*Note di Matematica* **24,** n. 1, 2004/2005, 85–109.

# Symplectic flock spreads in $PG(3, q)$ [i]

**Mauro Biliotti**
*Dipartimento di Matematica, Università di Lecce,*
*Via Arnesano, 73100 Lecce, Italy*
biliotti@ilenic.unile.it

**Vikram Jha**
*Department of Mathematics, Caledonian University,*
*Cowcaddens Road, Glasgow, Scotland*
v.jha@gcal.ac.uk

**Norman L. Johnson**
*Department of Mathematics , University of Iowa,*
*Iowa City, Iowa 52242, USA*
njohnson@math.uiowa.edu

Received: 27/5/2004; accepted: 27/5/2004.

**Abstract.** It is shown that any symplectic spread of symplectic dimension 2 corresponding to a flock of a quadratic cone is either Desarguesian or the Kantor-Knuth flock spread. The results of Thas-Payne, Ball and Brown connecting semifield flocks and symplectic spreads are also obtained algebraically.

**Keywords:** flock, symplectic spread, semifield flock

**MSC 2000 classification:** primary 51E23, secondary 51A40

## 1 Introduction

Let $\pi$ be a semifield flock of a quadratic cone translation plane with spread in $PG(3, q)$. In 1987, Johnson [6], showed that semifield flocks of quadratic cones; semifields of order $q^2$ that commute over a left nucleus $GF(q)$; are equivalent to semifields of order $q^2$ that commute over a right or middle nucleus isomorphic to $GF(q)$. Since semifields of order $q^2$ that commute over a middle nucleus $GF(q)$ are commutative, there is an implicit connection with commutative semifields of order $q^2$ with middle nucleus $GF(q)$ and semifield flocks.

[i]The ideas for this paper were conceived when the second two authors were visiting the University of Lecce during May and June of 2003. The authors are grateful to the university and to the MIUR for support on this research. Furthermore, part of this work was done at the Obermann Institute for Advanced Studies of the University of Iowa and the authors are indebted to the Institute for support both financial and collegial. The authors are also indebted to A. Maschietti for helpful conversations and suggestions regarding symplectic spreads.

In 1994, Thas and Payne [14] have shown geometrically that given a (Cohen-Ganley [2]) semifield flock, there is a corresponding semifield spread in $PG(3,q)$, whose dual lies in $Q(4,q)$; the semifield spread is symplectic with symplectic dimension 2. Even though Thas and Payne worked solely with the Cohen-Ganley flock, this result is also more generally valid by work of several authors Lunardon [10], Bloemen [4], Thas [16] (see also Payne and Johnson [8] (section 14)). The construction connects the associated generalized quadrangle of order $(q^2,q)$, forms the translation dual of order $(q,q^2)$, within which it is realized that there are subquadrangles isomorphic to $Q(4,q)$, and subsequently determines ovoids of isotropic points of a symplectic polarity within $PG(4,q)$. The dual of $Q(4,q)$ is $M(3,q)$, the set of isotropic points and lines of a symplectic polarity of $PG(3,q)$. Thus, the ovoids of $Q(4,q)$ determine symplectic spreads of $PG(3,q)$, that turn out to be semifield spreads, although this is not altogether completely obvious from this construction.

Recently, in 2003, relying on what we will call the Thas-Payne construction, Ball and Brown [1] have shown that there are six semifield spreads associated with a semifield flock spread, two of which are spreads in $PG(3,q)$. Furthermore, these two spreads in $PG(3,q)$ are isomorphic if and only if the semifield flock spread is Kantor-Knuth.

Also, in 2003, Kantor [9] has now connected symplectic semifield spreads with commutative semifields by the iterative construction process of transpose and dualization (from the commutative side).

Furthermore, it turns out that a commutative semifield of order $q^2$ with middle nucleus $GF(q)$ will construct a symplectic spread in $PG(3,q)$, with symplectic dimension two. Actually, we show that any symplectic semifield spread in $PG(3,q)$ must have symplectic dimension two.

If one tries to construct semifield spreads by an iteration of the construction processes of transpose and dualization, it turns out that there are not six but three possible spreads (see also Theorem 6). Using the Thas-Payne construction, there are three additional semifield spreads. But, also using the constructions of distortion, $t$-extension and derivation, it is possible to construct the same semifield spreads completely algebraically.

Hence, we see that there is now a complete equivalence of symplectic semifield spreads in $PG(3,q)$ and semifield flocks of quadratic cones in $PG(3,q)$ (see section 3), using only the algebraic methods of construction of various semifields of transpose, dualization, $t$-extension, derivation and $t$-distortion. In this way, we may obtain the results of Ball and Brown [1], independent of the Thas-Payne construction of ovoids of $Q(4,q)$. However, we also obtain generalized Hall planes in our sequence of spreads as well.

There are no semifields with spreads in $PG(3,q)$ that are symplectic but not of dimension two. However, this certainly raises the question whether any symplectic spread in $PG(3,q)$ must have symplectic dimension two. In particular, are there flock spreads in $PG(3,q)$ that are of symplectic dimension two? We show that the only such flocks are Desarguesian or Kantor-Knuth.

Finally, we include sections of the transpose of arbitrary finite semifields, which may be of independent use, both in the determination of semifields, commutative semifields and symplectic spreads, and of 'lifting', another construction procedure that produces arbitrarily long chains of semifield spreads from a given semifield spread in $PG(3,q)$.

## 2 Symplectic Flocks

As noted in the introduction, there is an interesting connection between commutative semifields and symplectic spreads, shown by Kantor in [9]; every semifield plane that has a coordinate commutative semifield produces, by transpose and dualization, a symplectic spread. Furthermore, Maschietti [12] points out that $2\times 2$ matrix spread sets consisting of symmetric matrices produce and are equivalent to symplectic spreads, considered over a 4-dimensional vector space.

We note subsequently that there is a conical flock spread, the Kantor-Knuth flock spread that admits a representation of symmetric $2\times 2$ matrices. In general, any flock spread has the following standard form:

$$(*): x=0, y=x\begin{bmatrix} u+G(t) & F(t) \\ t & u \end{bmatrix}, \forall t,u\in GF(q).$$

Clearly, we are not considering a representation where the matrices are $2\times 2$ and symmetric, unless $F(t)=t$ and we note below that this implies that the conical flock spread is Desarguesian when $q$ is even or Kantor-Knuth or Desarguesian when $q$ is odd.

The question then becomes: What are the flock spreads that are symplectic and can also be written as symmetric spread sets of $2\times 2$ matrices? We show that the only possibilities are the Desarguesian and Kantor-Knuth flock spreads.

For the benefit of the reader, we repeat the fundamental result proved by Maschietti in [12].

**1 Theorem.** *Let $\pi$ be a finite translation plane. Then, there is a matrix spread set $\mathcal{S}$ for $\pi$ that is symplectic if and only if there is a set of matrices such that if $M$ is in $\mathcal{S}$ then $M=M^t$, (transpose).*

PROOF. Let $f$ be the symplectic form and assume that $\mathcal{S}$ is a symplectic spread. Assume that $L$ is a component of $\mathcal{S}$ and change bases, regarding the kernel as the prime subfield so that $L$ is $y=0$. We may further change bases so that

$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ represents the skew-symmetric form determined by the symplectic polarity, since all forms are conjugate.

