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ESTIMATION OF MODIFIED MEASURE OF SKEWNESS

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***Abstract**: It is well known that the classical measures of skewness are not reliable and their sample distributions are not known for small samples. Therefore, we consider the modified measure of skewness that is defined in terms of cumulative probability function. The main adventage of this measure is that its sampling distribution is derived from sample data as the sum of dependent Bernoulli random variables. Moreover, it variance and confidence interval are obtained based on multiplicative binomial distribution. Comparison with classical measures using simulation and an application to actual data set are given.*

***Keywords**:* dependence, multiplicative-binomial distribution, maximum likelihood, under-dispersion, symmetry.

1. Introduction

Many statistical models often assume symmetric distributions. For example the behavior of stock market returns does not agree with the frequently assumed normal distribution. This disagreement is often highlighted by showing the large departures of the normal distribution; see, for example, [3], [11]. The role of skewness has become increasingly important because the need for symmetry test. It is known that the classical measures $\gamma_1 = \mu_3/\sigma^3$ and $\mathrm{sk} = (\mu - \mathrm{median})/\sigma$ are not reliable measures of skewness, μ population mean and σ sample standard deviation; see, for example, [16], [15], [9] and [2]. Many measures of skewness developed for continuous distributions follow a quantile pattern and letter values; see, for example, [10], [19], [5], and [12].
However [18] introduced a measure of skewness in terms of logarithm of the cumulative probability function and its modified measure of skewness in terms of cumulative probability

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function without giving any estimation or statistical inferences for these measures. The main purpose of this work is to estimate the modified measure of skewness from data and derives its sampling distribution. The simulation study is shown that the modified measure of skewness outperforms some good classical measures of skewness for a wide range of distributions.
In Section 2 we review the population modified measure of skewness K and study its properties. An estimator, the sampling distribution and the variance estimation are derived in Section 3. The confidence interval is obtained in Section 4. An application to data set is investigated in Section 5. Comparisons with other methods are given in Section 6.

2. Population modified measure of skewness

Let a vector X of random variables, $X_1, \dots, X_n$, from a continuous distribution with cumulative distribution function (cdf) $F(x) = F_X = F$, density function $f(x)$, quantile function $x(F) = F^{-1}(x)$ and $\mu = \int_{-\infty}^{\infty} x f(x)dx$ is the mean of the distribution and $f(x)$ is normalized $\int_{-\infty}^{\infty} f(x)dx = 1$. [18] defined the population modified measure of skewness about μ:

$$\mathrm{K} = 2F(\mu) - 1 = 2\int_{-\infty}^{\mu} f(x)dx - 1$$

as the twice the area to the left side from the mean minus one. Under the assumption of no ties between any X and μ, the measure K could be rewritten in the following two alternative forms:

$$\mathrm{K} = \frac{F(\mu) - (1 - F(\mu))}{1} = \frac{\mathrm{K}_1 - \mathrm{K}_2}{\mathrm{K}_1 + \mathrm{K}_2}$$

$\mathrm{K}_1 = P(x < \mu)$ and $\mathrm{K}_2 = P(x > \mu)$. This can be explained as the ratio of the difference between the probability of the X less than the mean and greater than the mean to their total. In terms of the conditional expectations as:

$$\mathrm{K} = \frac{E[(X-\mu)|X > \mu] - E[(\mu - X)|X < \mu]}{E[(X-\mu)|X > \mu] + E[(\mu - X)|X < \mu]}$$

This can be explained as the ratio of the difference between the conditional expectations of the deviation about mean given $X > \mu$ and $X < \mu$ to their total. These two expressions can be compared in their forms with the [4] measure:

$$B = \frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{(Q_3 - Q_2) + (Q_2 - Q_1)}$$

For symmetric distributions about μ we have:

$$\mathrm{K} = \frac{\mathrm{K}_1 - \mathrm{K}_2}{\mathrm{K}_1 + \mathrm{K}_2} = 0$$

The measure K will reflect some degree of skewness or symmetry of the distribution about μ. Since the area under the curve ranges from 0 to 1, the nature symmetric point for this measure is 0. If the distribution is skewed to the left, the value of $\mathrm{K} < 0$. If the distribution skewed to the right, the value of $\mathrm{K} > 0$. The upper limit of K is 1 where $\int_{-\infty}^{\infty} f(x)dx = 1$ and the lower limit is -1 where $\int_{a}^{a} f(x)dx = 0$ with -1≤K≤1.

2.1 Properties of the measure K

Groeneveld *et al.* [8] have suggested some properties that any reasonable measure of skewness should satisfy. The measure K has the following properties:

1. The measure K is symmetric about 0.
2. For any$c > 0$ and d, $\mathrm{K}(cX + d) = \mathrm{K}(X)$.
3. $\mathrm{K}(-X) = 1 - \mathrm{K}(X)$.
4. The distribution $f(x)$ is more skewed to the right than the distribution $g(y)$ with interval support if $\mathrm{K}(X) > \mathrm{K}(Y) > 0.5$.

