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Length-Biased Loai Distribution: Statistical Properties and Application

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A new distribution is proposed in this paper using the length-biased distribution as a special case of the weighted distributions. It is called the length-Bias Loai distribution. The properties of this distribution are investigated, including moments, moment generating function, and the reliability functions and many others. Various numerical studies are carried out, they show that the distribution right skewed and leptokurtic. Different methods of estimation are used to estimate the distribution parameters. A simulation study is carried out to see the efficiency of the estimation methods, it shows that the distribution's parameters are approximately unbiased and consistent. An application to a real data set is conducted to show the goodness of fit for the suggested distribution. It illustrates that the proposed distribution fits this data better than the other competence distributions.

keywords: Loai distribution, length biased, moments, reliability analysis, Rényi entropy, methods of estimation.

1 Introduction

It is not correct to use the original distribution for observations recorded from a random process, because the probability of these observations are not equal. The idea of weighted distributions introduced by Fisher (1934) and developed by Rao (1965) can be applied in this case. The weighted distribution is defined for a random variable X with probability density function (pdf) $g(x)$ as:

$$g_w(x) = \frac{w(x)g(x)}{E(w(X))}, \tag{1.1}$$

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where $w(x)$ is a non-negative weighted function and $E(w(X))$ is exist. Using $w(x) = x$, we have the length biased distribution. Thus the *pdf* of the length biased distribution is defined by Patil and Ord (1976) as:

$$g_w(x) = \frac{xg(x)}{E(X)} \tag{1.2}$$

Length biased distributions have been used by many authors to generate new distributions. For example, Al-Omari and Alsmairan (2019) generated the length biased Suja distribution. Gharaibeh (2022) used the idea of length biased distributions to propose weighted Gharaibeh distribution. Al-Omari et al. (2019a) proposed size-biased Ishita distribution and applied it to real data. Al-Omari et al. (2023) studied the asymmetric right-skewed size-biased Bilal distribution with mathematical properties. Usman et al. (2019) proposed the Marshall-Olkin Length-Biased exponential distribution. Alidamat and Al-Omari (2021) suggested the extended length biased two parameters Mirra distribution, they applied it to engineering data. Sharma et al. (2018) introduced length and area-biased Maxwell distribution. Al-Omari et al. (2019b) suggested power length-biased Suja distribution as a new extension of the length-biased Suja distribution. Shen et al. (2009) used semi-parametric transformations to model the length-biased data. Al-Omari and Alanzi (2021) suggested and studied the properties of the one parameter inverse length biased Maxwell distribution. Das and Roy (2011) suggested the length-biased form of weighted Weibull distribution.

Loai distribution is a new life time two-parameter distribution proposed by Alzoubi et al. (2022) as a mixture of $gamma(3, \theta)$ and Lindley with parameter θ with mixture proportions $\frac{1}{\alpha+1}$ and $\frac{\alpha}{\alpha+1}$. This distribution will be modified using the idea of length biased distribution. The *pdf* of Loai distribution is defined as:

$$g(x|\alpha, \theta) = \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta x^2 + \frac{(1+x)}{\theta+1}\right]e^{-\theta x},\ x > 0,\ \theta > 0,\ \alpha > 0, \tag{1.3}$$

with mean

$$E(X) = \frac{3\alpha(\theta+1)+\theta+2}{\theta(\theta+1)(\alpha+1)} \tag{1.4}$$

2 Length Biased Loai Distribution

This section will define the pdf and cdf of the length biased Loai distribution (LBLD).

Definition 2.1 *The random variable X is said the LBLD if its pdf is given by*

$$g_l(x) = \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^3 + \frac{x(x+1)}{\theta+1}\right]e^{-\theta x},\ x > 0, \theta,\ \alpha > 0, \tag{2.1}$$

Corollary 2.1 *The function defined in (2.1) is a pdf.*

Proof 2.1

$$\begin{aligned}\int_0^{\infty} g_l(x)dx &= \int_0^{\infty} \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^3+\frac{x(x+1)}{\theta+1}\right]e^{-\theta x}dx \\ &= \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{3\alpha}{\theta^3}+\frac{t+2}{t^3(t+1)}\right] \\ &= \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{3\alpha(t+1)+\theta+2}{\theta^3(\theta+1)}\right] \\ &= 1 \qquad \square\end{aligned}$$

The corresponding *cdf* of LBLD can be derived as:

$$\begin{aligned}G_l(X) &= \int_0^{x} g_l(x)dx = \int_0^{x} \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta u^3+\frac{u(u+1)}{\theta+1}\right]e^{-\theta u}du \\ &= \frac{\begin{pmatrix}(-\alpha(\theta+1)(\theta x(\theta x(\theta x+3)+6)+6) \\ -2(\theta x(\theta x+\theta+2)+\theta+2))e^{-\theta x}+6a(\theta+1)+2(\theta+2)\end{pmatrix}}{2(3\alpha(\theta+1)+\theta+2)} \\ &= 1-\begin{pmatrix}(\alpha(\theta+1)(\theta x(\theta x(\theta x+3)+6)+6) \\ +2(\theta x(\theta x+\theta+2)+\theta+2))e^{-\theta x}\end{pmatrix} \qquad (2.2)\end{aligned}$$

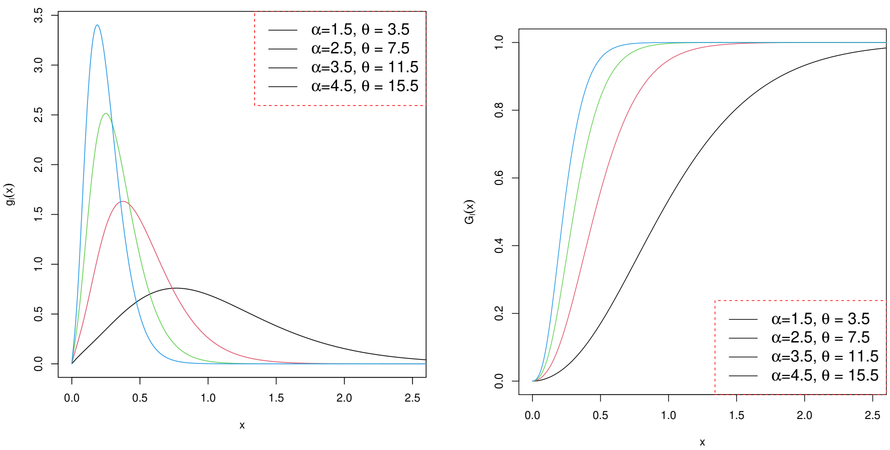


Figure 1: The *pdf* and *cdf* of LBLD for different values of α and θ.

3 Moments and Related Measures

This section introduces the moments and related measures and the moment generating function of LBLD.

3.1 Moments

The r^{th} moment of a random variable X is defined by

$$E(X^r) = \int_x x^r g(x)dx \tag{3.1}$$

Theorem 3.1 *Let X be an LBLD random variable with pdf defined in (2.1), then the r^{th} moment of X is*

$$E(X^r) = \frac{\alpha(\theta+1)\Gamma(r+4)+\theta\Gamma(r+2)+\theta^2\Gamma(r+1)}{2\theta^r(3\alpha(\theta+1)+\theta+2)} \tag{3.2}$$

Proof 3.1

$$\begin{aligned}
E(X^r) &= \int_0^\infty \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}x^r\left[\frac{1}{2}\alpha\theta x^3+\frac{x(x+1)}{\theta+1}\right]e^{-\theta x}dx \\
&= \int_0^\infty \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^{r+3}+\frac{x^{r+1}(x+1)}{\theta+1}\right]e^{-\theta x}dx \\
&= \int_0^\infty \frac{\theta^3}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta(\theta+1)x^{r+3}+x^{r+2}+x^{r+1}\right]e^{-\theta x}dx \\
&= \frac{\theta^3}{3\alpha(\theta+1)+\theta+2}\left[\frac{\alpha(\theta+1)\Gamma(r+4)}{2\theta^{r+3}}+\frac{\Gamma(r+3)}{\theta^{r+3}}+\frac{\Gamma(r+2)}{\theta^{r+2}}\right] \qquad \square
\end{aligned}$$

For r =1, we get the first moment (mean) of the LBLD random variable. The second, third and fourth moments can be calculated by substituting r = 2, 3 and 4 in (3.2). Thus, we have

$$E(X)=\mu = \frac{12\alpha(\theta+1)+2\theta+6}{\theta(3\alpha(\theta+1)+\theta+2)} \tag{3.3}$$

$$E(X^2) = \frac{60\alpha(\theta+1)+6\theta+24}{\theta^2(3\alpha(\theta+1)+\theta+2)} \tag{3.4}$$

$$E(X^3) = \frac{360\alpha(\theta+1)+24\theta+120}{\theta^3(3\alpha(\theta+1)+\theta+2)} \tag{3.5}$$

$$E(X^4) = \frac{2520\alpha(\theta+1)+120\theta+720}{\theta^4(3\alpha(\theta+1)+\theta+2)} \tag{3.6}$$

