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ORDER AND SYMMETRY OF SIMPLE GAMES

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SUMMARY. *The aim of the paper is to use some known results of the theory of boolean functions and of the theory of finite groups for the classification and construction of simple games. Simple games can be seen as monotone boolean functions. In the introductory part the properties defining Post's classes are translated to game-theoretical properties. The second part gives a complete analysis of the non-Postian game-theoretical classes of half-half games, dual-equivalent games, ordered and weighted majority games with respect to set-inclusion and intersection. Symmetric boolean functions and symmetric games can be classified according to their symmetry group. In the third part the conditions of the existence of multiple-transitive games that are not fully symmetric are discussed. The last part gives a construction for those exceptional games that are sharply multiple-transitive without being trivial. The paper closes with an explicit construction of the family of the 13 most complex games; the corresponding automorphism groups are Mathieu groups.*

1. NOTATION

Let me call (simple) **game** a monotonic non-constant boolean function v for some finite set $N = \{1, 2, \ldots, n\}$, i.e. $c : 2^N \to 2 := \{0, 1\}$, $v(0, \ldots, 0) = 0$, $v(1, \ldots, 1) = 1$ and for all subset S and T $S \leq T$ implies $v(S) \leq v(T)$.

Let us identify n-vectors and subsets of N; the subsets S, T of N are called «**coalitions**» elements of N / coordinates are called **players**.

Let V denote the set of all games.

A game can be (uniquely) represented by

– the set of «**winning coalitions**» $W = W(v) = v^{\leftarrow}(1)$;

– the set of «**minimal winning coalitions**» $M = M(v) = \{S \in W; T < S \to v(T) = 0\}$;

– the «**incidence matrix**» $X = X(v)$ with rows $S \in W$ ordered lexicographically/ according to their binary number;

– the «**minimal polynomial**» $p(x_1, \ldots, x_n) = \sum_{S \in M} \prod_{i \in S} x_i$ ($\sum$ for union, $\prod$ for intersection).

Example. The game maj is defined as follows

$$n = \# N = 3, v(S) = 1 \text{ iff } \# S \geq 2$$

Now $W = \{123, 12, 13, 23\}, M = \{12, 12, 23\}, p = x_1x_2 + x_1x_3 + x_2x_3$ and

$$x = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(We drop the brackets and commata for coalitions).

2. POST'S CLASSES

In 1941 Post classified all classes of boolean functions that are closed with respect to four basic operations. As the set of monotonic non-constant boolean functions, i.e. of games is closed, we can use the corresponding part of the classification as classification of games. Post uses the following operations:

a) **Permutations** of N induce isomorphic games $(\pi v)(S) = v(\pi S)$

b) **Adding a dummy** $(dv)(S_1, \ldots, S_{n+1}) = v(S_1, \ldots, S_n)$

c) **Aggregation** $(av)(S_1, \ldots, S_{n-1}) = v(S_1, \ldots, S_{n-1}, S_{n-1})$

d) **Composition** $v, v_1, \ldots, v_n$ are games with players sets $N, N_1, \ldots, N_n$

$v[v_1, \ldots, v_n]$ is the game with players set $\sum N_i$ (disjoint union) defined by $(v[v_1, \ldots, v_n])(S) = v(v_1(SN_1), \ldots, v_n(SN_n))$ (product $\approx$ intersection)

Operations a), b), d) are often discussed in game theory. Operation c) is also relevant for applications on committees when two parties join.

In the following parts permutations $\pi = (a_0, \ldots, a_r)$ are defined as usual ($\pi a_i = a_{i+1 \bmod r}$ and $\pi b = b$ for b not element of $\{a_i; i = 0, \ldots, r\}$).

Let us denote the **complementary set** of S by $\neg S$, and define the **dual game** $^*v$ of v by

$$(^*v)(S) = 1 - v(\neg S)$$

Lemma. *V is closed under operations a) - d) and under* *.

Easy calculations show that the generated boolean functions are monotone and not constant. Let K be a subset of V. «K» be the set of all games generated by repeated use of operations a) - d).

We define some more games by their incidence matrix:

- id $= (1)$
- et $= (1\ 1)$, vel $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- veto $= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Calculations: a et=id=a vel, a^2 veto= a et=id, a^2 maj=$a(12)d$ id = id

Theorem. *(Post 1941): Every closed class K = «K» is one of the following list*
- «id»
- P =«et» and its dual P_* = «vel»
- D =«maj»
- F^∞ =«veto» and its dual F_*^∞
- $F^k = \{v \in V;$ any k-set of W has nonempty intersection $\}$, $k = 2, 3, \ldots,$ and their duals F_*^k

Remarks. «id» is known as the class of «**dictator games**».

P is known as the set of «**unanimity games**», i.e. $v \in P$ fulfills $v = u_T, u_T(S) = 1$ iff $T \leq S$. $D = {}^*D$ is known as the set of «**constants-sum games**», i.e. $v \in D$ fulfills $v(-S) + v(S) = 1$. By definition of the dual game we can characterize D as «**selfdual games**», i.e. ${}^*v = v$.

F^∞ is known as «**veto games**», i.e. $v \in F^\infty$ fulfills $\cap\{S; S \in V\}$ not empty.

F^2 is known as «**superadditive games**», i.e. $v \in F^2$ fulfills $[ST = 0 \rightarrow v(S+T) \geq v(S) + v(T)]$.

I shall write $[p]$ for the set of all games **with property** p, sometimes I use [s.a.] instead of F^2 and [c.s.] instead of D.

Shapley 1962 contains a list of all simple games for $n \leq 4$ (up to an isomorphism and dropping dummies).

Example. Game (1) of the list. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = (1234)\,a(15432)\,\text{maj}[\text{id,id,veto}] \in F^2$

By multiple use of permutations, aggregation and composition we can generate every nondisjoint composition, for example game (1)= maj[[id$_1$, id$_2$,veto$_{314}$]] if you allow for the nondisjoint composition [[...]].