Example

Table 1 gives some values of K from some known distributions. The Weibull distribution used with density:

$$f(x) = \frac{k}{\lambda}\left(\frac{\mathrm{x}}{\lambda}\right)^{\mathrm{k}-1} \mathrm{e}^{-\left(\frac{\mathrm{x}}{\lambda}\right)^{\mathrm{k}}}$$

$x > 0$, $\lambda > 0$ the scale parameter and k is the shape parameter. The value of K is:

$$\mathrm{K} = 1 - e^{-(\Gamma(1+1/k))^{k}}.$$

Table 1. Values of *K* for some known distributions

Distribution	K	Weibull	k	K
Uniform	0		0.50	0.514
Normal	0		1	0.264
Laplace	0		1.5	0.152
Exponential	0.264		2.5	0.046
Gumbel	0.140		3.5	-0.002

3. Estimation and the sampling distribution

3.1 Estimation

The estimate of *K* is:

$$s = \frac{2\sum_{i=1}^{n} I(x_i < \bar{x})}{n} - 1 = \frac{2\sum_{i=1}^{n} z_i}{n} - 1 = \frac{2y_n}{n} - 1$$

$\bar{x} = \sum_{i=1}^{n} x_i/n$ and $z_i = I(x_i < \bar{x}) = 1$ or 0. Also, we assume that there is no tie between any x_i and $\bar{x}$ i.e. ($\bar{x} \neq x_i$, $\forall i$). It is known that if the indicator variates, $Z_i, i = 1,..n,$ are independent, then Z_i has a Bernoulli distribution and $Y_n = \sum_{i=1}^{n} Z_i$ has a standard binomial distribution

$$P(Y = y) = \binom{n}{y} \pi^y (1-\pi)^{n-y}, \quad y = 0,1,..,n$$

Since $\bar{x}$ is estimated from the sample and each x_i is influenced by the same sample mean, $Z_i, i = 1,..n$, are not independent. Therefore, we need to study the sampling distribution of $Y_n = \sum_{i=1}^{n} Z_i$ under dependence between $Z_i, i = 1,2,..,n$. Different models for this dependence provide a wider range of models than are provided by the binomial distribution. Among these, [14] had derived the multiplicative binomial distribution of the sum of such variables from a log-linear representation for the joint distribution of n binary-dependent variables introduced by [7] an alternative to Altham's multiplicative-binomial distribution [1].

3.2 Sampling distribution

Lovison [14] introduced the multiplicative binomial distribution as the sum of dependent Bernoulli random variables. Let Z be a binary response, which measures whether some event of interest is present 'success' or absent 'failure' for sample units, n, and $Y_n = \sum_{i=1}^{n} Z_i$ denote the sample frequency of successes. [14] studied the Cox's log-linear model:

$$P(Z = z) = Ce^{\left[\sum_k^n \lambda_k w_k + \sum_{k<h}^n \lambda_{kh} w_k w_h + \cdots + \sum_{k_1 < \cdots < k_n}^n \lambda_{k_1 \ldots k_n} w_{k_1} \cdots w_{k_n}\right]}$$

to accommodate for the possible dependence between Z_i he introduced the log-linear representation:

$$P(Z = z) = Ce^{\left[v \sum_k^n w_k + \lambda \sum_{k<h}^n w_k w_h\right]}$$

$W_k = 2Z_k - 1$, and C is a normalizing constant. This representation is introduced under the assumption that the units are exchangeable i.e., they have the same 0 and 1 order interaction parameters ($\lambda_k = v$, $\lambda_{kh} = \lambda$), and all interactions of order higher than 1 are zeros. Under the above log-linear representation Lovison had derived the distribution of Y_n as:

$$P(Y_n = y) = \frac{\binom{n}{y} \psi^y (1-\psi)^{n-y} \omega^{y(n-y)}}{\sum_{t=0}^{n} \binom{n}{t} \psi^t (1-\psi)^{n-t} \omega^{t(n-t)}}, \quad y = 0,1,\ldots,n$$

ψ and ω are the parameters. This distribution provides a wider range of distributions than are provided by the binomial distribution. The binomial distribution is obtained for $\omega = 1$ with $E(Y_n) = n\psi = n\pi$ and $V(Y_n) = n\psi(1-\psi) = n\pi(1-\pi)$. For $\psi = 0.5$ and $n = 10$, the distribution of Y_n for different values of ω is given in Figures 1 and 2. When $\omega > 1$, the distribution is unimodal.