3.2 Related measures

The variance and the standard deviation of the random variable X that follows an LBLD distribution is defined using (3.3) and (3.4) by

$$\begin{aligned}
\sigma^2 = Var(X) &= \left(E(X^2)-\mu^2\right) \\
&= \frac{60\alpha(\theta+1)+6\theta+24}{\theta^2(3\alpha(\theta+1)+\theta+2)} - \left(\frac{12\alpha(\theta+1)+2\theta+6}{\theta(3\alpha(\theta+1)+\theta+2)}\right)^2 \\
&= \frac{\begin{pmatrix} 144\alpha^2+144\alpha^2\theta^2+288\alpha^2\theta+660\alpha\theta+480\alpha \\ +144\alpha\theta^2-\theta^4+16\theta^2+24\theta-36\alpha\theta^3 \end{pmatrix}}{\theta^2\left(3\alpha\left(\theta+1\right)+\theta+2\right)^2}
\end{aligned}$$

$$\sigma = \frac{\sqrt{\begin{pmatrix} 144\alpha^2 + 144\alpha^2\theta^2 + 288\alpha^2\theta + 660\alpha\theta + 480\alpha \\ +144\alpha\theta^2 - \theta^4 + 16\theta^2 + 24\theta - 36\alpha\theta^3 \end{pmatrix}}}{\theta\left(3\alpha\left(\theta+1\right)+\theta+2\right)} \tag{3.7}$$

The coefficient of variation (cv) is defined using (3.3) and (3.7) as

$$cv = \frac{\sigma}{\mu} = \frac{\sqrt{\begin{pmatrix} 144\alpha^2 + 144\alpha^2\theta^2 + 288\alpha^2\theta + 660\alpha\theta + 480\alpha \\ +144\alpha\theta^2 - \theta^4 + 16\theta^2 + 24\theta - 36\alpha\theta^3 \end{pmatrix}}}{12\alpha(\theta+1) + 2\theta + 6}$$

Using (3.3), (3.4), (3.5) and (3.7), the skewness is defined to be

$$\begin{aligned} sk(X) &= \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{\sigma^3} \\ &= \frac{\begin{pmatrix} \frac{360\alpha(\theta+1)+24\theta+120}{3\alpha(\theta+1)+\theta+2} - 3\left(\frac{12\alpha(\theta+1)+2\theta+6}{3\alpha(\theta+1)+\theta+2}\right) \\ \times\left(\frac{60\alpha(\theta+1)+6\theta+24}{3\alpha(\theta+1)+\theta+2}\right) + 2\left(\frac{12\alpha(\theta+1)+2\theta+6}{3\alpha(\theta+1)+\theta+2}\right)^3 \end{pmatrix}}{\frac{\left(\sqrt{\begin{pmatrix} 144\alpha^2 + 144\alpha^2\theta^2 + 288\alpha^2\theta + 660\alpha\theta + 480\alpha \\ +144\alpha\theta^2 - \theta^4 + 16\theta^2 + 24\theta - 36\alpha\theta^3 \end{pmatrix}}\right)^3}{((3\alpha(\theta+1)+\theta+2))^3}} \end{aligned}$$

The kurtosis is defined using (3.3), (3.4), (3.5), (3.6) and (3.7) as

$$\begin{aligned} ku(X) &= \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{\sigma^4} \\ &= \frac{\begin{pmatrix} \frac{2520\alpha(\theta+1)+120\theta+720}{3\alpha(\theta+1)+\theta+2} - 4\left(\frac{12\alpha(\theta+1)+2\theta+6}{3\alpha(\theta+1)+\theta+2}\right)\left(\frac{360\alpha(\theta+1)+24\theta+120}{3\alpha(\theta+1)+\theta+2}\right) \\ +6\left(\frac{12\alpha(\theta+1)+2\theta+6}{3\alpha(\theta+1)+\theta+2}\right)^2\left(\frac{60\alpha(\theta+1)+6\theta+24}{3\alpha(\theta+1)+\theta+2}\right) - 3\left(\frac{12\alpha(\theta+1)+2\theta+6}{3\alpha(\theta+1)+\theta+2}\right)^4 \end{pmatrix}}{\frac{\left(\begin{pmatrix} 144\alpha^2 + 144\alpha^2\theta^2 + 288\alpha^2\theta + 660\alpha\theta + 480\alpha \\ +144\alpha\theta^2 - \theta^4 + 16\theta^2 + 24\theta - 36\alpha\theta^3 \end{pmatrix}\right)^2}{(3\alpha(\theta+1)+\theta+2)^4}} \end{aligned}$$

Table 1: Related moments measures for LBLD for different values of α and θ

α	θ	μ	σ	Sk	$Ekur$	cv	α	θ	μ	σ	Sk	$Ekur$	cv
3	**2**	**1.0536**	**1.2089**	**1.0294**	**0.631**	**1.1474**	**3**	**3.5**	**0.3478**	**0.6038**	**1.8134**	**2.9844**	**1.7359**
4	**2**	**1.0405**	**1.213**	**1.043**	**0.6367**	**1.1657**	**4**	**3.5**	**0.3426**	**0.6029**	**1.8366**	**3.0644**	**1.76**
5	**2**	**1.0326**	**1.2154**	**1.0516**	**0.6411**	**1.177**	**5**	**3.5**	**0.3394**	**0.6023**	**1.8509**	**3.1141**	**1.7748**
6	**2**	**1.0273**	**1.217**	**1.0575**	**0.6445**	**1.1847**	**6**	**3.5**	**0.3373**	**0.6019**	**1.8605**	**3.148**	**1.7848**
3	**2.5**	**0.6786**	**0.9301**	**1.3237**	**1.3016**	**1.3705**	**3**	**4**	**0.2665**	**0.5045**	**2.0225**	**3.8948**	**1.8932**
4	**2.5**	**0.6692**	**0.9309**	**1.3423**	**1.3375**	**1.3912**	**4**	**4**	**0.2624**	**0.5035**	**2.0472**	**3.9933**	**1.9187**
5	**2.5**	**0.6634**	**0.9314**	**1.3538**	**1.3602**	**1.4039**	**5**	**4**	**0.2599**	**0.5028**	**2.0623**	**4.0542**	**1.9343**
6	**2.5**	**0.6596**	**0.9317**	**1.3616**	**1.3759**	**1.4126**	**6**	**4**	**0.2583**	**0.5023**	**2.0725**	**4.0956**	**1.9449**
3	**3**	**0.4727**	**0.7391**	**1.5826**	**2.1103**	**1.5634**	**3**	**4.5**	**0.2106**	**0.4294**	**2.2146**	**4.8275**	**2.0389**
4	**3**	**0.4658**	**0.7387**	**1.604**	**2.1699**	**1.586**	**4**	**4.5**	**0.2074**	**0.4283**	**2.2405**	**4.9431**	**2.0655**
5	**3**	**0.4615**	**0.7384**	**1.6171**	**2.2072**	**1.5999**	**5**	**4.5**	**0.2054**	**0.4276**	**2.2563**	**5.0144**	**2.0819**
6	**3**	**0.4587**	**0.7382**	**1.626**	**2.2326**	**1.6093**	**6**	**4.5**	**0.2041**	**0.4272**	**2.2669**	**5.0628**	**2.093**

Table 1 shows the numerical results of the mean, standard deviation, coefficient of skewness, coefficient of excess kurtosis and coefficient of variation of the LBLD. The shape of the LBLD is skewed to right because all values of coefficient of skewness are positive which confirms the plot of the LBLD *pdf* (Figure 1 (left)). It shows that the mean values are decreasing as the values of both distribution parameters are increasing. The standard deviation, coefficient of skewness, coefficient of excess kurtosis and coefficient of variation vales are positively related with the values of distribution parameters.