Suppose we have a spread, written as a set of matrices $\mathcal{S}$ whose differences are non-singular or identically 0, and containing the zero matrix (but we do not here require that the identity is in $\mathcal{S}$), and note that the zero matrix is automatically isotropic. Then a subspace $y = xM$ is isotropic if and only if

$$(x, xM) \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} x^t \\ M^t x^t \end{bmatrix} = 0$$

if and only if

$$x(-M + M^t)x^t = 0,$$

for all $x$, if and only if $M = M^t$. QED

**1 Definition.** A symplectic spread $S$ will be said to have 'symplectic dimension $k$' if and only if some matrix spread set representing $S$ is elementwise self transpose, the matrices are $k \times k$ and $k$ is minimum in any self transpose representation. If $S$ has order $p^t$ then $1 \le k \le t$.

Note the symplectic spreads of symplectic dimension 1 are exactly the Desarguesian spreads.

**1 Remark.** In the above representation, it may not be the case that the matrix spread set of symmetric matrices contains the identity matrix. If this is required, we would need to adjust the symplectic form more generally to be:

$$\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix},$$

where $A = A^t$.

**1 Example.** The Kantor-Knuth conical flock spreads are symplectic spreads of symplectic dimension 2.

PROOF. A Kantor-Knuth spread may be represented as follows:

$$x = 0, y = x \begin{bmatrix} u & \gamma t^\sigma \\ t & u \end{bmatrix};$$

where $u, t \in GF(q)$,$q$ odd,$\gamma$ non-square, $\sigma$ a non-identity automorphism.

Apply a basis change $\begin{bmatrix} I & 0_2 \\ 0_2 & A \end{bmatrix}$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, to change the spread set in the form:

$$x = 0, y = x \begin{bmatrix} t & u \\ u & \gamma t^\sigma \end{bmatrix}; u, t \in GF(q),\ q \text{ odd}, \gamma \text{ non-square}.$$

Since the associated spread is elementwise self-transpose, the Kantor-Knuth spreads are symplectic of symplectic dimension 2.

Note that this amounts to taking the symplectic form:

$$\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}, \text{ for } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

QED

We now show that the Kantor-Knuth planes are the only non-Desarguesian translation planes with symplectic dimension $\leq 2$.

We begin with a fundamental lemma. We shall take the representation for a conical flock spread as indicated above.

**1 Lemma.** *For a flock spread of the form above, the spread is a Kantor-Knuth spread provided one of the following occur:*

(1) $F(t) = tf_0$, for all $t \in GF(q)$, $f_0$ a constant.

(2) $G(t) = tf_1 + F(t)f_2$, for all $t \in GF(q)$, for constants $f_1$ and $f_2$.

PROOF. Let the plane of the associated flock of a quadratic cone $x_0x_2 = x_1^2$, be represented in the form:

$$x_0t + x_1G(t) - x_2F(t) + x_3 = 0.$$

If the planes have a common point, it is known by Thas [15] that the plane is a Kantor-Knuth spread.

In situation (1), then we have

$$x_ot + x_1G(t) - x_2tf_o + x_3 = 0.$$

The planes then have the common point $(x_2f_o, 0, x_2, 0)$.

Now consider situation (2) above. Then we have

$$x_0t + x_1(tf_1 + F(t)f_2) - x_2F(t) + x_3 = 0.$$

We then have the common point $(-x_1f_1, x_1, x_1f_2, 0)$. This completes the proof of the lemma. QED

**2 Theorem.** *A flock spread is symplectic of dimension* 1 *or* 2 *if and only if it is either Desarguesian or Kantor-Knuth.*

PROOF. We assume that the dimension is 2. Therefore, we may represent the symplectic spread $\mathcal{S}$ so that $M = M^t$, for all $M$ in $\mathcal{S}$ and $\mathcal{S}$ is a set of $2 \times 2$ matrices over $GF(q)$. Hence, $M = \begin{bmatrix} t & u \\ u & f(t,u) \end{bmatrix}$, for all $u, t \in GF(q)$.

Now change bases by $\begin{bmatrix} A & 0_2 \\ 0_2 & I_2 \end{bmatrix}$, such that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This transforms the spread into the form

$$x = 0, y = x \begin{bmatrix} u & f(t,u) \\ t & u \end{bmatrix} = M_{t,u}.$$

Since the spread is a conical flock spread, it follows that there is a set of reguli sharing a component. Since the symplectic group is transitive on ordered pairs of totally isotropic subspaces of the spread, we may assume that the $x = 0$ is an elation axis for the associated flock spread.

We know that the elation group $E$ is of order $q$ and each orbit of components union the axis is a regulus. If the plane $\pi$ is non-Desarguesian, there is a unique regulus containing any non-axis component. Let $R$ denote the unique regulus containing $y = 0$. So, the regulus has the following form:

$$x = 0, y = 0, y = x \begin{bmatrix} u_i & f(t_i,u_i) \\ t_i & u_i \end{bmatrix} = M_i$$

for $i = 1, 2, \ldots, q-1$. Choose any of these, say $M_j$ and change bases by $\begin{bmatrix} I_2 & 0_2 \\ 0_2 & M_j^{-1} \end{bmatrix}$, so that we have $x = 0, y = 0, y = x$ represented in the regulus $R$. When this occurs, the regulus then has the form:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}; u \in GF(q).$$

Our spread now has the form

$$x = 0, y = 0, y = x \begin{bmatrix} u & f(t,u) \\ t & u \end{bmatrix} M_j^{-1}.$$

Also, when this occurs, the elation group $E$ has the following form:

$$\left\langle \begin{bmatrix} I_2 & U \\ 0_2 & I_2 \end{bmatrix}; U = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}; u \in GF(q) \right\rangle.$$

Furthermore, it follows that the form for the conical flock spread is now:

$$x = 0, y = x \begin{bmatrix} u + G(t) & F(t) \\ t & u \end{bmatrix} \forall t, u \in GF(q),$$

where $G$ and $F$ are functions on $GF(q)$, as well as

$$x = 0, y = 0, y = x \begin{bmatrix} u & f(t,u) \\ t & u \end{bmatrix} M_j^{-1}.$$

Let $M_j^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Hence, we have the following equations that must hold:

$$\begin{bmatrix} ua + f(t,u)c & ub + f(t,u)d \\ ta + uc & tb + ud \end{bmatrix} = \begin{bmatrix} v + G(s) & F(s) \\ s & v \end{bmatrix},$$

for all $u, t$, where $v$ and $s$ are functions of $u$ and $t$.

Hence, we have:

$$\begin{aligned} tb + ud + G(ta + uc) &= ua + f(t,u)c, \text{ and} \\ F(ta + uc) &= ub + f(t,u)d. \end{aligned}$$

Assume that $a = 0$. Then, $cb$ is not zero and

$$\begin{aligned} F(uc) &= ub + f(t,u)d, \\ G(uc) &= f(t,u)c - ud - tb. \end{aligned}$$

If $d = 0$ then $F(uc) = ub$ and we have situation (1) of the lemma. If $d$ is not zero then letting $t = 0$, we have $G(uc) = u(-b - d) + F(uc)d^{-1}c$

Letting $t = 0$, we have

$$G(uc) = u(-bd^{-1}c - d) + F(uc)d^{-1}d,$$

which is situation (2) of the lemma.

Hence, $a$ is not zero. But letting $u = 0$ we then obtain

$$\begin{aligned} G(ta) &= -tb + f(0,u)c \\ F(ta) &= f(t,0)d. \end{aligned}$$

and we are finished as before unless $d = 0$. But then $F$ is identically zero, and we may apply situation (1) of the lemma. This completes the proof. QED

Note that our proof of the above result did not make specific reference to having the transforming matrix be orthogonal.

