Note di Matematica Vol. 19 - n. 1, 65-70 (1999)

ON THE EUCLIDEAN GEOMETRY OF THE DOUBLE-EDGED RULER

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The aim of this note is to show that plane Euclidean geometry can be axiomatised by quantifier-free axioms in languages containing operations that correspond to the geometric constructions that can be performed by a double-edged ruler.

The double-edged ruler is an instrument which can be used in the following ways: (i) as a ruler (to draw the line joining two points, as well as to construct the point of intersection of two intersecting lines); (ii) to draw the two parallels to a given line that lie at unit distance from that line; (iii) to draw two parallel lines at unit distance apart, one of which passes through A, the other of which passes through B, whenever the distance between the points A and B, $d(A,B)$, is ≥ 1 (there are two such pairs of parallels whenever $d(A,B) > 1$).

The subject of quantifier-free (universal) axiomatisations of elementary geometry goes back to [4], where plane Euclidean geometry of ruler and segment-transporter constructions was axiomatised by universal axioms in a language containing two quaternary operation symbols and three individual constants, with 'points' as individual variables. It was further expanded in [2], [11], and [6]-[10].

The possibility of such axiomatisations is suggested by the well known fact (going back to Adler [1], see also [3]) that the set of points that can be constructed in the standard Euclidean plane by using a double-edged ruler in all three possible ways coincides with the set of points that can be constructed with ruler and compass. However, this result does not indicate if and how a quantifier-free axiom system may be set up, for results on the constructibility of points always assume the underlying geometry as 'given', whereas in a quantifier-free axiomatisation the geometry itself needs to be determined by axioms that refer to geometric constructions.

1 The axiomatisations

We first consider a language for constructions with a restricted double-edged ruler, which can be used only in ways (i) and (ii). Let **S** be a bi-sorted first order language **S**, whose two sorts of variables will be referred to as 'points' (upper case) and 'lines' (lower case), that contains: three individual constants A_0, A_1, A_2, standing for three non-collinear points; φ, a binary operation symbol with points (say A and B) as arguments and the line joining them, $\varphi(A,B)$, as value (whenever $A \neq B$, an arbitrary line, otherwise); ι, a binary operation symbol with lines (say g and h) as arguments and the intersection point of the lines g and h, $\iota(g,h)$, as value (whenever g and h are not parallel and not equal, an arbitrary point, otherwise); α_i (for $i = 1,2$), a unary operation symbol with a line (say l) as variable and the line $\alpha_i(l)$, such that $\{\alpha_1(l), \alpha_2(l)\}$ are the two lines that are parallel to l at unit distance from l, as value.[2]

[1]Partially supported by a TRIS grant from ASU West.
[2]For logical issues related to many-sorted languages see [5].

In [9] we have axiomatised Euclidean planes of characteristic $\neq 2$ in a bi-sorted first-order language $\mathbf{L}_1$, that contains $A_0, A_1, A_2, \varphi, \iota, \gamma$ as non-logical constants, with $\gamma(P,l)$ standing for the line perpendicular to l in P whenever $l = \varphi(P,Q)$ for some $Q \neq P$, arbitrary, otherwise. We shall prove that an operation equivalent to γ can be defined without using quantifiers by means of the primitive notions in $\mathbf{S}$.

