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Existence and uniqueness theorem for Frenet frame supercurves

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Abstract. In the first part of this paper, using the Banach Grassmann algebra B_L given by Rogers in her paper [10], a new scalar product and a new definition of the orthogonality are introduced on the (m,n)-dimensional total supereuclidean space ${B_L}^{m+n}$. Using the GH^∞ functions given by Rogers in [10], the new definitions of the supercurve, of the supersmooth supercurve, of the supersmooth supercurve in general position and of the Frenet frame associated to a supersmooth supercurve in general position are given.In the second part of this paper, using the classical results described in [9], the new existence and uniqueness theorem for some supercurves which admit Frenet frame is proved.

Keywords: (m, n)-dimensional total supereuclidean space ${B_L}^{m+n}$, the (m, n)-dimensional supereuclidean space $B_L^{m,n}$, the GH^∞ functions, supersmooth supercurve, supersmooth supercurve in general position, Frenet frame associated to a supersmooth supercurve, Frenet formulas for the supersmooth supercurve.

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1 Supersmooth supercurve in general position and Frenet frame associated to a supersmooth supercurve in general position

Let us consider only algebras over the reals. For each positive integer L, B_L [10] will denote the Grassmann algebra over the reals with generators $1^{(L)}$,$\beta_1^{(L)}, \dots, \beta_L^{(L)}$ and relations

$$1^{(L)}\beta_i^{(L)} = \beta_i^{(L)}1^{(L)} = \beta_i^{(L)} \quad i = 1, \dots, L, \tag{1.1}$$

$$\beta_i^{(L)}\beta_j^{(L)} = -\beta_j^{(L)}\beta_i^{(L)} \quad i, j = 1, \dots, L. \tag{1.2}$$

B_L is a graded algebra [12] and can be written as a direct sum

$$B_L = (B_L)_0 \oplus (B_L)_1$$

where $(B_L)_0$ and $(B_L)_1$ are the even and odd part of (B_L) respectively. Let M_L denote (due to Kostant [7]) the set of finite sequences of positive integers

$\mu = (\mu_1, \ldots, \mu_k)$ with $1 \leq \mu_1 < \cdots < \mu_k \leq L$. M_L includes the sequence with no elements, denoted ϕ. As it follows in [10] for each μ in M_L,

$$\beta_\mu^{(L)} := \beta_{\mu_1}^{(L)} \cdots \beta_{\mu_k}^{(L)}, \tag{1.3}$$

and

$$\beta_\phi^{(L)} := 1^{(L)} \tag{1.4}$$

a typical element b of B_L may be expressed as

$$b - \sum_{\mu \in M_L} b^\mu \beta_\mu^{(L)}, \tag{1.5}$$

where the coefficient b^μ are real numbers. We consider the body map (in DeWitt's terminology [4])

$$\varepsilon^{(L)} : B_L \to \mathbf{R}$$

given by

$$\varepsilon^{(L)}(b) = b^\phi. \tag{1.6}$$

With the norm on B_L defined by

$$\|b\| := \sum_{\mu \in M_L} |b^\mu|, \tag{1.7}$$

B_L is a Banach algebra [11]. Considering L' also a positive integer, with $L \geq L'$, then there is a natural injection $i_{L',L} : B_{L'} \to B_L$ [10], which is the unique algebra homomorphism satisfying

$$i_{L',L}(\beta_i^{(L')}) = \beta_i^{(L)} \quad i = 1, \ldots, L, \quad i_{L',L}(1^{(L')}) = 1^{(L)}. \tag{1.8}$$

B_L naturally has a $B_{L'}$ module structure [10] with, given $a \in B_{L'}$ and $b \in B_L$,

$$ab := i_{L',L}(a)b. \tag{1.9}$$

We define the $(m, n)-$dimensional total supereuclidean space ${B_L}^{m+n}$ [1] as the space which is the cartesian product of $m + n$ copies of B_L and has the graduation

$${B_L}^{m+n} = ({B_L}^{m+n})_0 \oplus ({B_L}^{m+n})_1$$

A typical element of ${B_L}^{m+n}$ is written $(x^1, \ldots, x^m. 6^1, \ldots, 6^n)$ or simply $(x, 6)$, an element of $({B_L}^{m+n})_0$ is called $c-$type or even element and is written in the form $(x'^1, \ldots, x'^m, 6'^1, \ldots, 6'^n)$ with $x'^1, \ldots, x'^m \in (B_L)_0$ and $6'^1, \ldots, 6'^n \in (B_L)_1$ and an element of $({B_L}^{m+n})_1$ is called $a-$type or odd element and is written in the form $(x''^1, \ldots, x''^m, 6''^1, \ldots, 6''^n)$ with $x''^1, \ldots, x''^m \in (B_L)_1$ and $6''^1, \ldots, 6''^m \in (B_L)_0$. An even element has the parity 0 and an odd element has the parity 1.

1 Definition. [4] W is called the supervector space over the space B_L if and only if W is the supervector space given by the $1)-5)$ conditions from the DeWitt's definition (see DeWitt's book pages 14-15) where instead of Λ_∞ we put B_L.

2 Example. The $(m,n)-$dimensional total supereuclidean space ${B_L}^{m+n}$ with the above graduation

$${B_L}^{m+n} = ({B_L}^{m+n})_0 \oplus ({B_L}^{m+n})_1$$

is a supervector space of dimension (m,n).

3 Remark. Let us consider the $(m,n)-$dimensional supereuclidean space $B_L^{m,n}$ given by Rogers in [10]. We note that the space $B_L^{m,n}$ is not a supervector space over B_L as in Definition 1.

A useful map is [10]

$$\varepsilon^{(L)}_{m,n} : ({B_L}^{m+n})_0 \to \mathbf{R^m}$$

with

$$\varepsilon^{(L)}_{m,n}(x'^1, \dots, x'^m, 6'^1, \dots, 6'^n) := (\varepsilon^{(L)}(x'^1), \dots, \varepsilon^{(L)}(x'^m)) \tag{1.10}$$

and another useful map is

$$\varepsilon'^{(L)}_{m,n} : ({B_L}^{m+n})_1 \to \mathbf{R^n}$$

with

$$\varepsilon'^{(L)}_{m,n}(x''^1, \dots, x''^m, 6''^1, \dots, 6''^n) := (\varepsilon^{(L)}(6''^1), \dots, \varepsilon^{(L)}(6''^n). \tag{1.11}$$

4 Remark. Let us consider the $(1,1)-$dimensional total supereuclidean space ${B_L}^2$, $(2,0)$-dimensional total supereuclidean space ${B_L}^2$ and $(0,2)$-dimensional total supereuclidean space ${B_L}^2$ and the element $(1,0)$ which belongs to these three spaces. We note that the element $(1,0)$ is c-type for the first two spaces and is a-type for the last space. We may write the supervector $(1,0)$ in a standard basis [4] in the form:

$$(1,0) = 1 \cdot (1,0) + 0 \cdot (0, \beta^1)$$

for the $(1,1)-$dimensional total supereuclidean space ${B_L}^2$ and $(0,2)$-dimensional total supereuclidean space ${B_L}^2$ where a standard basis [4] in these spaces is $\{(1,0),(0,\beta^1)\}$ with $(1,0)$ $c-$type supervector and $(0,\beta^1)$ $a-$type supervector and the supervector $(1,0)$ can be written in a standard basis [4] in the form:

$$(1,0) = 1 \cdot (1,0) + 0 \cdot (0,1)$$

for the (2, 0)-dimensional total supereuclidean space ${B_L}^2$, where a standard basis [4] in this space is $\{(1,0),(0,1)\}$ with $(1,0)$ c-type supervector and $(0,1)$ a-type supervector