## 3 The Equivalence of Symplectic Semifield Spreads and Semifield Flock Spreads

**3 Theorem.** *Let $S$ be a semifield flock spread in $PG(3,q)$.*

*(1) Then applying the following sequence of construction operations produces a symplectic semifield spread $S_{sym}$ in $PG(3,q)$:*

$$\begin{aligned} S(flock) &\longmapsto \textit{dualize} \longmapsto \textit{distort} \longmapsto \textit{derive} \\ &\longmapsto \textit{transpose} \longmapsto \textit{dualize} \longmapsto S_{sym}(\textit{symplectic}) \end{aligned}$$

*(2) Let $S_{sym}$ be a symplectic semifield whose spread is in $PG(3,q)$. The applying the following sequence of constructions produces a semifield flock spread in $PG(3,q)$.*

$$\begin{aligned} S_{sym}(symplectic) &\longmapsto dualize \longmapsto transpose \longmapsto derive \\ &\longmapsto extend \longmapsto dualize \longmapsto S(flock). \end{aligned}$$

PROOF. By Johnson [6] (2.7),

$$S(flock) \longmapsto dualize \longmapsto distort \longmapsto derive$$

produces a commutative semifield plane $S(commutative)$ of order $q^2$ with middle nucleus isomorphic to $GF(q)$. Using Kantor [9] (3.8),

$$S(commutative) \longmapsto transpose \longmapsto dualize \longmapsto S_{sym}(symplectic).$$

It remains to show that $S_{sym}(symplectic)$ has its spread in $PG(3,q)$. Since $S(commutative)$ has middle nucleus $GF(q)$, the transpose has right nucleus $GF(q)$ and transpose-dual has left nucleus $GF(q)$. This proves part (1).

Part (2) is similar:

$$S_{sym}(symplectic) \longmapsto dualize \longmapsto transpose$$

produces a commutative semifield plane $S(commutative)$ using Kantor [9] (3.8). Since $S_{sym}(symplectic)$ has its spread in $PG(3,q)$, the left nucleus is $GF(q)$, the dual has right nucleus $GF(q)$ and dual-transpose has middle nucleus $GF(q)$. Then

$$S(commutative) \longmapsto derive \longmapsto extend \longmapsto dualize \longmapsto S(flock),$$

by Johnson [6] (2.7). QED

**2 Definition.** Let $S$ be a semifield flock spread and let $S_{sym}$ denote the associated symplectic semifield spread constructed and connected as in the above theorem. In either case, the remaining semifield is the '5th-cousin' of the former. Thus, the 5th-cousin of a symplectic semifield spread in $PG(3,q)$ is a semifield flock spread and the 5th-cousin of a semifield flock spread is a symplectic semifield spread. More generally, any two spreads constructed from one another by $i$-iterations of the construction techniques of dualization, transpose, derivation, extension, dualization are said to be '$ith$-cousins'.

## 4 The Symplectic Semifield Flock Spreads

When we have a semifield flock spread in $PG(3,q)$, there is a corresponding symplectic flock spread in $PG(3,q)$, the 5th-cousin. We intend to show that a semifield flock spread is symplectic if and only if it is Kantor-Knuth or Desarguesian.

Our proof is based on the fact that if the semifield flock is symplectic then the dual transpose is a commutative semifield spread. We are not assuming anything regarding the symplectic dimension. So, there is a problem dealing with transposed spreads, since in the standard model the transposed spreads may be obtained via the transpose of matrices spread spreads, which might be large dimensional.

We choose a different symplectic form for which a more convenient model is possible when the original spread in in $PG(3,q)$. We begin with a more general analysis of the dual transpose of a semifield spread in $PG(3,q)$.

**2 Lemma.** *Let*

$$x = 0, y = x \begin{bmatrix} g_1(t) + g_2(u) = g(t,u) & f_1(t) + f_2(u) = f(t,u) \\ t & u \end{bmatrix}; t, u \in GF(q),$$

*be a semifield spread in $PG(3,q)$, where $g_i$ and $f_i$ are additive functions from $GF(q)$ to $GF(q)$, $i = 1,2$.*

*Define the associated pre-semifield $(S - GF(q) \times GF(q), +, \cdot)$ as follows:*

$$(c,d) \cdot (t,u) = (c(g_1(t) + g_2(u)) + dt, c(f_1(t) + f_2(u)) + du).$$

*Assume that the spread is non-Desarguesian, then this defines a pre-semifield with left nucleus $\{(0,\alpha); \alpha \in GF(q)\}$.*

PROOF. Note that for additive functions $g(t,u)$ and $f(u,t)$, since the spread is additive, it follows that $g(t,u)$ and $g(t,u)$ can be written in terms of two functions, each of one variable. The pre-semifield will be a semifield if the original matrix spread set contains $y = x$. QED

**3 Lemma.** *Define the dual pre-semifield $(S, +, *)$ by taking*

$$(a,b) * (c,d) = (c,d) \cdot (a,b) = (cg(a,b) + da, cf(a,b) + db),$$

*which defines a pre-semifield with right nucleus $\{(0,\alpha); \alpha \in GF(q)\}$.*

PROOF. Since we are merely interchanging left and right, the left nucleus becomes the right nucleus. QED

**4 Lemma.** *Let $p^r = q$, for $p$ a prime, so that $g_i(x) = \sum_{j=0}^{r-1} g_{i,j} x^{p^j}$ and $f_i(x) = \sum_{j=0}^{r-1} f_{i,j} x^{p^j}$. Then $(S, +, \diamond)$, with*

$$\begin{aligned}(s,t)\diamond(c,d) &= (\sum_{j=0}^{r-1} g_{2,j}^{1/p^j}(ct)^{1/p^j} + \sum_{j=0}^{r-1} f_{2,j}^{1/p^j}(cs)^{1/p^j} + \\ &\quad +ds, \sum_{j=0}^{r-1} g_{1,j}^{1/p^j}(ct)^{1/p^j} + \sum_{j=0}^{r-1} f_{1,j}^{1/p^j}(cs)^{1/p^j} + dt),\end{aligned}$$

*defines a pre-semifield which coordinatizes the dual transpose of the original semifield plane.*

Proof. We take the symplectic from from $GF(q)^4$ to the prime field $GF(p)$ defined by $\langle (a,b,c,d),(s,t,u,v)\rangle = T(av + bu - ct - ds)$, there $T$ is the trace function to $GF(p)$. We consider

$$(a,b)*(c,d) = (c,d)\cdot(a,b) = (cg(a,b)+da, cf(a,b)+db),$$

Let $c^* = cg(a,b) + da$ and $d^* = cf(a,b) + db$. We wish to describe $(s,t)\diamond(c,d)$. Hence, we obtain

$$T(av + bu - c^*t - d^*s) = T(av + bu - (cg(a,b)+da)t - (cf(a,b)+db)s) = 0$$

for all $a, b$. Let $h(x) = \sum_{j=o}^{r-1} h_j x^{p^j}$. $T(eh(x)w) = T(x(\sum h_i^{1/p_i}(ew)^{1/p_i}))$. Letting $a = 0$, we obtain:

$$T(b(u - (\sum_{j=0}^{r-1} g_{2,j}^{1/p^j}(ct)^{1/p^j} + \sum_{j=0}^{r-1} f_{2,j}^{1/p^j}(cs)^{1/p^j} + ds)),$$

for all $b$ in $GF(q)$, and letting $b = 0$, we obtain:

$$T(a(v - (\sum_{j=0}^{r-1} g_{1,j}^{1/p^j}(ct)^{1/p^j} + \sum_{j=0}^{r-1} f_{1,j}^{1/p^j}(cs)^{1/p^j} + dt)),$$

for all $a$ in $GF(q)$. Hence it follows that

$$\begin{aligned}u &= (\sum_{j=0}^{r-1} g_{2,j}^{1/p^j}(ct)^{1/p^j} + \sum_{j=0}^{r-1} f_{2,j}^{1/p^j}(cs)^{1/p^j} + ds),\\ v &= (\sum_{j=0}^{r-1} g_{1,j}^{1/p^j}(ct)^{1/p^j} + \sum_{j=0}^{r-1} f_{1,j}^{1/p^j}(cs)^{1/p^j} + dt).\end{aligned}$$