Proposition 2.1. $v \in$ [s.a.] *iff* $v \leq^* v$

This statement is a simple consequence of the definitions of * and s.a.; the above formula and its dual formula $v \in^*$[s.a.] iff ${}^*v \leq v$ can be seen as a weakening of the folling property stated above: $v \in$ [c.s.] iff ${}^*v = v$. Observe that [c.s.] = [s.a.]*[s.a.]; remember: product means intersection.

Post's Theorem and Sharpley's list can be summarized in the diagrams figure 1 and 2.

The numbers given in the following two diagrams mean the number of games in Shapley's list contained in the respective class but not in a lower one. Shapley listed games without dummies up to an isomorphism. Only three games are elements of

– ([s.a.] +*[s.a.]) for $n \leq 4$, namely

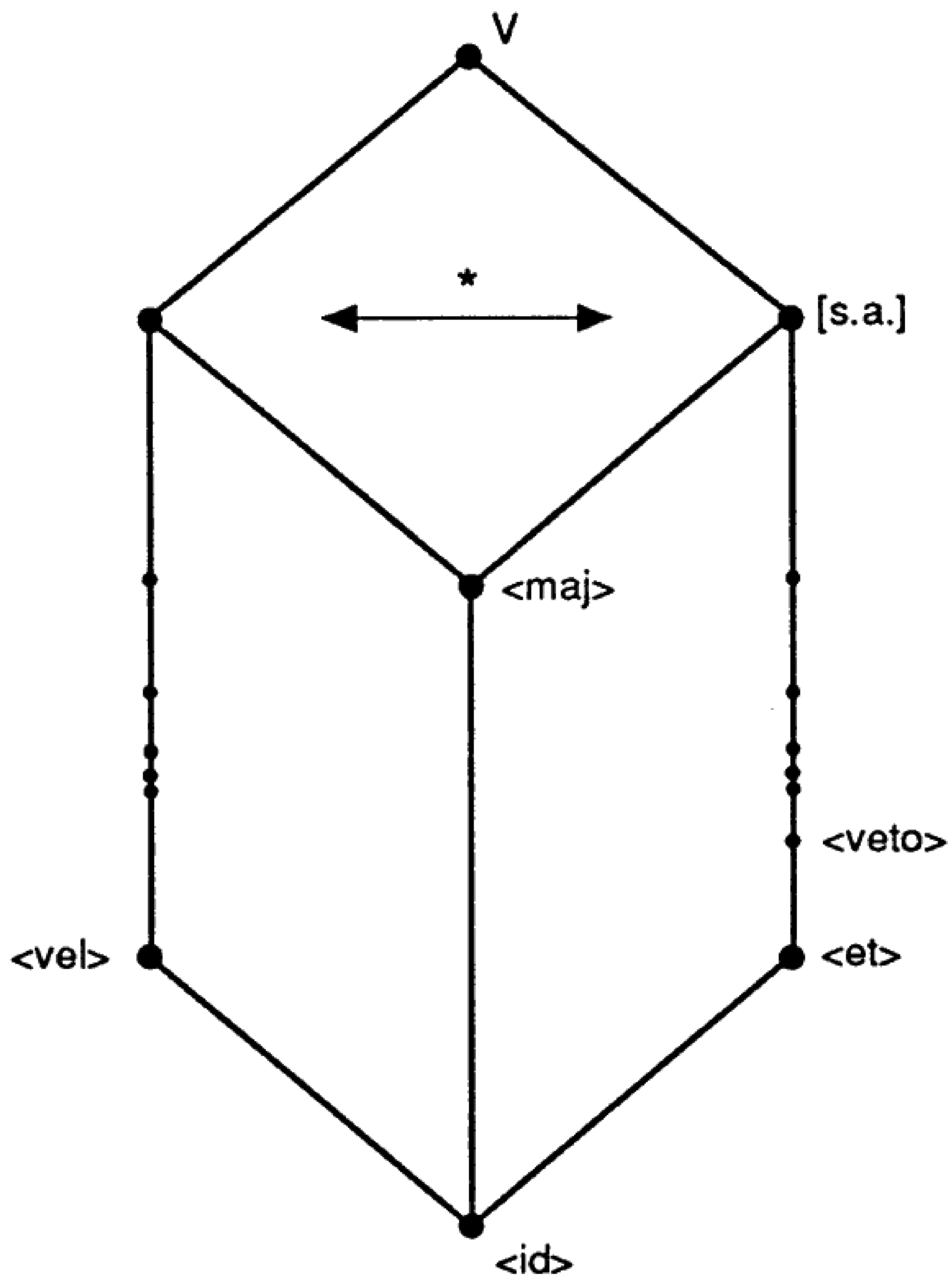

Figure 1

– game (o)= $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ = vel[et, et]

– its dual (* o)= $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ = et[vel,vel]

– and (p)= $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Observe: $^*(p)$ = (12)(34) (p), (p) is isomorphic to its dual.

Lemma 2.2. *The number of winning coalitions is equal to the number of coalitions not winning in the dual game, formally:* $\#W(v) = \#\neg W(^*v)$.

