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SYMMETRIES WHICH PRESERVE THE CHARACTERISTIC VECTOR FIELD OF K-CONTACT MANIFOLDS

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As is well-known locally symmetric K-contact manifolds are spaces of constant curvature ([13]). This means that having isometric local geodesic symmetries is a very strong restriction for K-contact manifolds. Thus other classes of isometries shall fit for contact geometry. For example, T. Takahashi [15] has introduced the notion of φ-geodesic symmetries on Sasaki manifolds and also on K-contact manifolds. Since then, manifolds with such isometries have been studied extensively.

In this paper we generalize the notion of φ-geodesic symmetries. Because we notice that our diffeomorphisms preserve the characteristic vector field ξ of K-contact manifolds, we call them symmetries which preserve the characteristic vector-field, or ξ-preserving symmetries. Our idea for a construction of such local diffeomorphisms on K-contact manifolds is the lifting of symmetries on almost Kähler manifolds through the local fibering $p : M \to M/\xi$ of K-contact manifolds.

After recalling elementary facts on contact geometry in Section 1, we devote Section 2 to our definition of symmetries which preserve the characteristic vector field. Also we construct such a family of symmetries, which is an example of local S-rotations around curves in the sense of L. Nicolodi and L. Vanhecke [11]. In Section 3 we give some examples of our symmetries.

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1. CONTACT GEOMETRY

A $(2n+1)$-dimensional manifold M is said to be an *almost contact manifold* if the structure group of its tangent bundle is reducible to $U(n) \times 1$. This is equivalent to the existence of a tensor field φ of type (1,1), a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1.1}$$

From these conditions we can deduce that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

Moreover, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

for all $X, Y \in \mathfrak{X}(M)$. Here $\mathfrak{X}(M)$ denotes the Lie algebra of all vector fields on M. Such a metric g is called a *compatible metric* of the almost contact manifold $(M; \varphi, \xi, \eta)$. With respect to this metric g, the 1-form η is metrically equivalent to ξ, that is, $\eta(X) = g(X,\xi)$ for all $X \in \mathfrak{X}(M)$. Hence ξ is a unit vector field with respect to g. A structure (φ,ξ,η,g) on M is called an *almost contact Riemannian structure* on M, and a manifold M together with these structure tensors is said to be an *almost contact Riemannian manifold.* Further, we can define a 2-form Φ on M by

$$\Phi(X,Y) = g(X,\varphi Y)$$

for all $X,Y \in \mathfrak{X}(M)$. The 2-form Φ is called the *fundamental 2-form* of $(M;\varphi,\xi,\eta,g)$. If Φ satisfies $\Phi = d\eta$, then M is said to be a *contact Riemannian manifold* with *contact Riemannian structure* (φ,ξ,η,g). On a contact Riemannian manifold $(M;\varphi,\xi,\eta,g)$, the 1-form η is called the *contact form* on M, and the vector field ξ is called the *characteristic vector field* on M with respect to the contact form η.

Next, if the characteristic vector field ξ of a contact Riemannian manifold M is a Killing vector field with respect to the metric g, then M is said to be a *K-contact manifold.* This is characterized by the condition

$$\nabla_X\xi = -\varphi X \tag{1.3}$$

for all $X \in \mathfrak{X}(M)$. Here ∇ denotes the Levi-Civita connection of (M,g). It is obvious that each integral curve of ξ is a unit speed geodesic on M.

The following lemma is a result of the *conservation lemma* in Riemannian geometry (cf. O'Neill [14], p. 252).

Lemma 1.A. *Let $(M;\varphi,\xi,\eta,g)$ be a K-contact manifold. Then, for any geodesic γ on M, the restriction $\xi|_\gamma$ is a Jacobi field and $g(\gamma',\xi)$ is constant along γ.*

Lemma 1.A implies that geodesics which are initially orthogonal to ξ remain orthogonal to ξ. Such a geodesic is called a *φ-geodesic.*

If an almost contact Riemannian manifold $(M;\varphi,\xi,g)$ satisfies

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X$$

for all $X,Y \in \mathfrak{X}(M)$, then $(M;\varphi,\xi,\eta,g)$ is said to be a *Sasaki manifold.* A Sasaki manifold is a K-contact manifold.