This completes the proof of the lemma. QED

**5 Lemma.** *Under the assumptions of Lemma 4,* $(0,1)\diamond(0,1)=(0,1)$ *and defining a multiplication* $\odot$ *by*

$$((s,t)\diamond(0,1))\odot((0,1)\diamond(c,d)=(s,t)\diamond(c,d),$$

*will produce the dual transpose semifield which has middle nucleus* $\{(0,\alpha);\alpha\in GF(q)\}$. *Let* $\widetilde{g}_i(x)=\sum_{j=0}^{r-1}g_{i,j}^{1/p^j}x^{1/p^j}$ *and* $\widetilde{f}_i(x)=\sum_{j=0}^{r-1}f_{i,j}^{1/p^j}x^{1/p^j}$, *for* $i=1,2$.
*Then*

$$(s,t)\odot(\widetilde{g}_2(c),\widetilde{g}_1(c)+d)=(\widetilde{g}_2(ct)+\widetilde{f}_2(cs)+ds,\widetilde{g}_1(ct)+\widetilde{f}_1(cs)+dt)),$$

*is the semifield* $(S,+,\odot)$, *with middle nucleus* $\{(0,\alpha);\alpha\in GF(q)\}$, *and identity* $(0,1)$.

PROOF. The proof is immediate by calculation. QED

**6 Lemma.** *Any semifield* $(S,+,\odot)$ *of order* $q^2$ *and middle nucleus* $GF(q)$ *that commutes over* $\{(0,\alpha);\alpha\in GF(q)\}$ *may be written in the form*

$$(s,t)\odot(c,d)=(H(sc)+ct+sd,M(sc)+dt)$$

*and constructs by transpose and dualization a symplectic spread of dimension two*

$$x=0,y=x\begin{bmatrix}\widetilde{M}(t)+\widetilde{H}(u) & u\\ u & t\end{bmatrix};t,u\in GF(q).$$

*Assume an isotope is commutative. Then the original semifield may be re-coordinatized by functions* $g_i$ *and* $f_i$ *such that* $g_1(t)=0$, *and* $g_2(u)=u$, *for all* $t,u\in GF(q)$. *In this case, the original semifield spread is symplectic of dimension two defined by*

$$x=0,y=x\begin{bmatrix}u & f_1(t)+f_2(u)\\ t & u\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}f_1(t)+f_2(u) & u\\ u & t\end{bmatrix};t,u\in GF(q),$$

We adopt the notation developed in the previous lemmas. The previous results produce specific functions when we have semifield flock spreads. First to construct the associated commutative semifield:

**1 Proposition.** *When* $(S,+,\cdot)$ *defines a semifield flock spread, we may assume that* $g_2(x)=x$ *and* $f_2(x)=0$. *Hence,* $\widetilde{g}_2(x)=x$ *and* $\widetilde{f}_2(x)=0$. *Then*

$$(s,t)\odot(c,d)=(ct+(d-\widetilde{g}_1(c))s,\widetilde{g}_1(ct)+\widetilde{f}_1(cs)+(d-\widetilde{g}_1(c))t)),$$

*is the dual transposed semifield with middle nucleus* $\{(0,\alpha);\alpha\in GF(q)\}$.

*When $(S,+,\cdot)$ defines a semifield flock spread with additive functions $g(t)$ and $f(t)$ so that*

$$x=0, y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}; t,u\in GF(q).$$

*is the associated spread then the commutative semifield $(S,+,\odot)$ obtained by*

$$S(flock)\longmapsto dualize\longmapsto distort\longmapsto derive=(commutative\ semifield)$$

*may be defined by*

$$(s,t)\odot(c,d)=(-g(sc)+ct+sd, f(sc)+dt).$$

Proof. This follows by tracing through the functions in the proof of this result in Johnson [**6**]. QED

We now produce the form for the symplectic semifield spread associated with the semifield flock spread.

**2 Proposition.** *When $(S,+,\cdot)$ defines a semifield flock spread with additive functions $g(t)$ and $f(t)$ so that*

$$x=0,\quad y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}; t,u\in GF(q),$$

*then the symplectic spread in $PG(3,q)$ constructed using*

$$\begin{aligned} S(flock) &\longmapsto dualize\longmapsto distort\longmapsto derivation \\ &\longmapsto transpose\longmapsto dualize\longmapsto S_{sym}(symplectic) \end{aligned}$$

*is*

$$x=0,\quad y=x\begin{bmatrix} -\widetilde{g}(u)+\widetilde{f}(t) & u \\ u & t \end{bmatrix}; t,u\in GF(q).$$

Proof. Apply our previous lemmas. QED

**3 Proposition.** *Any semifield spread in $PG(3,q)$ which is symplectic is symplectic of dimension two.*

Proof. We know by dualization and transpose, we may construct a commutative semifield plane of order $q^2$ and middle nucleus isomorphic to $GF(q)$. By recoordinatizing, if necessary, we write the commutative semifield in the manner of the Lemma 6) and re-construct the original symplectic semifield spread with symplectic dimension two. QED

As corollaries to our analysis, we have:

**1 Corollary.** *Let $\pi$ be a semifield flock spread in $PG(3,q)$. Then $\pi$ is symplectic if and only if $\pi$ is a Kantor-Knuth plane or Desarguesian.*

**2 Corollary.** *The symplectic 5th-cousin of a semifield flock spread is a flock spread if and only if the semifield flock is Kantor-Knuth.*

### 4.1 The Roman Son

In Payne and Thas [14], there is a construction of a semifield spread in $PG(3,q)$, using the Cohen-Ganley semifield flock spread. Basically, one considers the associated generalized quadrangle $Q(CG)$, of type $(q^2,q)$ and forms the translation dual of type $(q,q^2)$, the 'Roman generalized quadrangle'. This generalized quadrangle is shown to have subquadrangles isomorphic to $Q(4,q)$, the set of totally isotropic points and totally isotropic lines with respect to a quadric in $PG(4,q)$. Payne and Thas [14], show that there are ovoids in $Q(4,q)$. Since the dual of $Q(4,q)$ is $M(3,q)$, the set of totally isotropic points and totally isotropic lines of a symplectic polarity of $PG(3,q)$, we obtain from any such ovoid a symplectic spread in $PG(3,q)$ (given by a symplectic form over the associated 4-dimensional vector space $V_4$, 'symplectic dimension 2'). It is furthermore determined by Lunardon [10] that these symplectic spreads are semifield spreads. We call any such semifield spread a 'Roman Son'. In fact, it is noted in Payne and Thas [14] that all Roman Sons are isomorphic.

Hence, we obtain:

**4 Theorem.** *The Roman Son (Thas-Payne spread) and the Cohen-Ganley flock spread are non-isomorphic 5th-cousins.*

### 4.2 Penttila-Williams' 5th-Cousin

There is a sporadic ovoid in $Q(4,3^5)$ and hence a symplectic spread of symplectic dimension 2 in $PG(3,3^5)$, due to Penttila and Williams [13]. This turns out to be a semifield spread, so the 5th-cousin is a semifield flock spread in $PG(3,3^5)$. This is the semifield flock spread constructed in Bader, Lunardon,and Pinneri [11]

**5 Theorem.** *The symplectic Penttila-Williams semifield spread is not isomorphic to its 5th-cousin, the associated semifield flock spread of Bader, Lunardon and Pinneri.*

**2 Remark.** (1) Ball and Brown [1] show, using the Thas-Payne construction, that there are six semifield spreads corresponding to a semifield flock. Furthermore, they show that the symplectic semifield spread and semifield flock spread are isomorphic if and only if the semifield flock spread is Desarguesian or Kantor-Knuth (our version of this did not require that the symplectic semifield spread had symplectic dimension two).