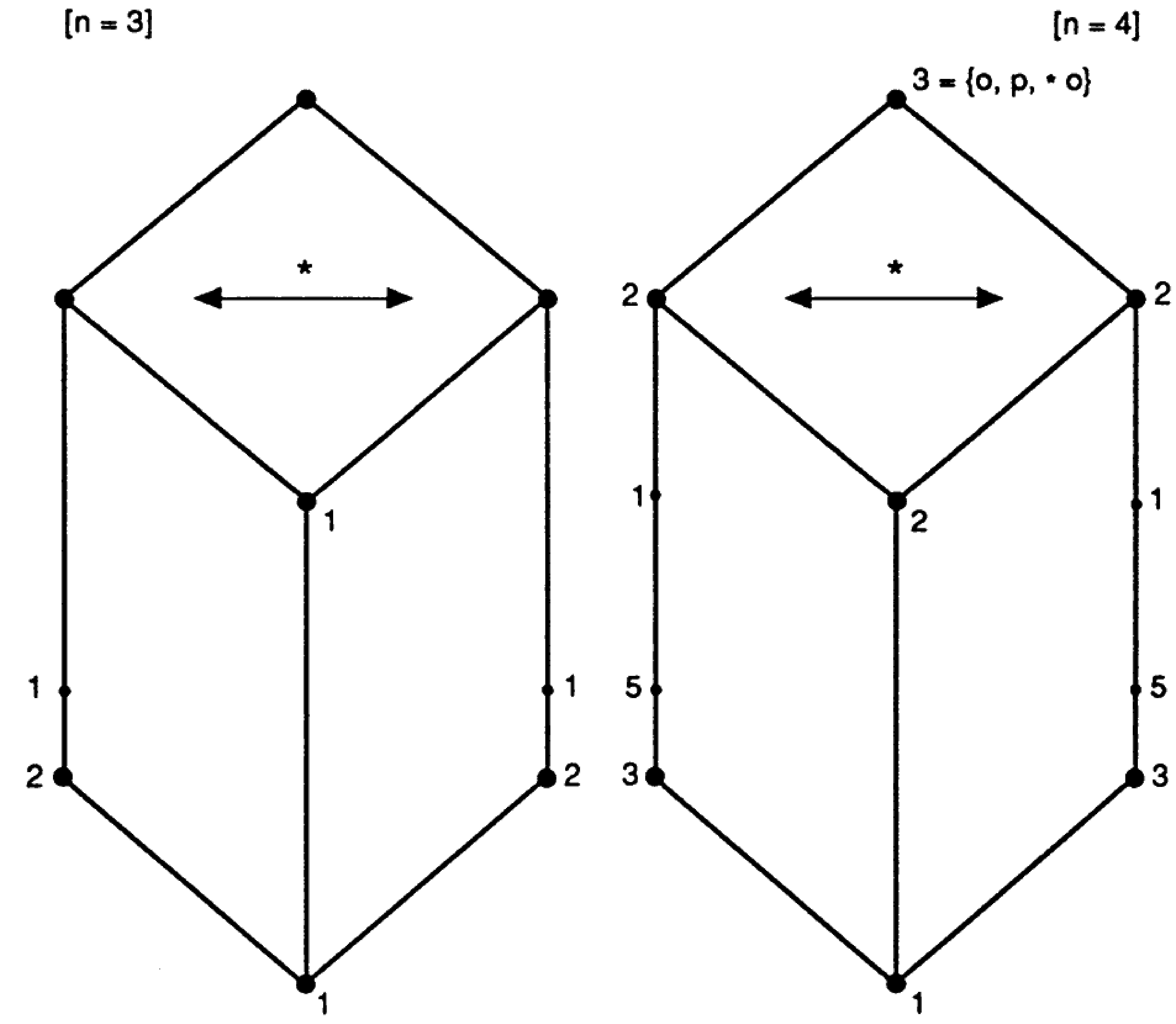

Figure 2

Games v that are isomorphic to their duals (i.e. there exists a permutation π of N such that ${}^*v = \pi v$) are called «**dual-equivalent**»; write d.e. and [d.e.]. Constant-sum games are d.e., but not all d.e. games are c.s., see game (p).

Games v that fulfill $\#W = \#\neg W$ are called «**half-half games**»; write h.h. and [h.h.]. Dual-equivalent games are half-half, but not all h.h. games are d.e. (an example will be given later, $n = 6$).

3. WEIGHTED MAJORITIES AND ORDERED GAMES

A weighted majority game v (write $v \in$ [w.m.]) is a game such that exist a measure m and a level μ such that

$$v(S) = 1 \text{ iff } m(S) \geq \mu$$

For $v \in$ [w.m.] we use the notation $v = (\mu; m_1, \ldots, m_n)$.

All games of the list except the three games (o), (${}^*o$), and (p) are w.m..

The notion of «ordered games» is based on the (following) **desirability relation** on the players set N:

$$i \succeq j \text{ iff } [j \in S \rightarrow v((ij)S) \geq v(S)]$$

Remember: (ij) is the permutation exchanging players i and j.

Let $i \succ j$ and $i \sim j$ be the asymmetric resp. the symmetric part of the relation.

The desirability relation is transitive but generally not complete (Maschler/Peleg 1966, Th. 9.2); thus define

$$i || j \text{ iff } [\text{not } i \succeq j] \text{ and } [\text{not } j \succeq i]$$

A game v is **ordered** (write $v \in$ [ord] iff $||$ is empty (desirability is complete).

Lemma 3.1. *[w.m.] is a proper subset of [ord].*

a) A player i with a higher weight as j is a substitute for j in any winning coalition,

b) up to $n = 5$ all ordered games are w.m., for $n = 6$ there are many ordered non-w.m. games (see Ostmann 1987); one of these is the game $e4k$ («the parents and there four children»: losing means «staying at home», winning means «travelling around»): $M(e4k) = \{$two parents and one child$\} + \{$one parent and two children$\}$.

For ordered games an easy **representation** is common use: players are numbered with decreasing desiderability (strongest first). $e4k$ is represented by <110001,010011> - this means that by shifting the two coalitions to the left - player by player - M is generated.

Theorem 3.2. *The only Post-classes fully contained in [w.m.] (resp. in [ord]) are the class of unanimity games, the class of their duals and the class of dictator games.*

It is enough to construct a game $w \in$ [c.s.]¬[ord], because et[id, w]∈ <veto> and its dual is element of *<veto>. Such a game is proj7, the game with 7 players and the 7 projective lines as minimal winning coalitions.

Proposition 3.3. *[w.m.] and [ord] are closed with respect to *, and [w.m.] is a subset of [s.a.]+*[s.a.].*

It is known that $^*(\mu; m) = (m(N) + 1 - \mu; m)$ and that $\succeq$ of v and $\succeq$ of $^*v$ are identical (cf. Ostmann 1985, 4.2 and 3.8).

Proposition 3.4. *[w.m.] is closed w.r.t. a, but [ord] is not; moreover [ord] is not a subset of [s.a.]+*[s.a.].*

– $m_n(av) = m_n(v) + m_{n+1}(v)$

– consider Aumann's game <10011001,01100110> and an aggregate player 34; in the new game we get 1|| 34; 110 0 110 is winning but 010 1 110 is not (i.e. non $3 \succeq 1$), 011 1 000 is winning but 111 0 000 is not (i.e. non $1 \succeq 3$).