We first introduce some abbreviations that will improve the readability of the axioms:
$a_{i,j} = \varphi(A_i, A_j)$,
$\lambda(A,B,C) :\leftrightarrow A = B \vee A = C \vee \varphi(A,B) = \varphi(A,C)$,
$\overline{\lambda}(A,B,C) :\leftrightarrow A \neq B \wedge B \neq C \wedge C \neq A \wedge \varphi(A,B) = \varphi(A,C)$,
and, for A,B,P with $\neg\lambda(A,B,P)$ and $i \in \{1,2\}$ (see Figures 1 and 2):
$\mu_i(A,B,P) := \iota(\varphi(A,B), \varphi(P, \iota(\varphi(A, \iota(\alpha_i(\varphi(A,B)), \varphi(B,P))), \varphi(B, \iota(\alpha_i(\varphi(A,B), \varphi(A,P)))))$,
$\delta_i(A,B,P) := \iota(\varphi(A,P), \alpha_i(\varphi(A,B))$,
$\varepsilon(A,B,P) := \iota(\varphi(B, \delta_1(A,B,P)), \alpha_2(\varphi(A,B)))$,
$\omega(A,B,P) := \iota(\varphi(A, \varepsilon(A,B,P)), \alpha_1(\varphi(A,B))$,
$\sigma(A,B,P) := \iota(\varphi(A,B), \varphi(\omega(A,B,P), \delta_2(A,B,P)))$.
The abbreviation $\lambda(A,B,C)$ may be read as 'A, B, C are collinear points', $\overline{\lambda}(A,B,C)$ as 'A,B,C are three different collinear points', $\mu_i(A,B,P)$ stands for the midpoint of AB, and $\sigma(A,B,P)$ stands for the reflection of B in A (i. e. A is the midpoint of $B\sigma(A,B,P)$). We further define (see Fig. 3):
$\rho(A,B,P) := \varphi(P, \sigma(\mu_1(P,B,A), A, B))$,
$\zeta(A,B,P) := \sigma(\iota(\varphi(A,B), \alpha_1(\varphi(A,P))), A, P)$,
$\tau(A,B,P) := \iota(\alpha_1(\varphi(A,B)), \alpha_1(\varphi(A,P)))$,
$\nu(A,B,P) := \iota(\alpha_1(\varphi(A,B)), \alpha_1(\varphi(\tau(A,B,P), A)))$,
$\psi(A,B,P) := \iota(\varphi(A,B), \alpha_1(\varphi(\tau(A,B,P), A)))$,
$\kappa(A,B,P) := \iota(\varphi(\zeta(A,B,P), \tau(A,B,P)), \varphi(A, \nu(A,B,P)))$,
and have that $\rho(A,B,P)$ is the parallel through P to AB in case $\neg\lambda(A,B,P)$, and that $\psi\kappa$ is perpendicular to AB. With these abbreviations we are now ready to define our main abbreviation, which is
$\Gamma(A, \varphi(A,B), P) := \rho(\kappa(A,B,P), \psi(A,B,P), A)$
and which may be read as 'the line perpendicular in A to AB'.

We have provided in [9] a quantifier-free axiom system for Euclidean planes of characteristic $\neq 2$ in the language $\mathbf{L}_1$ containing $A_0, A_1, A_2, \varphi, \iota, \gamma$ as primitive notions, where $\gamma(P,l)$ denotes the perpendicular in P on l whenever $l = \varphi(P,Q)$ with $P \neq Q$, an arbitrary line, otherwise. Having defined $\Gamma(A, \varphi(A,B), P)$, which is a variant of $\gamma(A, \varphi(A,B))$ for although the former contains an extra variable as argument, the values of the two are the same, we can now closely follow the axiom system presented in [9] to obtain the underlying Euclidean plane structure of our geometry. Adding to the axiom system thus obtained an axiom stating that the distance between l and $\alpha_1(l)$ is constant, we obtain a complete description of the operations in $\mathbf{S}$ and an axiom system for Euclidean planes with a unit length, in which all angles are bisesctable (or, in other words, which have free mobility).

We first introduce some additional abbreviations, for line intersection (χ), line parallelity ($\|$), line parallelity or equality ($\|'$), line perpendicularity ($\vdash$), segment congruence (Pieri's relation π), η, and θ_i:
$\chi(\varphi(A,B), \varphi(C,D)) :\leftrightarrow \lambda(A,B,\iota(\varphi(A,B), \varphi(C,D))) \wedge \lambda(C,D,\iota(\varphi(A,B), \varphi(C,D)))$,
$\varphi(X,Y) \parallel \varphi(U,V) :\leftrightarrow \varphi(X,Y) \neq \varphi(U,V) \wedge \neg\chi(\varphi(X,Y), \varphi(U,V))$,