5 Definition. [10] Suppose $U \subset \mathbf{R}^m$ is open and L' is a positive integer with $L' \leq L$. Let $C^\infty(U, B_{L'})$ denote the $B_{L'}$ module of C^∞ functions of U into $B_{L'}$; (recall that $B_{L'}$ is a Banach algebra, and hence *a fortiori* a Banach space). Then the map

$$Z_{L',L} : C^\infty(U, B_{L'}) \to [\varepsilon^{(L)}_{m,0}(U)]^{B_L}$$

is defined by [10]

$$Z_{L',L}(f)(x^1,\ldots,x^m) = \sum_{i_1=0\ldots i_m=0}^{L} \frac{1}{i_1!\cdots i_m!}\cdot$$
$$\cdot\, i_{L',L}(\partial_1^{i_1}\ldots\partial_m^{i_m} f(\varepsilon^{(L)}(x^1),\ldots,\varepsilon^{(L)}(x^m))) \times s(x^1)^{i_1}\cdots s(x^m)^{i_m} \tag{1.12}$$

where [10]

$$s(x^i) = x^i - \varepsilon^{(L)}(x^i)1, \quad i = 1,\ldots,m. \tag{1.13}$$

6 Definition. [10] Suppose V is open in $B_L^{m,n}$ (with respect to its usual finite-dimensional vector space topology) and $U = \varepsilon^{(L)}_{m,n}(V)$, suppose $L > 2n$ and $L' = [\frac{1}{2}L]$, the last integer not less than $\frac{1}{2}L$. $GH^\infty(V)$ denotes the set of functions $f : V \to B_L$ for which there exist [10] $f_\mu \in C^\infty(U, B_{L'})$ such that

$$f(x,6) = \sum_{\mu\in M_n} Z_{L',L}(f_\mu)(x)6^\mu \tag{1.14}$$

where

$$6^\mu = 6^{\mu_1}\cdots 6^{\mu_k} \tag{1.15}$$

and

$$6^\phi = 1^{(L)}. \tag{1.16}$$

7 Definition. [10] With the notation of Definition 5, let f be an element of $GH^\infty(V)$, with expansion (1.12). Then, for $i = 1,\ldots,m$

$$G_i f : V \to B_L$$

is defined by [10]

$$G_i f(x;6) = \sum_{\mu\in M_n} Z_{L',L}(\partial_i f_\mu)(x)6^\mu. \tag{1.17}$$

Also, for $j = 1,\ldots,n$,

$$G_{j+m} f : V \to B_L$$

is defined by [10]

$$G_{j-m}f(x;6) = \sum_{\mu \in M_n} Z_{L',L}(f_\mu)(x) 6^{\mu/j} \times (-1)^{|f_\mu(x)|}, \tag{1.18}$$

where $|f_\mu(x)|$ is the Grassmann parity of $f_\mu(x)$, and

$$6^{\mu/j} = 6^{\mu_1} \cdots 6^{\mu_{i-1}} 6^{\mu_{i+1}} \cdots 6^{\mu_k} (-1)^{i-1},$$

if $j = \mu_i$ for some i, $1 \le i \le k$, $6^{\mu/j}$ otherwise.

For the first time, I have introduced on ${B_L}^{m+n}$ with $n = 2r$, the scalar product

$$\langle v, w \rangle = x^1 y^1 + \cdots + x^m y^m + 6^1 {6_1}^{r+1} + \cdots + 6^r {6_1}^n - 6^{r+1} {6_1}^1 - \cdots - 6^n {6_1}^r$$

$(\forall)$ $v = (x^1, \dots, x^m, 6^1, \dots, 6^n), w = (y^1, \dots, y^m, {6_1}^1, \dots, {6_1}^n) \in {B_L}^{m+n}$ which has these properties:

a) $\langle v, w \rangle = (-1)^{|v| \cdot |w|} \langle w, v \rangle$ (supersymmetry)

b) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ $(\forall)$ $u, v, w \in {B_L}^{m+n}$ (linearity)

c) $\langle v, \cdot \rangle = 0$ if and only if $v = 0$.

8 Proposition (Number of scalar products on ${B_L}^{m+n}$). *On ${B_L}^{m+n}$ with $n = 2r$, we may give $r!$ different scalar products with $r \ge 1$.*

PROOF. There are $r!$ one-to-one functions $f : \{1, \dots, r\} \to \{r+1, \dots, 2r\}$. For each function f we have the following scalar product between $v = (x^1, \dots, x^m, 6^1, \dots, 6^n)$ and $w = (y^1, \dots, y^m, {6_1}^1, \dots, {6_1}^n)$

$$\langle v, w \rangle_f = \sum_{k=1}^{m} x^k y^k + \sum_{j_1=1}^{r} (6^{j_1} {6_1}^{f(j_1)} - 6^{f(j_1)} {6_1}^{j_1}).$$

One can easily verify the $a), b)$ and c relations of a scalar product. Let us prove $a)$ in the case when v and w are odd elements of ${B_L}^{m+n}$, that is, v and w belong to $({B_L}^{m+n})_1$. Then, we shall have $v = (x''^1, \dots, x''^m, 6''^1, \dots, 6''^n)$ and $w = (y''^1, \dots, y''^m, 6''^1_1, \dots, 6''^n_1)$. When we compute the scalar product between v and w, we shall get

$$\langle v, w \rangle_f = \sum_{k=1}^{m} x''^k \cdot y''^k + \sum_{j_1=1}^{r} (6''^j_1 6''^{f(j_1)}_1 - 6''^{f(j_1)} 6''^{j_1}_1) =$$

$$= - \sum_{k=1}^{n} y''^k \cdot x''^k + \sum_{j_1=1}^{r} (6''^{f(j_1)}_1 6''^{j_1} - 6''^{j_1}_1 6''^{f(j_1)})$$

Now, the scalar product between v and w becomes

$$\langle v, w\rangle_f = -(\sum_{k=1}^{n} y''^k \cdot x''^k + \sum_{j_1=1}^{r}(6_1''^{j_1} 6''^{f(j_1)} - 6_1''^{f(j_1)} 6''^{j_1}))$$

Thus, we get

$$\langle v, w\rangle_f = -\langle w, v\rangle_f.$$

This ends the proof. QED

9 Definition (Orthogonality on $B_L{}^{m+n}$). We say that the supervector v of $B_L{}^{m+n}$ is orthogonal to the supervector w of $B_L{}^{m+n}$ if and only if $\varepsilon^{(L)}(\langle v, w\rangle) = 0$.

The column supervectors $E_1 = (1, \dots, 0)$, ..., $E_m = (0, \dots, 1, 0, \dots, 0)$, where 1 is written on the m^{th} place, $E_{m+1} = (0, \dots, 0, -1, 0, \dots, 0)$, where -1 is written on the $(m + r + 1)^{th}$ place, ..., $E_{m+r} = (0, \dots, 0, -1)$, where -1 is written on the $(m + n)^{th}$ place, $E_{m+r+1} = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 is written on the $(m + 1)^{th}$ place, ..., $E_{m+n} = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 is written on the $(m + r)^{th}$ place, form the standard basis on $B_L{}^{m+n}$ [4] where the first m supervectors are c—type and the last n supervectors are a—type.

10 Definition (Supersmooth supercurve). [10] Let suppose $L > 2n$ and let $B_L{}^{m+n}$ be an (m, n)—dimensional total supereuclidean space, let V be an open set in $B_L^{1,1}$, let $c : V \subset B_L^{1,1} \to B_L{}^{m+n}$ be a function, and for every $6 \in V \cap (B_L)_1$ let define

$$c_{6,0} : V \cap (B_L)_0 \to (B_L{}^{m+n})_0$$

given by

$$c_{6,0}(t) = (c(t, 6))_0$$

where $(c(t, 6))_0$ is the even part of the supervector $c(t, 6)$ and

$$c_{6,B} : V \cap (B_L)_0 \to \mathbf{R}^m$$

given by

$$c_{6,B}(t) = \varepsilon^{(L)}_{m,n} \circ c_{6,0}(t)$$

for all $t \in V \cap (B_L)_0$. The function c will be said to be supercurve if and only if $c_{6,B}|_{V\cap\mathbf{R}}$ will be a curve. The function c is called supersmooth if and only if $c^i \in GH^\infty(V)$ $(\forall)$ $i \in \{1, \dots, m\}$ and $c^{j+m} \in GH^\infty(V)$ $(\forall)$ $j \in \{1, \dots, n\}$ where $c^i = x^i \circ c$ $(\forall)$ $i \in \{1, \dots, m\}$ and $c^{j+m} = 6^j \circ c$ $(\forall)$ $j \in \{1, \dots, n\}$.

11 Definition (Supercurve in general position). Let suppose $L > 2n$ and let ${B_L}^{m+n}$ be an $(m,n)-$dimensional total supereuclidean space, let V be an open set in $B_L^{1,1}$ and let $c : V \subset B_L^{1,1} \to {B_L}^{m+n}$ be a supersmooth supercurve. We say that the supercurve c is in general position if and only if

$$\{G_1c(t,6), \dots, G_1^{(m-1)}c(t,6), G_2c(t,6), G_1G_2c(t,6), \dots \\ \dots, G_1^{(n-1)}G_2c(t,6)\}$$

are linear independent $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$ and where by $G_1c(t,6)$ we understand the supervector

$$(G_1c^1(t,6), \dots, G_1c^m(t,6), G_1c^{m+1}(t,6), \dots, G_1c^{m+n}(t,6)) \\ (\forall)\ (t,6) \in V \subset B_L^{1,1}$$

by $G_2c(t,6)$ we understand the supervector

$$(G_2c^1(t,6), \dots, G_2c^m(t,6), G_2c^{m+1}(t,6), \dots, G_2c^{m+n}(t,6)) \\ (\forall)\ (t,6) \in V \subset B_L^{1,1},$$

with $G_1^{(s)}c(t,6) = G_1 \cdots G_1c(t,6)$, where $G_1^{(0)}c(t,6) = c(t,6)$ and $G_1^{(1)}c(t,6) = G_1c(t,6)$ and "$\cdots$" means that G_1 is applied by s times.