Aumann's game, call it **au8**, is neither s.a. nor dual s.a.

Proposition 3.5. *[ord][c.s.] is not a subset of [w.m.].*

Two examples ($n = 13$), call them **os13**$_i$, were given in Ostmann 1985.

Lemma 3.6. *(for d.e. games): If* $^*v = \pi v$*, a player* i *and his image* πi *are either equivalent* $(i \sim \pi i)$ *or incomparable.*

Proof. $i \succeq j$ induces $\pi i \succeq \pi j$ $(i \in S, i \in \neg\pi S : v(\pi S) = (^*v)(S) \geq (^*v)((ij)S) = v(\pi(ij)S) = v((\pi i\ \pi j)\pi S))$ because the dual game has an identical desiderability relation. So $i \succeq \pi i \succeq \pi^2 i \succeq \ldots \succeq i$ and $i \sim \pi i$ or $i \| \pi i$ for all $i \in N$.

Proposition 3.7. *[h.h.][w.m.] and [d.e.][ord] are subsets of [c.s.].*

Proof, first part. According to Prop. 3.3 weighted majority games are either s.a. or dual s.a.; such games fulfill $v \leq {}^*v$ resp. $v \geq {}^*v$ (Prop. 2.1); Lemma 2.2 gives $\#W(v) = \#\neg W(^*v)$; since $v \in$ [h.h.], i.e. $\#W(v) = \#\neg W(v)$, we get $v = {}^*v$.

Second part. Let $^*v = \pi v$, $T = \pi S$; $v(T) + v(\neg T) = v(\pi S) + v(\pi\neg S) = (^*v)(S) - (^*v)(\neg S) = 2 - v(S) - v(\neg S)$; if $S = T$ (this includes the cases $S = \neg N$ or $S = N$) $v(S) + v(\neg S) = 1$; since all players are equivalent to their image we get $v(T) = v(S)$ and $v(\neg T) = v(\neg S)$. It follows that $v(S) + v(\neg S) = 1$, and we get $^*v = v$.

3.8 The following game is element of [h.h.][ord]¬ [c.s.]

– <101001,010110>

The game is ordered by construction. To count the winning coalitions, define $^* = \{0, 1\}$ and observe $W = 111^{***} + 1101^{**} + 11001^{*} + 110001 + 1011^{**} + 10101^{*} + 101001 + 10011^{*} + 0111^{**} + 01101^{*} + 01011^{*}$. Thus $\#W = 8 + 4 + 2 + 1 + 4 + 2 + 1 + 2 + 4 + 2 + 2 = 32$. But both 100110 and 011001 are winning, and the game is not constant sum. (All other six-person ordered h.h. not-c.s. games have more shift-minimal winning coalitions. These games are reported in Ostmann 1987).

3.9 The following game is element of [h.h.]¬ [ord]¬ [d.e.]

Define the **game ho** by use of the octahedron; the players are the six vertices, minimal winning coalitions have size three and M contains all non-faces minus two coalitions that form a partition of N.

With the conventional numbering $i + j = 7$ for an antipodal pair, we can get the following incidence matrices for ho and * ho:

110010	111000	
101100	110100	
101001	110001	← is not a face
100101	101101	← contains 4 players
100011	101010	
011100	100110	
011010	011001	
010110	010101	
010011	001110	← is not a face
001101	001011	
	000111	

The game ho is not dual equivalent (consider the incidence matrix). An easy calculation shows that ho and * ho are not ordered, but they are half-half.

We summarize the findings in the following diagram (the example games given in the diagram are games in the respective set but not in a lower/smaller one):