$\varphi(X,Y) \parallel' \varphi(U,V) :\leftrightarrow \varphi(X,Y) = \varphi(U,V) \vee \varphi(X,Y) \parallel \varphi(U,V)$.
For A,B,P with $\neg\lambda(A,B,P)$ and $C \neq D$ we define $\vdash$ by[3] $\vdash (\varphi(A,B),\varphi(C,D),P) :\leftrightarrow \varphi(A,B) \neq \varphi(C,D) \wedge \chi(\varphi(A,B),\varphi(C,D)) \wedge \Gamma(\iota(\varphi(A,B),\varphi(C,D)),\varphi(A,B),P) = \varphi(C,D)$.
For points A,B,C,P with A,B,P not collinear, with $A \neq B$ and $A \neq C$, we define the relation $\pi(A,B,C,P)$ to be read as 'segment AB is congruent to segment AC' by:
$\pi(A,B,C,P) :\leftrightarrow \sigma(A,B,P) = C$
$\vee[\neg\lambda(A,B,C) \wedge \varphi(A,\mu_1(B,C,A)) = \Gamma(\mu_1(B,C,A),\varphi(\mu_1(B,C,A),C),A)]$.
We next introduce η by
$\eta(g,A,B) \leftrightarrow g = \varphi(A,B) \wedge A \neq B$,
to be read as 'g is the line joining the different points A and B', and $\theta_i(A,B,P)$ for $i \in \{1,2\}$ by
$\theta_i(A,B,P) := \iota(\Gamma(A,\varphi(A,B),P),\alpha_i(\varphi(A,B)))$.
We are now ready to state the axioms.[4]

A 1 $\varphi(A,B) = \varphi(B,A)$,

A 2 $A \neq B \wedge B \neq C \wedge D \neq B \wedge \varphi(A,B) = \varphi(C,D) \rightarrow \varphi(D,B) = \varphi(B,C)$,

A 3 $\bigvee_{0 \leq i,j \leq 2, i \neq j, i \neq j+1} (\eta(g,\iota(a_{i,j},g),\iota(a_{i,j+1},g)) \vee \eta(g,A_i,\iota(\alpha_1(A_i,A_{i+1}),g))$,

A 4 $\neg\lambda(A_0,A_1,A_2)$,

A 5 $\neg\lambda(A,B,P) \rightarrow \varphi(A,B) \parallel \rho(A,B,P) \wedge P \neq \sigma(\mu_1(P,B,A),A,B)$,

A 6 $\neg\lambda(A,B,X) \rightarrow \iota(\varphi(A,X),\varphi(B,X)) = X$,

A 7 $C \neq D \wedge \varphi(A,B) \parallel \varphi(C,D) \wedge \neg\lambda(A,B,X) \rightarrow \chi(\varphi(A,X),\varphi(C,D))$,

A 8 $C \neq D \wedge C' \neq D' \wedge \neg\lambda(A,B,P) \wedge \neg\lambda(A',B',P') \wedge \vdash (\varphi(A,B),\varphi(C,D),P)$
$\wedge \varphi(A,B) \parallel' \varphi(A',B') \wedge \varphi(C,D) \parallel' \varphi(C',D') \rightarrow \vdash (\varphi(A',B'),\varphi(C',D'),P')$,

A 9 $\neg\lambda(P_1,P_2,P_3) \wedge \neg\lambda(P_1,P_2,B) \wedge B \neq P_3 \vdash (\varphi(P_2,P_3),\varphi(P_1,B),P_1)$
$\vdash (\varphi(P_1,P_3),\varphi(P_2,B),P_2) \rightarrow \vdash (\varphi(P_1,P_2),\varphi(P_3,B),P_3)$,

A 10 $\neg\lambda(A,B,P) \rightarrow \Gamma(A,\varphi(A,B),P) \neq \varphi(A,B)$,

as well as the minor Desargues axiom (**des**) and Fano's axiom in the following form:

A 11 $\neg\lambda(A,B,C) \wedge \varphi(A,B) \parallel \varphi(C,D) \wedge \varphi(B,C) \parallel \varphi(A,D)$
$\rightarrow \chi(\varphi(A,C),\varphi(B,D)) \wedge \mu_1(A,C,B) = \iota(\varphi(A,C),\varphi(B,D))$.