In our sequence of constructions

$$\begin{aligned} S(flock) &\longmapsto \textit{dualize} \longmapsto \textit{distort} \longmapsto \textit{derive} \\ &\longmapsto \textit{transpose} \longmapsto \textit{dualize} \longmapsto S_{sym}(symplectic) \end{aligned}$$

and

$$\begin{array}{lll} S_{sym}(symplectic) & \longmapsto & dualize \longmapsto transpose \longmapsto derive \\ & \longmapsto & extend \longmapsto dualize \longmapsto S(flock), \end{array}$$

we see that there are also five possible semifields and a generalized Hall quasifield and these are mutually non-isomorphic provided the flock is not Desarguesian or Kantor-Knuth; two each with left, right nucleus isomorphic to $GF(q)$ and one with middle nucleus isomorphic to $GF(q)$. Furthermore, if one considers the sequence

$$S(flock) \longmapsto dualize \longmapsto transpose,$$

another semifield with middle nucleus $GF(q)$ is constructed. The six semifield spreads in our sequences are isomorphic to the six semifield spreads of Ball and Brown, who also included dual and transpose versions of the various semifield spreads.

(2) If one considers an iteration of transpose-dual-transpose, etc. it is potentially possible to construct twelve semifield spreads. To see that there are, in fact, only six, Ball and Brown [1] show that any transpose of a semifield flock spread is isomorphic to the original, using the Klein quadric.

More generally, one could ask if the transpose of a flock spread is isomorphic to itself. And, in fact, this is valid as well, and an algebraic proof using transposed matrices is all that is required.

**6 Theorem.** *The dual spread of a flock spread is isomorphic to the flock spread.*

PROOF. Represent the flock spread as follows:

$$x = 0, y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix}; u, t \in GF(q),$$

where $g$ and $f$ are functions on $GF(q)$ (not necessarily additive). The dual spread is

$$x = 0, y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix}^T = \begin{bmatrix} u + g(t) & t \\ f(t) & u \end{bmatrix}; u, t \in GF(q),$$

Now note that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u + g(t) & t \\ f(t) & u \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -u & f(t) \\ t & -u - g(t) \end{bmatrix},$$

and letting $-u - g(t) = v$, the previous matrix is

$$\begin{bmatrix} v + g(t) & f(t) \\ t & v \end{bmatrix}.$$

Since the operations listed merely correspond to a basis change, we have the proof. QED

**3 Remark.** It also is now trivial to note that there are no non-Desarguesian symplectic semifield spreads in $PG(3,q)$, when $q$ is even. That is, the symplectic dimension must be two so there is an associated flock semifield plane of even order $q^2$, which is necessarily Desarguesian by Johnson [6].

**4 Remark.** A direct comparison of our matrix spread sets and those of Ball and Brown will show that there is a sign difference on the functions $\widetilde{g}$. However, the spread

$$x=0, y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}; u,t \in GF(q)$$

is isomorphic to the spread

$$x=0, y=x\begin{bmatrix} u-g(t) & f(t) \\ t & u \end{bmatrix}; u,t \in GF(q),$$

which shows that the sign on the functions $\widetilde{g}$ is not critical.

PROOF.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} u & t \\ f(t) & u+g(t) \end{bmatrix}.$$

Letting $u+g(t)=v$, we have the spread

$$x=0, y=x\begin{bmatrix} v-g(t) & t \\ f(t) & v \end{bmatrix}; u,t \in GF(q),$$

but this spread is isomorphic to its transpose

$$x=0, y=x\begin{bmatrix} v-g(t) & f(t) \\ t & v \end{bmatrix}; u,t \in GF(q).$$

QED

## 5 Lifting and More Cousins

Given any semifield spread in $PG(3,q)$, the plane defined by dualizing the semifield may be derived to obtain a non-semifield spread corresponding to the semifield spread. Furthermore, any semifield flock spread may be derived using one of the base reguli, to obtain a spread of order $q^2$ admitting a Baer group of order $q$. The reader is referred to the paper of Jha and Johnson [7], where these derived planes are discussed in greater detail. Any of these might be called a 'cousin'. If cousins of semifields may only be semifields, then there is the matter of 'lifting'.

Given any spread in $PG(3,q)$, there is an associated spread in $PG(3,q^2)$, obtained by a process called 'lifting'. In fact, the construction is not spread dependent but quasifield dependent, in that it is possible to construct two different lifted spreads from the same spread.

The following is from the authors' text [3], 29.5, p. 450. Let $K$ be isomorphic to $GF(q)$ and let $K^+$ be a quadratic extension of $K$, by $\theta$, where $\theta^2 = \theta\alpha + \beta$, for $\alpha, \beta \in K$. Let

$$\pi : x = 0, y = x \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix}; u, t \in K,$$

then

$$\pi^L : x = 0, y = x \begin{bmatrix} (\theta s + w)^q & -\theta g(t,u) + (f(t,u) + \alpha g(t,u)) \\ (\theta t + u) & (\theta s + w) \end{bmatrix}; s, w, t, u \in K$$

is a spread in $PG(3,K^2)$, called a spread 'lifted' from $\pi$. $\pi$ is called a 'contraction' of $\pi^L$. Of course, this process may be repeated indefinitely to obtain an infinite chain of lifted planes.

It is clear from the construction that if the original spread set is additive then the lifted spread set is additive; semifields lift to semifields. In particular, this means that a semifield flock or a symplectic semifield spread constructs an infinite chain of lifted planes. But, also note that this means that a semifield flock spread in $PG(3,q)$ could be the '$nth$-cousin' of a semifield spread in $PG(3,q^{2^n})$, obtained by a series of $n$ contractions. To complicate matters, we note that we may derive the net

$$x = 0, y = x \begin{bmatrix} (\theta s + w)^q & 0 \\ 0 & (\theta s + w) \end{bmatrix}; s, w \in K,$$

to produces spreads in $PG(7,q)$. Are these 'cousins'?

## 5.1 Lifting Semifield Flocks and Their 5th Cousins

As an example of the enormous variety of semifield spreads associated with a given one, we consider a semifield flock spread

$$x = 0, y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix}; u, t \in GF(q),$$

where $g$ and $f$ are additive functions. Hence, we obtain the lifted spread

$$\pi^L : x = 0, y = x \begin{bmatrix} (\theta s + w)^q & -\theta(g(t) + u) + (f(t) + \alpha g(t)) \\ (\theta t + u) & (\theta s + w) \end{bmatrix}; s, w, t, u \in K.$$

Consider the 5th-cousin of the semifield spread

$$x=0, y=x\begin{bmatrix} -\widetilde{g}(u)+\widetilde{f}(t) & t \\ t & u \end{bmatrix}; u,t\in GF(q).$$

This spread lifts to

$$x=0, y=x\begin{bmatrix} (\theta s+w)^q & -\theta(-\widetilde{g}(u)+\widetilde{f}(t))+(t+\alpha(-\widetilde{g}(u)+\widetilde{f}(t))) \\ (\theta t+u) & (\theta s+w) \end{bmatrix};$$

where $s,w,t,u\in K$.

Obviously, we have a longer chain connecting a semifield spread and its 5th-cousin, if we begin with one of the lifted versions. But, what is a direct connection between the two lifted 5th-cousins?

So, the main question seems to be: **Are all spreads in $PG(3,q)$ cousins?**

# 6 Appendix: Transpose of Semifields

In the following, we consider the general transpose of semifields, with particular attention to commutative semifields and their transpose-duals. Most of the results, especially when considering the commutative case, are straightforward generalizations of ideas implicit in Kantor [9].

