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THE RANK 3 GEOMETRIES OF THE SIMPLE SUZUKI GROUPS $Sz(q)$

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Abstract. *We determine all possible rank three geometries on which a Suzuki simple group $Sz(q)$, with q an odd power of two, acts residually weakly primitively (*RWPRI*). We observe that if we impose the $(2T)_1$ property, there is no* RWPRI *geometry of rank ≥ 4 on which $Sz(q)$ acts.*

1 Introduction

In 1954, Jacques Tits gave a geometric interpretation of the exceptional complex Lie groups (see [24] and [27]). Francis Buekenhout generalized in [2] and [3] certain aspects of this theory in order to achieve a combinatorial understanding of all finite simple groups. Since then, two main traces have been developed in diagram geometry. One is to try to classify geometries over a given diagram, mainly over diagrams extending buildings (see e. g. [6] chap. 22, for a survey and [26] for the theory of buildings). Another trace is to classify coset geometries for a given group under certain conditions. Rules for such classifications have been stated by Buekenhout in [4] and [5]. These guidelines led Michel Dehon to present in [14] a set of CAYLEY programs in order to classify all firm, residually connected and flag-transitive geometries of a given group G with an additional restriction on the subgroups forming the geometries: each stabilizer of some element is a maximal subgroup of G. Several groups were investigated as for example $U_4(2)$ [14], M_{11} [9] and a collection of projective groups G such that $PSL(2,q) \leq G \leq Aut(PSL(2,q))$ with $5 \leq q \leq 19$ [8]. This experimental work led to new rules for such classifications. In 1993, Francis Buekenhout and Michel Dehon changed the restriction of the subgroups forming the geometries, taking a residually weakly primitive condition (RWPRI). Again, experimental work was accomplished in that way. In 1994, an atlas of residually weakly primitive geometries for small groups was achieved by Buekenhout, Dehon and Leemans [10]. In 1995, Harald Gottschalk determined all geometries of the group $PSL(3,4)$ is his Diplomarbeit [16]. In 1996, Dehon and Miller determined in [15] all geometries of M_{11} satisfying these new conditions, Gottschalk and Leemans classified the geometries of J_1 [17] and Leemans determined all those geometries for $Sz(8)$ in [21]. During that period, several theoretical works on the subject were also made: Buekenhout, Dehon and Leemans showed in [11] that the Mathieu group M_{12} does not have RWPRI and $(IP)_2$ geometries of rank ≥ 6. Buekenhout, Cara and Dehon described in [7] a class of inductively minimal geometries which satisfy the RWPRI condition, and Buekenhout and Leemans showed in [12] that the O'Nan sporadic simple group does not have RWPRI geometries of rank ≥ 5 (resp. six) with one of the subgroups forming the geometry isomorphic to J_1 (resp. M_{11}). All these results tended to show that the residually weakly primitive condition on the subgroups was a "good one".

The experience that we acquired led us to be more ambitious. We wanted to look at an infinite class of groups and classify all geometries satisfying some conditions for all groups of this class. We choosed to study the Suzuki simple groups. This choice was motivated by the fact that the structure of these groups is particularly easy compared to other simple groups. We started the classification by determining, up to isomorphism, all rank two firm, residually connected, flag-transitive geometries on which a Suzuki simple group $Sz(q)$, with q an odd power of two, acts residually weakly primitively [18]. This purely theoretical work pointed out some very surprising results, as for example the fact that a Suzuki simple group which does not have proper subgroups of Suzuki type (i.e. subgroups which are also Suzuki simple groups) gives rise to much more geometries than the others. We also showed in [18] that by adding just one condition, namely the $(2T)_1$ condition, we reduced the number of geometries arising to one for every Suzuki group except for the smallest one, i.e. $Sz(8)$ which has three such geometries. The next step was the classification of all rank three residually weakly primitive geometries of the Suzuki simple groups. To do this, we first needed to know all the RWPRI geometries of the dihedral groups since some of the maximal subgroups of a Suzuki group are dihedral groups, so we classified in [20] all RWPRI geometries of the dihedral groups.

In the present paper, we determine all RWPRI geometries of rank three, on which a Suzuki simple group $Sz(q)$, with q an odd power of 2, acts residually weakly primitively. The reader is warned that the RWPRI property as it is defined in the present paper is slightly different from the original one that was first given in [10]. The reason of this change is that in [10], the authors define the RWPRI property for flag-transitive geometries. Here we plan to determine geometries that satisfy the RWPRI condition, be them flag-transitive or not. The main reason of this choice is that RWPRI is a property much easier to test than flag-transitivity. Also, it seems much more restrictive, so it is natural to test it first. Moreover, we see that even without imposing the flag-transitivity condition, it is possible to get a very good control on the results.

Our main results are theorems 5.3, 5.4 and 5.5 which give all possible residually weakly primitive geometries of rank three for any Suzuki simple group $Sz(q)$, with $q = 2^{2e+1}$ and e a positive integer. As in the rank two classification [18], we subdivide the results into three cases, namely the case $q-1$ prime (and hence $2e+1$ prime), the case $q-1$ not prime and $2e+1$ prime, and the case $2e+1$ not prime (and hence $q-1$ not prime).

As a corollary to the classification theorems for the rank three residually weakly primitive geometries of the Suzuki groups, we show in theorem 6.1 that there is no pre-geometry (and hence no geometry) of rank ≥ 4, satisfying the residually weakly primitive condition and the $(2T)_1$ condition. Moreover, we show that the only rank three geometries satisfying that property are thin, and that they arise only for Suzuki groups $Sz(q)$ with $q-1$ a prime number. Thus, as in the rank two case, the $(2T)_1$ property seems to be a good candidate to enter our set of axioms. Anyway, we show in [19] that it is possible to classify all residually weakly primitive pre-geometries of rank ≥ 4 for any Suzuki group $Sz(q)$.

The results obtained in this paper strenghten our belief that the residually weakly primitive condition is an efficient one to impose on the subgroups forming the geometries since control is obtained even without asking the flag-transitivity condition. Moreover, the nice classification obtained in theorem 6.1 makes us believe that the locally two-transitive property should enter our set of axioms.

The paper is organised as follows. In section 2, we recall some basic definitions and we fix notation. In section 3, we state some preliminary lemmas used in the next sections for the classification process. In section 4, we give some lemmas that tend to restrict the possible maximal parabolic subgroups of the geometries we want to classify. In section 5, we classify all rank three, RWPRI geometries of a Suzuki group $Sz(q)$. In section 6, we show that for most of the geometries obtained, we can say whether the group from which they arise acts flag-transitively or not and we prove that there is no RWPRI and $(2T)_1$ pre-geometry of rank strictly greater than 3 for any Suzuki simple group $Sz(q)$ and that rank three geometries appear only when $q-1$ is prime.

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2 Definitions and notation

The basic concepts about geometries constructed from a group and some of its subgroups are due to Tits [25] (see also [6], chapter 3).

Let I be a finite set and let G be a group together with a family of subgroups $(G_i)_{i\in I}$. We define the *pre-geometry* $\Gamma = \Gamma(G,(G_i)_{i\in I})$ as follows. The set X of *elements* of Γ consists of all cosets gG_i, $g \in G$, $i \in I$. We define an *incidence relation* $*$ on X by :

$$g_1G_i * g_2G_j \text{ iff } g_1G_i \cap g_2G_j \text{ is non-empty in } G.$$

The *type function* t on Γ is defined by $t(gG_i) = i$. The *type* of a subset Y of X is the set $t(Y)$; its *rank* is the cardinality of $t(Y)$ and we call $\mid t(X) \mid$ the *rank* of Γ.

A *flag* is a set of pairwise incident elements of X and a *chamber* of Γ is a flag of type I. An element of type i is also called an *i-element*.

The group G acts on Γ as an automorphism group, by left translation, preserving the type of each element.

As in [14], we call Γ a *geometry* provided that every flag of Γ is contained in some chamber and we call Γ *flag-transitive* (FT) provided that G acts transitively on all chambers of Γ, hence also on all flags of any type J, where J is a subset of I. It is obvious that any rank two pre-geometry $\Gamma(G;G_0,G_1)$ is a flag-transitive geometry.

Lemma 2.1 *Let $\Gamma(G;G_0,G_1,G_2)$ be a rank 3 pre-geometry. Then Γ is a geometry (not necessarily flag-transitive).*

This lemma permits us to talk about geometries instead of pre-geometries in the rank three case.