Finally, we recall the basic facts about the local Riemannian fibering of a contact manifold for later use. Let x be a point of a K-contact manifold $(M;\varphi,\xi,\eta,g)$. Then we can take a sufficiently small neighborhood U of x on which ξ is a regular vector field. Since the structure tensors φ,η and g are invariant under the action of the local 1-parameter group of transformations generated by ξ, the fibering

$$p : U \to \bar{U} = U/\xi \tag{1.4}$$

induces an almost Kähler structure on the base manifold $\bar{U}$. It is defined by

$$(1.5) \qquad J\bar{X} = p_*\varphi\bar{X}^*, \quad \bar{g}(\bar{X},\bar{Y}) \circ p = g(\bar{X}^*,\bar{Y}^*)$$

for all $\bar{X},\bar{Y} \in \mathfrak{X}(\bar{U})$, where $*$ indicates the horizontal lift of vector fields on $\bar{U}$ with respect to η. From now on, a bar(-) is used systematically to distinguish objects in $\bar{U}$ from the corresponding objects in U.

Sasaki manifolds are characterized by the local fibering as follows.

Proposition 1.B. *A K-contact manifold is a Sasaki manifold if and only if each base manifold of a local fibering (1.4) is a Kähler manifold.*

For more details see Blair [2] and Ogiue [12].

2. SYMMETRIES WHICH PRESERVE THE CHARACTERISTIC VECTOR FIELD

In this section we shall introduce the notion of ξ-preserving symmetries which are a natural generalization of the φ-geodesic symmetries in the sense of Takahashi [15].

Definition 2.1. *Let $(M;\varphi,\xi,\eta,g)$ be a K-contact manifold, x a point of M and α the geodesic on M with initial velocity ξ_x. A ξ-preserving symmetry s_x (with base point $x \in M$) is a local diffeomorphism such that*

(1) the map s_x fixes every point on α, that is, $s_x \circ \alpha = \alpha$;

(2) for each point y on α, s_x sends any φ-geodesic through y to a φ-geodesic through y;

(3) in a small neighborhood of x, the points on α are the only fixed points of s_x.

If there exists a least integer $k(\geq 2)$ such that $(s_x)^k$ is the identity map, then s_x is said to be of order k. The geodesic α with initial velocity ξ_x is called the axial geodesic of s_x (cf. lemma 1.A).

Remark. A φ-geodesic symmetry in the sense of Takahashi [15] is a ξ-preserving symmetry of order 2 in our definition.

For ξ-preserving symmetries of order 2, we have the following result.

Proposition 2.2. *Let (M,φ,ξ,η,g) be a K-contact manifold. Then the following two conditions are equivalent:*

(1) Each base manifold of a local fibering (1.4) is a locally symmetric Kähler manifold.

(2) There exists an isometric ξ-preserving symmetry of order 2 for every point $x \in M$.

In this case each ξ-preserving symmetry is an automorphism, and the structure (φ,ξ,η,g) is Sasakian.

Proof. The result follows immediately from Theorem 2.1 in [15], Theorem 7 in [4] and Definition 2.1. *q.e.d.*

Proposition 2.2 gives a characterization of isometric ξ-preserving symmetries of order 2 in terms of the local fibering. This proof is based on tensor calculations. In the rest of this section we consider ξ-preserving-symmetries of arbitrary order. First we discuss the existence of such symmetries on K-contact manifolds with local symmetries.

Theorem 2.3. *Let $(M;\varphi,\xi,\eta,g)$ be a K-contact manifold, x a point of M, U a sufficiently small normal neighborhood of x on which ξ is regular, $p : U \to \bar{U} = U/\xi$ the local fibering, and $(J,\bar{g})$ the almost Kähler structure of $\bar{U}$ induced by p. If there exists a family of isometric symmetries $\bar{s}_{\bar{x}}$ on $(\bar{U},\bar{g})$ with base point $\bar{x} = p(x)$, then there exists a family of ξ-preserving symmetries on (U,g). Moreover, if $\bar{s}_{\bar{x}}$ is of order $k \geq 2$, then s_x is also of order k.*

Proof. Let γ be any φ-geodesic from a point y on the axial geodesic α in U. Then $p\circ\gamma$ is a geodesic in $\bar{U}$. Hence $\bar{s}_{\bar{x}}\circ p\circ\gamma$ is a geodesic in $\bar{U}$. Let $s_x\circ\gamma$ be the horizontal lift of $\bar{s}_{\bar{x}}\circ p\circ\gamma$ starting at y. Then it is a φ-geodesic in U (cf. pp. 244-245 of Besse [1]).

The map s_x clearly fixes every point on α. Further, in U, the points on α are only fixed points of s_x because $\bar{x}$ is an isolated fixed point of $\bar{s}_{\bar{x}}$. It is clear that s_x is of order k if $\bar{s}_{\bar{x}}$ is of order k.

This completes the proof of Theorem 2.3.