3.3 Moment generating function

The moment generating function of a random variable X that has an LBLD is defined as:

$$\begin{aligned}
M_X(t) &= E(e^{tX}) = \int_0^\infty e^{tx} g_l(x)dx \\
&= \int_0^\infty e^{tx} \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^3 + \frac{x(x+1)}{\theta+1}\right] e^{-\theta x}dx \\
&= \int_0^\infty \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^3 + \frac{x(x+1)}{\theta+1}\right] e^{-(\theta-t)x}dx \\
&= \frac{3\alpha\theta^4 + (2+\theta-t)(\theta-t)}{(\theta-t)^4(2+\theta+3\alpha(1+\theta))}, \quad t<\theta
\end{aligned}$$

4 Reliability Analysis

The reliability functions for the length biased Loai distribution that will be derived in the section are: survival, hazard rate, cumulative hazard function, reversed hazard rate and odds rate functions. They are derived as

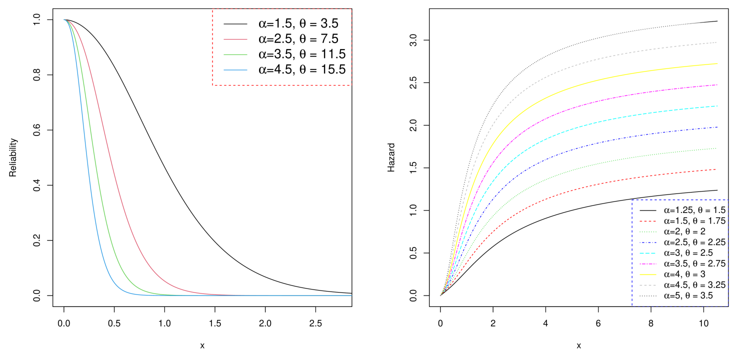
$$R_l(t) = 1 - G_l(t) = \frac{\begin{pmatrix} \theta^4t^3\alpha + 3\theta^3t^2\alpha + 6\theta^2t\alpha + 6\theta\alpha + \theta^3t^3\alpha + 3\theta^2t^2\alpha \\ +6\theta t\alpha + 6\alpha + 2\theta^2t^2 + 2\theta^2t + 4\theta t + 2\theta + 4 \end{pmatrix} e^{-\theta t}}{6\theta\alpha + 6\alpha + 2\theta + 4}$$

$$h_l(t) = \frac{g_l(t)}{1 - G_l(t)} = \frac{2\theta^3(\theta+1)\left[\frac{1}{2}\alpha\theta t^3 + \frac{t(t+1)}{\theta+1}\right] e^{-\theta t}}{\begin{pmatrix} \theta^4t^3\alpha + 3\theta^3t^2\alpha + 6\theta^2t\alpha + 6\theta\alpha + \theta^3t^3\alpha + 3\theta^2t^2\alpha \\ +6\theta t\alpha + 6\alpha + 2\theta^2t^2 + 2\theta^2t + 4\theta t + 2\theta + 4 \end{pmatrix} e^{-\theta t}}$$

$$rh_l(t) = \frac{g_l(t)}{G_l(t)} = \frac{2\theta^3(\theta+1)\left[\frac{1}{2}\alpha\theta t^3 + \frac{t(t+1)}{\theta+1}\right] e^{-\theta t}}{1 - \begin{pmatrix} \theta^4t^3\alpha + 3\theta^3t^2\alpha + 6\theta^2t\alpha + 6\theta\alpha + \theta^3t^3\alpha + 3\theta^2t^2\alpha \\ +6\theta t\alpha + 6\alpha + 2\theta^2t^2 + 2\theta^2t + 4\theta t + 2\theta + 4 \end{pmatrix} e^{-\theta t}}$$

$$CH_l(t) = -ln(1 - G_l(t)) = -ln \begin{pmatrix} \theta^4t^3\alpha + 3\theta^3t^2\alpha + 6\theta^2t\alpha + 6\theta\alpha \\ +\theta^3t^3\alpha + 3\theta^2t^2\alpha + 6\theta t\alpha + 6\alpha \\ +2\theta^2t^2 + 2\theta^2t + 4\theta t + 2\theta + 4 \end{pmatrix} e^{-\theta t} + \theta t + ln(6\theta\alpha + 6\alpha + 2\theta + 4)$$

$$O_l(t) = \frac{G_l(t)}{1 - G_l(t)} = \frac{6\theta\alpha + 6\alpha + 2\theta + 4 - \begin{pmatrix} \theta^4t^3\alpha + 3\theta^3t^2\alpha + 6\theta^2t\alpha + 6\theta\alpha \\ +\theta^3t^3\alpha + 3\theta^2t^2\alpha + 6\theta t\alpha + 6\alpha \\ +2\theta^2t^2 + 2\theta^2t + 4\theta t + 2\theta + 4 \end{pmatrix} e^{-\theta t}}{\begin{pmatrix} \theta^4t^3\alpha + 3\theta^3t^2\alpha + 6\theta^2t\alpha + 6\theta\alpha + \theta^3t^3\alpha + 3\theta^2t^2\alpha \\ +6\theta t\alpha + 6\alpha + 2\theta^2t^2 + 2\theta^2t + 4\theta t + 2\theta + 4 \end{pmatrix} e^{-\theta t}}$$



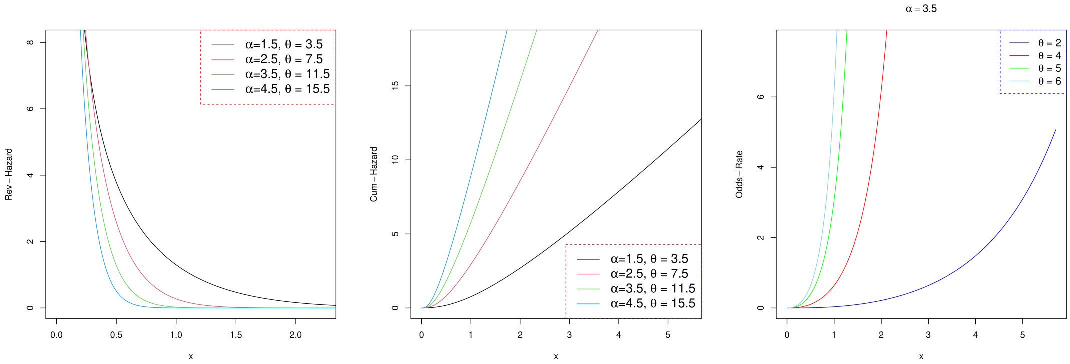


Figure 2: The reliability functions of LBLD for different values of α and θ.

5 Order Statistics and Quantile Function

In statistics, order statistics are playing a very important role in many areas, like the detection of outliers and quality control and many other areas. This section will provide the *pdf* of the j^{th}, the minimum, and maximum order statistics. Also, we will derive the quantile function of the LBLD.

5.1 Order statistics

Consider the random sample $X_1, X_2, \cdots, X_n$ selected from LBLD with *pdf* $g_l(x)$ defined in (2.1). Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the order statistics. Then the *pdf* of the j^{th} order statistic (David and Nagaraja (2003)) is defined using (2.1), (2.2) and (5.1) as:

$$\begin{aligned} g_{(j)}(x) &= j\binom{n}{j} g_l(x)[G_l(x)]^{j-1}[1-G_l(x)]^{n-j} = \frac{j\binom{n}{j}\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^3 + \frac{x(x+1)}{\theta+1}\right] \\ &\times\left[1-\frac{\begin{pmatrix}\theta^4x^3\alpha+3\theta^3x^2\alpha+6\theta^2x\alpha+6\theta\alpha+\theta^3x^3\alpha+3\theta^2x^2\alpha\\+6\theta x\alpha+6\alpha+2\theta^2x^2+2\theta^2x+4\theta x+2\theta+4\end{pmatrix}e^{-\theta x}}{6\theta\alpha+6\alpha+2\theta+4}\right]^{j-1} \\ &\times\left[\frac{\begin{pmatrix}\theta^4x^3\alpha+3\theta^3x^2\alpha+6\theta^2x\alpha+6\theta\alpha+\theta^3x^3\alpha+3\theta^2x^2\alpha\\+6\theta x\alpha+6\alpha+2\theta^2x^2+2\theta^2x+4\theta x+2\theta+4\end{pmatrix}}{6\theta\alpha+6\alpha+2\theta+4}\right]^{n-j} e^{-\theta(n-j+1)x} \end{aligned}$$

The first and last order statistics of LBLD can be calculated using $j = 1$ and $j = n$; respectively. Thus we have

$$
\begin{aligned}
g_{(1)}(x) &= \frac{n\begin{pmatrix} \theta^4x^3\alpha+3\theta^3x^2\alpha+6\theta^2x\alpha+6\theta\alpha+\theta^3x^3\alpha+3\theta^2x^2\alpha \\ +6\theta x\alpha+6\alpha+2\theta^2x^2+2\theta^2x+4\theta x+2\theta+4 \end{pmatrix}^{n-1}}{(6\theta\alpha+6\alpha+2\theta+4)^{n-1}} \\
&\times\left[\frac{\theta^3\left(\alpha\theta(\theta+1)x^3+2x(x+1)\right)}{2(3\alpha(\theta+1)+\theta+2)}e^{-n\theta x}\right] \\
g_{(n)}(x) &= n\left[1-\frac{\begin{pmatrix} \theta^4x^3\alpha+3\theta^3x^2\alpha+6\theta^2x\alpha+6\theta\alpha+\theta^3x^3\alpha+3\theta^2x^2\alpha \\ +6\theta x\alpha+6\alpha+2\theta^2x^2+2\theta^2x+4\theta x+2\theta+4 \end{pmatrix}e^{-\theta x}}{6\theta\alpha+6\alpha+2\theta+4}\right]^{n-1} \\
&\times\left[\frac{\theta^3\left(\alpha\theta(\theta+1)x^3+2x(x+1)\right)}{2(3\alpha(\theta+1)+\theta+2)}e^{-\theta x}\right]
\end{aligned}
$$