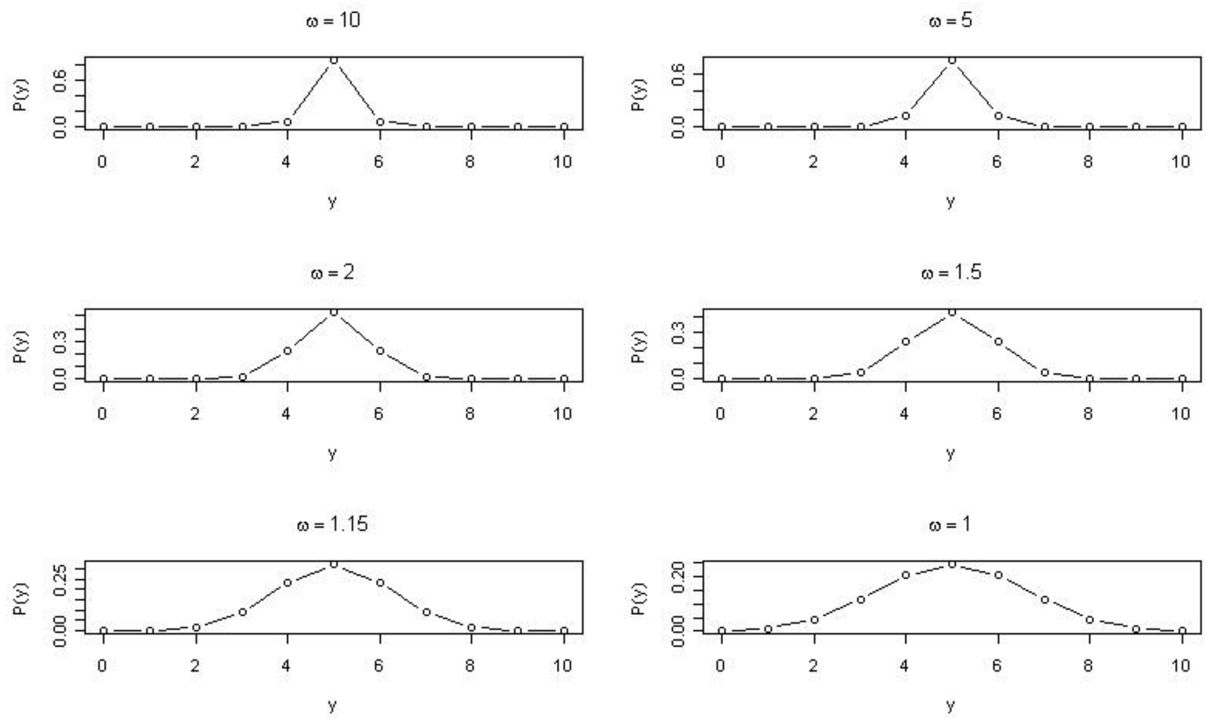

Figure 1. The distribution of Y_n for different values of ω, ψ=0.5 and n=10.

While for the values of $\omega < 1$, the distribution could take U, bimodal and unimodal shapes.

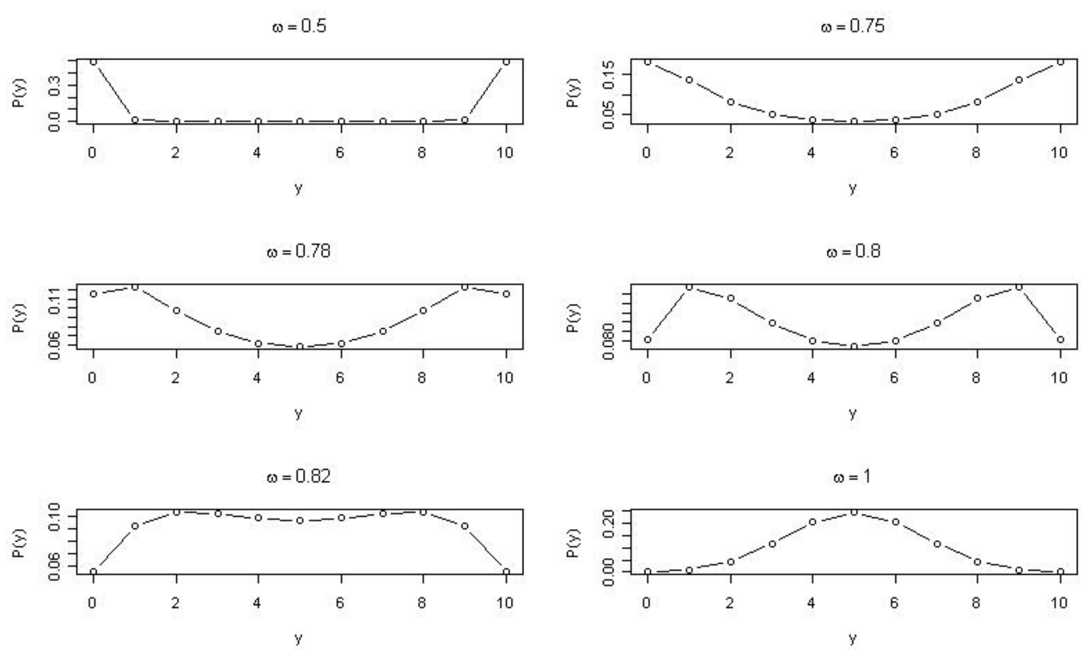

Figure 2. The distribution of Y_n for different values of ω, ψ=0.5 and n=10.