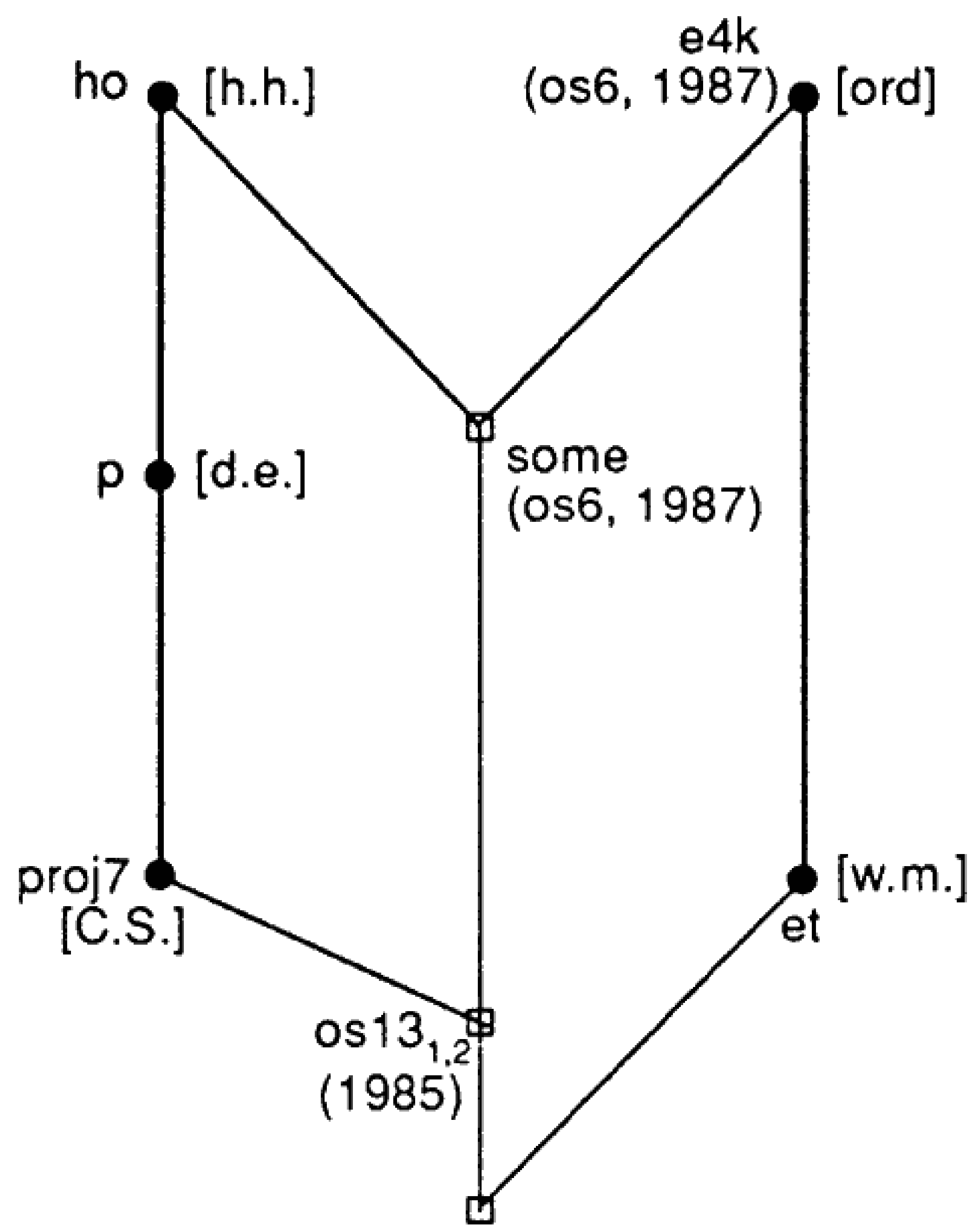

Figure 3

4. SYMMETRY

Let us define the full **automorphism group** of a game v by

$$\Gamma = \text{Aut } v := \{\gamma \in S_n ; \gamma v = v\}$$

S_n denotes the permutations of the player set N; Δ denotes a subgroup of Γ.

Orbits Γi of players $i \in N$ are called **types.** If $j \in \Gamma i$ then write $i \approx j$.

Lemma 4.1.

$$i \sim j \text{ induces } i \approx j$$

$$i \approx j \text{ induces } i \sim j \text{ or } i || j$$

$$i \sim j \text{ iff } (ij) \in \Gamma$$

Let $N^{(\tau)}$ the set of all τ-vectors of players corresponding to τ-sets (i.e. no two components are equal).

A permutation group $(\prod, X)$ is called transitive if $\prod x = X$ for some $x \in X$. A game is called τ- **transitive** if $(\Gamma, N^{(\tau)})$ is transitive. «Transitive» be short for 1-transitive.

Let us denote the corresponding classes of games by [t] resp. $[2t]$, $[3t]$,...,$[\tau t]$.

Observe: $[(\tau + 1)t]$ is a subset of $[\tau t]$.

Corollary 4.2. *Players in transitive games are either incomparable or equivalent.*
Proof. Let $\pi \in \Gamma$, then $i \succeq \pi i \succeq \pi^2 i \succeq \ldots \succeq i$ and $i \sim \pi i$ or $i || \pi i$ for all $i \in N$.

The games (o) and $({}^*o)$ are transitive. The 7 Pythagoraian games (N = vertices, M = faces of a P'n polyhedron; including the 5 Platonic games) are transitive but - except the tetrahedron game - not elements of $[2t]$. The game maj[maj,maj,maj] is element of $[t] \neg [2t] \neg$ [ord]. The game proj7 is 2-transitive and not ordered.

Theorem 4.3. $v \in [t][\text{ord}]$ *iff* $v = (\mu; 1, \ldots, 1)$.

Proof. Corollary 4.2 + ord: there is only one type. By lemma 4.1, for all i, j the permutation $(ij) \in \Gamma$, thus $\Gamma = S_n$ and v is a game with all sets with more elements than some number μ winning.

Let $B(i) := \{S \in B; i \in S\}$, $B(i,j) := \{S \in B; i; j \in S\}$, $B(i \neg j) := \{S \in B; i \in S, j \in \neg S\}$.

Define $i \gg j$ iff $W(j)$ is a proper subset of $W(i)$. Note that $i \gg j$ induces $i \succ j$.

Proposition 4.4. *If v is transitive the following holds for $B = M$ and $B = W$: $\# B(i) = \# B(j)$ and $\# B(i, \neg j) = \# B(j, \neg i)$.*

Proof. The first statement follows directly by transitivity. For the second statement consider the formulas $B(i, \neg j) + B(i, j) = B(i)$ and $B(j, \neg i) + B(i, j) = B(j)$.

Corollary 4.5. *If $M(i, \neg j)$ is empty and $v \in [t]$, then $i \sim j$.*

Proof by Prop. 4.4 and $W(i) = W(j)$. In this case M «does not separate» i and j.

Proposition 4.6. $v \in [t]$ *implies* **constant size of the equivalence classes** *of the desirability relation.*

Proof. For $\gamma \in \Gamma$: $\gamma\tilde{k} = \{\gamma i; i \sim k\} = \{j; i \sim \gamma k\} = (\gamma k)^{\sim}$ (because of: $i \sim k$ iff $\gamma i \sim \gamma k$)

Remember (4.3). $v \in [t]$ and one equivalence class is a w.m. game, namely $(\mu; 1, \ldots, 1)$. In this case we have $\Gamma = S_n$ and $v \in [nt]$.

Proposition 4.7. $v \in [2t]$ *implies one or n equivalence classes, for $v \in [2t] \neg [nt]$ all players are incomparable.*

Example: proj7.

Proposition 4.8. $v \in [2t]$ *implies for $B = W$ and $B = M$: $\# B(i, \neg j) =$ constant, and $\# B(i, j) =$ constant.*

Proposition 4.9. $v \in [2t] \neg \{u_N\}$ *implies* $\# M(i, \neg j) > 0$.

This mean all players are separated by M. It is well known that c.s. games exhibit the same property.

Proposition 4.10. *(Orbits of minimal winning coalitions): for $v \in [t] \neg \{u_N\}$ there is no fixed element of M (under the action of Γ on M); furthermore every orbit of some element of M contains all players.*

Let $\neg v$ the game with $M(\neg v) = \{S; \neg S \in M(v)\}$. Since $S \subset T$ iff $\neg T \subset \neg S$, the game is well-defined.