Let $(S,+,\cdot)$ be a semifield and let $L$, $R$, $M$ denote the left nucleus, right nucleus, and middle nucleus, respectively, isomorphic to $GF(q)$, $GF(w)$, and $GF(z)$, respectively. Assume that $q^n-w^m-z^k$. Then, $S$ is a left $n$-dimensional $L$-space, a right $m$-dimensional $R$-space and a right (by stipulation) $k$ dimensional $M$-space.

The 'transpose $S^T$' of $S$ is a semifield coordinatizing the dual spread of the spread coordinatized by $S$. The standard manner of obtaining the transpose is to write $S$ over the left nucleus in matrix form: Letting $x=(x_1,x_2,\dots,x_n)$ written over a basis for $S$ over $L$, and

$$\begin{aligned} x\cdot(s_1,s_2,\dots,s_n) &= xM_{(s_1,s_2,\dots,s_n)} \\ &= \begin{bmatrix} f_{11}(s_1,\dots,s_n) & f_{12}(s_1,\dots,s_n) & \cdots & f_{1n}(s_1,\dots,s_n) \\ f_{21}(s_1,\dots,s_n) & f_{22}(s_1,\dots,s_n) & \cdots & f_{2n}(s_1,\dots,s_n) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ s_1 & s_2 & \cdots & s_n \end{bmatrix}_{n\times n}, \end{aligned}$$

where $f_{ij}$ are additive functions over $L$ of $n$-variables.

Now, in this setting, the transposed semifield spreads may be easily described simply as

$$xM^t_{(s_1,s_2,\dots,s_n)} = \begin{bmatrix} f_{11}(s_1,\dots,s_n) & f_{12}(s_1,\dots,s_n) & \cdots & f_{1n}(s_1,\dots,s_n) \\ f_{21}(s_1,\dots,s_n) & f_{22}(s_1,\dots,s_n) & \cdots & f_{2n}(s_1,\dots,s_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ s_1 & s_2 & \cdots & s_n \end{bmatrix}^t_{n\times n}.$$

However, if the semifield is represented over the right or middle nucleus, the direct transpose of the associated matrices may not be convenient. Hence, we seek an alternative method for the construction of the transpose.

For $i = 1,2,\dots,n-1$, define

$$g_{ijk}(s_k) = f_{ij}(0,\dots,s_k,0,\dots,0), \text{ for } i = 1,\dots,n-1; j,k = 1,\dots,n.$$

Since we have a left vector space over $L$, we know that

$$f_{ij}(s_1,\dots,s_n) = \sum_{k=1}^{n} g_{ijk}(s_k), \text{ for } i = 1,2,\dots,n-1; j = 1,2,\dots,n.$$

Let $q = p^r$, where $p$ is a prime, then for $x$ in $L$,

$$g_{ijk}(x) = \sum_{s=0}^{r-1} g_{ijks}x^{p^s}.$$

Now define

$$\widetilde{g}_{ijk}(x) = \sum_{s=0}^{r-1} g_{ijks}^{1/p^s}x^{1/p^s}.$$

Then

$$\begin{aligned} \text{For, } f_{ij}(s_1,\dots,s_n) &= \sum_{k=1}^{n} g_{ijk}(s_k), \text{ for } i = 1,2,\dots,n-1; j = 1,2,\dots,n, \\ \text{define} & \\ \widetilde{f}_{ij}(s_1,s_2,\dots,s_n) &= \sum_{k=1}^{n} \widetilde{g}_{ijk}(s_k) \end{aligned}$$

Let $T$ denote the trace function from $L$ to $GF(p)$, defined by

$$T(a)=\sum_{s=0}^{r-1} a^{p^s}.$$

Then, we note that

$$T(bg_{ijk}(x))=T(x\tilde{g}_{ijk}(b)).$$

We now consider the subspace $y=x\cdot s=x\cdot(s_1,\ldots,s_n)$.
We see that

$$\begin{aligned}
&(c_1,c_2,\ldots,c_n)\cdot(s_1,\ldots,s_n)\\
=\ &(\sum_{i=1}^{n-1} c_i f_{1i}+c_n s_1, \sum_{i=1}^{n-1} c_i f_{2i}+c_n s_2,\ldots,\sum_{i=1}^{n-1} c_i f_{ni}+c_n s_n)\\
=\ &((\sum_{i=1}^{n-1} c_i(\sum_{k=1}^{n} g_{1ik}(s_k))+c_n s_1),\\
&\ldots,(\sum_{i=1}^{n} c_i(\sum_{k=1}^{n} g_{zik}(s_k))+c_n s_z),\ldots,\sum_{i=1}^{n} c_i(\sum_{k=1}^{n} g_{nik}(s_k))+c_n s_n).
\end{aligned}$$

Now dualize to the associated semifield $(S,+,\diamond)$, with right nucleus $L$ such that

$$(c_1,c_2,\ldots,c_n)\cdot(s_1,\ldots,s_n)=(s_1,\ldots,s_n)\diamond(c_1,c_2,\ldots,c_n)$$

We determine the transposed semifield $(S,+,\odot)$; to $(S,+,\diamond)$ by determining the lines $y=x\odot(c_1,c_2,\ldots,c_n)$ that as $GF(p)$ -subspaces are orthogonal to $y=x\diamond(c_1,\ldots,c_n)$. Since the associated spread arises from a $2n$-dimensional $L$-vector space, we use the following orthogonal form:

$$\begin{aligned}
&\langle(c_1,c_2,\ldots,c_n,c_{n+1},\ldots,c_{2n}),(s_1,s_2,\ldots,s_n,s_{n+1},\ldots,s_{2n})\rangle\\
&\qquad\qquad=T(\sum_{i=1}^{2n}(-1)^{i_\delta}c_i s_{n-i+1}),
\end{aligned}$$

where $i_\delta=+1$ for $i=1,2,\ldots,n$ and $-1$ for $i=n+1,\ldots,2n$.

We want to consider the lines (subspaces) $y=x\diamond(c_1,\ldots,c_n)$ and define an associated subspace on the transposed spread defined by a semifield $(S,+,\odot)$. Let

$$\begin{aligned}
&(s_1,\ldots,s_n)\diamond(c_1,c_2,\ldots,c_n)\\
&\qquad=(c_1,c_2,\ldots,c_n)\cdot(s_1,s_2,\ldots,s_n)=(s_1^*,s_2^*,\ldots,s_n^*),
\end{aligned}$$

so that

$$\begin{aligned} s_z^* &= \sum_{i=1}^{n-1} c_i f_{z,i}(s_1, s_2, \ldots, s_k) + c_n s_z \\ &= \sum_{i=1}^{n-1} c_i \sum_{k=1}^{n} g_{z,i,k}(s_k) + c_n s_z. \end{aligned}$$

We consider the vectors $(s_1, s_2, \ldots, s_n, s_1^*, s_2^*, \ldots, s_n^*)$ of $y = x \diamond (c_1, c_2, \ldots, c_n)$ and determine the vectors $(d_1, d_2, \ldots, d_n)$ on $y = x \odot (c_1, c_2, \ldots, c_n)$ that are orthogonal to $(s_1^*, s_2^*, \ldots, s_n^*)$, for all $s_i^*$, fixing $c_i$ and varying $s_i$, making $y = x \odot (c_1, c_2, \ldots, c_n)$ and $y = x \diamond (c_1, c_2, \ldots, c_n)$ orthogonal. Let the vectors on $y = x \odot (c_1, c_2, \ldots, c_n)$ be denoted by $(d_1, d_2, \ldots, d_n, d_1^*, d_2^*, \ldots, d_n^*)$, for all $d_i$, $i = 1, 2, \ldots, n$.