The expected value and the variance of this distribution are given by:

$$E(Y_n) = n\psi \frac{\kappa_{n-1}(\psi,\omega)}{\kappa_n(\psi,\omega)} = nP(Z_k = 1) = n\tau_1,$$
$$V(Y_n) = n\tau_1(1 - n\tau_1) + n(n-1)\tau_2$$

Where:

$$\kappa_{n-a}(\psi,\omega) = \sum_{x=0}^{n-a} \binom{n-a}{x} \psi^x (1-\psi)^{(n-a-x)} \omega^{(n-a-x)(x+a)}$$

and

$$\tau_2 = \psi^2 \frac{\kappa_{n-2}(\psi,\omega)}{\kappa_n(\psi,\omega)}$$

The expected value and variance of Y_n is nonlinearly on both ψ and ω. The nonlinear in the variance of Y_n is depicted in Figure 3 for some chosen values of ψ. For example, when $\psi = 0.5$, we have overdispersion for the values of $\omega < 1$ and underdispersion for the values of $\omega > 1$.

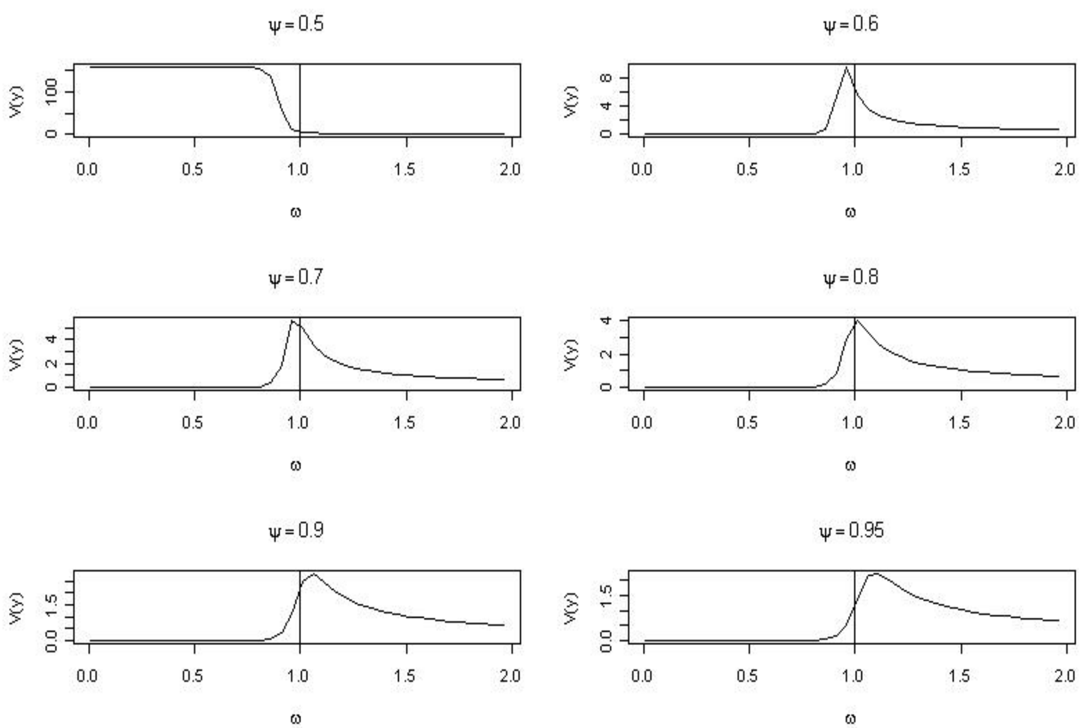

Figure 3. Variance of Y_n for various values of ω at each value of ψ and n=25.

The parameter ω is explained as a measure of intra-units association inversely related to the condition cross-product ratio (CPR):

$$\omega = 1/\sqrt{CPR(k,h\ |rest)}$$

where the conditional cross-product ratio of any two units given all others is given by:

$$CPR(k,h|\text{rest}) = \frac{P(Z_k = 0, Z_h = 0|\text{rest})P(Z_k = 1, Z_h = 1|\text{rest})}{P(Z_k = 0, Z_h = 1|\text{rest})P(Z_k = 1, Z_h = 0|\text{rest})} = e^{4\lambda}$$

This gives the λ the meaning of a measure of conditional pair-wise association between units and shows that $\omega = e^{-2\lambda}, \quad \omega > 0$. Also, ψ can be written as

$$\psi = P(Z_k = 1)\frac{\kappa_n(\psi,\omega)}{\kappa_{n-1}(\psi,\omega)} = \frac{e^{2v}}{1 + e^{2v}},$$

$0 < \psi < 1$; see, [14]. Then ψ can be thought as the probability of a particular outcome in other words the weighted probability of success that would be governing the binary response of the n units. This weighted probability of success becomes the probability of success when the binary responses are independent, i.e. $\psi = \pi = P(Z_k = 1)$ and $\kappa_{n-a}(\psi,\omega) = 1\ \forall a, \omega = 1$.
Under multiplicative binomial distribution we obtain

$$E(s) = \frac{2E(Y_n)}{n} - 1 = 2\frac{n\tau_1}{n} - 1 = 2\tau_1 - 1 = \mathrm{K}$$

and

$$V(s) = \frac{4V(Y_n)}{n^2} = 4\frac{n\tau_1(1 - n\tau_1) + n(n-1)\tau_2}{n^2} = 4\frac{\tau_1(1 - n\tau_1) + (n-1)\tau_2}{n}$$