5.2 Quantile function

Quantile function is another method to visualize order statistics and tolerate simple derivation of many of their important properties (Deshpande et al. (2017)). The quantile function of a probability distribution with *cdf*, $G_l(x)$, is defined by $x_q = G_l^{-1}(q)$ or $q = G_l(x_q)$, where $0 < q < 1$. Thus, for LBLD the quantile function is the real solution of the following equation:

$$
1-q = \frac{\begin{pmatrix} \theta^4x_q^3\alpha+3\theta^3x_q^2\alpha+6\theta^2x_q\alpha+6\theta\alpha+\theta^3x_q^3\alpha+3\theta^2x_q^2\alpha \\ +6\theta x_q\alpha+6\alpha+2\theta^2x_q^2+2\theta^2x_q+4\theta x_q+2\theta+4 \end{pmatrix}e^{-\theta x_q}}{6\theta\alpha+6\alpha+2\theta+4} \quad (5.1)
$$

The quantile function defined in (5.1) can not be solved explicitly. But Figure 3 shows that the quantile function has exactly one solution for $x_q > 0$. The quantile function defined in (5.1) can not be solve explicitly. But Figure 3 (right) shows that the quantile function has exactly one solution for $x_q > 0$. It shows the plot of the *pdf* of j^{th} order statistics from a sample of size n =10 for α of 3.5 and $\theta = 3$. We have selected j to be 1-10. It shows that the peak of the plot gets sharper for larger values of j.

6 Gini Index

Gini index (GI) (Corrado (1909)) is the most used measure in economics inequality. GI measures the amount or probability of a randomly selected variable to be classified in a wrong way (Giorgi and Gigliarano (2017)). Gini index is defined as:

$$
GI = 1 - \frac{1}{\mu}\int_0^\infty (1 - G_l(x;\alpha,\theta))^2 dx \quad (6.1)
$$

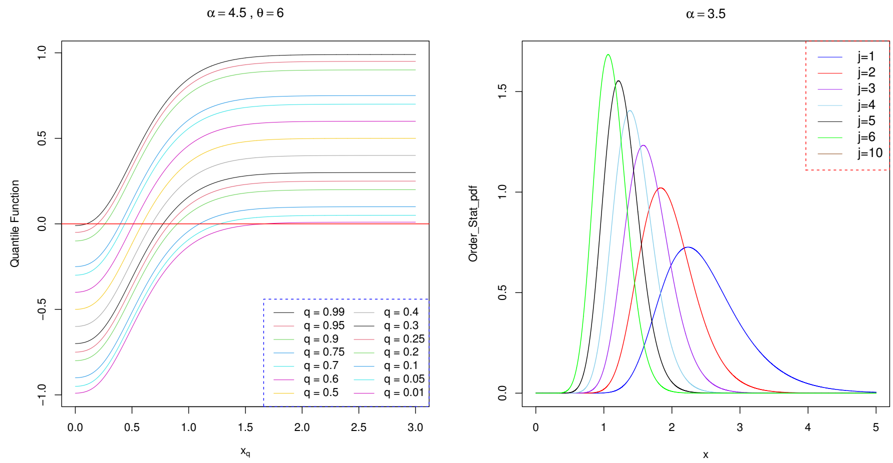


Figure 3: The pdf of order statistics and the quantile function of LBLD

For LBLD, using (6.1) and (2.2) it is given by:

$$
\begin{aligned}
GI &= 1-\frac{\theta(\theta+1)(\alpha+1)}{4(3\alpha(\theta+1)+\theta+2)^2}\int_0^\infty \begin{pmatrix} \theta^4x^3\alpha+3\theta^3x^2\alpha+6\theta^2x\alpha+6\theta\alpha \\ +\theta^3x^3\alpha+3\theta^2x^2\alpha+6\theta x\alpha+6\alpha \\ +2\theta^2x^2+2\theta^2x+4\theta x+2\theta+4 \end{pmatrix}^2 e^{-2\theta x}dx \\
&= 1-\frac{(\theta+1)(\alpha+1)}{4(3\alpha(\theta+1)+\theta+2)^2}\begin{pmatrix} \frac{45\alpha^2(\theta+1)^2}{8}+\frac{65(\theta+1)^2(3\alpha+2)}{4}+\frac{3(\theta+1)^2(3\alpha+2)^2}{4} \\ +\frac{3(\alpha(\theta+1)(6\alpha(\theta+1)+2\theta+4)+12(\theta+1)(3\alpha+2)(3\alpha(\theta+1)+\theta))}{4} \\ +\frac{(\theta+1)(3\alpha+2)(6\alpha(\theta+1)+2\theta+4)+2(\theta+3\alpha(\theta+1))^2}{2} \\ +(3\alpha(\theta+1)+\theta+2)(2\theta(\theta+3\alpha(\theta+1))+1) \end{pmatrix}
\end{aligned}
$$

Table 2 shows the values of the Gini index for some values α and θ. We have used the values of α of 3, 3.5, 4, 4.5, 5, 5.5 and the values of θ of 3, 3.5, 4, 4.5, 5, 5.5, 6. It shows that the values of the Gini index are all between 0 and 1.

7 Stochastic Ordering

To compare between two random variables X and Y, the most common procedure that can be used is through their means and variances. But the problem in that is the mean of X may be greater than the mean of Y and the median of Y is greater than the median of X. This problem can be solved through stochastic ordering (Khaledi and Kochar (1999)). The idea of Stochastic ordering has gained more attention in reliability analysis and statistics (Yaming (2009)).

Table 2: Gini index for some values of α and θ

θ	α	GI	θ	α	GI	θ	α	GI	θ	α	GI
3	**3**	**0.8981**	**4**	**3**	**0.673**	**5**	**3**	**0.538**	**6**	**3**	**0.4482**
3	**3.5**	**0.9069**	**4**	**3.5**	**0.6797**	**5**	**3.5**	**0.5435**	**6**	**3.5**	**0.4527**
3	**4**	**0.9138**	**4**	**4**	**0.6849**	**5**	**4**	**0.5477**	**6**	**4**	**0.4562**
3	**4.5**	**0.9192**	**4**	**4.5**	**0.689**	**5**	**4.5**	**0.551**	**6**	**4.5**	**0.4591**
3	**5**	**0.9237**	**4**	**5**	**0.6925**	**5**	**5**	**0.5538**	**6**	**5**	**0.4614**
3	**5.5**	**0.9275**	**4**	**5.5**	**0.6953**	**5**	**5.5**	**0.5561**	**6**	**5.5**	**0.4633**
3	**6**	**0.9306**	**4**	**6**	**0.6977**	**5**	**6**	**0.558**	**6**	**6**	**0.4649**
3.5	**3**	**0.7694**	**4.5**	**3**	**0.598**	**5.5**	**3**	**0.489**	**6.5**	**3**	**0.4136**
3.5	**3.5**	**0.777**	**4.5**	**3.5**	**0.604**	**5.5**	**3.5**	**0.4939**	**6.5**	**3.5**	**0.4178**
3.5	**4**	**0.7829**	**4.5**	**4**	**0.6086**	**5.5**	**4**	**0.4978**	**6.5**	**4**	**0.4211**
3.5	**4.5**	**0.7877**	**4.5**	**4.5**	**0.6124**	**5.5**	**4.5**	**0.5009**	**6.5**	**4.5**	**0.4237**
3.5	**5**	**0.7916**	**4.5**	**5**	**0.6154**	**5.5**	**5**	**0.5034**	**6.5**	**5**	**0.4258**
3.5	**5.5**	**0.7948**	**4.5**	**5.5**	**0.6179**	**5.5**	**5.5**	**0.5055**	**6.5**	**5.5**	**0.4276**
3.5	**6**	**0.7975**	**4.5**	**6**	**0.6201**	**5.5**	**6**	**0.5072**	**6.5**	**6**	**0.4291**

Consider the two random variables X and Y with probability density, cumulative distribution and reliability functions: $g_l(x)$, $g_l(y)$, $G_l(x)$, $G_l(y)$, $\bar{G}_l(x) = 1 - G_l(x)$ and $\bar{G}_l(y) = 1 - G_l(y)$; respectively. Then

1. Mean residual life order denoted by $X \leq_{MRLO} Y$, if $m_x(x) \leq m_y(y)$, $\forall\, x$.
2. Hazard rate order denoted as $X \leq_{HRO} Y$, if $\frac{\bar{G}_X(x)}{\bar{G}_Y(x)}$ is decreasing if $x \geq 0$.
3. Stochastic order denoted as $X \leq_{SO} Y$, if $\bar{G}(x) \leq_{SO} \bar{G}_Y(x)$, $\forall\, x$.
4. Likelihood ratio order denote as $X \leq_{LRO} Y$, if $\frac{f_X(x)}{f_Y(x)}$ is decreasing for $x \geq 0$.