Let $\Gamma(G;G_0,\dots,G_{n-1})$ be a rank n pre-geometry. We call $C = \{G_0,\dots,G_{n-1}\}$ the *maximal parabolic chamber associated to* Γ. Assuming that F is a subset of C, the *residue* of F is the pre-geometry

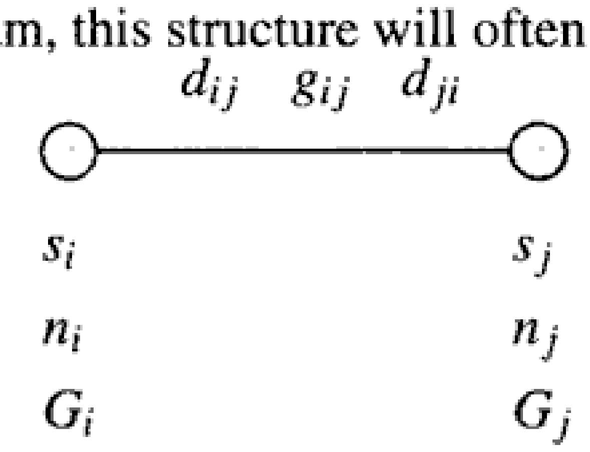
$$\Gamma_F = \Gamma(\cap_{j\in t(F)}G_j, (G_i \cap (\cap_{j\in t(F)}G_j))_{i\in I\setminus t(F)})$$

If $F = \{G_i\}$ for some $i \in I = \{0,\dots,n-1\}$ then Γ_F is also called the G_i*-residue* of Γ and denoted Γ_i. If Γ is flag-transitive and F is any flag of Γ, of type $t(F)$, then the residue Γ_F of Γ is isomorphic to the residue of the flag $\{G_i, i \in t(F)\} \subseteq C$.

Assume Γ is a pre-geometry. We call Γ *firm* (F) (resp. *thick*, *thin*) provided that every flag of rank $|I|-1$ is contained in at least two (resp. three, exactly two) chambers. We call Γ *residually connected* (RC) provided that the incidence graph of each residue of rank ≥ 2 is connected.
Let $\Gamma(G;G_0,\ldots,G_{n-1})$ be a pre-geometry and denote $I=\{0,\ldots,n-1\}$. As in [17], for any $\emptyset \subset J \subseteq I$, we set $G_J=\bigcap_{j\in J}G_j$, and $G_\emptyset = G$. The subgroup G_I is the *Borel subgroup* of Γ. We call $\mathcal{L}(\Gamma)=\{G_J : J\subseteq I\}$ the *sublattice* (of the subgroup lattice of G) *spanned* by the collection $(G_i)_{i\in I}$. The elements of the lattice are called the *parabolic subgroups* and the subgroups G_i's are the *maximal parabolic subgroups*. When the context is clear, we write "sublattice" instead of "sublattice spanned by ...".
We call Γ *residually weakly primitive* (RWPRI) provided that for any $\emptyset \subseteq J \subset I$ there exists at least one element $i\in I\backslash J$ such that $G_{J\cup\{i\}}$ is maximal in G_J. This definition of RWPRI differs slightly from the one given in [10]. If the pre-geometry Γ is a flag-transitive geometry, then the present definition is equivalent to the one given in [10].
The RWPRI condition implies that all subgroups of the sublattice are pairwise distincts and that $\cap_{j\in I}G_j$ is a maximal subgroup of $\cap_{j\in I\backslash\{i\}}G_j$ for all $i\in I$. Arranging the indices in suitable manner, we may also assume that $\cap_{j\in\{0,\ldots,i\}}G_j$ is a maximal subgroup of $\cap_{j\in\{0,\ldots,i-1\}}G_j$ for all $i=1,\ldots,n-1$.

If Γ is a geometry of rank 2 with $I=\{0,1\}$ such that each of its 0-elements is incident with each of its 1-elements, then we call Γ a *generalized digon*.
We call the pre-geometry Γ *locally 2-transitive* and we write $(2T)_1$ for this, provided that the stabilizer G_F of any flag $F\subset C$ of rank $|I|-1$ acts 2-transitively on the residue Γ_F.
Again here, if Γ is a flag-transitive geometry, then the $(2T)_1$ property is the same as the one given in [10].

Following [2] and [3], the *diagram* of a firm, residually connected, flag-transitive geometry Γ is a graph on the elements of I together with the following structure: to each vertex $i\in I$, we attach the *order* s_i which is $|\Gamma_F|-1$, where F is any flag of type $I\backslash\{i\}$, the *number* n_i *of varieties of type* i, which is the index of G_i in G, and the subgroup G_i. Elements i, j of I are not joined by an edge provided that a residue Γ_F of type $\{i,j\}$ is a generalized digon. Otherwise, i and j are joined by an edge endowed with three positive integers d_{ij}, g_{ij}, d_{ji}, where g_{ij} (the *gonality*) is equal to half the girth of the incidence graph of a residue Γ_F of type $\{i,j\}$ and d_{ij} (resp. d_{ji}), the *i-diameter* (resp. *j-diameter*) is the greatest distance from some fixed i-element (resp. j-element) to any other element in the incidence graph of Γ_F.
On a picture of the diagram, this structure will often be depicted as follows.



If $g_{ij}=d_{ij}=d_{ji}=n$, then Γ_F is called a *generalized n-gon* and we do not write d_{ij} and d_{ji} on the picture.

As to notation for groups, we follow the conventions of the Atlas [13] up to slight variations. The symbol ":" stands for split extensions, the "hat" symbol "$\hat{\cdot}$"stands for non split extensions and the symbol $\times$ stands for direct products. We write E_q for an elementary

Structure	Order	Index	Description
$(E_q{\cdot}E_q){:}(q\text{-}1)$	$q^2 \cdot (q-1)$	q^2+1	Normalizer of a 2-Sylow, stabilizer of a point of Ω.
$D_{2(q-1)}$	$2 \cdot (q - 1)$	$\frac{(q^2+1)\cdot q^2}{2}$	Stabilizer of a pair of points of Ω.
$\alpha_q{:}4$	$\alpha_q{\cdot}4$	$\frac{q^2(q-1)}{4\beta_q}$	Normalizer of a cyclic group of order α_q
$\beta_q{:}4$	$\beta_q{\cdot}4$	$\frac{q^2(q-1)}{4\alpha_q}$	Normalizer of a cyclic group of order β_q
$Sz(2^{2f+1})$ with $2f+1 \mid_M 2e+1$	$(s^2+1)\cdot s^2 \cdot (s-1)$		

Table 1: The maximal subgroups of Sz(q)

abelian group of order q. A group is called *of Suzuki type* if it is a simple group isomorphic to $Sz(q)$ with q an odd power of two. Observe that in the present paper, we prefer not to consider the group $Sz(2) \cong AGL(1,5)$ as a group of Suzuki type. Its geometries can be found in [10].

When an integer n divides an integer m and $\frac{m}{n}$ is a prime number, we write $n \mid_M m$. When an integer n divides an integer m and $n \neq m$, we write $n \mid_P m$.

3 Preliminary lemmas

For a good introduction on the Suzuki groups, we refer to [23] (see also [18]). We remind the reader that the group $Sz(q)$ has order $q^2(q^2+1)(q-1)$.
Observe that $q^2+1 = (q+\sqrt{2q}+1)(q-\sqrt{2q}+1)$. We write α_q (resp. β_q) for $q+\sqrt{2q}+1$ (resp. $q-\sqrt{2q}+1$). Let Ω be a set of q^2+1 points on which $Sz(q)$ acts doubly transitively. Table 1 is taken from [22]. It gives the list of maximal subgroups of $Sz(q)$. These subgroups are studied more deeply in [18].

Lemma 3.1 *[18] Let $G \cong Sz(q)$ with q an odd power of 2.*
(1) The maximal subgroups of a $(E_q{\cdot}E_q) : (q-1)$-subgroup of G have one of the following structures:

- $(E_q{\cdot}E_q) : n$ *where n is a maximal divisor of $q-1$;*
- $E_q : (q-1)$.

(2) The maximal subgroups of a $n : 4$-subgroup of G with $n = \alpha_q$ or β_q, have the following structures:

- $m : 4$ *where m is a maximal divisor of n;*
- $n : 2$.