3.3 Estimation of the parameters

We could estimate the parameters ψ and ω as follows. In view of exchangeability and absence of second and higher order interaction results to be the same for all pairs of units and for any combination of categories taken by the other units and by noticing that in a vector of binary responses z there are $n(n-1)/2$ pairs of responses and if the order is irrelevant three type of pairs are distinguishable: there are $(n-y)(n-y-1)/2$ pairs of $(z_k = 0, z_h = 0)$, $y(y-1)/2$ pairs of $(z_k = 1, z_h = 1)$, and $(n-y)y$ pairs of $(z_k = 0, z_h = 1)$, or $(z_k = 1, z_h = 0)$, for $y = 0, \dots, n$ and $y = \sum_{k=1}^{n} z_k$. Therefore, the estimate of ω is

$$\hat{\omega} = 1/\sqrt{\widehat{CPR}}$$

and

$$\widehat{CPR} = \frac{0.25y(y-1)(n-y)(n-y-1)}{\neq (z_k = 0, z_h = 1)\ \#(z_k = 1, z_h = 0)}$$

provided $\#(0,1)$ and $\#(1,0) > 0$. To find estimate, $\hat{\psi}$, of ψ we could use the maximum likelihood method for $P(Y = y)$ as:

$$L(\psi|n, y, \omega) = \frac{\binom{n}{y}\psi^y(1-\psi)^{n-y}\omega^{y(n-y)}}{\sum_{t=0}^{n}\binom{n}{t}\psi^t(1-\psi)^{n-t}\omega^{t(n-t)}}$$

$0 < \psi < 1$. We looking for the value of ψ which maximize L in the range $(0,1)$. This value is the solution of the function $\frac{d}{d\psi}\log L(\psi|n, y, \omega) = 0$:

$$\frac{y}{\psi} - \frac{n-y}{1-\psi} - \frac{\sum_{t=0}^{n}\binom{n}{t}\psi^t(1-\psi)^{n-t}\omega^{t(n-t)}\left[\frac{t}{\psi} - \frac{n-t}{1-\psi}\right]}{\sum_{t=0}^{n}\binom{n}{t}\psi^t(1-\psi)^{n-t}\omega^{t(n-t)}} = 0$$

and $\frac{d^2}{d\psi^2}\log L(\psi|n, y, \omega) < 0$. Then, we have:

$$\hat{E}(s_1) = 2\hat{\tau}_1 - 1$$

and

$$\hat{V}(s) = 4\,\frac{\hat{\tau}_1(1 - n\hat{\tau}_1) + (n-1)\hat{\tau}_2}{n}$$

where

$$\hat{\tau}_1 = \hat{\psi}\,\frac{\kappa_{n-1}(\hat{\psi}, \hat{\omega})}{\kappa_n(\hat{\psi}, \hat{\omega})}$$

and

$$\hat{\tau}_2 = \hat{\psi}^2\,\frac{\kappa_{n-2}(\hat{\psi}, \hat{\omega})}{\kappa_n(\hat{\psi}, \hat{\omega})}$$

Note that, if $\#(0,1)$ or $\#(1,0)$ are zero, $\widehat{CPR}$ will be undefined. In this case, we may adjust the estimate by adding 0.5 to each cell count; see, [13].

Example:
In this example we find an estimation of ψ, ω, $E(s)$ and $V(s)$ from simulated data from beta distribution with shape parameters 1, 1 of size $n = 10$.
Simulated data from beta distribution with shape parameters 1, 1 and *n*=10.

- x_i: 0.156, 0.569, 0.976, 0.136, 0.162, 0.997, 0.793, 0.174, 0.124, 0.559
- $\bar{x}$ = 0.465, therefore the values of are:

- 1, 0, 0, 1, 1, 0, 0, 1, 1, 0. Then *n*=10, *y*=5 , and
- $\neq (z_k = 0, z_h = 1) = 12, \quad \neq (z_k = 1, z_h = 0) = 13$,. Hence, $\widehat{CPR} = 0.641$, $\widehat{\omega} = 1.249$, and the maximum likelihood estimate from figure 4 is $\widehat{\psi} = 0.5$.

Therefore, $s_1 = 0$, $\hat{E}(s) = 0$, and $\hat{V}(s) = 0.0485$ if we use the binomial distribution we have $\hat{E}(s_b) = 0$, and $\hat{V}(s_b) = \frac{4(.5)(1-.5)}{10} = 0.10$ which has more variance than $\hat{V}(s)$.

To find estimate of ψ we use

$$L(\psi|n, y, \omega = \widehat{\omega}) = \frac{\binom{10}{5}\psi^5(1-\psi)^5 1.249^{25}}{\sum_{t=0}^{n}\binom{10}{t}\psi^t(1-\psi)^{10-t}\omega^{t(10-t)}}$$

From Figure 4 we find that $\widehat{\psi} = 0.5$.