Shaked and Shanthikumar (1994) showed that:

$$\begin{array}{ccccc} X \leq_{LRO} Y & \Rightarrow & X \leq_{HRO} Y & \Rightarrow & X \leq_{MRLO} Y \\ & & \Downarrow & & \\ & & X \leq_{SO} Y & & \end{array}$$

Theorem 7.1 *Let X and Y be two independent random variable with probability density functions $g_X(x, \alpha, \theta)$ and $g_Y(x, \beta, \zeta)$; respectively. If $\beta < \theta$ and $\zeta < \alpha$, then $X \leq_{LRO} Y$, $X \leq_{HRO} Y$, $X \leq_{MRLO} Y$ and $X \leq_{SO} Y$.*

Proof 7.1 *Consider* $\Xi = \frac{g_X(x,\alpha,\theta)}{g_Y(x,\beta,\zeta)}$. *Thus,*

$$\begin{aligned}
\Xi &= \frac{\frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^3+\frac{x(x+1)}{\theta+1}\right]e^{-\theta x}}{\frac{\beta^3(\beta+1)}{3\zeta(\beta+1)+\beta+2}\left[\frac{1}{2}\zeta\beta x^3+\frac{x(x+1)}{\beta+1}\right]e^{-\beta x}} \\
&= \frac{\theta^3(3\zeta(\beta+1)+\beta+2)(\theta+1)\left[\frac{1}{2}\alpha\theta x^3+\frac{x(x+1)}{\theta+1}\right]}{\beta^3(3\alpha(\theta+1)+\theta+2)(\beta+1)\left[\frac{1}{2}\zeta\beta x^3+\frac{x(x+1)}{\beta+1}\right]}e^{-(\theta-\beta)x}
\end{aligned}$$

$$\begin{aligned}
\therefore\ ln(\Xi) &= ln\left[\frac{\theta^3(3\zeta(\beta+1)+\beta+2)(\theta+1)\left[\frac{1}{2}\alpha\theta x^3+\frac{x(x+1)}{\theta+1}\right]}{\beta^3(3\alpha(\theta+1)+\theta+2)(\beta+1)\left[\frac{1}{2}\zeta\beta x^3+\frac{x(x+1)}{\beta+1}\right]}e^{-(\theta-\beta)x}\right] \\
&= ln\left[\frac{\theta^3(3\zeta(\beta+1)+\beta+2)(\theta+1)}{\beta^3(3\alpha(\theta+1)+\theta+2)(\beta+1)}\right]+ln\left[\frac{1}{2}\alpha\theta x^3+\frac{x(x+1)}{\theta+1}\right] \\
&\quad -ln\left[\frac{1}{2}\zeta\beta x^3+\frac{x(x+1)}{\beta+1}\right]-(\theta-\beta)x
\end{aligned}$$

Deriving with respect to x, we get:

$$\frac{\partial ln(\Xi)}{\partial x} = \frac{3\alpha\theta(\theta+1)x^2+4x+2}{\alpha\theta x^3+2x(x+1)}-\frac{3\zeta\beta(\beta+1)x^2+4x+2}{\zeta\beta x^3+2x(x+1)}-(\theta-\beta)$$

$\frac{\partial ln(\Xi)}{\partial x} < 0$ *if* $\beta < \theta$, $\zeta < \alpha$. *Thus,* $X \leq_{LRO} Y$, $X \leq_{HRO} Y$, $X \leq_{MRLO} Y$ *and* $X \leq_{SO} Y$.

8 Bonferroni and Lorenz Curves

As well as the Gini index, the Bonferroni and Lorenz curves are very important in economics, demography (Kakwani and Podder (1973)). The Bonferroni and Lorenz curves for a LBLD random variable X are, respectively, defined as:

$$\begin{aligned}
B &= \frac{1}{p\mu}\int_0^q x g_l(x)dx = \frac{(\alpha+1)}{p\theta^3(3\alpha(\theta+1)+\theta+2)} \\
&\times\begin{pmatrix} 0.5\alpha(\theta+1)(e^{-q\theta}(-q\theta(q\theta(q\theta(q\theta+4)+12)+24)-24)+24) \\ +\theta(e^{-q\theta}(-q\theta(q\theta+2)-2)+2)+e^{-q\theta}(-q\theta(q\theta(q\theta+3)+6)-6)+6 \end{pmatrix} \\
Z &= \frac{1}{\mu}\int_0^q x g_l(x)dx = \frac{(\alpha+1)}{\theta^3(3\alpha(\theta+1)+\theta+2)} \\
&\times\begin{pmatrix} 0.5\alpha(\theta+1)(e^{-q\theta}(-q\theta(q\theta(q\theta(q\theta+4)+12)+24)-24)+24) \\ +\theta(e^{-q\theta}(-q\theta(q\theta+2)-2)+2)+e^{-q\theta}(-q\theta(q\theta(q\theta+3)+6)-6)+6 \end{pmatrix},
\end{aligned}$$

where $\mu = E(X)$

9 Entropy

Shannon (1948) introduced the entropy in a general theory of communication. It is an accurate measure of uncertainty, which makes the second law of thermodynamics understandable. In statistics, it is the measure of uncertainty of the probability distribution of a random variable X Wang (2008). The Shannon (Shannon (1948)), Rényi (Rényi (1961)) and Tsallis (Tsallis (1988)) entropies of LBLD random variable X are defined as:

$$
\begin{aligned}
S_l^{\rho} &= -\int_0^{\infty} g_l(x) log(g_l(x))dx = -\int_0^{\infty} \frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^3 + \frac{x(x+1)}{\theta+1}\right]e^{-\theta x} \\
&\quad \times log\left(\frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^3 + \frac{x(x+1)}{\theta+1}\right]e^{-\theta x}\right)dx \\
R_l^{\rho} &= \frac{\rho}{1-\rho} log\int_0^{\infty}[g_l(x)]^{\rho}dx \\
&= \frac{\rho}{1-\rho} log\int_0^{\infty}\left[\frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\left[\frac{1}{2}\alpha\theta x^3 + \frac{x(x+1)}{\theta+1}\right]e^{-\theta x}\right]^{\rho} dx;\ \rho>0,\ \rho\neq 1 \\
&= \frac{\rho}{1-\rho} log\left[\frac{\binom{\rho}{i}\binom{\rho-i}{j}\theta^{3\rho}(\theta+1)^{\rho}\left(\frac{1}{2}\alpha\theta\right)^{\rho-i}\Gamma(\rho+2i+j+1)}{(3\alpha(\theta+1)+\theta+2)^{\rho}(\theta+1)^{\rho-i}(\rho\theta)^{\rho+2i+j+2}}\right] \\
T_l^{\rho} &= \frac{1}{\rho-1}\left[1-\int_0^{\infty}[g_l(x)]^{\rho}dx\right] \\
&= \frac{1}{\rho-1}\left[1-\left[\frac{\binom{\rho}{i}\binom{\rho-i}{j}\theta^{3\rho}(\theta+1)^{\rho}\left(\frac{1}{2}\alpha\theta\right)^{\rho-i}\Gamma(\rho+2i+j+1)}{(3\alpha(\theta+1)+\theta+2)^{\rho}(\theta+1)^{\rho-i}(\rho\theta)^{\rho+2i+j+2}}\right]\right];\ \rho>0,\ \rho\neq 1
\end{aligned}
$$

Table 3 shows some results of Shannon, Rènyi and Tsallis entropies for the values of α of 1, 1.5, 2, 2.5, 3 and 3.5 and values of θ of 1.5, 2, 2.5, 3 and 3.5. It shows that all entropy values are decreasing as the values of θ are increasing. Entropy values are increasing as the values of α are increasing.

10 Stress-Strength Reliability

Consider the two independent random variables X and Y from Loai distribution, where X represents the strength of the system and Y is the stress applied to this system (Al-marashi et al. (2020)). The component failed to work at the moment that the stress applied to it exceeds the strength and the component will function satisfactorily when-

Table 3: Numerical results for entropy using different values of α and θ with ρ=5.