(3) The maximal subgroups of a $D_{2(q-1)}$-subgroup of G have the following structure:

- *the cyclic group of order $q-1$;*
- *D_{2n} where n is a maximal divisor of $q-1$. If $q-1$ is a prime, D_2 denotes the cyclic group of order 2.*

Lemma 3.2 *[18] Let $q = 2^{2e+1}$ with e a positive integer. The numbers q - 1, α_q and β_q are pairwise coprime.*

Observe that if H, I and J are subgroups of $Sz(q)$ with $H \cong D_{2(q-1)}$, $I \cong \alpha_q : 4$ and $J \cong \beta_q : 4$, lemma 3.2 yields $| H \cap I | \leq 2$, $| H \cap J | \leq 2$, and $| I \cap J | \leq 4$. This fact is used very often in section 5.

Lemma 3.3 *The group $Sz(q)$ has a unique class of maximal subgroups isomorphic to a Suzuki group if and only if $q = 2^{2e+1}$, $2e+1 = p^n$ with p a prime number, and $n > 1$ an integer.*

Proof. The claim immediately follows from the classification of maximal subgroups of $Sz(q)$ given in [22]. □

Lemma 3.4 *Let $Sz(s)$ be a maximal subgroup of $Sz(q)$. Then $s-1 \mid_M q-1$ implies $s = 2^{p^n}$ and $q = s^p$, with p a prime and n a positive integer.*

Proof. Suppose that there is no prime p and positive integer n such that $q = 2^{p^n}$. Then, by lemma 3.3, there are at least two different classes of maximal subgroups of Suzuki type in $Sz(q)$, say $Sz(s)$ and $Sz(t)$. Thus $q = s^p = t^{p'}$ with $p \neq p'$ two primes. Then both $s-1$ and $t-1$ divide $q-1$, and $(s-1)\cdot(t-1) < (q-1)$. This implies that $\frac{q-1}{s-1}$ cannot be a prime, because $q-1 = (s-1)\cdot\frac{t-1}{(s-1,t-1)}\cdot n$ with $n > 1$ an integer. □

Lemma 3.5 *Let $Sz(s)$ be a maximal subgroup of $Sz(q)$. If $Sz(q)$ has at least two different classes of maximal subgroups of Suzuki type, then $s-1$ is not a prime divisor of $q-1$.*

Proof. Obvious thanks to lemmas 3.3 and 3.4. □

Lemma 3.6 *Let $s \geq 8$ be a divisor of $q = 2^{2e+1}$ with e a positive integer. Then*
(1) 5 divides q^2+1;
(2) either $\alpha_s \mid \alpha_q$ and $\beta_s \mid \beta_q$ or $\alpha_s \mid \beta_q$ and $\beta_s \mid \alpha_q$;
(3) $5 \mid \alpha_q$ (resp. β_q) if and only if $2e+1 = 0$ or 3 (resp. 1 or 2) modulo 4.

Lemma 3.7 *Let m and n be odd positive integers. Then $(2^{2m}+1, 2^{2n}+1) = 2^{2(m,n)}+1$.*

Lemma 3.8 *Let $\Gamma(G; G_0, G_1, G_2)$ be a rank 3 geometry. If the diagram of Γ is linear, then Γ is a flag-transitive geometry.*

Observe that this lemma is a corollary of the main theorem proved in [1].

The next three lemmas summarize arguments that are used several times in section 5 to determine the rank three residually weakly primitive geometries of $Sz(q)$.

Lemma 3.9 *Let $q = 2^{2e+1}$ with e a positive integer and let $G \cong Sz(q)$.*
(i) Let H be a $n : 2$-subgroup of G with $1 \neq n \mid q^2+1$. There is a unique $n : 4$-subgroup of G containing H.
(ii) Let H be a $n : 4$-subgroup of G with $1 \neq n \mid q^2+1$. For every prime $p \mid \frac{q^2+1}{n}$, there is a unique $(np) : 4$-subgroup of G containing H.

Proof. (ii) Every $x : 4$-subgroup of G with $1 \neq x$ a divisor of q^2+1 is self-normalizing in G. Moreover, all $x : 4$-subgroups are conjugate in G. There are $\frac{|G|}{|x:4|}$ $x : 4$-subgroups in G. A $(np) : 4$-subgroup contains p $n : 4$-subgroups. Hence the number of $(np) : 4$-subgroups containing H is $\frac{|G|}{|np:4|}p / \frac{|G|}{|n:4|} = 1$.
(i) Since H is a subgroup of α_q: 4 or β_q: 4, it must be contained in a $n : 4$-subgroup. Thus its normalizer must contain a $n : 4$-subgroup. It cannot be a subgroup of Suzuki type since these are simple groups. Hence, looking at table 1, we see that $N_G(H) \cong n : 4$. The number of $n : 4$-subgroups in G is $\frac{|G|}{|n:4|}$. Each $n : 4$-subgroup contains exactely one $n : 2$-subgroup. Hence there is only one $n : 4$-subgroup containing H. □

Lemma 3.10 *The only subgroups of $G \cong Sz(q)$ containing $G_{01} \cong q-1$ are $N_G(G_{01}) \cong D_{2(q-1)}$, the two Sylow 2-normalizers containing G_{01} and their $E_q : (q-1)$-subgroups.*

Proof. Obvious thanks to the structures of the maximal subgroups of G (see table 1) and lemma 3.2. □

Lemma 3.11 *Let $G \cong Sz(q)$ and $G_0 \cong Sz(s)$ be a maximal subgroup of G. Let $G_{01} \cong D_{2(s-1)}$ be a maximal subgroup of G_0. The only subgroups of G containing G_{01} are G_0 and D_{2n}-subgroups with $s-1 \mid n \mid q-1$.*

Proof. Let H be a subgroup of $Sz(q)$ containing G_{01}. Then $2(s-1)$ divides the order of H. Table 1 and lemma 3.2 imply that H must be of Suzuki type or H is (a subgroup of) a $D_{2(q-1)}$-subgroup. Suppose H is of Suzuki type. Then, $H \cong G_0$. Since G_{01} is self-normalizing in G, and since there is only one conjugacy class of $Sz(s)$-subgroups in G, $G_0 = H$. Suppose H is (a subgroup of) a $D_{2(q-1)}$-subgroup. Since it contains G_{01}, it must be a D_{2n}-subgroup with $s-1 \mid n \mid q-1$. □

4 Restricting the maximal parabolic subgroups

In this section, we prove lemmas that restrict the possible maximal parabolic subgroups to construct RWPRI pre-geometries of rank ≥ 3. By lemma 2.1, we know that every rank 3 pre-geometry is indeed a geometry, so we can use these lemmas for the classification of rank 3 RWPRI geometries of $Sz(q)$. We prefer to talk about pre-geometries here because the results obtained in this section are used also in [19] to construct all RWPRI pre-geometries of rank ≥ 4.
Since we want to construct RWPRI pre-geometries, we know that at least one of the maximal parabolic subgroups of the pre-geometry must be a maximal subgroup of $Sz(q)$. Not every maximal subgroup of $Sz(q)$ can give rise to RWPRI pre-geometries of rank ≥ 3. We give here lemmas that discard some of them.