It follows from [9, p. 12-14] that all models of $\Sigma = \{A1 - A11, \mathbf{des}\}$ are Euclidean planes associated to a quadratic field extension where the field characteristic is $\neq 2$, and that $\varphi,\iota,\Gamma,\mu_1$ have the desired interpretation therein. To obtain the desired interpretation for σ, α_1 and α_2 we add the following axioms ($i \in \{1,2\}$):

[3] In this definition, by $\Gamma(\iota(\varphi(A,B),\varphi(C,D)),\varphi(A,B),P)$ we mean $\Gamma(\iota(\varphi(A,B),\varphi(C,D)),\varphi(\iota(\varphi(A,B),\varphi(C,D),X),P)$, where X is A or B, whichever is $\neq \iota(\varphi(A,B),\varphi(C,D))$.

[4] Addition in the indices of the A_i is mod 3.

A 12 $\neg\lambda(A,B,P) \to \lambda(A,B,\sigma(A,B,P)) \wedge \mu_1(B,\sigma(A,B,P),P) = A$,

A 13$_i$ $A \neq B \to \varphi(A,B) \parallel \alpha_i(\varphi(A,B))$,

A 14 $\neg\lambda(A,B,P) \to \mu_1(\theta_1(A,B,P),\theta_2(A,B,P),B) = A$,

A 15 $\neg\lambda(A,B,A') \wedge \neg\lambda(A',B',A) \wedge \varphi(A,B) \neq \varphi(A',B') \to [\theta_1(A,B,A') = \mu_1(A,A',B)$
$\wedge\pi(A',\theta_1(A',B',A),\mu_1(A,A',B),B')] \vee [\theta_1(A,B,A') \neq \mu_1(A,A',B)$
$\wedge((\pi(A',\theta_1(A',B',A),\sigma(\mu_1(A,A',B),\theta_1(A,B,A'),A),B') \wedge \neg\lambda(A,A',\theta_1(A,B,A')))$
$\vee(\pi(A',\theta_1(A',B',A),\sigma(\mu_1(A,A',B),\theta_1(A,B,A'),B),B') \wedge \lambda(A,A',\theta_1(A,B,A'))))]$.

From axiom A12 it follows that σ has the desired interpretation, A13 and A14 imply that for every line l $\alpha_1(l)$ and $\alpha_2(l)$ are two parallel lines equidistant from l, and A15 ensures that for different lines l and l', the distances between $\alpha_1(l)$ and l, and $\alpha_1(l')$ and l' are the same. It follows that

Representation Theorem 1 . *$\mathfrak{M}$ is a model of $\Sigma \cup \{A12 - A15\}$ if and only if $\mathfrak{M}$ is isomorphic to a Euclidean plane associated to a quadratic field extension with characteristic $\neq 2$ in which all angles are bisectable, and in which the primitive notions of* **S** *have the desired interpretation.*

We can obtain ordered Euclidean geometry with free mobility and with a unit distance by adding a ternary relation symbol ξ for the betweenness relation to **S** and the following axioms:

A 16 $\xi(A,B,C) \to \bar{\lambda}(A,B,C)$

A 17 $\bar{\lambda}(A,B,C) \to \xi(A,B,C) \vee \xi(B,C,A) \vee \xi(C,A,B)$,

A 18 $\xi(A,B,C) \to \xi(C,B,A)$,

A 19 $\neg\lambda(A,B,P) \to \xi(A,\mu_1(A,B,P),B)$,

A 20 $\neg(\lambda(P_1,P_2,P_3) \bigvee_{n=1}^{3} \lambda(P_n,M,N)) \wedge \xi(P_1,M,P_2) \to \bigvee_{i=1}^{2}(\chi(\varphi(M,N), \varphi(P_i,P_3)) \wedge\xi(P_i, \iota(\varphi(M,N),\varphi(P_i,P_3)),P_3))$.