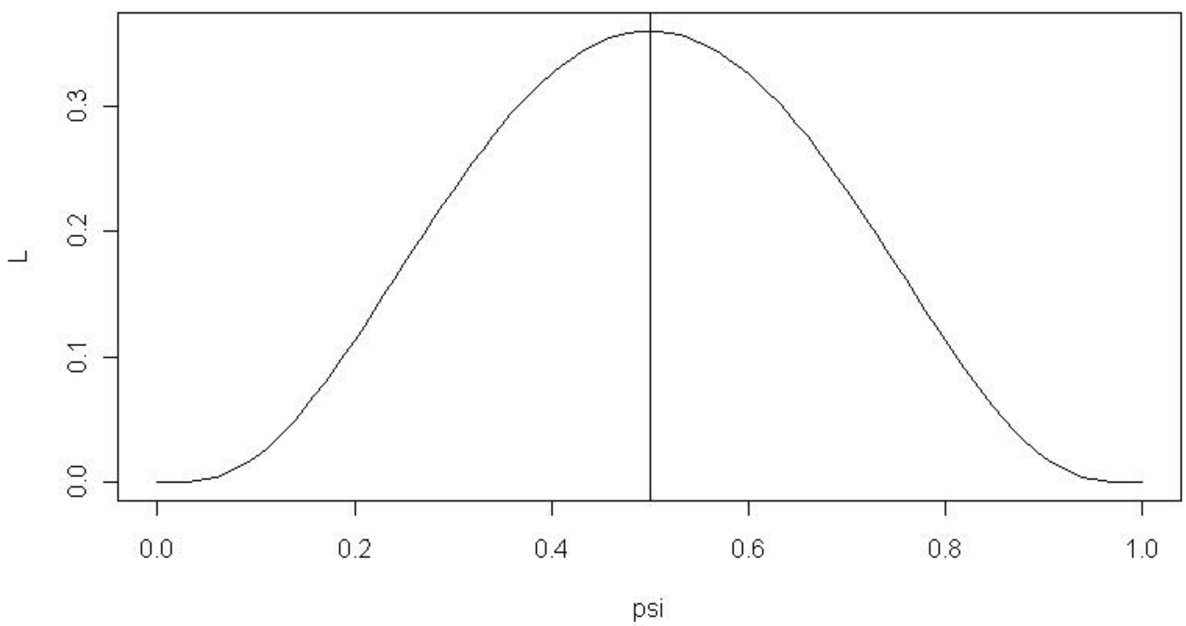

Figure 4. The likelihood function *L*(*ψ*) with *n*=10, *y*=5 and *ω* =1.249.

4. The confidence interval

The multiplicative binomial distribution is used to construct a two-sided confidence interval at the 100(1 - α)% confidence level for K given n, y and ω from the sample rather than the normal approximation. we first find the confidence interval for ψ and then obtain the confidence interval of $\mathrm{K} = 2P(Z_k = 1) - 1 = 2\psi\frac{\kappa_{n-1}(\psi,\omega)}{\kappa_n(\psi,\omega)} - 1$ as follows. Following [6] method, the desired upper limit ψ_U so that if $Y = y$ was observed we would just barely reject H_0 when testing $H_0: \psi = \psi_U$ against $H_1: \psi < \psi_U$ using level of significant $\alpha/2$. However "just barely reject H_0" translates to

$p-\text{value}=\alpha/2$. But the $p-\text{value}$ for the left tail is given by $P(Y\le y)$. Therefore, by solving the equation:

$$\sum_{j=0}^{y}\frac{\binom{n}{j}\psi_u^j(1-\psi_u)^{n-j}\omega^{j(n-j)}}{\sum_{t=0}^{n}\binom{n}{t}\psi_u^t(1-\psi_u)^{n-t}\omega^{t(n-t)}}=\frac{\alpha}{2}$$

for ψ_U, we obtain an upper limit for ψ_U then $\mathrm{K}_U=2\psi_U\frac{\kappa_{n-1}(\psi_u,\omega)}{\kappa_n(\psi_u,\omega)}-1$.

Next, the desired lower limit ψ_L so that if $Y=y$ was observed we would just barely reject H_0 when testing $H_0:\psi=\psi_L$ against $H_1:\psi>\psi_L$ using level of significant $\alpha/2$. However "just barely reject H_0" translates to $p-\text{value}=\alpha/2$. But the $p-\text{value}$ for the right tail is given by $P(Y\ge y)=1-P(Y<y)$. Therefore, by solving the equation:

$$\sum_{j=0}^{y-1}\frac{\binom{n}{j}\psi_L^j(1-\psi_L)^{n-j}\omega^{j(n-j)}}{\sum_{t=0}^{n}\binom{n}{t}\psi_L^t(1-\psi_L)^{n-t}\omega^{t(n-t)}}=1-\frac{\alpha}{2}$$

for ψ_L, we obtain a lower limit for ψ_L then $\mathrm{K}_L=2\psi_L\frac{\kappa_{n-1}(\psi_L,\omega)}{\kappa_n(\psi_L,\omega)}-1$. These two equations can be easily solved using function "uniroot" in R-software given n, y and ω. Then a $(1-\alpha)100\%$ confidence interval for K is given by:

$(\mathrm{K}_L,\ \mathrm{K}_U)$

Note that, ψ and K have one-to-one correspondence for given ω. Figure 5 shows the relation between ψ and $P(Z_k)=1$ for specified values of ω. The relation is linear when ω is 1. Note also, the interval $(\mathrm{K}_L,\ \mathrm{K}_U)$ is random.