Lemma 4.1 *[18] There is no* RWPRI *pre-geometry of rank ≥ 2 with some G_i isomorphic to $(E_q{:}E_q){:}(q-1)$.*

Lemma 4.2 *There is no* RWPRI *pre-geometry of rank ≥ 3 with some G_i isomorphic to $\alpha_q{:}4$ or $\beta_q{:}4$.*

Proof. Suitably arranging the ordering of the indices, we may assume that $G_0 \cong n:4$ where $n = \alpha_q$ or β_q, and that G_{01} is maximal in G_0. Then G_{01} is isomorphic to D_{2n} or $(n/p):4$ where p is a prime divisor of n.
If $G_{01} \cong D_{2n}$, then G_1 must be a subgroup of $n:4$ but this does not happen, as there is a unique subgroup of G containing G_{01} isomorphic to $n:4$ by lemma 3.9.
The case $G_{01} \cong (n/p):4$ requires a little bit longer argument. By lemma 3.9, G_1 cannot be $n:4$. Since the order of G_{01} must divide the order of G_1, we know by lemma 3.2 and table 1 that G_1 must be a $Sz(s)$-subgroup for some s. Also, by lemma 3.6(2) and the fact that p is a prime, G_1 must be maximal in $Sz(q)$. Since G_{01} is of prime index in G_0, it must be maximal in G_1 and $(n/p) = \alpha_s$ or β_s.
Because the case $G_{01} \cong n:2$ has been already considered, we may assume that $G_{02} \cong (n/r):4$ with r, a prime divisor of n and $r \neq p$ (for otherwise $G_{02} \leq G_{01}$), which implies $G_2 \cong Sz(t)$. This already means that n cannot be a prime number and that $q-1$ must have at least two prime divisors. We also know that $G_{012} \cong \frac{n}{pr}:4$ because it is a subgroup of G_{02} which is maximal in G_{01}.
Suppose $q = 2^{2e+1}$, $s = 2^{2f+1}$ and $t = 2^{2g+1}$. Since G_1 and G_2 are maximal subgroups of $Sz(q)$, we have $q = s^{p_s} = t^{p_t}$ with p_s and p_t two distinct primes. Thus, $2e+1 = (2f+1)\cdot p_s = (2g+1)\cdot p_t$ and $(p_s, p_t) = 1$. This allows us to write $2e+1 = (2h+1)\cdot p_s \cdot p_t$ for some positive integer h. We know that n is either α_q or β_q and that n/p is either α_s or β_s. Then, $p = \frac{n}{n/p} > \alpha_t > \beta_t$. But p must divide α_t or β_t. So we have a contradiction. □

We show that if G_0 is a $D_{2(q-1)}$, then the pre-geometry has rank at most 3.

Lemma 4.3 *In every* RWPRI *pre-geometry of rank $n > 3$ with $G_0 \cong D_{2(q-1)}$, at least two G_{0i} are dihedral groups.*

Proof. We rely on the classification of the RWPRI geometries for the dihedral groups given in [20], since all RWPRI pre-geometries are flag-transitive RWPRI geometries for these groups. The proof becomes then obvious thanks to that classification. □

Lemma 4.4 *Suppose $q-1$ is not a prime. In a* RWPRI *pre-geometry of rank $n > 2$ with $G_0 \cong D_{2(q-1)}$, there is at most one G_{0i} isomorphic to a dihedral group.*

Proof. Since G_0 is a dihedral group, its residue is given in [20]. Moreover, this residue has the property that every of its maximal parabolic subgroups must be maximal in G_0. Assume, without loss of generality, that G_{01} and G_{02} are dihedral groups and that G_{01} is maximal in G_0. Then $G_{01} \cong D_{2x}$ with $x \,|_M\, q-1$. This implies $G_{02} \cong D_{2y}$ with $y \,|_M\, q-1$. Hence G_1 and G_2 cannot be subgroups of $\alpha_q{:}4$ or $\beta_q{:}4$ by lemma 3.2. Moreover, since G_{01} and G_{02} are self-normalizing in G, G_1 and G_2 cannot be subgroups of $D_{2(q-1)}$ either. Then G_1 (resp. G_2) must be a $Sz(s)$ (resp. $Sz(t)$). These two subgroups have to be maximal in $Sz(q)$ and G_{01} (resp. G_{02}) has to be maximal in $Sz(s)$ (resp. $Sz(t)$). We have $q = 2^{(2e+1)p_s p_t}$, $s = 2^{(2e+1)p_t}$

and $t = 2^{(2e+1)p_s}$ for some primes p_s and p_t. We also have $\frac{q-1}{s-1} = p$ a prime, which must be a divisor of $t-1$. But

$$p = \frac{2^{(2e+1)p_s p_t} - 1}{2^{(2e+1)p_t} - 1} = \sum_{i=1}^{p_t} 2^{(2e+1)p_s(i-1)} > 2^{(2e+1)p_s} - 1$$

So p cannot divide $t-1$, a contradiction. □

Lemma 4.5 *The maximal rank of a* RWPRI *pre-geometry with $G_0 \cong D_{2(q-1)}$ is three.*

Proof. Trivial by lemmas 4.3 and 4.4. □

This is a fairly strong result. It permits us to concentrate only on the rank three when we construct RWPRI pre-geometries with $G_0 \cong D_{2(q-1)}$.

Lemma 4.6 *If Γ is a* RWPRI *pre-geometry of rank ≥ 4 of a Suzuki group $Sz(q)$, then each of the maximal parabolic subgroups that is maximal in $Sz(q)$ must be isomorphic to a Suzuki group $Sz(s)$ for some s.*

Proof. Apply lemmas 4.1, 4.2 and 4.5. □

5 The rank 3 geometries

We determine now all possible rank three RWPRI geometries of a Suzuki group $Sz(q)$ with $q = 2^{2e+1}$ and e a positive integer. We subdivide our discussion in three cases:

- $q-1$ is a prime.
- $q-1$ is not a prime and $2e+1$ is prime.
- $2e+1$ is not a prime.

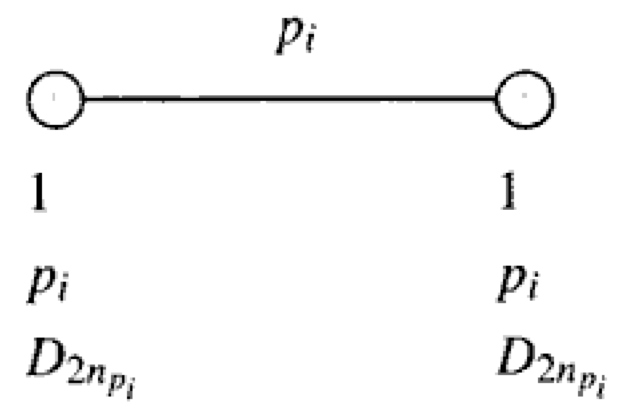
Since we want to construct rank three RWPRI geometries, we have to find triples of subgroups of $Sz(q)$ whose sublattice satisfies the RWPRI condition. Some of these subgroups must be maximal in $Sz(q)$. By lemmas 4.1 and 4.2, the only maximal subgroups of $Sz(q)$ that can be used to construct a RWPRI geometry of rank three are $D_{2(q-1)}$-subgroups and groups of Suzuki type. The following two theorems give all firm, residually connected, flag-transitive and residually weakly primitive rank two geometries of a D_{2n}-group and of a group of Suzuki type. Some of these geometries appear as rank two residues of rank three geometries for $Sz(q)$.

Theorem 5.1 *[20] Let $G \cong 2^2$. Up to isomorphism, G has only one rank 2 geometry satisfying F, RC, FT, and* RWPRI. *It has two maximal parabolic subgroups isomorphic to cyclic groups of order 2 and it is a generalized digon.*
Let $G \cong D_{2n}$, with $n \geq 3$ an integer, be a dihedral group and suppose $n = p_1^{e_1} \dots p_m^{e_m}$. Up to isomorphism, the group G has $2m + \frac{m(m-1)}{2}$ geometries of rank 2 satisfying F, RC, FT, and RWPRI. *They are given below.*

- *m geometries with the following diagram ($i = 1, \dots, m$):*

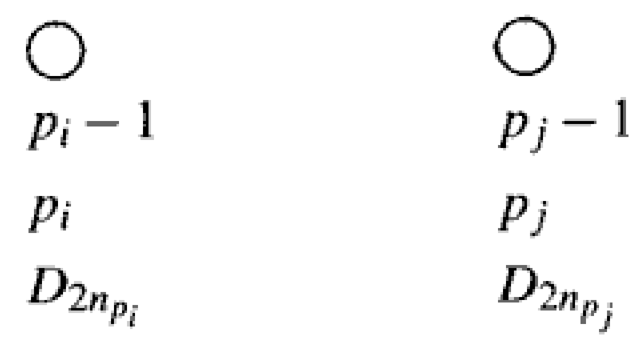
$B = Z_{n_{p_i}}$

- *m geometries with the following diagram* ($i = 1, \ldots, m$)*:*



$B = Z_{n_{p_i}}$

- $\frac{m(m-1)}{2}$ *geometries with the following diagram:* ($\{i, j\} \subseteq \{1, \ldots, m\}$ *and* $i \neq j$)



$B = D_{2n_{p_i p_j}}$

Theorem 5.2 *[18] Let* $e > 0$ *be a positive integer. Let* $s = 2^{2f+1}$ *and let Sz(s) be the Suzuki simple group over GF(s).*
If $2f + 1$ *and* $s - 1$ *are primes, the F, RC, FT, RWPRI rank two geometries of Sz(s) are isomorphic to one of the geometries appearing in table 2.*
If $2f + 1$ *is a prime and* $s - 1$ *is not a prime, the F, RC, FT, RWPRI rank two geometries of Sz(s) are isomorphic to one of the geometries appearing in table 3.*
If $2f + 1$ *is not a prime, the F, RC, FT, RWPRI rank two geometries of Sz(q) are isomorphic to one of the geometries appearing in table 4.*

5.1 First case: $q - 1$ is a prime

If $q - 1$ is a prime, *Sz(q)* cannot have subgroups isomorphic to *Sz(s)*. Observe that by lemma 4.6, $Sz(q)$ does not have RWPRI pre-geometries of rank ≥ 4 in that case. The only candidate for G_0 is $D_{2(q-1)}$ by lemmas 4.1 and 4.2. Thanks to theorem 5.1, we know that there are two possible residues for G_0.