So,

$$T\left(\sum_{i=1}^{n} s_i d_{n-i+1}^* - \sum_{i=1}^{n} s_i^* d_{n-i+1}\right) = 0.$$

for all $s_i$.

We need to determine $d_{n-i+1}^*$, for $i = 1, 2, \ldots, n$.

Consider all $s_i = 0$ but $s_z$. Then, we obtain:

$$T\left(s_z d_{n-z+1}^* - \sum_{i=1}^{n} s_i^* d_{n-i+1}\right) = 0.$$

$$\begin{aligned} \sum_{i=1}^{n} s_i^* d_{n-i+1} &= \sum_{i=1}^{n} d_{n-i+1}\left(\sum_{j=1}^{n-1} c_j \left(\sum_{k=1}^{n} g_{j,i,k}(s_k)\right) + c_n s_i\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{k=1}^{n} d_{n-i+1} c_j g_{j,i,k}(s_k) + \sum_{i=1}^{n} d_{n-i+1} c_n s_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n-1} d_{n-i+1} c_j g_{j,i,k}(s_z) + d_{n-z+1} c_n s_z. \end{aligned}$$

Note that

$$\begin{aligned} & T\left(s_z d_{n-z+1}^* - \sum_{i=1}^{n} \sum_{j=1}^{n-1} d_{n-i+1} c_j g_{j,i,k}(s_z) + d_{n-z+1} c_n s_z\right) \\ = \; & T\left(s_z \left(d_{n-z+1}^* - \sum_{i=1}^{n} \sum_{j=1}^{n-1} \tilde{g}_{j,i,z}(d_{n-i+1} c_j) + d_{n-z+1} c_n\right)\right) = 0, \\ & \forall s_z. \end{aligned}$$

Thus, we have:

$$\begin{aligned} d^*_{n-z+1} &= \sum_{i=1}^{n}\sum_{j=1}^{n-1} \widetilde{g}_{j,i,z}(d_{n-i+1}c_j) + d_{n-z+1}c_n, \\ so\ d^*_e &= \sum_{i=1}^{n}\sum_{j=1}^{n-1} \widetilde{g}_{j,i,n-e+1}(d_{n-i+1}c_j) + d_e c_n. \end{aligned}$$

## 6.1 The Middle Nucleus Semifield $(S,+,\odot)$ transposes to the Right Nucleus Semifield $(S,+,\diamond)$

Thus, we obtain the transpose of the middle nucleus semifield $(S,+,\odot)$ is the right nucleus $(S,+,\diamond)$, so

$$\begin{aligned} &(d_1,d_2,\ldots,d_n)\odot(c_1,c_2,\ldots,c_n) \\ &\quad = ((\sum_{i-1}^{n}\sum_{j-1}^{n-1} \widetilde{g}_{j,i,n-e+1}(d_{n-i+1}c_j) + d_e c_n))_{1\times n}, \text{ for } e = 1,2,\ldots,n \end{aligned}$$

where

$$\begin{aligned} (d_1,d_2,\ldots,d_n)\diamond(c_1,c_2,\ldots,c_n) &= (c_1,c_2,\ldots,c_n)\cdot(d_1,d_2,\ldots,d_n) \\ &= (((\sum_{i=1}^{n-1} c_i \sum_{k=1}^{n} g_{z,i,k}(s_k)) \mid c_n s_z))_{n\times 1}, \text{ for } z = 1,2,\ldots,n. \end{aligned}$$

Notice that we are connecting pre-semifields, not necessarily semifields.

## 6.2 Commutative Semifields

Now if we have a commutative pre-semifield $S$ of dimension $n$ over the middle nucleus $M$, then $S(transpose - dual)$ defines a symplectic spread given by a symplectic pre-semifield of dimension $n$ over the left nucleus $M$. We point out that if the transpose-dual is re-coordinatized by the backward identity permutation matrix, we obtain a matrix spread set of symmetric matrices. Of course, Kantor [9] showed that if a commutative semifield is written over its prime field then using the standard symplectic form, the transpose-dual spreads consist of symmetric matrices. It is sometimes more convenient to write the semifield as a vector space over its middle nucleus. Thus, one nice feature of this procedure is that the commutative pre-semifields written over their middle nuclei give rise directly to symmetric matrix spread sets.

**7 Theorem.** *If a pre-semifield $(S,+,\odot)$ is written over its middle nucleus, then the pre-semifield is commutative if and only if the transpose-dual pre-semifield $(S,+,\cdot)$, followed by the backward identity permutation matrix, is a symplectic pre-semifield, whose spread in matrix form consists of symmetric matrices.*

Proof. Assume that $(S,+,\odot)$ is commutative. Then

$$\sum_{i=1}^{n}\sum_{j=1}^{n-1}\widetilde{g}_{j,i,n-e+1}(d_{n-i+1}c_j)+d_ec_n = \sum_{i=1}^{n}\sum_{j=1}^{n-1}\widetilde{g}_{j,i,n-e+1}(c_{n-i+1}d_j)+c_ed_n \quad \forall e\ \forall c_i\ \forall d_j.$$

Let $z=n-i+1$ to transform

$$\sum_{i=1}^{n}\sum_{j=1}^{n-1}\widetilde{g}_{j,i,n-e+1}(c_{n-i+1}d_j) \text{ to } \sum_{j=1}^{n-1}\sum_{z=1}^{n-1}\widetilde{g}_{j,n-z+1,z}(c_zd_j).$$

Similarly, let $k=n-i+1$ to transform

$$\sum_{i=1}^{n}\sum_{j=1}^{n-1}\widetilde{g}_{j,i,n-e+1}(d_{n-i+1}c_j) \text{ to } \sum_{j=1}^{n-1}\sum_{k=1}^{n}\widetilde{g}_{j,n-k+1,n-e+1}(d_kc_j).$$

Hence, we obtain

$$\begin{aligned}&\sum_{j=1}^{n-1}\sum_{z=1}^{n-1}\widetilde{g}_{j,n-z+1,n-e+1}(c_zd_j)+\sum_{j=1}^{n-1}\widetilde{g}_{j,1,n-e+1}(c_nd_j)+c_ed_n\\ =\ &\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\widetilde{g}_{j,n-k+1,n-e+1}(d_kc_j)+\sum_{j=1}^{n-1}\widetilde{g}_{j,1,n-e+1}(d_nc_j)+d_ec_n.\end{aligned}$$

Now in order to equate, we let $(z,j)\longmapsto(j,k)$ in the second expression. This produces

$$\begin{aligned}&\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\widetilde{g}_{k,n-j+1,n-e+1}(c_jd_k)+\sum_{j=1}^{n-1}\widetilde{g}_{j,1,n-e+1}(c_nd_j)+c_ed_n\\ =\ &\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\widetilde{g}_{j,n-k+1,n-e+1}(d_kc_j)+\sum_{j=1}^{n-1}\widetilde{g}_{j,1,n-e+1}(d_nc_j)+d_ec_n.\end{aligned}$$

Since this expression is valid for all $d_k$ and for all $c_j$, we obtain the following identities:

$$\begin{aligned} \widetilde{g}_{k,n-j+1,n-e+1} &= \widetilde{g}_{j,n-k+1,n-e+1}, \text{ for } j \neq n,\ k \neq n, \\ \widetilde{g}_{j,1,n-e+1} &= 0 \text{ for } j \neq e \text{ or } n, \\ \widetilde{g}_{j,1,1} &= 0, \text{ for } j \neq n \\ \widetilde{g}_{e,1,n-e+1} &= 1, \text{ for } e \neq n. \end{aligned}$$

