \mathbb{N} \setminus W(\beta+1)$. This does not prevent us to finish the proof, since the assertion $\mathbf{A}^{\text{new}}$ concerns only those h_n for which $n \in \mathbb{N} \setminus W(\beta+1)$.

The proof of the assertion of $\mathbf{B}^{\text{new}}$ is the same as the proof of the assertion $\mathbf{B}$ of lemma 1.

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