Proposition 4.11. *Aut* $^{*}v =$ *Aut* $v =$ *Aut* $\neg v$

Proof. For $\gamma \in \Gamma$: $v(S) = v(\gamma S), \gamma(\neg S) = \neg \gamma S$, insert into the definition of the dual game.

This proposition has the following simple consequence:

Corollary 4.12. $^{*}[\tau t] = [\tau t]$ *for all* τ

Proposition 4.13 (Principle of construction). *In order to construct a game $v \in [\tau t]$, take a τ-transitive permutation group (Δ, N) and a set A of coalition of N containing all players $i \leq \tau$ (call them base blocks). The game v with $W(v) = \{T; T \supset \gamma S$ for some $\gamma \in \Delta, S \in A\}$ fulfills $\Delta \subset$* Aut v *and* $v \in [\tau t]$.

Let us call the property $S, T \in M \to \# S = \# T$ **«constant block size»** (c.b.) and write $v \in [c.b.]$ for the corresponding games v. It is easy to see that $v \in [\tau t]$ is a union of c.b. games ($v = \sum v_i, v_i \in [c.b.][\tau t]$).

Proposition 4.14. *If $v \in [\tau t]$ and there exists a coalition $S \in M$ such that $\# S \leq \tau$ or $\# S \geq n - \tau$ then the game is weigthed majority.* **For $n \leq 2\tau + 1$ the set $[\tau t]\neg$[w.m.] is empty.**

Proof. Let $k := \# S$. Observe that for $k \leq \tau$ and for $k \geq n - \tau$ the orbit of S contains all coalitions of size k (τ-transitivity of the game). To find a coalition with a nontrivial orbit is necessary that $k > \varrho$ and $k < n - \tau$; thus $2\tau + 1 < n$.

Remarks. For $n \leq 2\tau + 1$ we found $[\tau t] = [nt]$. Remember that by Corollary 4.7 symmetry in the sense of 2-transitivity causes a game to be fully symmetric or «completely unordered» (= n equivalence classes of players). The smallest n for a game in $[2t]\neg$[w.m.] according to proposition 4.14 is 6. Indeed the condition is sharp and we can find such a game. The construction uses the permutation group (PSL(2,5),GF(5)+∞). Take the base block $S = \{0, 1, 4, \infty\}$. The set of of minimal winning coalitions is given by $M = \{\pi S; \pi \in PSL(2,5)\}$. The action of π is given by

$$\pi(i) = (ai + b)/(ci + d),$$

$ad - bc$ is a square, a, b, c, d are elements of the Galois field GF(5), calculations with ∞ as usual.

Ordering the players by $0, 1, 2, 3, 4, 5, \infty$ we can construct the following incidence matrix

$$X = \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{matrix}$$

The game is constant-sum.

5. ON BLOCK DESIGN GAMES AND SHARPLY τ-TRANSITIVE GAMES

Remark. Group theoretists say they know how to get all τ-transitive permutation groups (Δ, N) for $\tau > 1$; but construction is difficult. The construction is easy if Δ acts sharply τ-transitive.

We define: (Δ, N) is **sharply τ-transitive** iff the only element of Δ fixing one element of $N^{(\tau)}$ is the identity.

Let us call a game **sharply τ-transitive** if there is a subgroup Δ of Γ that acts sharply τ-transitive on N; write $v \in [sh - \tau t]$.

A **τ-design** (N, B) is an incidence structure with constant block size k such that the number of blocks that contain a τ-set of points for every choice of the τ-set is constant;

Formally: (0) $B \subset 2^N$, (1) $\# S =: k$ (for all $S \in B$), (2) $\# \{S \in B; Q \subset S\} =: \lambda$ for all τ-subsets Q of N.

A τ-design is denoted as a $S_\lambda(\tau, k; n)$. Let $r := \# \{S \in B; i \in B\} = \# B(i)$ («repetitions»), r is well-defined/independent of the choice of i.

Corollary. *For $v \in [\tau t]$ [c.b.] the game induces the τ-design (N, M).*

If there is lack of constant block size transitive games do not induce τ-designs.

On the other hand τ-designs (N, B) can lack of the «monotonicity» $S, T \in B \rightarrow$ non $(S \supset T)$ and even of transitivity.

Remark. [c.b.] and [c.b.][t] are not closed with respect to * (examples are the Platonic games). Furthermore: [c.b.] and [c.b.][t] are closed w.r.t. $\neg$.

If a τ-design (N, B) induces a game v directly, i.e. $M(v) = B$, then the so-called dual design, i.e. the design with the **transposed incidence matrix** also directly induces a game (otherwise the transposed incidence matrix X^T contains two rows $s < t$, so $M(s)$ is a proper subset of $M(t)$, but the number $\# M(.)$ has to be constant, because v is a design game).

Hoffman/Richardson show that the only 2-designs with $\lambda = 1$ and the transposed inducing a c.s. game are maj and proj7. They fulfill $X = X^T$.

Theorem 5.1. *A game $v \in [sh - \tau t]\neg[w.m.]$ has one of the following parameter pairs* (τ, n)

- $(1, n)$,
- $(2, p^r)$ or $(3, p^r + 1)$ for p^r being a prime power,
- $(4, 11)$ or $(5, 12)$.

For $n = 4, 5$ the symmetry groups are isomorphic to the small Mathieu groups M_{11} resp. to M_{12}.

The main part of this theorem is a well-known theorem on highly transitive permutation groups: a sharply τ-transitive group is either trivial or has the parameters given above; a proof of all its parts can be found in the book of Beth/ Jungnickel/Lenz: Design Theory 1985, part V. The crucial parts go back to Jordan 1873 and Zassenhaus 1935. The trivial sharply τ-transitive groups are $A_{\tau+2}, S_{\tau+1}, S_\tau$. It is easy to see that they generate w.m. games (S_n clear, for A_n use proposition 4.13; cf. Lapidot 1970). For $\tau = 1$ note that every group can be considered as acting sharply transitive on itself. The case of $\tau = 2$ and 3 are near analogues of the affine group $A\Gamma L(2, p^r)$ of a Galois field $GF(p^r)$, respectively of the projective group $PGL(2, p^r)$.