This implies that

$$\begin{aligned} g_{k,n-j+1,n-e+1} &= g_{j,n-k+1,n-e+1}, \text{ for } j \neq n,\ k \neq n, \\ g_{j,1,n-e+1} &= 0 \text{ for } j \neq e \text{ or } n, \\ g_{j,1,1} &= 0, \text{ for } j \neq n \\ g_{e,1,n-e+1} &= 1, \text{ for } e \neq n. \end{aligned}$$

This means that

$$\begin{aligned} f_{i,j}(s_1,\dots,s_n) &= \sum_{k=1}^{n} g_{i,j,k}(s_k) = \\ \sum_{k=1}^{n} g_{n-j-1,n-i+1,k}(s_k) &= f_{n-j+1,n-i+1}(s_1,\dots,s_n), \text{ for } i,j \neq n. \\ \textit{Also, } f_{i,n}(s_1,\dots,s_n) &= g_{i,n,n-i+1}(s_{n-i+1}) = s_{n-i+1}, \text{ for } i \neq n. \end{aligned}$$

Before we give the general conclusion, we pause to consider $n = 2, 3, 4$.
If $n - 2$, we obtain

$$\begin{bmatrix} f_{11}(s_1,s_2) = g_{1,1,1}(s_1) + g_{1,1,2}(s_2) = s_2 & f(1,2) = g_{1,2,1}(s_1) + g_{1,,2,2}(s_2) \\ s_1 & s_2 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} g_{1,2,1}(s_1) + g_{1,2,2}(s_2) & s_2 \\ s_2 & s_1 \end{bmatrix},$$

a matrix spread set of $2 \times 2$ symmetric matrices.

If $n = 3$, we obtain: (we now write $f_{i,j}(s_1,\dots,s_n)$ merely as $f_{i,j}$). We use the identities to obtain:

$$\begin{bmatrix} s_3 & f_{1,2} & f_{1,3} \\ s_2 & f_{2,2} & f_{2,3} = f_{1,2} \\ s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} f_{1,3} & f_{1,2} & s_3 \\ f_{1,2} & f_{2,2} & s_2 \\ s_3 & s_2 & s_1 \end{bmatrix},$$

a matrix spread set of $3 \times 3$ symmetric matrices.

When $n = 4$, using the identities, we obtain:

$f_{1,2} = f_{3,4}, f_{1,3} = f_{2,4}, f_{2,2} = f_{3,3}$ and

$$\begin{bmatrix} s_4 & f_{1,2} & f_{1,3} & f_{1,4} \\ s_3 & f_{2,2} & f_{2,3} & f_{2,4} = f_{1,3} \\ s_2 & f_{3,2} & f_{3,3} = f_{2,2} & f_{3,4} = f_{1,2} \\ s_1 & s_2 & s_3 & s_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} f_{1,4} & f_{1,3} & f_{1,2} & s_4 \\ f_{1,3} & f_{2,3} & f_{2,2} & s_3 \\ f_{1,2} & f_{2,2} & f_{3,2} & s_2 \\ s_4 & s_3 & s_2 & s_1 \end{bmatrix},$$

a matrix spread set of $4 \times 4$ symmetric matrices.

Generally, for arbitrary $n$, using $f_{i,j} = f_{n-j+1,n-i+1}$, for $i, j \neq n$, we obtain:

$$\begin{bmatrix} s_n & f_{1,2} & f_{1,3} & \cdots & f_{1,n-1} & f_{1,n} \\ s_{n-1} & f_{2,2} & f_{2,3} & \cdots & f_{2,n-1} & f_{2,n} \\ s_{n-2} & \cdots & \cdots & \cdots & \cdots & f_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ s_2 & f_{n\ 1,2} & f_{n\ 1,3} & \cdots & f_{n\ 1,n\ 1} & f_{n\ 1,n} \\ s_1 & s_2 & s_3 & \cdots & s_{n-1} & s_n \end{bmatrix}$$
$$\cdot \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & ..0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} f_{1,n} & f_{1,n-1} & f_{1,n-2} & \cdots & f_{1,2} & s_n \\ f_{1,n-1} = f_{2,n} & f_{2,n-1} & f_{2,n-2} & \cdots & f_{2,2} & s_{n-1} \\ f_{1,n-2} = f_{3,n} & f_{3,n-1} = f_{n-2} & f_{3,n-2} & \cdots & f_{2,3} & s_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{1,2} = f_{n-1,n} & \cdots & \cdots & \cdots & f_{n-1,2} & s_2 \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_2 & s_1 \end{bmatrix}$$

a matrix spread set of $n \times n$ symmetric matrices. QED

**3 Corollary.** *Any finite symplectic semifield of dimension $n$ over its left nucleus has symplectic dimension $n$.*

PROOF. By Kantor [9], the dual-transpose produces a commutative semifield plane that has dimension $n$ over the middle nucleus. Re-coordinatize so that we have a commutative semifield coordinatizing the semifield plane. Since the middle nucleus is an invariant, this commutative semifield will be of dimension

$n$ over the middle nucleus. Now apply the previous theorem to obtain a symplectic pre-semifield defining a matrix spread set of $n \times n$ symmetric matrices; the symplectic semifield has symplectic dimension $n$. QED

## References


[1] S. Ball, M.R. Brown: *The six semifield planes associated with a semifield flock* (to appear).

[2] S.D. Cohen, M.J. Ganley: *Commutative semifields, two-dimensional over their middle nuclei*, J. Algebra, **75** (1982), n. 2, 373–385.

[3] M. Biliotti, V. Jha, Norman L. Johnson: Foundations of Translation Planes, Pure and Applied Mathematics, Vol. 243, Marcel Dekker, New York, Basel, 2001.

[4] I. Bloemen: *Substructures and characterizations of finite generalized polygons*, Ph.D. thesis, University of Ghent, 1995.

[5] H. Gevaert, Norman L. Johnson: *Flocks of quadratic cones, generalized quadrangles, and translation planes*, Geom. Ded., **27** (1988), n. 3, 301–317.

[6] N.L. Johnson: *Semifield flocks of quadratic cones*, Simon Stevin, **61** (1987), n. 3-4, 313–326.

[7] V. Jha, N.L. Johnson: *Ostrom-Derivates, Innovations in incidence geometry*, to appear.

[8] N.L. Johnson, S.E. Payne: *Flocks of Laguerre planes and associated geometries*, in Mostly finite geometries, Lecture Notes in Pure and Applied Math., Vol. 190, Marcel Dekker, New York–Basil–Hong Kong, 1997, 51–122.

[9] W.M. Kantor: *Commutative semifields and symplectic spreads*, Journal of Algebra, **270** (2003), 96–114.

[10] G. Lunardon: *Flocks, ovoids of $Q(4,q)$ and designs*, Geom. Ded., **66** (1997), 163–173.

[11] L. Bader, G. Lunardon, Guglielmo, I. Pinneri: *A new semifield flock*, J. Combin. Theory Ser. A, **86** (1999), n. 1, 49–62.

[12] A. Maschietti: *Symplectic Translation Planes*, pp. 46–103 in Scuola Estiva di Geometrie Combinatorie, Giuseppe Tallini, Potenza, Italy, 1-6 September 2003.

[13] T. Penttila, B. Williams: *Ovoids of parabolic spaces*, Geom. Ded., **82** (2000), 1–19.

[14] J.A. Thas, S.E. Payne: *Spreads and Ovoids in Finite Generalized Quadrangles*, Geom. Ded., **52** (1994), 227–253.

[15] J.A. Thas: *Generalized quadrangles and flocks of cones*, European J. Combin., **8** (1987), 441–452.

[16] J.A. Thas: *Generalized quadrangles of order $(s, s^2)$, II*, J. Combin. theory Ser. A, **79** (1997), 223–254.