Remark. The above theorem gives an example of local S-rotations around curves in the sense of Nicolodi-Vanhecke [11]. In fact, let $S(t)$ be a linear transformation on $T_{\alpha(t)}M$ given by

$$S(t)\xi = \xi, \quad S(t)\bar{X}^* = (\bar{S}\bar{X})^* \quad \text{for all} \quad \bar{X} \in T_{p\circ\alpha(t)}M,$$

where $\bar{S}$ is the tangent tensor of $\bar{s}_{p\circ\alpha(t)}$. Then, since the horizontal lift $(\bar{S}(p\circ\gamma)'(0))^*$ is the initial velocity of $s_x\circ\gamma$, the local diffeomorphism s_x satisfies the following defining condition of local S-rotations :

$$s_\alpha = \exp_\alpha \circ S \circ \exp_\alpha^{-1} .$$

Here $\exp_\alpha$ denotes the exponential map of the normal bundle of α.

Conversely, if there exists a ξ-preserving symmetry s_x with base point x then the local fibering $p : U \to \bar{U} = U/\xi$ induces a local symmetry $\bar{s}_{\bar{x}}$ with base point $\bar{x} = p(x)$.

Theorem 2.4. *Let $(M;\varphi,\xi,\eta,g)$ be a K-contact manifold, x a point of M, U a sufficiently small neighborhood of x such that $(U;\varphi,\xi,\eta,g)$ is a regular K-contact manifold, $p : U \to \bar{U} = U/\xi$ the local fibering, and $(J,\bar{g})$ the almost Kähler structure of $\bar{U}$ induced by p. If* *there exists a ξ-preserving isometric symmetry s_x with base point x then there exists a local isometric symmetry $\bar{s}_{\bar{x}}$ on $\bar{U}$ with base point $\bar{x} = p(x)$. Further, the order of $\bar{s}_{\bar{x}}$ is k if s_x is of order k.*

Proof. Since s_x is a fibre-preserving map, we can define a map $\bar{s}_{\bar{x}} : \bar{U} \to \bar{U}$ by

$$\bar{s}_{\bar{x}}(\bar{z}) = p \circ s_x(z), \tag{2.1}$$

where z is any point on the fibre $p^{-1}(\bar{z})$.

Let $\bar{X}_{\bar{z}}$ be a tangent vector to $\bar{U}$, and ζ the φ-geodesic in U with initial velocity $\bar{X}^*_z$. Then, clearly, $\bar{X}_{\bar{z}}$ is the initial velocity of a geodesic $\bar{\zeta} = p \circ \zeta$. Here we have

$$p_* s_{x*} \bar{X}^*_z = \bar{s}_{\bar{x}*} \bar{X}_{\bar{z}}. \tag{2.2}$$

In fact, by (2.1),

$$p_* s_{x*} \bar{X}^*_z = p_*(s_x \circ \zeta)'(0) = (p \circ s_x \circ \zeta)'(0)$$
$$= (\bar{s}_{\bar{x}} \circ p \circ \zeta)'(0) = (\bar{s}_{\bar{x}} \circ \bar{\zeta})'(0) = \bar{s}_{\bar{x}*} \bar{X}_{\bar{z}}.$$

Now, since s_x is a local isometry, s_x sends a φ-geodesic to a φ-geodesic because of the conservation lemma. Hence s_{x*} sends any horizontal vector field to a horizontal vector field. Thus (2.2) implies that $s_{x*} \bar{X}^*_z$ is the horizontal lift of $\bar{s}_{\bar{x}*} \bar{X}_{\bar{z}}$, that is,

$$s_{x*} \bar{X}^*_z = (\bar{s}_{\bar{x}*} \bar{X}_{\bar{z}})^*.$$

Hence $\bar{s}_{\bar{x}}$ is an isometry of $(\bar{U}, \bar{g})$ because s_x is an isometry of (U, g).

By the condition (3) of Definition 2.1, the points on α are the only fixed points of s_x in a small neighborhood of x. Hence $\bar{x}$ is a isolated fixed point of $\bar{s}_{\bar{x}}$ because the axial geodesic α is the fibre over $\bar{x}$.

Thus $\bar{s}_{\bar{x}}$ is a symmetry with base point $\bar{x}$.

It is clear that $\bar{s}_{\bar{x}}$ is of order $k (k \geq 2)$ if s_x is of order k.

This completes the proof of Theorem.