12 Definition (Frenet Frame associated to a supersmooth supercurve). Let suppose $L > 2n$ and let ${B_L}^{m+n}$ be an $(m,n)-$dimensional total supereuclidean space, let V be an open set in $B_L^{1,1}$ and let $c : V \subset B_L^{1,1} \to {B_L}^{m+n}$ be a supersmooth supercurve. By a Frenet frame associated to a supersmooth supercurve $c : V \subset B_L^{1,1} \to {B_L}^{m+n}$ we shall mean a system of $m+n$ supervector fields $\{e_1, \dots, e_{m+n}\}$ along to the supersmooth supercurve c such that $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$ we have the following properties:

$$\langle e_k(t,6), e_h(t,6)\rangle = \delta_{kh} \qquad (\forall)\ k,h \in \{1,\dots,m\} \tag{1.19}$$

$$\langle e_{m+r-j_1}(t,6), e_{m+j_2}(t,6)\rangle = -\delta_{j_1j_2} \qquad (\forall)\ j_1,j_2 \in \{1,\dots,r\} \tag{1.20}$$

$$\langle e_{m+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle = \delta_{j_1j_2} \qquad (\forall)\ j_1,j_2 \in \{1,\dots,r\} \tag{1.21}$$

$$\langle e_{m-j_1}(t,6), e_{m+j_2}(t,6)\rangle = 0, \qquad (\forall)\ j_1,j_2 \in \{1,\dots,r\} \tag{1.22}$$

$$\langle e_{m+r+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle = 0, \qquad (\forall)\ j_1,j_2 \in \{1,\dots,r\} \tag{1.23}$$

$$\langle e_i(t,6), e_{m+j}(t,6)\rangle = \langle e_{m+j}(t,6), e_i(t,6)\rangle = 0, \tag{1.24}$$

$(\forall)$ $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$

$$\text{span}(G_1c(t,6), \dots, G_1^{(k)}c(t,6)) = \text{span}(e_1(t,6), \dots, e_k(t,6)) \\ (\forall)\ k \in \{1,\dots,m-1\} \tag{1.25}$$

where $G_1^{(1)}c(t,6) - G_1c(t,6)$ and

$$\operatorname{span}(G_2c(t,6), G_1G_2c(t,6), \ldots, G_1^{(j-1)}G_2c(t,6)) = \\ = \operatorname{span}(e_{m+1}(t,6), \ldots, e_{m+j}(t,6)) \qquad (\forall)\ j \in \{1, \ldots, n\} \quad (1.26)$$

where $G_1^{(0)}G_2c(t,6) = G_2c(t,6)$ and $G_1^{(1)}G_2c(t,6) = G_1G_2c(t,6)$ and we shall mean by $\operatorname{span}(e_1(t,6), \ldots, e_k(t,6))$ the supervector space [4], spanned by $e_1(t,6)$, …, $e_k(t,6)$ $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$. The systems of supervectors

$$\{G_1c(t,6), \ldots, G_1^{(k)}c(t,6)\} \quad \text{and} \quad \{e_1(t,6), \ldots, e_k(t,6)\} \qquad (1.27)$$

are directed in the same way $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$ and $(\forall)$ $k \in \{1, \ldots, m-1\}$, that is, $\varepsilon^{(L)}(\det M_1) > 0$ where M_1 is the matrix when we change from the frame $\{G_1c(t,6), \ldots, G_1^{(k)}c(t,6)\}$ to the frame $\{e_1(t,6), \ldots, e_k(t,6)\}$ and the systems of supervectors

$$\{G_2c(t,6), G_1G_2c(t,6), \ldots, G_1^{(j-1)}G_2c(t,6)\} \\ \text{and} \quad \{e_{m+1}(t,6), \ldots, e_{m+j}(t,6)\} \quad (1.28)$$

are directed in the same way $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$ $(\forall)$ $j \in \{1, \ldots, n\}$,that is, $\varepsilon^{(L)}(\det M_2) > 0$ where M_2 is the matrix when we change from the frame

$$\{G_2c(t,6), G_1G_2c(t,6), \ldots, G_1^{(j-1)}G_2c(t,6)\}$$

to the frame

$$\{e_{m+1}(t,6), \ldots, e_{m+j}(t,6)\}.$$

The system of supervectors

$$\{e_1(t,6), \ldots, e_{m+n}(t,6)\} \qquad (1.29)$$

is positive directed $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$, that is, $\varepsilon^{(L)}(\operatorname{sdet}(e_q^s(t,6))) > 0$ where by $\operatorname{sdet}(e_q^s(t,6))$ we understand $\det(A - C \cdot B^{-1} \cdot D) \cdot (\det B)^{-1}$ [2], [8], [4], [1], [5] where

$$(e_q^s(t,6))_{1 \le s,q \le m+n} = \begin{pmatrix} A & C \\ D & B \end{pmatrix},$$

where A, B, C and D are $m \times m$, $n \times n$, $m \times n$ and $n \times m$ matrices with elements in B_L, respectively.

2 Existence and Uniqueness Theorem for Frenet Frame Supercurves

13 Theorem (Existence and Uniqueness Theorem for Frenet Frame Supercurves). *Let suppose $L > 2n$ and let ${B_L}^{m+n}$ be an $(m,n)-$dimensional total supereuclidean space, let V be an open set of $B_L^{1,1}$ and let $c : V \subset B_L^{1,1} \to {B_L}^{m+n}$ be a supersmooth supercurve in general position which satisfy the following relation:*

$$\varepsilon^{(L)}(\langle G_2c(t,6), G_1^{(r)}G_2c(t,6)\rangle) > 0 \tag{2.1}$$

$$\varepsilon^{(L)}(\langle G_1^{(j_1)}G_2c(t,6)), G_1^{(r+j_1)}G_2c(t,6)\rangle) > 0 \tag{2.2}$$

$(\forall)\ j_1 \in \{1,\dots,r-1\}$, *and* $(\forall)\ (t,6) \in V \subset B_L^{1,1}$,

$$\varepsilon^{(L)}(\langle G_2c(t,6), G_1^{(j)}G_2c(t,6)\rangle) = 0, \tag{2.3}$$

$(\forall)\ j \in \{0,\dots,n-1\}$, *with* $j \neq r$ *and* $(\forall)\ (t,6) \in V \subset B_L^{1,1}$,

$$\varepsilon^{(L)}(\langle G_1^{(j')}G_2c(t,6)), G_1^{(j)}G_2c(t,6)\rangle) = 0 \tag{2.4}$$

$(\forall)\ j' \subset \{1,\dots,n\quad 1\}$, $(\forall)\ j \subset \{1,\dots,n\quad 1\}$ *with* $j \neq j' \mid r$ *and* $j' < j$ *and* $(\forall)\ (t,6) \in V \subset B_L^{1,1}$. *Then there exists a unique Frenet frame* $\{e_1,\dots,e_{m+n}\}$ *associated to the supercurve c and we have the following formulas* $(\forall)\ (t,6) \in V \subset B_L^{1,1}$*:*

$$G_1e_k(t,6) - \sum_{h=1}^{m} a_{kh}(t,6)\cdot e_h(t,6) \qquad (\forall)\ k \in \{1,\dots,m\}, \tag{2.5}$$

where

$$a_{kh}(t,6) + a_{hk}(t,6) = 0 \qquad (\forall)\ k,h \in \{1,\dots,m\} \tag{2.6}$$

and

$$a_{kh}(t,6) = 0 \qquad \text{if} \qquad h > k+1, \qquad (\forall)\ k,h \in \{1,\dots,m\} \tag{2.7}$$

$$G_1e_{m+j}(t,6) = \sum_{l=1}^{n} a_{m+j\ m+l}(t,6)\cdot e_{m+l}(t,6) \qquad (\forall)\ j \in \{1,\dots,n\}, \tag{2.8}$$

where

$$a_{m+j_1\ m+j_2}(t,6) + a_{m+r+j_2\ m+r+j_1}(t,6) = 0 \qquad (\forall)\ j_1, j_2 \in \{1,\dots,r\}, \quad (2.9)$$

$$a_{m+r+j_1\ m+j_2}(t,6) - a_{m+r-j_2\ m+j_1}(t,6) = 0 \qquad (\forall)\ j_1, j_2 \in \{1,\dots,r\}, \quad (2.10)$$

$$a_{m+j_1\ m+r+j_2}(t,6) - a_{m+j_2\ m+r+j_1}(t,6) = 0 \qquad (\forall)\ j_1, j_2 \in \{1,\dots,r\}, \quad (2.11)$$

and

$$a_{i\ m+j}(t,6) = 0, \quad a_{m+j\ i}(t,6) = 0 \quad (\forall)\ i \in \{1,\dots,m\}, \qquad (2.12)$$