In the first case, we have $G_{01} \cong q - 1$, $G_{02} \cong 2$ and $G_{012} \cong 1$. Since $G_{012} \cong 1$, G_{12} must be a cyclic simple group. Here there are two possibilities for G_{12}. They are $q - 1$ and 2. By lemma 3.10 and lemma 4.1 we may assume $G_1 \cong E_q : (q - 1)$.
Suppose first that $G_{12} \cong q - 1$. Then G_2 must be isomorphic to either $E_q : (q - 1)$ or $D_{2(q-1)}$. Let Ω be a set of $q^2 + 1$ points on which $Sz(q)$ acts doubly transitively. By lemma 3.10 and lemma 4.1, $G_1 \cong E_q : (q - 1)$ in both cases.

No.	H_0	H_1	H_{01}	$(2T)_1$
1	$D_{2(s-1)}$	$D_{2(s-1)}$	2	NO
2		D_{2n} where n is a prime divisor of α_s or β_s	2	NO
3		2^2	2	NO
4		4	2	NO
5		$E_s:(s-1)$	$s-1$	YES
	α_s a prime			
6	$\alpha_s:4$	$\alpha_s:4$	4	NO
7		n:4 where n is a prime divisor of β_s	4	NO
8		4×2	4	NO
	β_s a prime			
9	$\beta_s:4$	$\beta_s:4$	4	YES iff s=8
10		n:4 where n is a prime divisor of α_s	4	NO
11		4×2	4	YES iff q=8

Table 2: The case when $2f+1$ is a prime and $s-1$ is a prime

No.	H_0	H_1	H_{01}	$(2T)_1$
5	$D_{2(s-1)}$	$E_s:(s-1)$	s-1	YES
	α_s a prime			
6	$\alpha_s:4$	$\alpha_s:4$	4	NO
7		n:4 where n is a prime divisor of β_s	4	NO
8		4×2	4	NO
	β_s a prime			
9	$\beta_s:4$	$\beta_s:4$	4	NO
10		n:4 where n is a prime divisor of α_s	4	NO
11		4×2	4	NO

Table 3: The case when $2f+1$ is a prime and $s-1$ is not

No.	H_0	H_1	H_{01}	$(2T)_1$
12	$D_{2(s-1)}$	$E_s:(s-1)$	s-1	YES
13		*Sz(t)* if t-1 is a maximal divisor of s-1	$D_{2(t-1)}$	NO
14	$\alpha_s:4$	*Sz(t)* if α_t or β_t is a maximal divisor of α_s	α_t:4 or β_t:4	NO
15	$\beta_s:4$	*Sz(t)* if α_t or β_t is a maximal divisor of β_s	α_t:4 or β_t:4	NO
16	*Sz(t)*	$(E_s \dot{\cap} E_t):(t-1)$	$(E_t \dot{\cap} E_t):(t-1)$	NO
17		*Sz(t')* when it is a maximal subgroup and t' $\neq$ t	*Sz(u)*	NO
18		D_{2n} where t-1 $\|_M$ n $\|_P$ s-1	$D_{2(t-1)}$	NO
19		n:4 where α_t $\|_M$ n $\|_P$ α_s or β_s	α_t:4	NO
20		n:4 where β_t $\|_M$ n $\|_P$ α_s or β_s	β_t:4	NO

Table 4: The case when $2f+1$ is not a prime (and thus $s-1$ is not a prime)

Let $G_2 \cong E_q : (q-1)$. Then G_1 (resp. G_2) has one fixed point $\alpha \in \Omega$ (resp. $\beta \in \Omega$) and one orbit A (resp. B) of q points of Ω such that $(A \cup \{\alpha\}) \cap (B \cup \{\beta\}) = \{\alpha, \beta\}$. Now, G_0 is the stabilizer of a pair of points $\{\alpha, \gamma\}$ where $\gamma \in A \backslash B$. Since the involution of G_{02} fixes β, it must exchange α and γ. So $\gamma \in B$, a contradiction.
Let $G_2 \cong D_{2(q-1)}$. Then G_2 is the stabilizer of a pair of points $\{\alpha, \beta\} \subset \Omega$ and G_1 is the stabilizer of one of these two points, say α. Also, G_{01} fixes α and another point, say $\gamma \in \Omega$. Then G_0 is the stabilizer of the pair $\{\alpha, \gamma\}$. The involution in G_{02} exchanges α and β, but also α and γ. So $\beta = \gamma$ and hence $G_0 = G_2$, a contradiction.
Suppose then that $G_{12} \cong 2$. Then G_2 contains two cyclic groups of order 2, one of which is maximal in G_2. Thus $G_2 \cong D_{2p}$ with p a prime dividing α_q or β_q or $q-1$. The case $p = 2$ is not possible because G_{02} and G_{12} do not have the same fixed point.

In the second case, we have G_{01} and G_{02} isomorphic to cyclic groups of order 2. There, either G_1 is one of $p : 2$ (with $p \mid \alpha_q, \beta_q, 2$ or $q-1$) and G_2 is one of $p' : 2$ (with $p' \mid \alpha_q, \beta_q$ or $q-1$), where p and p' are primes, or $G_1 \cong E_q : (q-1)$ and $G_2 \cong E_q : (q-1)$. Observe that there are no two maximal parabolic subgroups isomorphic to 2^2. Indeed, this should mean that all the G_{ij}'s have a common fixed point, and thus the group generated by G_0, G_1 and G_2 cannot be *Sz(q)*.

We summarize the preceding discussion in the following theorem.

Theorem 5.3 *Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1$ and $q-1$ are primes. Then every rank 3* RWPRI *geometry of Sz(q) has a sublattice isomorphic to one of the following.*

-

G_0 $D_{2(q-1)}$	G_1 $E_q : (q-1)$	G_2 $E_q : (q-1)$
G_{01} 2	G_{02} 2	G_{12} $q-1$
	G_{012} 1	

-

G_0 $D_{2(q-1)}$	G_1 $E_q : (q-1)$	G_2 D_{2p}
G_{01} $q-1$	G_{02} 2	G_{12} 2
	G_{012} 1	

with p a prime dividing α_q, β_q or $q-1$.

-

G_0 $D_{2(q-1)}$	G_1 D_{2p}	G_2 $D_{2p'}$
G_{01} 2	G_{02} 2	G_{12} 2
	G_{012} 1	

with p, p' two primes dividing α_q, β_q, or $q-1$, including the possibility $p = 2$.

Observe that for the smallest Suzuki simple group, i.e. $Sz(8)$, whose geometries are available in [21], every possibility mentioned by theorem 5.3 gives at least one firm, residually connected, flag-transitive and residually weakly primitive geometry.

5.2 Second case: $q-1$ is not a prime and $2e+1$ is a prime

In this case again, we have $G_0 \cong D_{2(q-1)}$. Every possible rank two residue for G_0 has a dihedral subgroup as one of its maximal parabolic subgroups. Thus we may assume that $G_{01} \cong D_{2n}$ where $n \mid_M q-1$. Since $2e+1$ is a prime, $Sz(q)$ does not have proper subgroups of Suzuki type. Then, by lemma 3.11, the only proper subgroup of $Sz(q)$ containing G_{01} is G_0. Thus we cannot find a subgroup G_1 to complete the geometry. Observe that in that case as in the preceding one, lemma 4.5 tells us that $Sz(q)$ does not have a RWPRI pre-geometry of rank ≥ 4. Thus we can now state the following theorem.