α	θ	Shannon	Renyi	Tsallis	α	θ	Shannon	Renyi	Tsallis
1.0	**1.5**	**1.62047**	**1.31110**	**0.24868**	**2.5**	**1.5**	**1.63161**	**1.31374**	**0.24869**
1.0	**2.0**	**1.33634**	**1.02909**	**0.24592**	**2.5**	**2.0**	**1.34641**	**1.02905**	**0.24592**
1.0	**2.5**	**1.11571**	**0.81009**	**0.24021**	**2.5**	**2.5**	**1.12503**	**0.80807**	**0.24013**
1.0	**3.0**	**0.93525**	**0.63092**	**0.22996**	**2.5**	**3.0**	**0.94402**	**0.62739**	**0.22967**
1.0	**3.5**	**0.78254**	**0.47925**	**0.21324**	**2.5**	**3.5**	**0.79090**	**0.47452**	**0.21254**
1.5	**1.5**	**1.62864**	**1.31527**	**0.24870**	**3.0**	**1.5**	**1.63142**	**1.31238**	**0.24869**
1.5	**2.0**	**1.34415**	**1.03209**	**0.24597**	**3.0**	**2.0**	**1.34595**	**1.02724**	**0.24589**
1.5	**2.5**	**1.12328**	**0.81222**	**0.24030**	**3.0**	**2.5**	**1.12438**	**0.80593**	**0.24005**
1.5	**3.0**	**0.94265**	**0.63238**	**0.23008**	**3.0**	**3.0**	**0.94324**	**0.62500**	**0.22948**
1.5	**3.5**	**0.78982**	**0.48019**	**0.21338**	**3.0**	**3.5**	**0.79000**	**0.47194**	**0.21215**
2.0	**1.5**	**1.63108**	**1.31498**	**0.24870**	**3.5**	**1.5**	**1.63095**	**1.31112**	**0.24868**
2.0	**2.0**	**1.34620**	**1.03091**	**0.24595**	**3.5**	**2.0**	**1.34526**	**1.02564**	**0.24587**
2.0	**2.5**	**1.12506**	**0.81039**	**0.24022**	**3.5**	**2.5**	**1.12354**	**0.80408**	**0.23997**
2.0	**3.0**	**0.94423**	**0.63006**	**0.22989**	**3.5**	**3.0**	**0.94228**	**0.62296**	**0.22931**
2.0	**3.5**	**0.79123**	**0.47747**	**0.21298**	**3.5**	**3.5**	**0.78896**	**0.46975**	**0.21181**

ever $X > Y$. The stress strength model is defined as $p(Y < X)$ (Hassan (2017)).

$$
\begin{aligned}
p(Y<X) &= \left[\frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2}\right]^2 \int_0^\infty \int_0^x \left(\left[\frac{1}{2}\alpha\theta x^3 + \frac{x(x+1)}{\theta+1}\right] \right. \\
&\quad \left. \times \left[\frac{1}{2}\alpha\theta y^3 + \frac{y(y+1)}{\theta+1}\right] e^{-\theta(x+y)} dy dx \right) \\
&= \int_0^\infty \frac{\alpha\theta(\theta+1)x^3 + 2x(x+1)}{2(\theta+1)\left(3\alpha\left(\theta+1\right)+\theta+2\right)} \begin{pmatrix} \left(\left(3\alpha+1\right)\theta+3\alpha+2\right)e^{-\theta x} \\ -\left(\alpha\theta^4+\alpha\theta^3\right)x^3\frac{e^{-2\theta x}}{2} \\ -\left(3\alpha\theta^3+\left(3\alpha+2\right)\theta^2\right)x^2\frac{e^{-2\theta x}}{2} \\ -\left(\left(6\alpha+2\right)\theta^2+\left(6\alpha+4\right)\theta\right)x\frac{e^{-2\theta x}}{2} \\ -\left(\left(6\alpha+2\right)\theta+6\alpha+4\right)\frac{e^{-2\theta x}}{2} \end{pmatrix} dx \\
&= \frac{\left(3\alpha+1\right)\theta+3\alpha+2}{2\theta^3\cdot(\theta+1)}
\end{aligned}
$$

11 Parameters Estimation Methods

11.1 Maximum likelihood method

Let $X_1, X_2, ...X_n$ be a random sample from LBLD, then the likelihood function $L(x, \alpha, \theta)$ is defined by

$$\begin{aligned} L &= L(x,\alpha,\theta) = \prod_{i=1}^{n} g_l(x_i,\alpha,\theta) \\ &= \prod_{i=1}^{n} \left[\frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2} \left[\frac{1}{2}\alpha\theta x_i^3 + \frac{x_i(x_i+1)}{\theta+1} \right] e^{-\theta x_i} \right] \\ &= \left[\frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2} \right]^n \prod_{i=1}^{n} \left[\frac{1}{2}\alpha\theta x_i^3 + \frac{x_i(x_i+1)}{\theta+1} \right] e^{-\theta \sum_1^n x_i} \end{aligned}$$

Thus the log-likelihood function is

$$\begin{aligned} \ell &= ln(L) = ln \left\{ \left[\frac{\theta^3(\theta+1)}{3\alpha(\theta+1)+\theta+2} \right]^n \prod_{i=1}^{n} \left[\frac{1}{2}\alpha\theta x_i^3 + \frac{x_i(x_i+1)}{\theta+1} \right] e^{-\theta \sum_1^n x_i} \right\} \\ &= 3nln(\theta) + nln(\theta+1) - nln(3\alpha(\theta+1)+\theta+2) - \theta \sum_1^n x_i \\ &\quad + \sum_1^n \left[ln(\alpha(\theta+1)\theta x_i^3 + 2x_i(x_i+1)) \right] - ln(2(\theta+1)) \end{aligned}$$

The maximum likelihood estimates (MLEs) of LBLD parameters can be obtained by equating the following derivatives to zero and solving with respect to the parameters.

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{-3n(\theta+1)}{3\alpha(\theta+1)+\theta+2} + \sum_1^n \left[ln(\alpha(\theta+1)\theta x_i^3 + 2x_i(x_i+1)) - ln(2(\theta+1)) \right] \\ \frac{\partial \ell}{\partial \theta} &= \frac{3n}{\theta} + \frac{n}{\theta+1} - \frac{n(3\alpha+1)}{3\alpha(\theta+1)+\theta+2} + \sum_1^n \left[\frac{\alpha(2\theta+1)}{\alpha(\theta+1)\theta x_i^3 + 2x_i(x_i+1)} - \frac{1}{\theta+1} - x_i \right] \end{aligned}$$

There is no exact solution for the system of equations $\{\frac{\partial \ell}{\partial \alpha} = 0,\ \frac{\partial \ell}{\partial \theta} = 0\}$. Therefore, we can solve it numerically.

11.2 Ordinary and weighted least square methods

Swain et al. (1988) suggested the ordinary least square (OLS) and weighted least square (WLS) methods of estimation to estimate the parameters of beta distributions. Consider that $G_l(x_{(k)})$ be the *cdf* of k^{th} order statistic of the order statistics $X_{(1)}$, $X_{(2)}$, ..., $X_{(n)}$. The OLS and WLS estimators can; respectively be obtained by minimizing the following functions with respect to the parameters (Yılmaz et al. (2021)).

$$R_{OLS} = \sum_{k=1}^{n} \left[G_l(x_{(k)}) - \frac{k}{n+1} \right]^2,\ W_{WLS} = \sum_{k=1}^{n} \frac{(n+1)^2(n+2)}{k(n+1-k)} \left[G_l(x_{(k)}) - \frac{k}{n+1} \right]^2$$

Thus, the OLS can be defined using (2.2) as:

$$R_{OLS} = \sum_{k=1}^{n}\left[1-\frac{\begin{pmatrix}(\alpha(\theta+1)(\theta x_{(k)}(\theta x_{(k)}(\theta x_{(k)}+3)+6)+6)\\ +2(\theta x_{(k)}(\theta x_{(k)}+\theta+2)+\theta+2))e^{-\theta x_{(k)}}\end{pmatrix}}{2(3\alpha(\theta+1)+\theta+2)}-\frac{k}{n+1}\right]^2$$

$$= \sum_{k=1}^{n}\left[\frac{n+1-k}{n+1}-\frac{\begin{pmatrix}\alpha(\theta+1)\left(\theta^3x_{(k)}^3+3\theta^2x_{(k)}^2+6\theta x_{(k)}+6\theta\right)\\ (2\theta^2(x_{(k)}^2+x_{(k)})+4\theta x_{(k)}+2\theta+4)e^{-\theta x_{(k)}}\end{pmatrix}}{2(3\alpha(\theta+1)+\theta+2)}\right]^2$$

Thus, the OLS estimators of α and θ are the solutions of the following equations

$$\frac{\partial R_{OLS}}{\partial\alpha} = 0, \quad \frac{\partial R_{OLS}}{\partial\theta}=0$$

The WLS of LBLD is defined as

$$W_{WLS} = \sum_{k=1}^{n}\frac{(n+1)^2(n+2)}{k(n+1-k)}\left[\frac{n+1-k}{n+1}-\frac{\begin{pmatrix}\alpha(\theta+1)\left(\theta^3x_{(k)}^3+3\theta^2x_{(k)}^2+6\theta x_{(k)}+6\theta\right)\\ (2\theta^2(x_{(k)}^2+x_{(k)})+4\theta x_{(k)}+2\theta+4)e^{-\theta x_{(k)}}\end{pmatrix}}{2(3\alpha(\theta+1)+\theta+2)}\right]^2$$

Again, the WLS estimators of α and θ are the solutions of the following equations

$$\frac{\partial W_{OLS}}{\partial\alpha} = 0, \quad \frac{\partial W_{OLS}}{\partial\theta}=0$$

11.3 Method of maximum product of spacings

Maximum product spacing (MPS) method of estimation is an alternative to the maximum likelihood method. It is proposed by Cheng and Amin (1979, 1983). This method depends on maximizing the geometric mean of the spacings of the data with respect to the parameters. The MPS method provides consistent and asymptotically efficient estimators whether MLE exists or not. The uniform spacings is defined as:

$$\Psi_k(\alpha,\theta) = G_l(x_{(k)}|\alpha,\theta)-G_l(x_{(k-1)}|\alpha,\theta),\ k=1,...,n,$$

where $G_l(x_{(k)}|\alpha,\theta)=0$ at $k=0$ and 1 at $k=n+1$. It is clear that $\sum_{i=1}^{n+1}\Psi_k(\alpha,\theta)=1$.