$(\forall)\ j \in \{1,\dots,n\}$,

$$a_{m+j\ m+l}(t,6) = 0 \qquad \text{if} \qquad l \neq j+1 \qquad (\forall)\ j,l \in \{1,\dots,n\}, \qquad (2.13)$$

where

$$a_{m+j_1\ m+j_2}(t,6) = \langle G_1 e_{m+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle, \qquad (2.14)$$

$$a_{m+j_1\ m+r+j_2}(t,6) = -\langle G_1 e_{m+j_1}(t,6), e_{m+j_2}(t,6)\rangle, \qquad (2.15)$$

$$a_{m+r+j_1\ m+j_2}(t,6) = \langle G_1 e_{m+r+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle, \qquad (2.16)$$

$$a_{m+r+j_1\ m+r+j_2}(t,6) = -\langle G_1 e_{m+r+j_1}(t,6), e_{m+j_2}(t,6)\rangle, \qquad (2.17)$$

$(\forall)\ j_1, j_2 \in \{1,\dots,r\}$,

$$a_{kh}(t,6) = \langle G_1 e_k(t,6), e_h(t,6)\rangle, \qquad (\forall)\ k,h \in \{1,\dots,m\}, \qquad (2.18)$$

$$a_{k\ m+j}(t,6) = \langle G_1 e_k(t,6), e_{m+j}(t,6)\rangle, \qquad (\forall)\ k \in \{1,\dots,m\} \qquad (2.19)$$

and $(\forall)\ j \in \{1,\dots,n\}$,

$$a_{m+j\ k}(t,6) = \langle G_1 e_{m+j}(t,6), e_k(t,6)\rangle, \qquad (\forall)\ k \in \{1,\dots,m\} \qquad (2.20)$$

and $(\forall)\ j \in \{1,\dots,n\}$.

PROOF. We use the proof of the existence and uniqueness theorem for Frenet frame curves from [9] in our proof. From the (2.1) and (2.2) relations we have:

$$\varepsilon^{(L)}(\langle G_2 c(t,6), G_1^{(r)} G_2 c(t,6)\rangle) \neq 0$$

and

$$\varepsilon^{(L)}(\langle G_1^{(j_1)} G_2 c(t,6)), G_1^{(r+j_1)} G_2 c(t,6)\rangle) \neq 0$$

$\forall\ j_1 \in \{1,\dots,r-1\}$. Let us consider

$$\lambda_1(t,6) = \langle G_2 c(t,6), G_1^{(r)} G_2 c(t,6)\rangle$$

and

$$\lambda_{j_1}(t,6) = \langle G_1^{(j_1-1)} G_2 c(t,6), G_1^{(r+j_1-1)} G_2 c(t,6)\rangle$$

$(\forall)\ j_1 \in \{2,\dots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Because of $\varepsilon^{(L)}(\lambda_{j_1}(t,\theta)) \neq 0\ (\forall)\ j_1 \in \{1,\dots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ it results that there exist $(\lambda_{j_1}(t,\theta))^{-1}$. Let us consider

$$e_{m+1}(t,\theta) = (\lambda_1(t))^{-1} \cdot G_2 c(t,\theta)$$

and

$$e_{m+j_1}(t,\theta) = (\lambda_{j_1}(t))^{-1} \cdot G_1^{(j_1-1)} G_2 c(t,\theta)$$

$(\forall)\ j_1 \subset \{2,\dots,r\}$ and $(\forall)\ (t,\theta) \subset V \subset B_L^{1,1}$ and let us consider

$$e_{m+r+j_2}(t,\theta) = G_1^{(r+j_2-1)} G_2 c(t,\theta)$$

$(\forall)\ j_2 \in \{1,\dots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

The supervectors $\{e_{m+1}(t,\theta),\dots,e_{m+n}(t,\theta)\}$ are linear independent because the supersmooth supercurve c is general position. From the definition of these supervectors, we have:

$$\begin{gathered}
\operatorname{span}(G_1 c(t,\theta),\dots,G_1^{(k)} c(t,\theta)) = \operatorname{span}(e_1(t,\theta),\dots,e_k(t,\theta)) \\
(\forall)\ k \in \{1,\dots,m-1\} \\
\operatorname{span}(G_2 c(t,\theta), G_1 G_2 c(t,\theta),\dots,G_1^{(j-1)} G_2 c(t,\theta)) = \\
= \operatorname{span}(e_{m+1}(t,\theta),\dots,e_{m+j}(t,\theta)) \\
(\forall)\ j \in \{1,\dots,n\}
\end{gathered}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

From the (2.1) and (2.2) relation we have $\varepsilon^L((\lambda_{j_1}(t,\theta))) > 0\ (\forall)\ j_1 \in \{1,\dots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. The "body" of the determinants of the matrices when we change from the system $\{G_2 c(t,\theta), G_1 G_2 c(t,\theta),\dots,G_1^{(j-1)} G_2 c(t,\theta)\}$ to the system $\{e_{m+1}(t,\theta),\dots,e_{m+j}(t,\theta)\}\ (\forall)\ j \in \{1,\dots,n\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ is

$$\varepsilon^{(L)}\left(\begin{vmatrix}
(\lambda_1(t,\theta))^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & (\lambda_j(t,\theta))^{-1} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{vmatrix}\right) =$$

$$\begin{gathered}
= \varepsilon^{(L)}((\lambda_1(t,\theta))^{-1} \cdot \ldots \cdot (\lambda_j(t,\theta))^{-1}) = \\
= \varepsilon^{(L)}((\lambda_1(t,\theta))^{-1}) \cdot \ldots \cdot \varepsilon^{(L)}((\lambda_j(t,\theta))^{-1}) > 0.
\end{gathered}$$

We have $(\forall)\ j_1 \in \{1, \dots, r\}$ and $(\forall)\ t \in I$ that

$$\langle e_{m+j_1}(t), e_{m+r+j_1}(t)\rangle = \\ = (\lambda_{j_1}(t,6))^{-1} \cdot \langle G_1^{(j_1-1)} G_2 c(t,6)), G_1^{(r+j_1-1)} G_2 c(t,6)\rangle = \\ = (\lambda_{j_1}(t))^{-1} \cdot \lambda_{j_1}(t) = 1.$$

Thus, we get the (1.21) formula from the first part of this paper. The (1.20) formula, from the first part of this paper, results from the supersymmetry property of the scalar product $\langle , \rangle$ and the (1.23) and (1.22) formulas, from the first part of this paper, result from the (2.3) and (2.4) relations. We shall specify if a relation belongs to the first part of this paper and we shall not specify if a relation belongs to the second part of this paper.

Because the supersmooth supercurve c is in general position, it results that the supervector $f_1(t,6) = G_1 c(t,6)$ is nonzero $(\forall)\ (t,6) \in V \subset B_L^{1,1}$. Let us consider

$$||v||' = \sqrt{\varepsilon^{(L)}(\langle v, v\rangle)} \tag{2.21}$$

$(\forall)\ v \in {B_L}^{m+n}$.

We put

$$e_1(t,6) = f_1(t,6) \cdot (||f_1(t,6)||')^{-1} \tag{2.22}$$

$(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Let the supervector $f_2(t,6)$ be

$$G_1^{(2)} c(t,6) + (\varepsilon^{(L)}(A(t,6)) + s(A(t,6))) \cdot e_1(t,6) \tag{2.23}$$

We shall get the functions $\varepsilon^{(L)}(A(t,6))$ and $s(A(t,6))$ such that the supervectors $f_2(t,6)$ and $e_1(t,6)$ to be orthogonal, that is,

$$\varepsilon^{(L)}(\langle f_2(t,6), e_1(t,6)\rangle) = 0 \tag{2.24}$$

$(t,6) \in V \subset B_L^{1,1}$. From (2.23) and (2.24), we have

$$0 = \langle G_1^{(2)} c(t,6), e_1(t,6)\rangle + (\varepsilon^{(L)}(A(t,6)) + s(A(t,6))) \cdot \langle e_1(t,6), e_1(t,6)\rangle$$

from which we obtain

$$0 = \varepsilon^{(L)}(\langle G_1^{(2)} c(t,6), e_1(t,6)\rangle) + s(\langle G_1^{(2)} c(t,6), e_1(t,6)\rangle) + \\ + \varepsilon^{(L)}(A(t,6)) \cdot \varepsilon^{(L)}(\langle e_1(t,6), e_1(t,6)\rangle) + \varepsilon^{(L)}(A(t,6)) \cdot s(\langle e_1(t), e_1(t)\rangle) + \\ + s(A(t,6)) \cdot s(\langle e_1(t,6), e_1(t,6)\rangle) + s(A(t,6)) \cdot \varepsilon^{(L)}(\langle e_1(t,6), e_1(t,6)\rangle).$$