Theorem 5.4 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1$ is a prime but $q-1$ is not. Then G does not possess any* RWPRI *pre-geometry of rank ≥ 3.*

5.3 Third case: $2e+1$ is not a prime

This case is the most tricky one because the candidates for G_0 are $D_{2(q-1)}$ and $Sz(s)$ where $s = 2^{2f+1}$ and $\frac{2e+1}{2f+1}$ is a prime.

The case $G_0 \cong D_{2(q-1)}$

In this case, the maximal rank is 3 by lemma 4.5. Also, lemma 4.4 and theorem 5.1 imply that one of G_{01} or G_{02} must be a cyclic group. We may assume that G_{01} is maximal in G_0, and hence $G_{01} \cong (q-1)$. The sublattice is then partially given as follows, with $n \mid_M q-1$. $G_0 \cong D_{2(q-1)}$, $G_{01} \cong q-1$, $G_{02} \cong D_{2n}$ and $G_{012} \cong n$. The only subgroups that can contain G_{02} are then G_0 and subgroups isomorphic to a Suzuki group. Then G_2 is a Suzuki group and its residue is determined by theorem 5.2: we have $G_2 \cong Sz(s)$ with $q = s^p$ and p a prime, and $n = s-1$. This means $\frac{q-1}{s-1}$ is a prime. Lemma 3.4 yields then $2e+1 = p^a$. Now, examining the rank two geometries of $Sz(q)$ in theorem 5.2, we can determine the remaining parabolic subgroups : indeed, G_{12} must be a $E_s : (s-1)$-subgroup and by lemma 3.10, $G_1 \cong E_q : (q-1)$.

Lemma 5.1 *Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1$ is not a prime number. If $2e+1 = p^a$ and p is a prime, every* RWPRI *geometry of rank 3 with $G_0 = D_{2(q-1)}$ has a sublattice isomorphic to the following one.*

G_0 $D_{2(q-1)}$	G_1 $E_q : (q-1)$	G_2 $Sz(s)$
G_{01} $q-1$	G_{02} $D_{2(s-1)}$	G_{12} $E_s : (s-1)$
	G_{012} $s-1$	

If $2e+1 \neq p^a$ for some prime p, then there is no RWPRI *geometry of rank 3 with $G_0 \cong D_{2(q-1)}$.*

The case $G_0 \cong Sz(s)$

In order to have $Sz(s)$ maximal in $Sz(q)$, we must impose that $q = s^p$ with p a prime. Then $2e+1 = p(2f+1)$ for some positive integer f.

The rank two residue of G_0 is given by theorem 5.2. If $2f+1$ is a prime, table 2 gives the possible residues for G_0. When $s-1$ is not a prime, table 3 gives the possible residues for G_0. These residues appear also for the case when $s-1$ is a prime. Indeed, they do not depend

on the primality of $s-1$. Thus, we treat the case $s-1$ prime and the case $s-1$ not prime simultaneoulsy.
If $2f+1$ is not a prime, table 4 gives the possible residues for G_0.

(1) $2f+1$ is a prime :

In this case, the residue of G_0 is one of those appearing in table 2.

For each of these 11 possible residues, we examine whether or not it can be extended to a rank three residually weakly primitive geometry. We may assume without loss of generality that G_{01} is maximal in G_0. We recall that for residues 1 to 4, $s-1$ must be a prime.

Residue 1: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong D_{2(s-1)}$, $G_{02} \cong D_{2(s-1)}$ and $G_{012} \cong 2$. By lemma 3.11, G_1 must be a subgroup of $D_{2(q-1)}$. Its residue is then given by theorem 5.1. This yields $G_1 \cong D_{2(s-1)x}$ where $x \mid \frac{q-1}{s-1}$ is a prime. The same argument shows that $G_2 \cong D_{2(s-1)x'}$ where $x' \mid \frac{q-1}{s-1}$ is also a prime. Again by theorem 5.1, the subgroup $G_{12} \cong D_{2x}$ and $x = x'$. Since D_{2x} is self-normalizing in $Sz(q)$, we have $G_1 = G_2$, a contradiction.

Residue 2: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong D_{2(s-1)}$, $G_{02} \cong D_{2n}$ with n a prime divisor of s^2+1 and $G_{012} \cong 2$. Lemma 3.11 yields again $G_1 \leq D_{2(q-1)}$ and the residue of G_1 is given by theorem 5.1. This implies $G_1 \cong D_{2(s-1)x}$ with $x \mid q-1$ a prime and $G_{12} \cong D_{2x}$. Thus $G_2 = <G_{02}, G_{12}>$ is a $Sz(t)$-subgroup with $t-1$ a prime. Because of G_{02}, we must have $t \neq s$. This means $q = s^a = t^b$ and because $s-1$ (resp. $t-1$) is a prime, we know that $s = 2^{p_1}$ (resp. $t = 2^{p_2}$) with p_1 (resp. p_2) a prime. Thus $q = 2^{p_1 p_2}$, with p_1, p_2 two different odd primes. By lemma 3.7, we get $n = 5$.

Residue 3: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong D_{2(s-1)}$, $G_{02} \cong 2^2$ and $G_{012} \cong 2$. Again, by lemma 3.11, G_1 must be a subgroup of $D_{2(q-1)}$. Thus $G_1 \cong D_{2(s-1)x}$ and $G_{12} \cong D_{2x}$ with $x \mid q-1$ a prime as in the preceding case, and $G_2 = <2^2, D_{2x}>$ must be a subgroup of Suzuki type. So $G_2 \cong Sz(t)$ with $t \neq s$ and $x = t-1$. Also, $q = 2^{p_1 p_2}$, $s = 2^{p_1}$ and $t = 2^{p_2}$. Moreover, $s-1$ and $t-1$ must be primes.

Residue 4: Almost the same discussion as in the preceding case leads to $G_0 \cong Sz(s)$, $G_1 \cong D_{2(s-1)(t-1)}$, $G_2 \cong Sz(t)$, $G_{01} \cong D_{2(s-1)}$, $G_{02} \cong 4$, $G_{12} \cong D_{2(t-1)}$ and $G_{012} \cong 2$.

Residue 5: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong D_{2(s-1)}$, $G_{02} \cong E_s : (s-1)$ and $G_{012} \cong s-1$. Again, by lemma 3.11, G_1 must be a subgroup of $D_{2(q-1)}$. Thus, by theorem 5.1, $G_1 \cong D_{2(s-1)x}$ where $x \mid q-1$ is a prime. This implies that G_{12} may be isomorphic to either $(s-1)x$ or $D_{2(s-1)}$. Suppose $G_{12} \cong D_{2(s-1)}$. This yields $G_2 \cong Sz(s)$. Then, $G_{02} \cong E_s : (s-1)$ must be contained in two distinct $Sz(s)$, a contradiction. So $G_{12} \cong (s-1)x$. Then $G_2 = < E_s : (s-1), (s-1)x >$ is a subgroup of $E_q : (q-1)$. In order to have a residually weakly primitive geometry, we must impose $G_2 \cong E_q : ((s-1)x)$. A priori, the number x can be any prime divisor of $\frac{q-1}{s-1}$. So we get more than one geometry most of the time. Observe that when $x = \frac{q-1}{s-1}$ we get the geometry given in lemma 5.1.

Residue 6: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong \alpha_s{:}\,4$, $G_{02} \cong \alpha_s{:}\,4$ and $G_{012} \cong 4$. Since G_1 (resp. G_2) must contain a subgroup isomorphic to $\alpha_s{:}\,4$, it must either be a subgroup of Suzuki type or a $n:4$-subgroup by lemma 3.9. Assume first that $G_1 \cong Sz(t)$ for some t. Then two cases appear. The first one is with G_{01} maximal in G_1. This implies $s = t$ and thus $G_0 = G_1$, a contradiction. The second case is with G_{01} not maximal in G_1. Hence the residue of G_1 is of the form 7 or 10 in table 2 and G_{12} must be maximal in G_1. Moreover, by lemma 3.6 (2), one of G_{01} or G_{12} must be a subgroup of a $\alpha_q{:}\,4$ and the other must be a subgroup of a $\beta_q{:}\,4$. Also, $G_{02} \cong G_{01}$, so $G_2 = < G_{02}, G_{12} >$ must be a subgroup of

Suzuki type. Since we are in the case $2e+1=(2f+1)p$ where p and $2f+1$ are primes, it implies that either $G_0 \cong G_2$ or $G_1 \cong G_2$ and hence G_2 must be equal to either G_0 or G_1, a contradiction. So, the subgroup G_1 must be a subgroup of α_q: 4 or β_q: 4 depending whether α_s divides α_q or β_q. This means $G_1 \cong (\alpha_s x):4$ and $G_{12} \cong x:4$ with x a prime such that $x\alpha_s$ divides either α_q or β_q as α_s does, and $x \neq \alpha_s$. Thus $G_2 = <\alpha_s: 4, x:4>$ is isomorphic to G_1. Then by lemma 3.9, we have $G_2 = G_1$, a contradiction.