In the remaining part of this contribution all games for $\tau = 4$ and $\tau = 5$ are constructed.

Corollary 5.2. sharply 5 or 4-transitive games exhibit $\# S = 6$ *resp. 5 or 6.*

Proof follows from Corollary 4.14. $\tau < \# S < n - \tau$.

Lemma 5.3. *If the game* $v \in [sh - 6t]$ *then it can be constructed via* $(PSL(2, 11), GF(11) + \infty)$.

Proof. We observe that $(PSL(2, 11), N^{(6)})$ has only three orbits, namely the orbit of the squares $SQ =< 0\ 1\ 3\ 4\ 5\ 9 >$, the orbit of the non-zero squares $NZSQ =< \infty\ 1\ 3\ 4\ 5\ 9 >$ and the cyclic family $C =< \infty\ 0\ 1\ 2\ 3\ 4 >$. On $N^{(6)}$ the group M_{12} has the same orbits as $PSL(2, 11)$. According to $\# \Delta = \# \Delta_x \# \Delta x$ and $\Delta = PSL(2, 11)$, $\# \Delta = 11 \times 10 \times 6$, we get 132 coalitions in each of the squares families and 660 of the cyclic family/summing up to the total of 12 over 6 = 924 coalitions. The design $NSZQ$ is called the Witt design $S_1(5, 6; 12)$. In 1938 Witt sketched the uniqueness proof for this (and the $S_1(4, 5; 11)$ Witt design used below); a detailed proof was given by Lüneburg 1969 (cf. Beth/Jungnickel/Lenz). So we know that the games corresponding to SQ and $NZSQ$ are isomorphic. Call the corresponding games $\pi w11$ and $w11$.

Lemma 5.4. *If the game* $v \in [sh - 5t]$ *then it can be constructed via* $((M_{12})_\infty, GF(11) + \infty)$, $N = GF(11)$. *Each of* $((M_{12})_\infty, N^{(5)})$ *and* $((M_{12})_\infty, N^{(6)})$ *has three orbits induced by the above orbits* SQ, $NZSQ$ *and* C.

– The stabilizer $(M_{12})_\infty$ equals to M_{11}. Sets in $N^{(5)}$ are complementary to sets in $N^{(6)}$.
The following theorem gives **the 13 exceptional highly symmetric games.**

Theorem 5.3. $[sh - \tau t]\neg[w.m.]$ *contains only 13 elements for* $\tau \geq 4$, *namely (according to the number* k *of members of a minimal winning coalition:*