Thus we have

$$\varepsilon^{(L)}(\langle G_1^{(2)}c(t,6), e_1(t,6)\rangle) + \varepsilon^{(L)}(A(t,6)) \cdot \varepsilon^{(L)}(\langle e_1(t,6), e_1(t,6)\rangle) = 0$$

and

$$s(\langle G_1^{(2)}c(t,6), c_1(t,6)\rangle) + \varepsilon^{(L)}(A_B(t,6)) \cdot s(\langle c_1(t,6), c_1(t,6)\rangle) + \\ + s(A(t,6)) \cdot s(\langle c_1(t,6), c_1(t,6)\rangle) + s(A(t,6)) \cdot \varepsilon^{(L)}(\langle c_1(t,6), c_1(t,6)\rangle) = 0.$$

Therefore

$$\varepsilon^{(L)}(A(t,6)) = -\varepsilon^{(L)}(\langle G_1^{(2)}c(t,6), e_1(t,6)\rangle) \tag{2.25}$$

and

$$s(A(t,6)) = (-s(\langle G_1^{(2)}c(t,6), e_1(t,6)\rangle) - \varepsilon^{(L)}(\langle G_1^{(2)}c(t,6), e_1(t,6)\rangle) \cdot \\ \cdot s(\langle e_1(t), e_1(t)\rangle)) \cdot (1 + s(\langle e_1(t), e_1(t)\rangle))^{-1}. \tag{2.26}$$

From (2.23), (2.25) and (2.26) we get

$$f_2(t,6) = G_1^{(2)}c(t,6) + (-\varepsilon^{(L)}(\langle G_1^{(2)}c(t,6), e_1(t,6)\rangle) + s(A(t,6))) \cdot e_1(t,6). \tag{2.27}$$

Because the supervectors $G_1c(t,6)$ and $G_1^{(2)}c(t,6)$ are linearly independent $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$ from (2.27) we have $\varepsilon^{(L)}(\langle f_2(t,6), f_2(t,6)\rangle) \neq 0$ $(\forall)(t,6) \in V \subset B_L^{1,1}$. Therefore $||f_2(t,6)||' \neq 0 (\forall)$ $(t,6) \in V \subset B_L^{1,1}$.

We set

$$e_2(t,6) = f_2(t,6) \cdot (||f_2(t,6)||')^{-1} \tag{2.28}$$

We note that $||e_1(t,6)||' = ||e_2(t,6)||' = 1$ but $\langle e_1(t,6), e_1(t,6)\rangle = 1+$ $+s(\langle e_1(t,6), e_1(t,6)\rangle)$ where $s(\langle e_1(t,6), e_1(t,6)\rangle)$ has not importance and we have the same for $\langle e_2(t,6), e_2(t,6)\rangle$. From the formulas (2.22), (2.27) and (2.28) formulas we have:

$$()G_1c(t,6) = ||G_1c(t,6)||' \cdot e_1(t,6) \tag{2.29}$$

and

$$G_1^{(2)}c(t,6) = \varepsilon^{(L)}(\langle G_1^{(2)}c(t,6), e_1(t,6)\rangle) - s(A(t,6)) \cdot e_1(t,6) + \\ + ||f_2(t,6)||' \cdot e_2(t,6). \tag{2.30}$$

The relations (2.29) and (2.23) show us that

$$\mathrm{span}(G_1c(t,6), G_1^{(2)}c(t,6)) = \mathrm{span}(e_1(t,6), e_2(t,6)).$$

Because of

$$\varepsilon^{(L)}\left(\begin{vmatrix} \|G_1c(t,6)\|' & 0 \\ \varepsilon^{(L)}(\langle G_1^{(2)}c(t,6), e_1(t,6)\rangle) - s(A(t,6)) & \|f_2(t,6)\|' \end{vmatrix}\right) > 0,$$

$(t,6) \in V \subset B_L^{1,1}$ it results that the systems of supervectors $\{G_1c(t,6), G_1^{(2)}c(t,6)\}$ and $\{e_1(t,6), e_2(t,6)\}$ are directed in the same way $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$.

We assume that we have constructed the unit, orthogonal two by two supervectors $e_1(t,6), \ldots, e_{h-1}(t,6)(h < m)$ with the properties that:

$$\text{span}(G_1c(t,6), G_1^{(2)}c(t,6), \ldots, G_1^{(h-1)}c(t,6)) = \text{span}(e_1(t,6), \ldots, e_{h-1}(t,6))$$

and the systems of supervectors $\{G_1c(t,6), G_1^{(2)}c(t,6), \ldots, G_1^{(h-1)}c(t,6)\}$ and $\{e_1(t,6), \ldots, e_{h-1}(t,6)\}$ are directed in the same way $(\forall)(t,6) \in V \subset B_L^{1,1}$. Thus we construct the supervector $f_h(t,6)$:

$$f_h(t,6) = G_1^{(h)}c(t,6) + \sum_{k=1}^{h-1}(\varepsilon^{(L)}(A_k(t,6)) + s(A_k(t,6))) \cdot e_k(t,6), h < m \quad (2.31)$$

where $(t,6) \to A_k(t,6)$ are supersmooth functions $(\forall)$ $k \in \{1, \ldots, h-1\}$ which will be determined by the conditions:

$$\langle f_h(t,6), e_i(t,6)\rangle = 0, \quad h < m, \quad i \in \{1, \ldots, h-1\}. \quad (2.32)$$

By (31) and (32) we get

$$\langle G_1^{(h)}c(t,6), e_i(t,6)\rangle + \\ + \sum_{k-1}^{h-1}(\varepsilon^{(L)}(A_k(t,6)) + s(A_k(t,6))) \cdot \langle e_k(t,6), e_i(t,6)\rangle = 0 \quad (2.33)$$

$(\forall)$ $i \in \{1, \ldots, h-1\}$ or equivalent to

$$\langle G_1^{(h)}c(t,6), e_i(t,6)\rangle + \\ + (\varepsilon^{(L)}(A_i(t,6)) + s(A_i(t,6))) \cdot (1 + s(\langle e_i(t,6), e_i(t,6)\rangle)) = 0 \quad (2.34)$$

$(\forall)$ $i \in \{1, \ldots, h-1\}$. From (2.34) we have:

$$\varepsilon^{(L)}(\langle G_1^{(h)}c(t,6), e_i(t,6)\rangle) + \varepsilon^{(L)}(A_i(t,6)) = 0 \quad (2.35)$$

and

$$s(\langle G_1^{(h)}c(t,6), e_i(t,6)\rangle) + \varepsilon^{(L)}(A_i(t,6)) \cdot s(\langle e_i(t,6), e_i(t,6)\rangle) + \\ + s(A_i(t,6)) \cdot (1 + s(\langle e_i(t,6), e_i(t,6)\rangle)) = 0 \qquad (2.36) \\ (\forall)\ h < m,\ i \in \{1, \dots, h-1\}.$$

Because the supervectors $G_1c(t,6), G_1^{(2)}c(t,6), \dots, G_1^{(h)}c(t,6)$ $(h < m)$ are linear independent it results that $f_h(t,6) \neq 0$.

We put

$$e_h(t,6) = f_h(t,6) \cdot (||f_h(t,6)||')^{-1}, \quad h < m. \qquad (2.37)$$

Thus, we have constructed the unit and orthogonal two by two supervectors $e_1(t,6), \dots, e_{m-1}(t,6)$. On the other hand, from (2.31) and (2.37) we get the following relations for all $h < m$:

$$G_1^{(h)}c(t,6) = (\varepsilon^{(L)}(\langle G_1^{(h)}c(t,6), e_1(t,6)\rangle) - s(A(t,6))) \cdot e_1(t,6) + \cdots \\ \cdots + (\varepsilon^{(L)}(\langle G_1^{(h)}c(t,6), e_{h-1}(t)\rangle) - s(A_{h-1}(t,6))) \cdot e_{h-1}(t,6) + \\ + ||f_h(t,6)||' \cdot e_h(t,6). \qquad (2.38)$$

From (2.29), (2.30) and (2.38) we obtain:

$$\mathrm{span}(G_1c(t,6), G_1^{(2)}c(t,6), \dots, G_1^{(h)}c(t,6)\}) = \mathrm{span}(e_1(t,6), \dots, e_h(t,6)).$$

Taking account of (2.29), (2.30), and (2.38) we get that the " body " of the determinant of the matrix of the linear transformation when we change from the basis $\{e_1(t,6), \dots, e_h(t,6)\}$ to the basis $\{G_1c(t,6), G_1^{(2)}c(t,6), \dots, G_1^{(h)}c(t,6)\}$ $(h < m)$ $\varepsilon^{(L)}(\Delta(t,6))$ is given by:

$$\varepsilon^{(L)}(\Delta(t,6)) = ||f_1(t,6)||' \cdot \ \cdots \cdot ||f_h(t,6)||' > 0 \quad (\forall)\ (t,6) \in V \subset B_L^{1,1}.$$

Therefore $\varepsilon^{(L)}(\Delta(t,6)) > 0$ $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$ and the systems of supervectors $\{e_1(t,6), \dots, e_h(t,6)\}$ and $\{G_1c(t,6), G_1^{(2)}c(t,6), \dots, G_1^{(h)}c(t,6)\}$ $(h < m)$ are directed in the same way $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$.