Residue 7: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong \alpha_s$: 4, $G_{02} \cong n:4$ with n a prime divisor of β_s, and $G_{012} \cong 4$. The same kind of arguments as in the preceding case lead to $G_1 \cong \alpha_s x:4$ and $G_{12} \cong x:4$, with x a prime number such that $x\alpha_s$ divides either α_q or β_q as α_s does and $x \neq \alpha_s$. Then, by lemma 3.6, $G_2 = <n:4, x:4> = Sz(q)$, a contradiction.

Residue 8: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong \alpha_s$: 4, $G_{02} \cong 4\times 2$ and $G_{012} \cong 4$. The same arguments as in the preceding case lead to $G_1 \cong \alpha_s x:4$ and $G_{12} \cong x:4$, with x a prime number such that $x\alpha_s$ divides α_q or β_q and $x \neq \alpha_s$. Then $G_2 = <4\times 2, x:4> \cong Sz(t)$ with $t \neq s$ and we must have $x = \alpha_t$ or β_t. Such a geometry exists, for example, with $s=2^3$, $t=2^5$ and $q=2^{15}$.

Residues 9 to 11: The discussion for residue 9 (resp. 10, 11) is almost the same as for residue 6 (resp. 7, 8). We just have to read β_s instead of α_s in these cases.

We summarize the preceding discussion at the end of this section in theorem 5.5.

(2) $2f+1$ is not a prime :

In this case, $s=2^{2f+1}$ and $2f+1$ is not a prime thus $s-1$ is not a prime either. The residue of G_0 is one of those appearing in table 4.

Residue 12: The same discussion as in the case of residue 5 leads to $G_0 \cong Sz(s)$, $G_1 \cong D_{2(s-1)x}$ with $x \mid \frac{q-1}{s-1}$ a prime, $G_2 \cong E_q:((s-1)x)$, $G_{01} \cong D_{2(s-1)}$ $G_{02} \cong E_s:(s-1)$, $G_{12} \cong (s-1)x$ and $G_{012} \cong s-1$. Observe that, when $x = \frac{q-1}{s-1}$, we get the geometry given in lemma 5.1.

Residue 13: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong D_{2(s-1)}$, $G_{02} \cong Sz(t)$ with $t=2^{2g+1}$ and $2g+1$ a maximal divisor of $2f+1$, and $G_{012} \cong D_{2(t-1)}$. Also, $t-1$ must be a maximal divisor of $s-1$. By lemma 3.4, we have $t=2^{p^a}$ and $s=t^p$. By lemma 3.11, G_1 is a subgroup of $D_{2(q-1)}$, thus $G_1 \cong D_{2(s-1)x}$ and $G_{12} \cong D_{2x}$ with x a prime that divides $\frac{q-1}{s-1}$. Then $G_2 = <Sz(t), D_{2x}>$ is a subgroup of Suzuki type. Let $G_2 \cong Sz(s')$. We must have $s' \neq s$, thus $q=2^{(p^a \cdot p')}$ and $s'=2^{(p^{a-1}p')}$. This implies that the residue of G_2 is of type 18, and $x \mid s'-1$. Since x can take more than one value, there might be more than one geometry in that case.

Residue 14: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong \alpha_s$: 4, $G_{02} \cong Sz(t)$ with $t=2^{2g+1}$ and $2g+1$ a maximal divisor of $2f+1$, and $G_{012} \cong n:4$ where n is either α_t or β_t. By lemma 3.6, $Sz(t)$ is a maximal subgroup of $Sz(s)$. This implies that $G_2 \cong Sz(s')$ must also contain $Sz(t)$ as maximal subgroup and that $s' \neq s$. Thus its residue is such that $G_{12} \cong n':4$ where n' divides s'^2+1 and n is a maximal divisor of n'. Also, $G_1 = \frac{\alpha_s n'}{n}:4$.

Residue 15: Let $q=2^{2e+1}=2^{(2g+1)pp'}$, $s=2^{2f+1}=2^{(2g+1)p}$ and $s'=2^{(2g+1)p'}$ Let $t=2^{2g+1}$ and n be either α_t or β_t. Take n' a divisor of s'^2+1 such that n is a maximal divisor of n'. Almost the same discussion as in the preceding case leads to $G_0 \cong Sz(s)$, $G_1 \cong (\frac{\beta_s n'}{n}):4$, $G_2 \cong Sz(s')$, $G_{01} \cong \beta_s$: 4, $G_{02} \cong Sz(t)$ with $t=2^{2g+1}$ and $2g+1$ a maximal divisor of $2f+1$, $G_{12} \cong n':4$ and $G_{012} \cong n:4$ where n is either α_t or β_t.

Residue 16: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$ with $t=2^{2g+1}$ and $2g+1$ a maximal divisor of $2f+1$, $G_{02} \cong E_s{\cdot}E_t:(t-1)$ and $G_{012} \cong E_t{\cdot}E_t:(t-1)$. Since G_{01} is self-normalizing in $Sz(q)$, the subgroup G_1 must be a subgroup of Suzuki type, say $Sz(s')$ with $s' \neq s$. Thus its residue is given by table 4. It implies that $G_{12} \cong E_{s'}{\cdot}E_t : (t-1)$. Then $G_2 = < E_s{\cdot}E_t : (t-1), E_{s'}{\cdot}E_t : (t-1) >$ is a subgroup of $E_q{\cdot}E_q : (q-1)$. In order to have a residually weakly primitive geometry, we must be sure that either G_{02} or G_{12} is a maximal subgroup of G_2. This implies $G_2 \cong E_q{\cdot}E_t : (t-1)$.

Residue 17: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$ with $t = 2^{2g+1}$ and $2g+1$ a maximal divisor of $2f+1$, $G_{02} \cong Sz(t')$ with $t' = 2^{2g'+1}$ and $2g'+1$ a maximal divisor of $2f+1$ and $G_{012} \cong Sz(u)$ with $u = 2^{2h+1}$ and $2h+1$ a maximal divisor of $2g+1$ and $2g'+1$. It is obvious that the only geometry we can obtain here is with all subgroups of the sublattice isomorphic to subgroups of Suzuki type.

Residue 18: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$ with $t = 2^{2g+1}$ and $2g+1$ a maximal divisor of $2f+1$, $G_{02} \cong D_{2(t-1)x}$ with x a prime divisor of $\frac{s-1}{t-1}$ and $G_{012} \cong D_{2(t-1)}$. Since G_{01} is of Suzuki type, we have $G_1 \cong Sz(s')$ with $s' \neq s$. Thus the residue of G_1 is well known. Then $G_{12} \cong D_{2m}$ where $t-1 \,|_M\, m \,|_P\, s'-1$. Also, $G_2 = < D_{2n}, D_{2m} > \cong D_{2\frac{nm}{t-1}}$. Here $n \neq m$ by theorem 5.1.

Residue 19: In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$ with $t = 2^{2g+1}$ and $2g+1$ a maximal divisor of $2f+1$, $G_{02} \cong n : 4$ with $t-1 \,|_M\, n \,|_P\, s-1$ and $G_{012} \cong \alpha_t : 4$. Again, $G_1 \cong Sz(s')$ with $s' \neq s$. Thus $G_{12} \cong m : 4$ with $\alpha_t \,|_M\, m \,|_P\, \alpha_{s'}$ or $\beta_{s'}$ and $G_2 \cong \frac{nm}{\alpha_t} : 4$.

