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# NOTE ON A PROBLEM IN GRAPH THEORY (*)

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**Abstract.** *In this paper, we give a solution for the last problem of [3].*

In his survey article [3], Nash Williams gives a list of unsolved problems. The last problem is the following. Let an $(n, \geq q)$-digraph denotes a digraph, without loops and parallel directed edges, on a set of $n$ vertices, such that the outdegree of every vertex is at least $q$. If $D$ is an $(m + n, \geq q + r)$-digraph, must there be some subdigraph of $D$, which is an $(m, \geq q)$ or an $(n, \geq r)$-digraph?

The following theorem shows that the answer is «No», even if we allow a somewhat weaker conclusion.

In the sequel, $\lceil x \rceil$ denotes the smallest integer greater than or equal to the real number $x$.

**Theorem.** *For every $k > 0$ and every prime $p \equiv 3$ (mod 4), that satisfies $p > k^2 \cdot 2^{2k-2}$, there exists a $(p, \geq (p-1)/2))$-digraph that contains neither $(k, \geq \lceil k/2 \rceil)$ nor $(p-k, \geq (p-1)/2 - k + 1)$-subdigraphs.*

*Proof.* Let $T = T_p = (V, E)$ denote the tournament whose vertices are the elements of $\mathbb{Z}_p$, where $(i, i+s)$ is a directed edge if and only if $s$ is a quadratic residue $(\neq 0)$ modulo $p$. Clearly, $T$ is a $(p, \geq (p-1)/2))$-digraph. Every subdigraph of $T$, on $k$ vertices, has exactly $\binom{k}{2}$ edges and, therefore, cannot be a $(k, \geq \lceil k/2 \rceil)$-digraph. Consider a subdigraph $D$ of $T$, on a set $W$ of $p - k$ vertices. By a theorem of Graham and Spencer [1], there exists some $v \in W$ that dominates all vertices of $V - W$ and, thus, in $D$ the outdegree of $v$ is $(p-1)/2 - k < (p-1)/2 - k + 1$. □

The assumption $p > k^2 \cdot 2^{2k-2}$ is not the best possible. Indeed, for $k = 2$, we can take $p = 7$. The tournament $T_7$ is a $(7, \geq 3)$-digraph, with neither $(2, \geq 1)$-nor $(5, \geq 2)$-subdigraph.

It is worth noting that using Difference Sets, we can construct many examples of $(n + 2, \geq q + 1)$-digraphs that are not tournaments and contain neither $(n, \geq q)$-nor $(2, \geq 1)$-subdigraphs. As an illustration, we use the difference set $A = \{2, 3, 8, 10, 20\}$, in $\mathbb{Z}_{21}$ (see [2, p. 131]), to construct a $(21, \geq 5)$-digraph, with neither $(19, \geq 4)$-nor $(2, \geq 1)$-subdigraphs. Let $D = (V, E)$ be the digraph whose vertices are the elements of $\mathbb{Z}_{21}$, where $(i, i+s)$ is a directed edge if and only if $s \in A$. Clearly, $D$ is a $(21, \geq 5)$-digraph and, since $A \cap (-A) = 0$ (in $\mathbb{Z}_{21}$), we conclude that it contains no $(2, \geq 1)$-subdigraph.

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Consider a subdigraph $H$ of $D$, on a set $W$ of 19 vertices, and put $V - W = \{i, j\}$. Since $A$ is a difference set, there exist $a, b \in A$, such that $a - b = i - j$, that is, $i - a = j - b$. Define $k = i - a = j - b$. By the definition of $D$, $(k, i)$ and $(k, j)$ are edges of $D$ and, thus, the outdegree of $k$, in $H$, is $3 < 4$.

REFERENCES

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