By our construction, the functions $(t,6) \to e_k(t,6) (\forall)\ k \in \{1, \dots, m-1\}$ and $(t,6) \to e_{m+j}(t,6)$ $(\forall)$ $j \in \{1, \dots, n\}$ are supersmooth.

We shall get $e_m(t,6)$ from the relations

$$\langle e_m(t,6), e_k(t,6)\rangle = 0 \quad (\forall)\ k \in \{1, \dots, m-1\} \qquad (2.39)$$

and

$$\langle e_m(t,6), e_{m+j}(t,6)\rangle = 0 \quad (\forall)\ j \in \{1, \dots, n\} \qquad (2.40)$$

Thus, we have:

$$
\begin{aligned}
&e_m^1(t,6)\cdot e_1^1(t,6)+\cdots+e_m^m(t,6)\cdot e_1^m(t,6)+e_m^{m+1}(t,6)\cdot e_1^{m+r+1}(t,6)+\cdots\\
&\cdots+e_m^{m+r}(t,6)\cdot e_1^{m+n}(t,6)-e_m^{m+r-1}(t,6)\cdot e_1^{m+1}(t,6)-\dots\\
&\dots-e_m^{m+n}(t,6)\cdot e_1^{m+r}(t,6)=0\\
&\ \vdots\\
&e_m^1(t,6)\cdot e_{m+n}^1(t,6)+\cdots+e_m^m(t,6)\cdot e_{m+n}^m(t,6)+e_m^{m+1}(t,6)\cdot e_{m+n}^{m+r+1}(t,6)+\cdots\\
&\cdots+e_m^{m+r}(t,6)\cdot e_{m+n}^{m+n}(t,6)-e_m^{m+r-1}(t,6)\cdot e_{m+n}^{m+1}(t,6)-\dots\\
&\dots-e_m^{m+n}(t,6)\cdot e_{m+n}^{m+r}(t,6)=0
\end{aligned}
\tag{2.41}
$$

where $e_k^1(t,6),\dots,e_k^{m+n}(t,6)$ are the components of the supervector $e_k(t,6)$ $(\forall)$ $k\in\{1,\dots,m\}$ and $e_{m+j}^1(t,6),\dots,e_{m+j}^{m+n}(t,6)$ are the components of the supervector $e_{m+j}(t,6)(\forall)$ $j\in\{1,\dots,n\}$.

Let us consider (2.41) as a linear and homogeneous system of $m+n-1$ equations with the $m+n$ unknowns $e_m^1(t,6),\dots,e_m^{m+n}(t,6)$. Because the supervectors $e_1(t,6),\dots,e_{m-1}(t,6),e_{m+1}(t,6),\dots,e_{m+n}(t,6)(\forall)$ $(t,6)\in V\subset B_L^{1,1}$ are linear independent, it follows that the rank of the matrix $M(t,6)$:

$$
\begin{pmatrix}
e_1^1 & \cdots & e_1^m & e_1^{m+r+1} & \cdots & e_1^{m+n} & -e_1^{m-1} & \cdots & -e_1^{m+r}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
e_{m-1}^1 & \cdots & e_{m-1}^m & e_{m-1}^{m+r+1} & \cdots & e_{m-1}^{m+n} & -e_{m-1}^{m-1} & \cdots & -e_{m-1}^{m+r}\\
e_{m+1}^1 & \cdots & e_{m+1}^m & e_{m+1}^{m+r+1} & \cdots & e_{m+1}^{m+n} & -e_{m-1}^{m-1} & \cdots & -e_{m+1}^{m+r}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
e_{m+n}^1 & \cdots & e_{m+n}^m & e_{m+n}^{m+r+1} & \cdots & e_{m+n}^{m+n} & -e_{m+n}^{m+1} & \cdots & -e_{m+n}^{m+r}
\end{pmatrix}
$$

is $m+n-1$, where e_q^s means $e_q^s(t,6)$ with $s\in\{1,\dots,m+n\}$ and $q\in\{1,\dots,m-1,m+1,\dots,m+n\}$. Let $\Delta_q(t,6)$ be the minor of order $m+n-1$ obtained by omitting the q column from the matrix $M(t,6)(\forall)$ $q\in\{1,\dots,m+n\}$ and $(\forall)$ $(t,6)\in V\subset B_L^{1,1}$. Then from (2.41), we get:

$$e_m^q(t,6)=(-1)^{q-1}\cdot\nu(t,6)\cdot\Delta_q(t,6),\qquad(\forall)\ q\in\{1,\dots,m+n\},\tag{2.42}$$

where $\nu(t,6)$ has to fulfil the condition:

$$||e_m(t,6)||'=1.\tag{2.43}$$

Because of $\operatorname{rank}(M(t,6))=m+n-1$ we have:

$$\varepsilon^{(L)}(\Delta(t,6))=\left(\varepsilon^{(L)}(\Delta_1(t,6))\right)^2+\cdots+\left(\varepsilon^{(L)}(\Delta_m(t,6))\right)^2>0.\tag{2.44}$$

From (2.42) and (2.43) we get:

$$\nu(t,6) = \eta \cdot \left(\sqrt{(\Delta(t,6))}\right)^{-1} \tag{2.45}$$

where $\eta = 1$ or $\eta = -1$; this resulting from the condition that the frame $\{e_1(t,6),\ldots,e_{m+n}(t,6)\}$ to be positively directed.

By (2.42) and (2.45) we get:

$$e_m^q(t,6) = (-1)^{q-1} \cdot \eta \cdot \Delta_q(t,6) \cdot \left(\sqrt{(\Delta(t,6))}\right)^{-1} (\forall)\ q \in \{1,\ldots,m+n\}, \tag{2.46}$$

where η verifies the conditions $|\eta| = 1$ and

$$\varepsilon^{(L)}(\text{sdet}\,(c_q^s(t,6))_{1\leq s,q\leq m+n}) > 0.$$

By (2.46) and (2.43) it follows that the functions $(t,6) \to e_m^s(t,6)$, $1 \leq s \leq m+n$ are supersmooth and the function $(t,6) \to e_m(t,6)$ is supersmooth.

From our construction results (2.19) and (1.24) from the first part of this paper.

The uniqueness of the Frenet frame results by our construction.

We fix an index $k \subset \{1,\ldots,m\}$. We express $G_1 e_k(t,6)$ in the frame $\{e_1(t,6),$ $\ldots,$ $e_{m+n}(t,6)\}$ and we have:

$$G_1 e_k(t,6) = \sum_{h=1}^{m} a_{kh}(t,6) \cdot e_h(t,6) + \sum_{j=1}^{n} a_{kj}(t,6) \cdot e_{m+j}(t,6). \tag{2.47}$$

Computing the scalar product between the (2.47) relations and $e_i(t,6)$ $(\forall)$ $i \in \{1,\ldots,m\}$ we get:

$$a_{ki}(t,6) = \langle G_1 e_k(t,6), e_i(t,6)\rangle \tag{2.48}$$

because $\langle e_{m+j}(t,6), e_i(t,6)\rangle = 0 (\forall)$ $j \in \{1,\ldots,n\}$ and $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$. Thus we proved the (2.18) relation $(\forall)$ $k,i \in \{1,\ldots,m\}$ and $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$.

Computing the scalar product between the (2.47) relations and $e_{m+l}(t,6)$ $(\forall)$ $l \in \{1,\ldots,n\}$ we get:

$$a_{k\ m+l}(t,6) = \langle G_1 e_k(t,6), e_{m+l}(t,6)\rangle \tag{2.49}$$

because $\langle e_h(t,6), e_{m+l}(t,6)\rangle = 0 (\forall)$ $h \in \{1,\ldots,m\}$ and $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$. Thus we proved the (2.19) relation $(\forall)$ $h \in \{1,\ldots,m\}$, $l \in \{1,\ldots,n\}$ and $(\forall)$ $(t,6) \in V \subset B_L^{1,1}$.