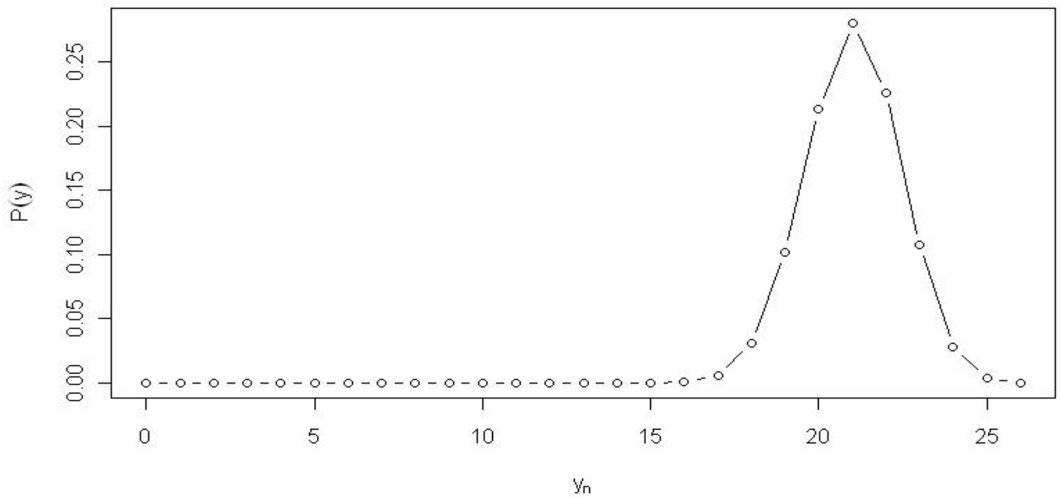

Figure 5. The sampling distribution of Y_n with n=26, $\hat{\psi}=0.97$ and $\hat{\omega}=1.139$.

5. Application

We consider a random sample of 26 measurements of the heat of sublimation of platinum from [17]. The 26 measurements are all attempts to measure the true heat of sublimation. Are these data symmetric? The data set is given in Table 2, also, the values of z_i, y_n, s, $\widehat{CPR}$ and $\hat{\omega}$.

Table 2. Heats of sublimation of platinum data and the estimation of K, CPR and ω.

				Data				
136.3	147.8	134.8	134.3	136.6	148.8	135.8	135.2	135.8
134.8	135	135.4	135.2	133.7	134.7	134.9	134.4	135
146.5	134.9	134.1	141.2	134.8	143.3	135.4	134.5	
$n=26$, $\bar{x}=137.046$								
				z_i				
1	0	1	1	1	0	1	1	1
1	1	1	1	1	1	1	1	1
0	1	1	0	1	0	1	1	
from the values of z_i: $y_n=21$, $s=0.6154$, $\widehat{CPR}=0.877$ and $\hat{\omega}=1.1393$								

The maximum likelihood function is:

$$L(\psi|n,y,\omega)=\frac{\binom{26}{21}\psi^{21}(1-\psi)^{5}1.1393^{105}}{\sum_{t=0}^{n}\binom{26}{t}\psi^{t}(1-\psi)^{26-t}\omega^{t(26-t)}}$$

The maximization of this function gives $\hat{\psi} = 0.9702$. Then, the estimated sampling distribution of Y_n is:

$$P(Y_n = y) = \frac{\binom{26}{y}\hat{\psi}^y(1-\hat{\psi})^{26-y}\hat{\omega}^{y(26-y)}}{\sum_{t=0}^{26}\binom{26}{t}\hat{\psi}^t(1-\hat{\psi})^{26-t}\hat{\omega}^{t(26-t)}}, \quad y = 0,1,\ldots,26$$

The graph of this distribution is depicted in Figure 6 and it seems almost symmetric about the value of success, $y = 21$.