$k = 6$ – the game $w12$ with M being the unique $S_1(5, 6; 12)$

– the game $2w12$ corresponding to $SQ + NZSQ$

– the game $c12 =^* 2w12$ corresponding to C

– the game $w12 + c12 =^* w12$ corresponding to $NZSQ + C$

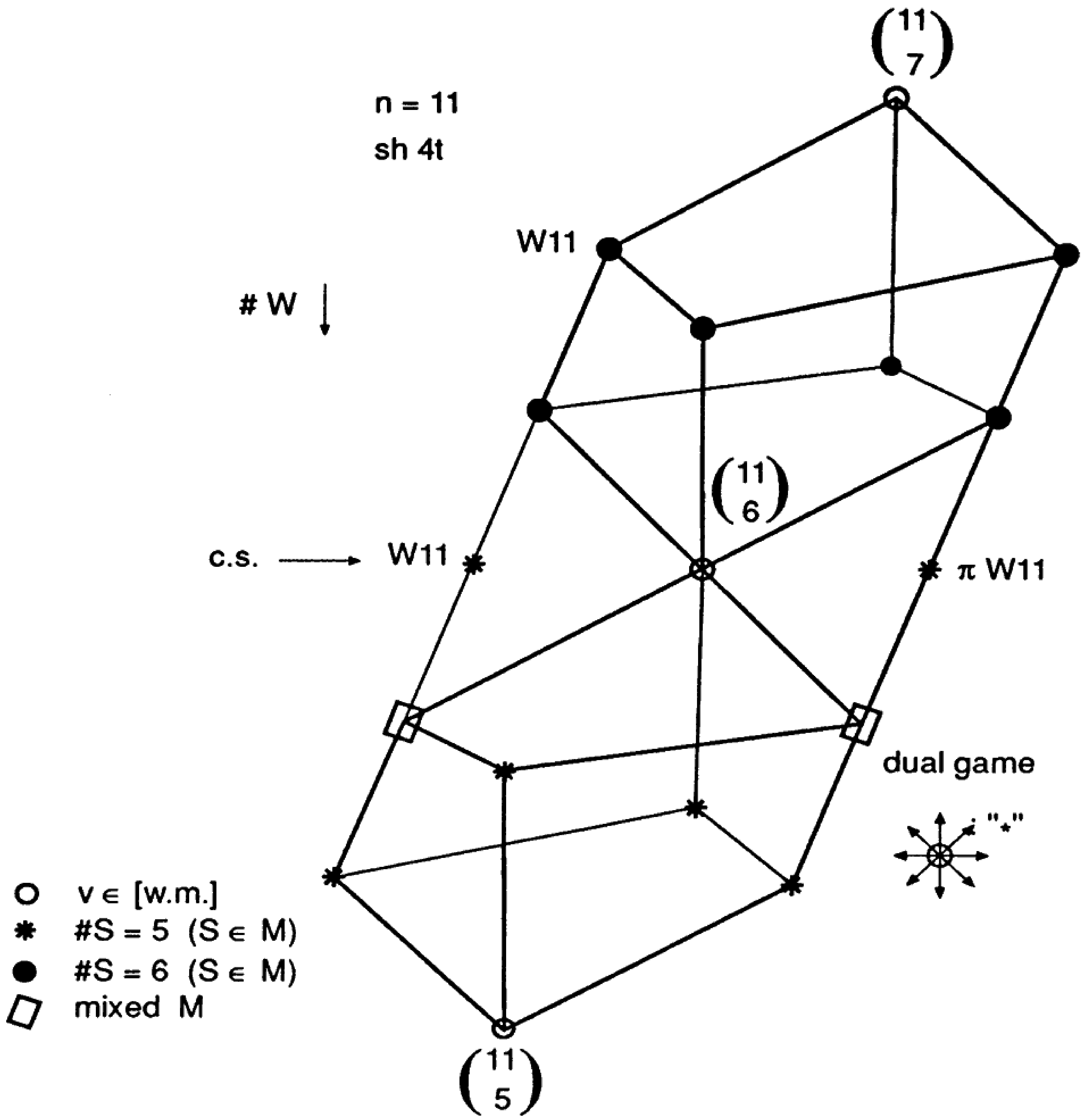

Figure 4

- the game $\neg w11$
- the game $\neg 2\,w11 =^{*} c11$
- the game $\neg c11$
- the game $\neg(c11 + w11)$

$k = 5$ – the game $w11$ with M being the unique $S_1(4, 5; 11)$.

- the game $2\,w11$ corresponding to $SQ + NZSQ$
- the game $c11$ corresponding to C

– the game $w11 + c11$

both– the only game not in [c.b.]: $w11 + \neg w11$

There are the following equations: $w12 = \neg w12, w12 +^* w12 = (6; 1, \ldots, 1)$, *and* $w11 =^* w11, \neg w11 +^* (w11 + \neg w11) = (6; 1, \ldots, 1)$.

The 11-player game $w11$ has 66 minimal winning coalitions and the large game $w12$ has 132 of them. Observe that the set W of winning coalitions does not contain all 6-resp. all 7-person coalitions. The mixed game $w11 + \neg w11$ contains all 6-person coalitions.

The following diagram gives the sublattice of the corresponding games for $n = 11$.

Concluding remark. For $n = 7$ there exists the first non-w.m. in $[sh - 2t]$. But these games are elements of [c.b.]. The smallest n I found for a game $v \in [sh - 2t] \neg$ [c.b.] is $n = 11$ ($\# S \in \{6, 7\}$ or $\{4, 5\}, \# M = 165$).

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APPENDIX

The following incidences matrices of the games w_{11} and w_{12} are computed in APL.

```
      W11
1 1 1 1 0 1 0 0 0 0 0
1 1 1 0 1 0 0 0 0 0 1
1 1 1 0 0 0 1 0 0 1 0
1 1 1 0 0 0 0 1 1 0 0
1 1 0 1 1 0 0 1 0 0 0
1 1 0 1 0 0 1 0 1 0 0
1 1 0 1 0 0 0 0 0 1 1
1 1 0 0 1 1 1 0 0 0 0
1 1 0 0 1 0 0 0 1 1 0
1 1 0 0 0 1 0 1 0 1 0
1 1 0 0 0 1 0 0 1 0 1
1 1 0 0 0 0 1 1 0 0 1
1 0 1 1 1 0 0 0 1 0 0
1 0 1 1 0 0 1 0 0 0 1
1 0 1 1 0 0 0 1 0 1 0
1 0 1 0 1 1 0 0 0 1 0
1 0 1 0 1 0 1 1 0 0 0
1 0 1 0 0 1 1 0 1 0 0
1 0 1 0 0 1 0 1 0 0 1
1 0 1 0 0 0 0 0 1 1 1
1 0 0 1 1 1 0 0 0 0 1
1 0 0 1 1 0 1 0 0 1 0
1 0 0 1 0 1 1 1 0 0 0
1 0 0 1 0 1 0 0 1 1 0
1 0 0 1 0 0 0 1 1 0 1
1 0 0 0 1 1 0 1 1 0 0
1 0 0 0 1 0 1 0 1 0 1
1 0 0 0 1 0 0 1 0 1 1
1 0 0 0 0 1 1 0 0 1 1
1 0 0 0 0 0 1 1 1 1 0
0 1 1 1 1 0 1 0 0 0 0
0 1 1 1 0 0 0 1 0 0 1
0 1 1 1 0 0 0 0 1 1 0
0 1 1 0 1 1 0 0 1 0 0
0 1 1 0 1 0 0 1 0 1 0
0 1 1 0 0 1 1 1 0 0 0
0 1 1 0 0 1 0 0 0 1 1
0 1 1 0 0 0 1 0 1 0 1
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0 0 1 1 1 1 0 1 0 0 0
0 0 1 1 1 0 0 0 0 1 1
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0 0 1 1 0 1 0 0 1 0 1
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0 0 1 0 1 1 1 0 0 0 1
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0 0 1 0 1 0 0 1 1 0 1
0 0 1 0 0 1 0 1 1 1 0
0 0 1 0 0 0 1 1 0 1 1
0 0 0 1 1 1 1 0 1 0 0
0 0 0 1 1 0 1 1 0 0 1
0 0 0 1 1 0 0 1 1 1 0
0 0 0 1 0 1 0 1 0 1 1
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0 0 0 0 1 1 1 1 0 1 0
0 0 0 0 1 1 0 0 1 1 1
0 0 0 0 0 1 1 1 1 0 1
```

```
      W12
1 1 1 1 1 0 0 0 0 1 0 0
1 1 1 1 0 1 0 0 0 0 0 1
1 1 1 1 0 0 1 1 0 0 0 0
1 1 1 1 0 0 0 0 1 0 1 0
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1 1 1 0 1 0 1 0 1 0 0 0
1 1 1 0 1 0 0 0 0 0 1 1
1 1 1 0 0 1 1 0 0 0 1 0
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1 1 1 0 0 0 1 0 0 1 0 1
1 1 1 0 0 0 0 1 1 0 0 1
1 1 1 0 0 0 0 1 0 1 1 0
1 1 0 1 1 1 0 0 1 0 0 0
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1 0 0 0 0 1 1 0 0 1 1 1
1 0 0 0 0 1 0 1 1 1 1 0
1 0 0 0 0 0 1 1 1 1 0 1
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