The MPS estimators, $\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$, of α and θ can be obtained by maximizing the geometric mean of the spacings, that is,

$$
\begin{aligned}
GM(\alpha,\theta|x) &= \left(\prod_{k=1}^{n+1}\Psi_i(\alpha,\theta)\right)^{\frac{1}{n+1}} \\
&= \left(\prod_{k=1}^{n+1}\left(\frac{\left(\begin{array}{c}(\alpha(\theta+1)(\theta x_{(k)}(\theta x_{(k)}(\theta x_{(k)}+3)+6)+6)\\ +2(\theta x_{(k)}(\theta x_{(k)}+\theta+2)+\theta+2))e^{-\theta x_{(k)}}\end{array}\right)}{2(3\alpha(\theta+1)+\theta+2)} - \frac{\left(\begin{array}{c}(\alpha(\theta+1)(\theta x_{(k-1)}(\theta x_{(k-1)}(\theta x_{(k-1)}+3)+6)+6)\\ +2(\theta x_{(k-1)}(\theta x_{(k-1)}+\theta+2)+\theta+2))e^{-\theta x_{(k-1)}}\end{array}\right)}{2(3\alpha(\theta+1)+\theta+2)}\right)\right)^{\frac{1}{n+1}}
\end{aligned}
$$

Now, the natural logarithm gives

$$
NL(\alpha,\theta|x) = \frac{1}{n+1}\sum_{k=1}^{n+1} ln\left(\frac{\left(\begin{array}{c}(\alpha(\theta+1)(\theta x_{(k)}(\theta x_{(k)}(\theta x_{(k)}+3)+6)+6)\\ +2(\theta x_{(k)}(\theta x_{(k)}+\theta+2)+\theta+2))e^{-\theta x_{(k)}}\end{array}\right)}{2(3\alpha(\theta+1)+\theta+2)} - \frac{\left(\begin{array}{c}(\alpha(\theta+1)(\theta x_{(k-1)}(\theta x_{(k-1)}(\theta x_{(k-1)}+3)+6)+6)\\ +2(\theta x_{(k-1)}(\theta x_{(k-1)}+\theta+2)+\theta+2))e^{-\theta x_{(k-1)}}\end{array}\right)}{2(3\alpha(\theta+1)+\theta+2)}\right)
$$

$\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$ can be obtained by solving the following nonlinear system of equations with respect to the parameters α and θ.

$$
\begin{aligned}
\frac{\partial NL(\alpha,\theta|x)}{\partial\alpha} &= \frac{1}{n+1}\sum_{k=1}^{n+1}\frac{\Delta_1(x_{(k)}|\alpha,\theta)-\Delta_1(x_{(k-1)}|\alpha,\theta)}{\Psi_i(\alpha,\theta)} = 0 \\
\frac{\partial NL(\alpha,\theta|x)}{\partial\theta} &= \frac{1}{n+1}\sum_{k=1}^{n+1}\frac{\Delta_2(x_{(k)}|\alpha,\theta)-\Delta_2(x_{(k-1)}|\alpha,\theta)}{\Psi_i(\alpha,\theta)} = 0,
\end{aligned}
$$

where

$$
\Delta_1(x_{(k)}|\alpha,\theta) = \frac{\partial G(x_{(k)}|\alpha,\theta)}{\partial\alpha},\ \ \Delta_2(x_{(k)}|\alpha,\theta) = \frac{\partial G(x_{(k)}|\alpha,\theta)}{\partial\theta} \tag{11.1}
$$

11.4 Methods of minimum distances

Wolfowitz (1957) proposed the method of minimum distance obtain strong consistent estimators. Consider the random sample of size n, say $X_1,\cdots,X_n$ with *cdf* $G(x|\alpha,\theta)$ and let $G_n(x)$ be the empirical distribution function based on the sample $\mathbf{x}=(x_1,\cdots,x_n)$. If $(\hat{\alpha},\hat{\theta})$ is the vector of estimators of (α,θ), then $G(x|\hat{\alpha},\hat{\theta})$ is an estimator of $G_l(x|\alpha,\theta)$. Assuming $(\hat{\alpha},\hat{\theta})$ is exist, such that

$$
d[G_l(x|\hat{\alpha},\hat{\theta}),G_n(x)] = inf\{d[G(x|\alpha,\theta),G_n(x)]\},
$$

where $d[.,.]$ is the distance between $G(x|\hat{\alpha},\hat{\theta})$ and $G_n(x)$, then $(\hat{\alpha},\hat{\theta})$ is called the minimum-distance estimate of (α,θ) (Drossos and Philippou (1980)).

11.5 Cramer-Von-Mises method

Cramer-Von-Mises method (Cramér (1928); Von Mises (1928)) usually called W^2, is a method used in one-sample applications to compare between the theoretical cumulative distribution function $G^*(x)$ of a random variable and a given empirical distribution $G_n(x)$ using the goodness of fit. It is also used as a part of the minimum distance method of estimation. It is defined as

$$\varrho^2 = \int_{-\infty}^{\infty} \left[G_n(x) - G_l^*(x)\right]^2 dG_l^*(x)$$

For a random sample of size n with observed values $x_1, \cdots, x_n$ sorted in an ascending order the Cramer-Von Mises test statistic value is (Stephens (1986)),

$$CVM^2 \;=\; \sum_{k=0}^{n}\left[G_l(x_{(k)},\alpha,\theta) - \frac{2k-1}{2n}\right]^2 + \frac{1}{12n}$$

Thus for a random sample of size n from Loai distribution with observed values $x_1, \cdots, x_n$ sorted in an ascending order the Cramér-von Mises test statistic value is

$$\begin{aligned} CVM^2 &= \frac{1}{12n} + \sum_{k=0}^{n}\left[G(x_{(k)},\alpha,\theta) - \frac{2k-1}{2n}\right]^2 \\ &= \frac{1}{12n} + \sum_{k=1}^{n}\left[1 - \frac{\begin{pmatrix} (\alpha(\theta+1)(\theta x_{(k)}(\theta x_{(k)}(\theta x_{(k)}+3)+6)+6) \\ +2(\theta x_{(k)}(\theta x_{(k)}+\theta+2)+\theta+2))e^{-\theta x_{(k)}} \end{pmatrix}}{2(3\alpha(\theta+1)+\theta+2)} - \frac{2k-1}{2n}\right]^2 \end{aligned}$$

The Cramer-von Mises estimators $\hat{\alpha}$ and $\hat{\theta}$ of α and θ can be obtained by minimizing W^2. These estimators are the solutions of the following system of nonlinear equations

$$\begin{aligned} \sum_{k=0}^{n}\left[2G_l(x_{(k)},\alpha,\theta) - \frac{2k-1}{n}\right]\Delta_1(x_{(k)}|\alpha,\theta) &= 0 \\ \sum_{k=0}^{n}\left[2G_l(x_{(k)},\alpha,\theta) - \frac{2k-1}{n}\right]\Delta_2(x_{(k)}|\alpha,\theta) &= 0, \end{aligned}$$

where Δ_1 and Δ_2 are defined in (11.1).