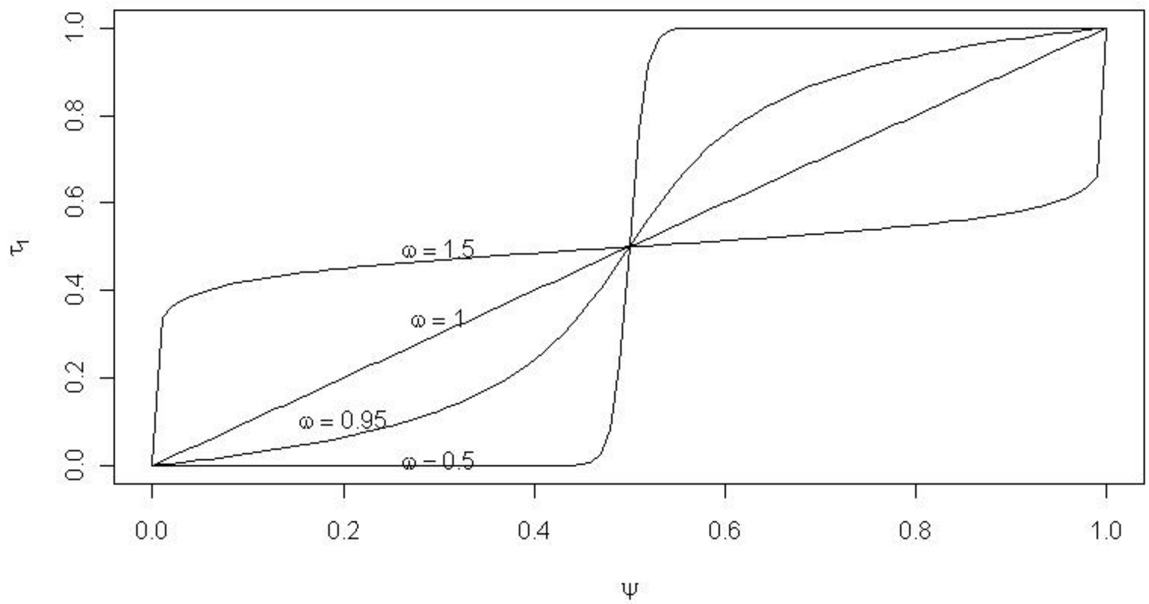

Figure 6. The relation between ψ , ω and $\tau_1=P(Z_k=1)$ using n=25.

The estimated mean and variance are:

$\hat{E}(s) = 0.64$ and $\hat{V}(s) = 0.0118$.

To obtain the 0.95 confidence interval we solve:

$$\sum_{j=0}^{21}\frac{\binom{26}{j}\psi_u^j(1-\psi_u)^{26-j}1.139^{j(26-j)}}{\sum_{t=0}^{n}\binom{26}{t}\psi_u^t(1-\psi_u)^{26-t}1.139^{t(26-t)}} = 0.025$$

to give the upper limit $\psi_U = 0.9943$, and solve the equation

$$\sum_{j=0}^{20} \frac{\binom{26}{j} \psi_l^j (1-\psi_l)^{26-j} 1.139^{j(26-j)}}{\sum_{t=0}^{n} \binom{26}{t} \psi_l^t (1-\psi_l)^{26-t} 1.139^{t(26-t)}} = 0.975$$

to give the lower limit $\psi_l = 0.8723$. Then, the 95% confidence interval for K is:

$$(0.360, 0.934)$$

Where 0 is not included in both intervals we could conclude that the data is not symmetric about mean.

6. Comparisons with other methods

We compare the measure of skewness s with two known measures of skewness. The Bowley's coefficient of skewness:

$$sk_b = \frac{(q_3 - q_2) - (q_2 - q_1)}{(q_3 - q_1)}$$

This measure bounded by -1 and 1; see, [12] and the measure which is given by [8]:

$$sk_{gm} = \frac{\bar{x} - Med}{\frac{1}{n}\sum_{i=1}^{n} |x_i - Med|}$$

bounded by -1 and 1, $\bar{x}$ is the sample mean, q_3, q_2 and q_1 are the third, second and first sample quartiles, sd sample standard deviation, and Med is the sample median.
The simulation results in Table 3 are shown that

1. The measure s has overall less bias and variance.
2. The measure sk_b has the largest variance among the three measures.
3. The measure sk_{gm} is better than sk_b in terms of variance.
4. The bias for the three measures decreases with increasing the sample size.

Table 3: Simulated mean (Est.), variance (Var.) for sk_p, $\mathrm{sk}_{\mathrm{gm}}$ and s using Weibull distribution with different values of k, the number of replication is 10000.

	sk_b			sk_{gm}			s		
n	Exact	Est.	Var.	Exact	Est.	Var.	Exact	Est.	Var
				$k = 0.75$					
10	0.375	0.287	0.123	0.598	0.507	0.064	0.360	0.311	0.037
20		0.328	0.072		0.556	0.032		0.337	0.020
30		0.342	0.051		0.567	0.021		0.342	0.011
50		0.348	0.031		0.581	0.013		0.354	0.007
100		0.363	0.016		0.588	0.006		0.362	0.004
				$k = 1$					
10	0.261	0.201	0.121	0.443	0.372	0.065	0.264	0.236	0.037
20		0.229	0.073		0.406	0.036		0.244	0.020
30		0.236	0.053		0.423	0.026		0.251	0.012
50		0.249	0.033		0.431	0.017		0.253	0.008
100		0.251	0.017		0.438	0.008		0.265	0.004
				$k = 2.5$					
10	0.037	0.031	0.121	0.077	0.062	0.068	0.046	0.041	0.035
20		0.034	0.074		0.072	0.039		0.044	0.017
30		0.033	0.056		0.070	0.027		0.044	0.011
50		0.035	0.034		0.077	0.017		0.047	0.007
100		0.037	0.017		0.073	0.008		0.045	0.003

7. Conclusion

We have studied modified measure of skewness about μ for the continuous distributions in terms of the incomplete density function. We have provided simple nonparametric estimator for computing the measure. The main advantage of this measure is the availability of its sampling distribution under a sum of dependent Bernoulli random variables for small and large sample sizes. Also, we used the maximum likelihood method to obtain an estimate to multiplicative binomial distribution parameters. Moreover, we have derived its confidence interval using multiplicative binomial distribution.

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