To obtain the ordered Euclidean geometry of ruler and compass constructions, i. e. Cartesian planes over ordered Euclidean fields, we extend the language of $\mathbf{S} \cup \{\xi\}$ by another pair of binary operation symbols, standing for the operation (iii) of the double edged ruler, β_1 and β_2. This language will be denoted by $\mathbf{S}'$. For points A and B with $d(A,B) \geq 1$, $(\beta_1(A,B),\beta_2(A,B))$ may be read as 'the pair of lines (different if $d(A,B) > 1$) that pass through A such that $\alpha_i(\beta_1(A,B))$ and $\alpha_j(\beta_2(A,B))$ pass through B for some $i,j \in \{1,2\}$'. For points A and B that are less than 1 apart, $\beta_i(A,B)$ is an arbitrary line. The axioms that ensure that the β_i have precisely the above meaning are (here i,k take values in $\{1,2\}$, so there are four axioms $A21_{i,k}$ and two axioms $A22_i$):

A 21$_{i,k}$ $\neg\lambda(A,B,P) \wedge \bar{\xi}(A,\iota(\varphi(A,B),\alpha_i(\Gamma(A,\varphi(A,B),P))),B)$
$\to \eta(\beta_k(A,B),A,\iota(\alpha_1(A,B),\beta_k(A,B))) \wedge (\bigvee_{j=1}^{2} \eta(\alpha_j(\beta_k(A,B)),B,\iota(\alpha_1(A,B),\alpha_j(\beta_k(A,B))),$

A 22$_i$ $\neg\lambda(A,B,P)\wedge\xi(A,\iota(\varphi(A,B),\alpha_i(\Gamma(A,\varphi(A,B),\rho))),B)$ $\to\beta_1(A,B)\neq\beta_2(A,B)$, where $\overline{\xi}(X,Y,Z):\leftrightarrow\xi(X,Y,Z)\vee Y=Z$. We now have the following

Representation Theorem 2 . *$\mathfrak{M}$ is a model of $\Sigma\cup\{A12-A22\}$ if and only if $\mathfrak{M}$ is isomorphic to a Cartesian plane over a Euclidean ordered field, in which the primitive notions of* $\mathbf{S}'$ *have the desired interpretation.*