11.6 Method of Anderson-Darling

Anderson and Darling (1952) introduced a method of estimating the distribution parameters. This method is called Anderson-Darling method of estimation, it is defined as

$$AD \;=\; -n - \frac{1}{n}\sum_{k=0}^{n}(2k-1)\left\{log[G_l(x_{(k)};\alpha,\theta)] + log[\bar{G}_l(x_{(n+1-k)};\alpha,\theta)]\right\} \qquad (11.2)$$

The estimators $\hat{\alpha}_{AD}$ and $\hat{\theta}_{AD}$ can be obtained by minimizing (11.2), or by solving the following nonlinear system of equations.

$$\frac{\partial AD(\alpha,\theta|x)}{\partial\alpha} = \sum_{k=0}^{n}(2k-1)\left\{\left[\frac{\Delta_1(x_{(k)}|\alpha,\theta)}{G(x_{(k)};\alpha,\theta)}\right] - \frac{\Delta_1(x_{(k)}|\alpha,\theta)}{\bar{G}(x_{(n+1-i)};\alpha,\theta)}\right\} = 0$$
$$\frac{\partial AD(\alpha,\theta|x)}{\partial\theta} = \sum_{k=0}^{n}(2k-1)\left\{log[\frac{\Delta_2(x_{(k)}|\alpha,\theta)}{G(x_{(k)})};\alpha,\theta)] - \frac{\Delta_2(x_{(k)}|\alpha,\theta)}{\bar{G}(x_{(n+1-k)};\alpha,\theta)}\right\} = 0,$$

where $\bar{G} = 1 - G$ and Δ_1 and Δ_2 are defined in (11.1).

12 Simulation Study

A simulation study is performed in this section to test the accuracy of the estimators of the LBLD distribution parameters with the help of *R* software R Core Team (2021). For this purpose, $N = 1500$ samples are generated, each of size 50, 100, 300, and 500 for values of $\alpha = 3$ and $\theta = 1.5$. For each sample, the estimators of the parameter space $\phi = (\alpha, \theta)$ using MLE, OLS, WLS, MPS, CVM, and AD methods of estimation with their mean square error (MSE) and the bias are obtained. Then, the average bias (AB) and the mean square error (MSE) are calculated as follows:

$$AB(\hat{\phi}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}-\phi),\ MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}-\phi)^2$$

Table 4 shows that the WLS method of estimation is the best method for estimating both parameters, regardless the sample size used.

13 Real Data Application

In this section, we will test the applicability of the proposed distribution by considering a real-life time data set and comparing its goodness of fit with some existing distributions. This data set is reported in Ross (2010) and represents 48 reaction times (in seconds) to a certain stimulus recorded by a psychologist. The data are given in Table 5. The goodness of fit of the proposed distribution is compared with the following distributions:

- Loai distribution (Loai) (Alzoubi et al. (2022)): (See (1.3))
- Exponential distribution (Exp) (Kingman (1982)).
- Transmuted Aradhana distribution (T. Arad) (Gharaibeh (2020)) $f(x) = \frac{\theta^3(1+x)^2}{\theta^2+2\theta+2}e^{-\theta x}\left(1-\lambda+2\lambda e^{-\theta x}\left(1+\frac{\theta x(\theta^2+2\theta+2)}{\theta x+2\theta+2}\right)\right),\ x,\ \theta>0,\ |\lambda|\le 1$
- Pranav distribution (Pran) (Shanker (2015)): $f(x) = \frac{\alpha^2(\alpha+x)}{\alpha^2+1}e^{-\alpha x},\ x,\ \alpha>0$
- Benrabia distribution (Br.) (Benrabia and Alzoubi (2022)): $f(x) = \frac{\theta}{\alpha+\theta}\left(\alpha+\frac{x^{\alpha-2}\theta^{\alpha-1}}{\Gamma(\alpha-1)}\right)e^{-\theta x},\ x,\theta>0,\ \alpha>1$
- Gamma distribution (Gam) (Johnson et al. (1970)): $f(x) = \frac{\theta^\alpha x^{\alpha-1}e^{-\theta x}}{\Gamma(\alpha)},\ x,\ \alpha,\ \theta>0.$
- Lindley distribution (Lind) (Ghitany et al. (2008)): $f(x) = \frac{\alpha^2(1+x)e^{-\alpha x}}{1+\alpha},\ x,\ \alpha>0$

Table 4: Parameter Estimates and their average biases and mean squares errors, when $\alpha = 0.5$.

Method	n	$\hat{\alpha}$	$\hat{\theta}$	$AB(\hat{\alpha})$	$MSE(\hat{\alpha})$	$AB(\hat{\theta})$	$MSE(\hat{\theta})$
MLE	50	3.152871	1.603173	0.152871	0.023369	0.103173	0.010645
OLS		3.080675	1.554248	0.080675	0.006508	0.054248	0.002943
WLS		3.028712	1.534770	0.028712	0.000824	0.034770	0.001209
cv		3.081520	1.509044	0.081520	0.006646	0.009044	0.000082
MPS		2.215823	1.544529	-0.784177	0.614934	0.044529	0.001983
AD		3.599086	1.500372	0.599086	0.358904	0.000372	0.000000
MLE	100	3.064931	1.542442	0.064931	0.004216	0.042442	0.001801
OLS		3.071232	1.543758	0.071232	0.005074	0.043758	0.001915
WLS		3.015435	1.514369	0.015435	0.000238	0.014369	0.000206
cv		3.217368	1.498033	0.217368	0.047249	-0.001967	0.000004
MPS		2.440163	1.519425	-0.559837	0.313418	0.019425	0.000377
AD		3.234412	1.485611	0.234412	0.054949	-0.014389	0.000207
MLE	300	3.027274	1.515767	0.027274	0.000744	0.015767	0.000249
OLS		3.051275	1.529172	0.051275	0.002629	0.029172	0.000851
WLS		3.016020	1.509483	0.016020	0.000257	0.009483	0.000090
cv		3.534389	1.494560	0.534389	0.285571	-0.005440	0.000030
MPS		2.687529	1.503691	-0.312471	0.097638	0.003691	0.000014
AD		3.139813	1.487234	0.139813	0.019548	-0.012766	0.000163
MLE	500	3.015758	1.511154	0.015758	0.000248	0.011154	0.000124
OLS		3.045811	1.528302	0.045811	0.002099	0.028302	0.000801
WLS		3.009473	1.507542	0.009473	0.000090	0.007542	0.000057
cv		2.962686	1.497238	-0.037314	0.001392	-0.002762	0.000008
MPS		2.764535	1.502655	-0.235465	0.055444	0.002655	0.000007
AD		3.136523	1.490340	0.136523	0.018638	-0.009660	0.000093

Table 5: reaction times (in seconds) to a certain stimulus recorded by a psychologist.

1.1, 2.1, 0.4, 3.3, 1.5, 1.3, 3.2, 2.0, 1.7, 0.6, 0.9, 1.6, 2.2, 2.6, 1.8, 0.9,
2.5, 3.0, 0.7, 1.3, 1.8, 2.9, 2.6, 1.8, 3.1, 2.6, 1.5, 1.2, 2.5, 2.8, 0.7, 2.3,
0.6, 1.8, 1.1, 2.9, 3.2, 2.8, 1.2, 2.4, 0.5, 0.7, 2.4, 1.6, 1.3, 2.8, 2.1, 1.5

For comparison, we consider the following goodness of fit criteria: -2lnL, Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov Statistic (KS-Statistic) and its p-value, where

$$AIC = -2lnL + 2k, \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$
$$BIC = -2lnL + kln(n), \qquad KS = \sup_x |F_n(x) - F_0(x)|,$$

where L is the likelihood function, k is the number of parameters, n is the sample size and $F_n(x)$ is the empirical distribution function.

Table 6: $-2lnL$, AIC, AICC, BIC, KS statistic and the p-values of the fitted distributions.

Dist	$-2ln(L)$	AIC	$CAIC$	BIC	$HQIC$	KS	pv	MLE	SE
LBLD	122.9	126.913	127.179	130.655	128.33	0.096	0.537	$\hat{\alpha}$=5.740	0.845
								$\hat{\theta}$=2.474	0.473
Loai	124.970	128.970	129.237	132.713	130.385	0.102	0.525	$\hat{\alpha}$=11.011	7.399
								$\hat{\theta}$=1.567	0.139
Exp	155.7	157.700	157.800	159.600	158.4 0	0.259	0.003	$\hat{\theta}$=0.537	0.077
T. Arad	268.4	272.400	272.500	272.200	274.30	0.112	0.245	$\hat{\theta}$=1.137	0.062
								$\hat{\lambda}$=0.930	0.081
Pran	268.3	272.254	272.403	277.116	274.21	0.110	0.260	$\hat{\alpha}$=1.476	0.052
								$\hat{\theta}$=0.986	0.068
Br	153.5	157.545	157.811	161.287	158.96	0.260	0.003	$\hat{\alpha}$=2.548	0.910
								$\hat{\theta}$=0.614	0.354
Gam	414.9	418.867	419.089	422.953	420.45	0.155	0.132	$\hat{\alpha}$=3.032	0.540
								$\hat{\theta}$=0.168	0.032
Lind	423.2	425.183	425.256	427.226	425.98	0.185	0.040	$\hat{\alpha}$=0.105	0.010

14 Conclusion

This article proposed a length biased Loai distribution (LBLD) and studied various properties of the distribution. The moments, mode, reliability analysis functions and the different methods of estimating the distribution parameters, have been examined. Applications of the new distribution have also been established with real life data. The results are compared with some distributions, showed that the LBLD provides a better fit than the other distributions.

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