Derivating the following relation by G_1

$$\langle e_k(t,6), e_h(t,6)\rangle = \delta_{kh} \qquad (\forall)\ k,h \in \{1,\ldots,m\},\ (\forall)(t,6) \in V \subset B_L^{1,1}$$

we get:

$$\langle G_1 e_k(t,6), e_h(t,6)\rangle + \langle e_k(t,6), G_1 e_h(t)\rangle = 0 \qquad (\forall)\ k,h \in \{1,\ldots,m\},$$

$(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

By the (2.48) relation we have:

$$a_{kh}(t,6) + a_{hk}(t,6) = 0$$

that means we proved the (2.6) relation $(\forall)\ k,h \in \{1,\ldots,m\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Because $\{e_1(t,6),\ldots,e_{m+n}(t,6)\}$ is a Frenet frame we have:

$$G_1^{(k)} c(t,6) \in \operatorname{span}(e_1(t,6),\ldots,e_k(t,6)), \quad (\forall)\ k \in \{1,\ldots,m-1\} \tag{2.50}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ and

$$e_k(t,6) \in \operatorname{span}(G_1 c(t,6),\ldots,G_1^{(k)} c(t,6)) \quad (\forall)\ k \in \{1,\ldots,m-1\} \tag{2.51}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

By (2.50) and (2.51) we get:

$$G_1 e_k(t,6) \in \operatorname{span}(G_1 c(t,6),\ldots,G_1^{(k)} c(t,6), G_1^{(k+1)} c(t,6)) \tag{2.52}$$

$(\forall)\ k \in \{1,\ldots,m-1\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

By (2.50) and (2.52) we have:

$$G_1 e_k(t) \in \operatorname{span}(e_1(t,6),\ldots,e_{k+1}(t,6)) \quad (\forall)\ k \in \{1,\ldots,m-1\} \tag{2.53}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

By (2.53), we note that in the writing

$$G_1 e_k(t,6) = \sum_{h=1}^{m} a_{kh}(t,6) \cdot e_h(t,6) + \sum_{j=1}^{n} a_{k\ m+j}(t,6) \cdot e_{m+j}(t,6)$$

$$(\forall)\ k \in \{1,\ldots,m\}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ the coefficients $a_{kh}(t,6)$ are zero if $h > k+1$ and $a_{k\ m+j}(t,6) = 0\ (\forall)\ k \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Thus, we get the (2.5) formula $(\forall)\ (t,6) \in V \subset B_L^{1,1}$:

$$G_1 e_k(t,6) = \sum_{h=1}^{m} a_{kh}(t,6) \cdot e_h(t,6) (\forall)\ k \in \{1,\ldots,m\}$$

and the (2.7) formula $a_{kh}(t,6) = 0$ if $h > k+1, (\forall)\ k,h \in \{1,\dots,m\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Because of our construction of the frame $\{e_1(t,6),\dots,e_{m+n}(t,6)\}$, we have:

$$G_1^{(j-1)}G_2c(t,6) \in \text{span}(e_{m+j}(t,6)), \tag{2.54}$$

and

$$e_{m+j}(t,6) \in \text{span}(G_1^{(j-1)}G_2c(t,6)) \tag{2.55}$$

$(\forall)\ j \in \{1,\dots,n\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

From now, we have:

$$G_1e_{m+j}(t,6) \in \text{span}(G_1^{(j)}G_2c(t,6)) \tag{2.56}$$

$(\forall)\ j \in \{1,\dots,n\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

By the (2.54) and (2.56) relations we get

$$G_1e_{m+j}(t,6) \in \text{span}(e_{m+j+1}(t,6)) \tag{2.57}$$

$(\forall)\ j \in \{1,\dots,n\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

By (2.57) we note that in writing

$$G_1e_{m-j}(t,6) = \sum_{k=1}^{m} a_{m+j\ k}(t,6)\cdot e_k(t,6) + \sum_{l=1}^{n} a_{m+j\ m+l}(t,6)\cdot e_{m+j}(t,6)$$

$(\forall)\ (t,6) \in V \subset B_L^{1,1}$, the coefficients $a_{m+j\ m+l}(t,6)$ are zero if $l \neq j+1$ and $a_{m+j\ k}(t,6) = 0\ (\forall)\ k \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$ $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Thus we proved the (2.8), (2.12) and (2.13) relations.

We fix an index $j_1 \in \{1,\dots,r\}$. We express $G_1e_{m+j_1}(t,6)$ in the frame $\{e_1(t,6),\ \dots,\ e_{m+n}(t,6)\}$. We have:

$$G_1e_{m+j_1}(t,6) = \sum_{h-1}^{m} a_{m+j_1\ h}(t,6)\cdot e_h(t,6) + \sum_{j=1}^{n} a_{m+j_1\ m+j}(t)\cdot e_{m+j}(t,6) \tag{2.58}$$

$(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Computing the scalar product between the (2.58) relation and $e_k(t,6)$ $(\forall)\ k \in \{1,\dots,m\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ we get:

$$a_{m+j_1\ k}(t) = \langle G_1e_{m+j_1}(t,6), e_k(t,6)\rangle \quad (\forall)\ k \in \{1,\dots,m\} \tag{2.59}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ because $\langle e_{m+j}(t,6), e_k(t,6)\rangle = 0 (\forall)\ k \in \{1,\dots,m\}$, $(\forall)\ j \in \{1,\dots,n\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$. Thus, we proved the (2.20) relation $(\forall)\ j_1 \in \{1,\dots,r\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Computing the scalar product between the (2.58) relation and $e_{m+j_2}(t,6)$ $(\forall)\ j_2 \in \{1,\dots,r\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$. We get:

$$a_{m+j_1\ m+r+j_2}(t,6) = -\langle G_1 e_{m+j_1}(t,6), e_{m+j_2}(t,6)\rangle \quad (\forall)\ j_2 \in \{1,\dots,r\} \tag{2.60}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$. Thus we proved the (2.15) relation $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Computing the scalar product between the (2.58) relation and $e_{m+r+j_2}(t,6)$ $(\forall)\ j_2 \in \{1,\dots,r\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ we get:

$$a_{m+j_1\ m+j_2}(t,6) = \langle G_1 e_{m+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle \quad (\forall)\ j_2 \in \{1,\dots,r\} \tag{2.61}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$. Thus, we proved the (2.14) relation $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

We fix an index $j_1 \subset \{1,\dots,r\}$. We express $G_1 e_{m+r+j_1}(t,6)$ in the frame $\{e_1(t,6),\dots,e_{m+n}(t,6)\}$. We have:

$$\begin{aligned} G_1 e_{m+r+j_1}(t,6) = \sum_{h=1}^{m} a_{m+r+j_1\ h}(t) \cdot e_h(t,6) + \\ + \sum_{j=1}^{n} a_{m+r+j_1\ m+j}(t,6) \cdot e_{m+j}(t,6) \\ (\forall)\ (t,6) \in V \subset B_L^{1,1}. \end{aligned} \tag{2.62}$$

Computing the scalar product between the (2.62) relation and $e_k(t,6)$ $(\forall)\ k \in \{1,\dots,m\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ we get:

$$a_{m+r+j_1\ k}(t,6) = \langle G_1 e_{m+r+j_1}(t,6), e_k(t,6)\rangle \quad (\forall)\ k \in \{1,\dots,m\} \tag{2.63}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ because $\langle e_{m+j}(t,6), e_k(t,6)\rangle = 0 (\forall)\ k \in \{1,\dots,m\}$, $(\forall)\ j \in \{1,\dots,n\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$. Thus, by (2.59) and (2.63), we proved the (2.20) relation $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Computing the scalar product between the (2.62) relation and $e_{m+j_2}(t,6)$ $(\forall)\ j_2 \in \{1,\dots,r\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ we get:

$$a_{m-r+j_1\ m-r+j_2}(t,6) = -\langle G_1 e_{m+r+j_1}(t,6), e_{m+j_2}(t,6)\rangle \quad (\forall)\ j_2 \in \{1,\dots,r\} \tag{2.64}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$. Thus we proved the (2.17) relation $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Computing the scalar product between the (2.62) relation and $e_{m+r+j_2}(t,6)$ $(\forall)\ j_2 \in \{1,\dots,r\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ we get:

$$a_{m+r+j_1\ m+j_2}(t,6) = \langle G_1 e_{m+r+j_1}(t,6), e_{m+r-j_2}(t,6)\rangle \quad (\forall)\ j_2 \in \{1,\dots,r\} \tag{2.65}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$. Thus, we proved the (2.16) relation $(\forall)\ (t,6) \in V \subset B_L^{1,1}$.

Derivating the following relation by G_1

$$\langle e_{m+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle = \delta_{j_1 j_2} \qquad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$$

we get

$$\langle G_1 e_{m+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle + \langle e_{m+j_1}(t,6), G_1 e_{m+r+j_2}(t,6)\rangle = 0$$

$(\forall)\ j_1, j_2 \in \{1,\ldots,r\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ which is equivalent to

$$\langle G_1 e_{m+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle - \langle G_1 e_{m+r+j_2}(t,6), e_{m+j_1}(t,6)\rangle = 0$$

$(\forall)\ j_1, j_2 \in \{1,\ldots,r\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$and using the (2.14) and (2.17) relations we get:

$$a_{m+j_1\ m+j_2}(t,6) + a_{m+r+j_2\ m+r+j_1}(t,6) = 0 \quad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ which is the (2.9) relation.