Residue 20: Almost the same discussion as in the preceding case leads to $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$ with $t = 2^{2g+1}$ and $2g+1$ a maximal divisor of $2f+1$, $G_{02} \cong n : 4$ with $t-1 \,|_M\, n \,|_P\, s-1$, $G_{012} \cong \beta_t : 4$, $G_1 \cong Sz(s')$ with $s' \neq s$, $G_{12} \cong m : 4$ with $\beta_t \,|_M\, m \,|_P\, \alpha_{s'}$ or $\beta_{s'}$ and $G_2 \cong \frac{nm}{\beta_t} : 4$.

This ends the classification of all RWPRI geometries of rank 3 for the Suzuki groups. We now summarize the preceding discussion in a theorem.

Theorem 5.5 *Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1 = p_1^{e_1} \dots p_n^{e_n}$ with $p_i \neq p_j$, $\forall i \neq j$, and $\sum_{i=1}^n e_i \geq 2$. Suppose $q = s^{p_i}$ for some $i \in \{1 \dots n\}$. If $s-1$ is a prime (which implies that $\sum_{i=1}^n e_i = 2$) then all rank 3 RWPRI geometries of G have a sublattice isomorphic to one of the following.*

1.

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(t)$
G_{01} $D_{2(s-1)}$	G_{02} D_{10}	G_{12} $D_{2(t-1)}$
	G_{012} 2	

with $s = 2^{p_1}$, $t = 2^{p_2}$, $2e+1 = p_1 \cdot p_2$, and $p_1 \neq p_2$ two primes.

2.

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(t)$
G_{01} $D_{2(s-1)}$	G_{02} 2^2	G_{12} $D_{2(t-1)}$
	G_{012} 2	

with $s = 2^{p_1}$, $s' = 2^{p_2}$, $2e+1 = p_1 \cdot p_2$, and $p_1 \neq p_2$ two primes.

3.

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(t)$
G_{01} $D_{2(s-1)}$	G_{02} 4	G_{12} $D_{2(t-1)}$
	G_{012} 2	

with $s = 2^{p_1}$, $s' = 2^{p_2}$, $2e+1 = p_1 \cdot p_2$, are $p_1 \neq p_2$ two primes.

4.

G_0 $Sz(s)$	G_1 $D_{2(s-1)p}$	G_2 $q:((s-1)p)$
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{12} $(s-1)p$
	G_{012} $s-1$	

with $p \mid \frac{q-1}{s-1}$ a prime.

5.

G_0 $Sz(s)$	G_1 $(ap):4$	G_2 $Sz(t)$
G_{01} $a:4$	G_{02} 4×2	G_{12} $p:4$
	G_{012} 4	

with $a = \alpha_s$ or β_s (resp. $p = \alpha_t$ or β_t) primes, $n = 2$, $q = s^{p_1} = t^{p_2}$ and $p_1 \neq p_2$.

If $s-1$ is not a prime but $\sum_{i=1}^{n} e_i = 2$ then all rank 3 RWPRI *geometries of G have a sublattice isomorphic to sublattices number 4 or 5 of the case when $s-1$ is a prime.*
Finally, if $\sum_{i=1}^{n} e_i > 2$ then all rank 3 RWPRI *geometries of G have a sublattice isomorphic to one of the following.*

6.

G_0 $Sz(s)$	G_1 $D_{2(s-1)p}$	G_2 $q:((s-1)p)$
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{12} $(s-1)p$
	G_{012} $s-1$	

with $p \mid \frac{q-1}{s-1}$ a prime.

7.

G_0 $Sz(s)$	G_1 $D_{2(t-1)pp'}$	G_2 $Sz(s')$
G_{01} $D_{2(t-1)p}$	G_{02} $Sz(t)$	G_{12} $D_{2(t-1)p'}$
	G_{012} $D_{2(t-1)}$	

with $p \mid \frac{s-1}{t-1}$, $p' \mid \frac{s'-1}{t-1}$ two distinct primes, $s = t^{p_i}$ and $s' = t^{p_j}$, with $i \neq j$.

8.

G_0 $Sz(s)$	G_1 $(mpp'):4$	G_2 $Sz(s')$
G_{01} $mp:4$	G_{02} $Sz(t)$	G_{12} $mp':4$
	G_{012} $m:4$	

with $m = \alpha_t$ *or* β_t, $mp \mid \alpha_s$ *or* β_s, $mp' \mid \alpha_{s'}$ *or* $\beta_{s'}$, *and* p, p' *two primes.*

9.

G_0 $Sz(s)$	G_1 $q{:}t:(t-1)$	G_2 $Sz(s')$
G_{01} $s{:}t:(t-1)$	G_{02} $Sz(t)$	G_{12} $s'{:}t:(t-1)$
	G_{012} $t{:}t:(t-1)$	

with $q = s^{p_i} = s'^{p_j} = t^{p_ip_j}$, *and* $i \neq j$.

10.

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $Sz(s")$
G_{01} $Sz(t)$	G_{02} $Sz(t')$	G_{12} $Sz(t")$
	G_{012} $Sz(u)$	

with $\sum_{i=1}^{n} e_i \geq 4$, $n \geq 3$, $q = s^{p_i} = s'^{p_j} = s"^{p_k} = t^{p_ip_j} = t'^{p_ip_k} = t"^{p_jp_k} = u^{p_ip_jp_k}$, *and* i, j, k *three pairwise distinct numbers.*

6 Some concluding remarks

About the flag-transitivity :

Most of the time, we are interested in knowing whether a RWPRI (pre-)geometry is a flag-transitive geometry or not. We do not intend to solve that problem in the present paper. We even have doubts about the complete solvability of that problem. Anyway, lemma 3.8 permits us to give the answer for a lot of geometries obtained in the previous section. For the case when $q-1$ is a prime, looking at theorem 5.3, we see that geometries corresponding to the second type, and geometries corresponding to the third type with $p=2$ are indeed flag-transitive. For the case when $2e+1$ is not prime, looking at theorem 5.5, we see that all geometries, except number 9 and 10, have a linear diagram and are thus surely flag-transitive.

About the $(2T)_1$ property :

If we look at the rank 3 RWPRI geometries that satisfy the $(2T)_1$ property, we see quite easily that in theorem 5.3, the only geometries that survive this test are the thin geometries, and in theorem 5.5, no geometry satisfies this test. This means that, if we want to construct a rank 4 RWPRI pre-geometry satisfying $(2T)_1$ for a Suzuki group $Sz(q)$, we have to choose $q = 2^{pp'}$ with p, p' two primes. Then G_0 is a subgroup of Suzuki type because of lemma 4.6. It means that $G_0 = Sz(s)$ must have a rank 3 RWPRI geometry satisfying $(2T)_1$. This implies that $s-1$ is a prime and we may assume that $G_{01} = D_{2(s-1)}$. Thus G_1 is either a group of Suzuki type or a dihedral group. It cannot be a dihedral group D_{2n} with n an odd number, because no such dihedral group has RWPRI pre-geometries of rank 3 or higher satisfying $(2T)_1$ thanks to the classification of the RWPRI geometries for these groups given in [20] (we remind again here that all RWPRI pre-geometries of a dihedral group are flag-transitive geometries). So it must be a $Sz(t)$ with $t \neq s$. But then $s-1=t-1$ and $s \neq t$, a contradiction. This permits to state the following theorem.

Theorem 6.1 *Let $G = Sz(q)$ with $q = 2^{2e+1}$.*
If $q-1$ is not a prime, $Sz(q)$ has only one RWPRI *and $(2T)_1$ pre-geometry up to isomorphism. It is a rank 2 F, RC, FT geometry with $G_0 = D_{2(q-1)}$, $G_1 = q:(q-1)$ and $G_{01} = q-1$.*
If $q-1 \neq 7$ is a prime, the RWPRI *and $(2T)_1$ pre-geometries of G are of rank ≤ 3, one of them only being of rank 2, the others being thin rank 3 geometries.*
Finally, if $q = 8$, the RWPRI *and $(2T)_1$ pre-geometries of G are of rank ≤ 3, three of them only being of rank 2, the others being thin rank 3 geometries.*

We would like to conclude this paper by pointing out that the results obtained in theorems 5.3, 5.4 and 5.5 give only the possible rank three RWPRI sublattices for a Suzuki simple group $Sz(q)$. We did not prove that for all these sublattices it is possible to find three subgroups of $Sz(q)$ meeting as wanted. We already observed that the group $Sz(8)$ has at least one geometry for every sublattice mentioned by theorem 5.3.

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