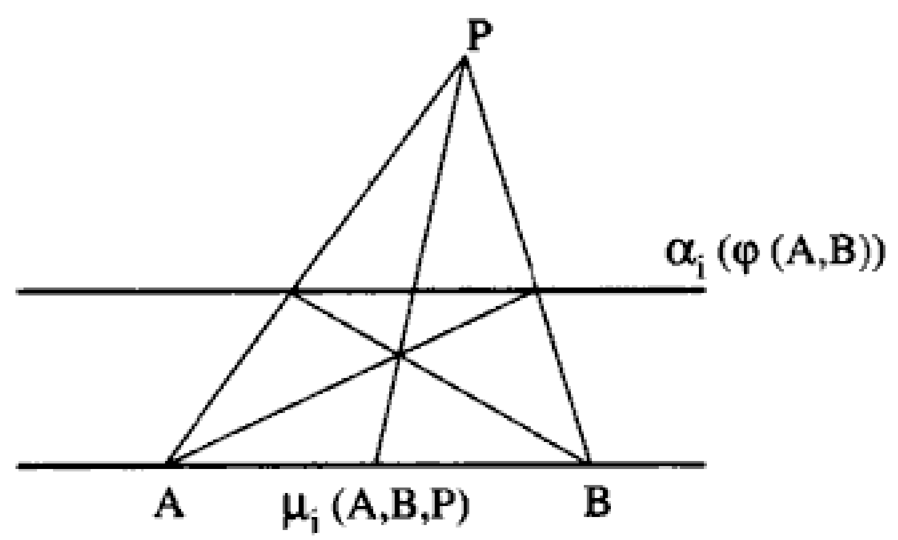

Figure 1

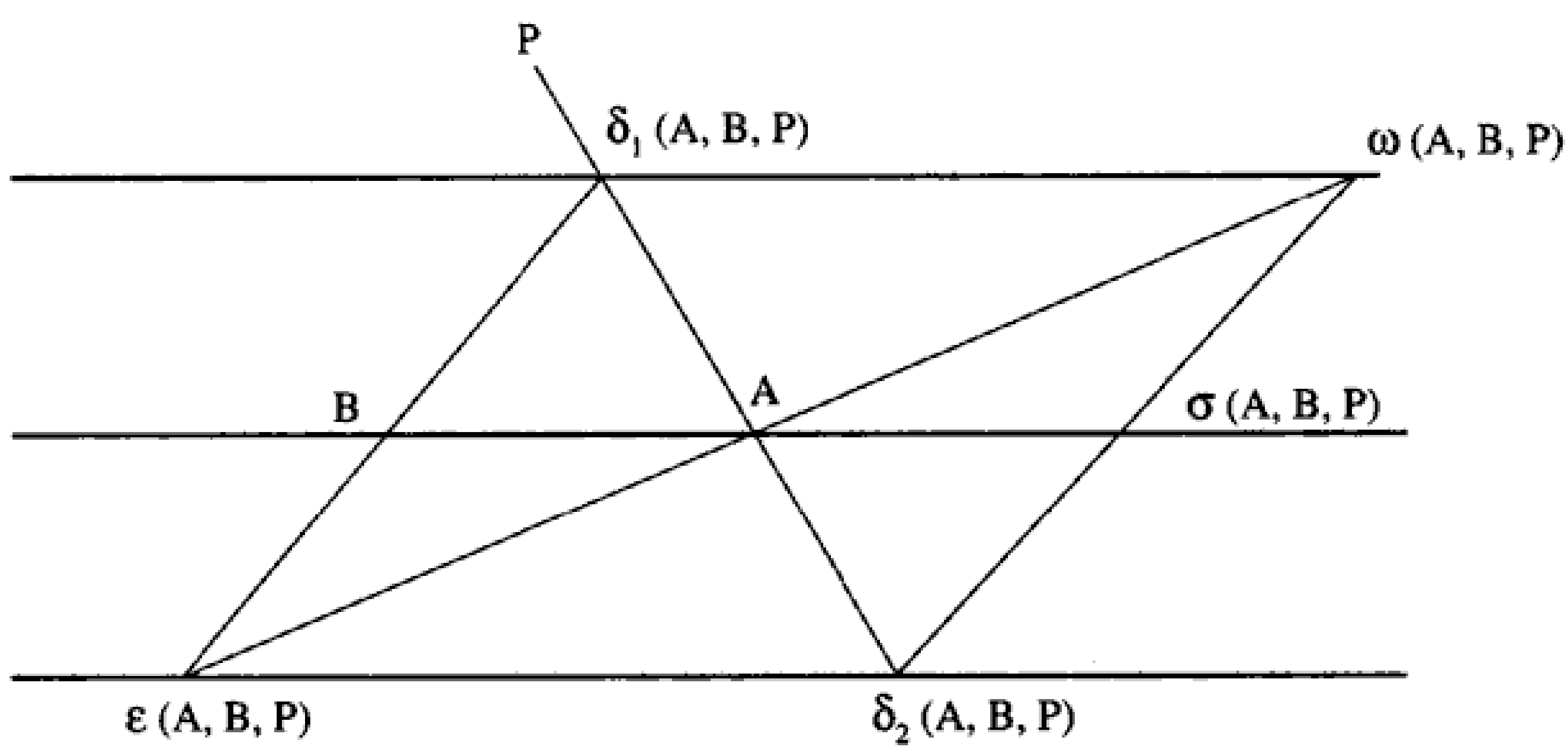

Figure 2

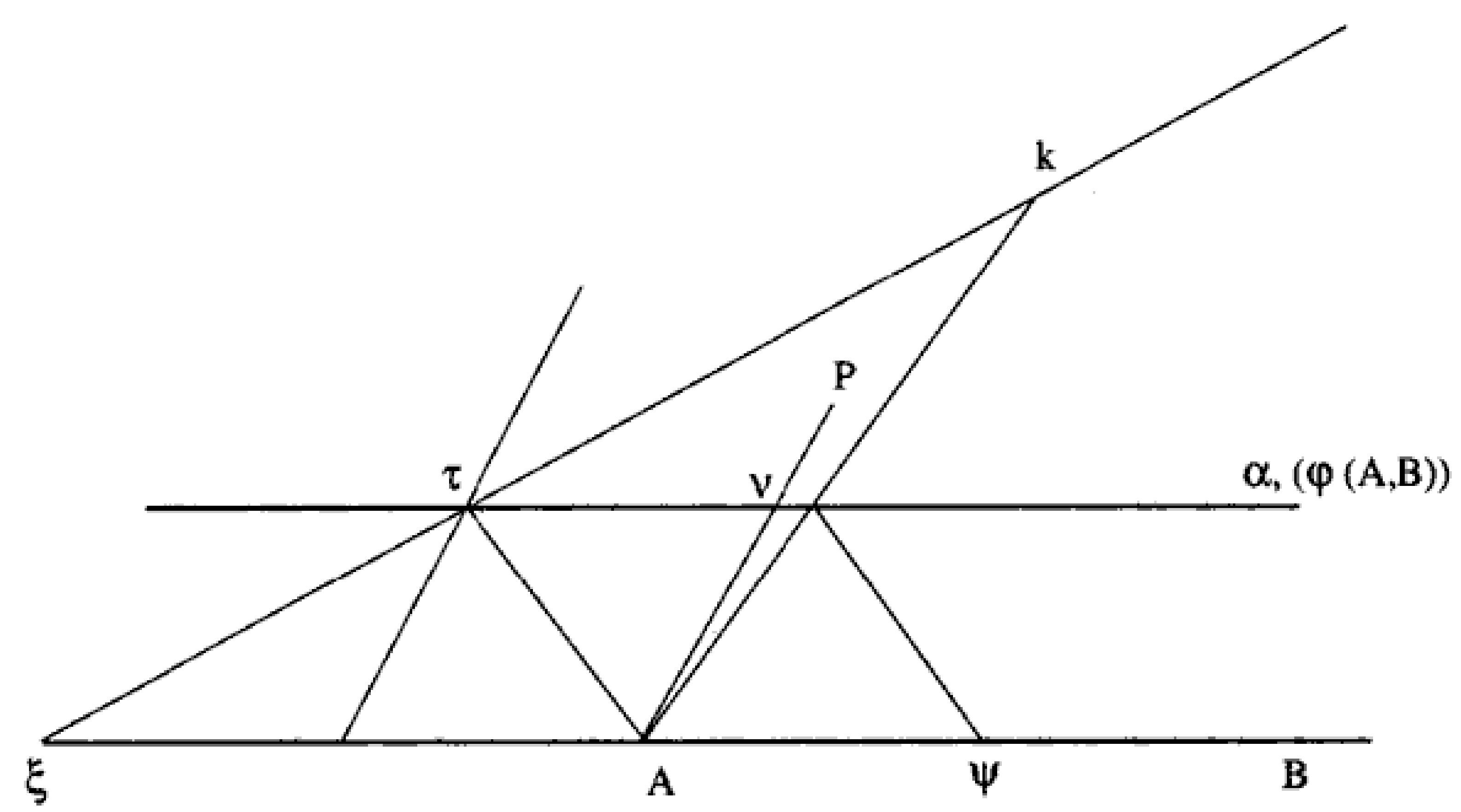

Figure 3

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Received February 5, 1999
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