Corollary 2.5. *Under the assumptions in Theorem 2.4, if a ξ-preserving symmetry s_x is φ-preserving, then $\bar{s}_{\bar{x}}$ is a holomorphic isometry.*

Proof. Because of a lemma due to Tanno [16], the φ-preserving map s_x is an isometry. Hence, by Theorem 2.4, $\bar{s}_{\bar{x}}$ is an isometry. Moreover, using (1.5), we have

$$J \bar{s}_{\bar{x}*} \bar{X}_{\bar{z}} = J p_* s_{x*} \bar{X}^*_z = p_* \varphi s_{x*} \bar{X}^*_z = p_* s_{x*} \varphi \bar{X}^*_z \tag{2.3}$$

for all vectors $\bar{X}_{\bar{z}}$ at $\bar{z}$. Since $\varphi \bar{X}^*_z$ is a horizontal vector, there exists a φ-geodesic ζ with initial velocity $\varphi \bar{X}^*_z$. Hence

$$p_* s_{x*} \varphi \bar{X}^*_z = (p \circ s_x \circ \zeta)'(0) = (\bar{s}_{\bar{x}} \circ p \circ \zeta)'(0) = \bar{s}_{\bar{x}*} p_* \varphi \bar{X}^*_z = \bar{s}_{\bar{x}*} J \bar{X}_{\bar{z}} \tag{2.4}$$

for all vectors $\bar{X}_{\bar{z}}$. The equations (2.3) and (2.4) imply that $\bar{s}_{\bar{x}}$ is holomorphic. *q.e.d.*

3. EXAMPLES OF ξ-PRESERVING SYMMETRIES

We shall apply our construction in the previous section to some examples of symmetries on base manifolds.

Example 3.1. A *locally φ-symmetric space* may be defined as a K-contact manifold with ξ-preserving isometric symmetries of order 2. In this case the structure is Sasakian and all the ξ-preserving symmetries are automorphisms (by Proposition 2.2). Well-known examples of locally φ-symmetric spaces are Sasaki manifolds of constant φ-sectional curvature, the Heisenberg group, the universal covering space of $SL(2;\mathbf{R})$ and $SU(2)$ (Blair-Vanhecke [3]). See Takahashi [15] for other examples, also Kowalski-Węgrzynowski [9] and Jiménez-Kowalşki [7] on classifications of φ-symmetric spaces.

Example 3.2. Let s_x be a local diffeomorphism on a K-contact manifold defined by

$$s_x = \exp_x \circ(\varphi_x + \eta_x \otimes \xi_x) \circ \exp_x^{-1} .$$

Then it is a ξ-preserving symmetry of order 4. This is an analogue of a J-rotation on an almost Hermitian manifold in the sense of L. Nicolodi-L. Vanhecke [10]. The first author of this paper has investigated this symmetry in [5].

Example 3.3. Let $(\bar{M}, \bar{g}, \{\bar{s}_{\bar{x}}\})$ be the space of dimension 4 given by Example III.53 and also Theorem VI.3 in Kowalski [8]. The underling Riemannian manifold $(\bar{M}, \bar{g})$ is the number space $\mathbf{R}^4(x, y, u, v)$ with metric

$$\bar{g} = \left(-x + \sqrt{x^2 + y^2 + 1}\right) du^2 + \left(x + \sqrt{x^2 + y^2 + 1}\right) dv^2 - 2y du dv \vartheta + \\ + \lambda^2 \frac{(1 + y^2) dx^2 + (1 + x^2) dy^2 - 2xy dx dy}{x^2 + y^2 + 1},$$

where λ is a positive constant. A typical symmetry of order 3 at the origin $(0,0,0,0)$ is the transformation

$$u' = \cos \frac{2\pi}{3} u - \sin \frac{2\pi}{3} v, \quad v' = \sin \frac{4\pi}{3} u + \cos \frac{4\pi}{3} v,$$
$$x' = \cos \frac{2\pi}{3} x - \sin \frac{2\pi}{3} y, \quad y' = \sin \frac{4\pi}{3} x + \cos \frac{4\pi}{3} y.$$

The space with this structure is a 3-symmetric almost Kähler manifold.

Let $M = \mathbf{R}^5(x, y, u, v, t)$ be the Riemannian product of $\bar{M}$ and $\mathbf{R}(t)$. Then M has an almost contact structure whose characteristic vector field is a Killing vector field (see p. 35 of

[2] for details). Since, as seen easily, Theorem 2.3 is still valid for a pair of such manifolds M and $\bar{M}$, we have ξ-preserving symmetries of order 3. Because M is neither Sasakian nor co-sympletic, M does not admit isometric φ-geodesic symmetries (of order 2) (see Proposition 2.2 and also [3]).

Remark. We can see, in Inoguti [6], that there also exist ξ-preserving symmetries of order 3 on nearly cosymplectic manifolds.

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