Derivating the following relation by G_1

$$\langle e_{m+j_1}(t,6), e_{m+j_2}(t,6)\rangle = 0 \qquad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$$

we get

$$\langle G_1 e_{m+j_1}(t,6), e_{m+j_2}(t,6)\rangle + \langle e_{m+j_1}(t,6), G_1 e_{m+j_2}(t,6)\rangle = 0$$

$$(\forall)\ j_1, j_2 \in \{1,\ldots,r\}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ which is equivalent to

$$\langle G_1 e_{m+j_1}(t,6), e_{m+j_2}(t,6)\rangle - \langle G_1 e_{m+j_1}(t,6), e_{m+j_1}(t,6)\rangle = 0$$

$$(\forall)\ j_1, j_2 \in \{1,\ldots,r\}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ and using the (2.15) relation we get:

$$-a_{m+j_1\ m+r+j_2}(t,6) + a_{m+j_2\ m+r+j_1}(t,6) = 0 \quad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ which is the (2.11) relation.

Derivating the following relation by G_1

$$\langle e_{m+r+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle = 0 \qquad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$$

we get

$$\langle G_1 e_{m+r+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle + \langle e_{m+r+j_1}(t,6), G_1 e_{m+r+j_2}(t,6)\rangle = 0$$

$(\forall)\ j_1, j_2 \in \{1,\ldots,r\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ which is equivalent to

$$\langle G_1 e_{m+r+j_1}(t,6), e_{m+r+j_2}(t,6)\rangle - \langle G_1 e_{m+r+j_2}(t,6), e_{m+r+j_1}(t,6)\rangle = 0$$

$(\forall)\ j_1, j_2 \in \{1,\ldots,r\}$ and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ and using the (2.16) relation we get:

$$a_{m+r+j_1\ m+j_2}(t,6) - a_{m+r+j_2\ m+j_1}(t,6) = 0 \quad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$$

and $(\forall)\ (t,6) \in V \subset B_L^{1,1}$ which is the (2.10) relation. QED

14 Corollary. *The (2.5) and (2.8) relations extend the Frenet formulas for the curves.*

15 Remark. By (2.6), (2.7), (2.12), (2.13), (2.9), (2.10) and (2.11) we get:

$$\mathcal{A} - (a_{sq})_{1\le s,q\le m+n} - \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 0 & a_{12} & \cdots & 0 & 0 \\ -a_{12} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{m-1\ m} \\ 0 & 0 & \cdots & -a_{m-1\ m} & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} a_{m+1\ m+1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{m+r\ m+r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -a_{m+1\ m+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -a_{m+r\ m+r} \end{pmatrix}$$

and

$$A_3 = A_4 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

16 Example. Let ${B_L}^{2+2}$ be the $(2,2)-$dimensional total supereuclidean space, V be an open subset of $B_L^{1,1}$ and $c : V \subset B_L^{1,1} \to {B_L}^{2-2}$ be a supersmooth supercurve given by

$$c(t,6) = (t^2, 6\cdot\beta^2, 6 + 2\cdot\beta^1\cdot t, 6\cdot t^2)$$

where $V \cap \mathbf{R}$ be the open set $(0,1)$.

We note that the supercurve c is supersmooth because the functions

$$c^1(t,6) = t^2, \qquad c^2(t,6) = 6\cdot\beta^2, \qquad c^3(t,6) = 6 + 2\cdot\beta^1\cdot t, \qquad c^4(t,6) = 6\cdot t^2$$

are supersmooth. Let us compute $G_1c(t,6)$, $G_2c(t,6)$, and $G_1G_2c(t,6)$:

$$G_1c(t,6) = (2\cdot t, 0, 2\cdot\beta^1, 2\cdot 6\cdot t)$$

$$G_2c(t,6) = (0, \beta^2, 1, t^2) \qquad \text{and} \qquad G_1G_2c(t,6) = (0,0,0,2\cdot t).$$

By

$$\varepsilon^{(L)}(\langle G_2c(t,6), G_1G_2c(t,6)\rangle) = \varepsilon^{(L)}(0\cdot 0 + \beta^2\cdot 0 + 1\cdot 2\cdot t - t^2\cdot 0) = 2\cdot t > 0$$

$(\forall)\ t \in (0,1)$,

$$\varepsilon^{(L)}(\langle G_2c(t,6), G_2c(t,6)\rangle) = \varepsilon^{(L)}(0\cdot 0 + \beta^2\cdot\beta^2 + 1\cdot t^2 - t^2\cdot 1) = 0$$

$(\forall)\ t \in (0,1)$ and

$$\varepsilon^{(L)}(\langle G_1G_2c(t,6), G_1G_2c(t,6)\rangle) = \varepsilon^{(L)}(0\cdot 0 + 0\cdot 0 + 0\cdot 2\cdot t - 2\cdot 0\cdot t) = 0$$

we conclude that the supercurve c fulfills the (2.1), (2.2), (2.3), (2.4), relations from the Theorem 13.

Computing $G_1c(t,6)$, $G_2c(t,6)$, and $G_1G_2c(t,6)$, we get that the supervectors $\{G_1c(t,6), G_2c(t,6), G_1G_2c(t,6)\}$ are linear independent. Thus we conclude that the supercurve c is in general position.

Let us get the Frenet frame of the supercurve c, $\{e_1(t,6), e_2(t,6), e_3(t,6), e_4(t,6)\}$ and the matrix $\mathcal{A} = (a_{sq})_{1\leq s,q\leq 4}$. Let $f_1(t,6)$ be $G_1c(t,6) = (2\cdot t, 0, 2\cdot \beta^1, 2\cdot 6\cdot t)$ and

$$\langle G_1c(t,6), G_1c(t,6)\rangle = 2\cdot t\cdot 2\cdot t + 0\cdot 0 + 2\cdot\beta^1\cdot 2\cdot 6\cdot t - 2\cdot 6\cdot 2\cdot\beta^1\cdot t =$$
$$= 4\cdot t^2 + 4\cdot\beta^1\cdot 6\cdot t - 4\cdot 6\cdot\beta^1\cdot t.$$

We have $\langle G_1c(t,6), G_1c(t,6)\rangle = 4\cdot t^2 + 8\cdot\beta^1\cdot 6\cdot t$ and $\varepsilon^{(L)}(\langle G_1c(t,6), G_1c(t,6)\rangle) = 4\cdot t^2$. Thus we get $||G_1c(t,6)||' = \sqrt{\varepsilon^{(L)}(\langle G_1c(t,6), G_1c(t,6)\rangle)} = \sqrt{4\cdot t^2} = 2\cdot t$.

By the above Theorem 13 we compute the Frenet frame of c, $\{e_1(t,6), e_2(t,6), e_3(t,6), e_4(t,6)\}$ and we get

$$c_1(t,6) = (1, 0, 2\cdot\beta^1\cdot(2\cdot t)^{-1}, 6),$$

$$e_3(t,6) = (0, \beta^2\cdot(2\cdot t)^{-1}, (2\cdot t)^{-1}, 2^{-1}\cdot t)$$

and

$$e_4(t,6) = (0,0,0,2\cdot t).$$

Let the matrix $M(t,6)$ be

$$\begin{pmatrix} 1 & 0 & 6 & -2\cdot\beta^1\cdot(2\cdot t)^{-1} \\ 0 & \beta^2\cdot(2\cdot t)^{-1} & 2^{-1}\cdot t & -(2\cdot t)^{-1} \\ 0 & 0 & 2\cdot t & 0 \end{pmatrix}$$

and computing $e_2(t,6)$ we get

$$e_2(t,6) = (-2\beta^2\cdot\beta^1\cdot(2\cdot t)^{-1}, 1, 0, -\beta^2).$$

Now, we may compute $a_{12}(t,6)$ and $a_{33}(t,6)$ and we have

$$a_{12}(t,6) = \langle G_1 e_1(t,6), e_2(t,6)\rangle = \beta^1\cdot\beta^2\cdot t^{-2}$$

where

$$G_1 e_1(t,6) = (0,0,-\beta^1\cdot t^{-2}, 0)$$

and

$$a_{33}(t,6) = \langle G_1 e_3(t,6), e_4(t,6)\rangle = -t^{-1},$$

where

$$G_1 e_3(t,6) = (0, -\beta^2\cdot 2^{-1}\cdot t^{-2}, -2^{-1}\cdot t^{-2}, 2^{-1}).$$

We conclude that

$$\mathcal{A} = (a_{sq})_{1\le s,q\le 4} = \begin{pmatrix} 0 & \beta^1\cdot\beta\cdot t^{-2} & 0 & 0 \\ -\beta^1\cdot\beta^2\cdot t^{-2} & 0 & 0 & 0 \\ 0 & 0 & -t^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}.$$

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