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The New Family of Exponentiated Half Logistic-Type I Heavy-Tailed-G Distributions with Applications

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In this research, we present and examine the exponentiated half logistic-type I heavy-tailed-G (EHL-TIHT-G) family of distributions (Fod). Properties and characteristics of this newly proposed Fod are investigated. Moments, stochastic ordering, order statistics, and Rényi entropy were derived. To estimate the parameters of the EHL-TIHT-G distribution, we used seven estimation methods and a Monte-Carlo simulation exercise to evaluate the performance of the parameter estimates. The simulation investigation showed that the maximum likelihood estimates (MLEs) were the best for this model. We also derive actuarial risk measures and look at their numerical simulations. Finally, we demonstrate the practical usefulness of the EHL-TIHT-G Fod through two real data examples. Our results suggest that the EHL-TIHT-G distribution can serve as a flexible and valuable model for a variety of applications.

keywords: Likelihood-Ratio Order, Maximum Likelihood Estimation, Monte-Carlo Simulations, Value at Risk.

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1 Introduction

There are several methods to enhance existing probability distributions to effectively model data across different domains such as finance, engineering, and medical sciences. These techniques have been developed to address the specific needs and challenges of diverse fields, allowing for improved modeling and analysis of data in these areas. One such technique involves the use of generators, and some of the recent generators include Topp-Leone-G generator by Al-Shomrani et al. (2016), T-X approach by Alzaatreh et al. (2013), beta-G generator by Eugene et al. (2002), Half Logistic-G Fod by Cordeiro et al. (2016), exponentiated Half Logistic-G (EHL-G) Fod by Cordeiro et al. (2014), and many others.

Recently, via the T-X generator, Zhao et al. (2020) proposed the type I heavy-tailed-G (TIHT-G) distribution and demonstrated its usefulness in modeling heavy-tailed data. They used the special case of this family, the type I heavy-tailed-Weibull (TIHT-W), and compared it to Pareto and Weibull distributions, and found that it outperformed these models.

This highlights the growing importance of heavy-tailed distributions in various fields due to their ability to model extreme values and capture significant variability in data. Significant research has since been conducted on heavy-tailed distributions, including the work by Zhao et al. Zhao et al. (2021) on the Heavy-Tailed Beta-Power Transformed Weibull (HTBPT–W) distribution, the Zubair-Weibull (Z-W) distribution by Ahmad (2020), the new heavy-tailed Weibull (NEHTW) distribution by Arif et al. (2021), the Weibull-Loss (W-Loss) model Ahmad et al. (2019), and the Topp-Leone Type I Heavy-Tailed-G Power Series class of distributions explored by Nkomo et al. Nkomo et al. (2024). Additionally, Amer et al. Amer (2020) investigated the Alpha-Power Transformed Lomax distribution. The cumulative distribution function (cdf) of the TIHT-G Fod is

$$F(x;\theta,\vartheta) = 1 - K_\theta(x;\vartheta), \tag{1}$$

where $K_\theta(x;\vartheta) = \left(\frac{1-G(x;\vartheta)}{1-(1-\theta)G(x;\vartheta)}\right)^\theta$ and probability density function (pdf) is

$$f(x;\theta,\vartheta) = \frac{\theta^2 g(x;\vartheta)\left(1-G(x;\vartheta)\right)^{\theta-1}}{\left(1-(1-\theta)G(x;\vartheta)\right)^{\theta+1}}, \tag{2}$$

for $\theta, x > 0$ and parameter vector ϑ. Cordeiro et al. (2014) developed the EHL-G Fod whose cdf is given by

$$F_{EHL-G}(x;\alpha,\vartheta) = \left[\frac{G(x;\vartheta)}{1+\overline{G}(x;\vartheta)}\right]^\alpha, \tag{3}$$

and the corresponding pdf is

$$f_{EHL-G}(x;\alpha,\vartheta) = 2\alpha g(x;\vartheta)(G(x;\vartheta))^{(\alpha-1)}\left(1+\overline{G}(x;\vartheta)\right)^{-(\alpha-1)}, \tag{4}$$

where $\alpha > 0$, and $G(x;\vartheta)$ is a parent or baseline cdf, and $\overline{G}(x;\vartheta) = 1 - G(x;\vartheta)$.

The EHL-TIHT-G Fod combines the desirable features of the TIHT-G and EHL-G families, resulting in a distribution that excels in capturing the intricate nature of heavy-tailed data. This combination allows for remarkable flexibility in modeling heavy-tailed data and effectively capturing their complex characteristics. Applications involving specific instances of the EHL-TIHT-G Fod have demonstrated its superiority over several non-nested models. This performance showcases its potential to enhance modeling capabilities in various domains, including engineering, medical sciences, and finance, where heavy-tailed data commonly arise. By utilizing the EHL-TIHT-G Fod, researchers and practitioners will gain access to an innovative tool that enables them to analyze and model heavy-tailed data with greater flexibility and accuracy. Moreover, the EHL-TIHT-G Fod offers the advantage of exploring hazard rate functions of various shapes including bathtub and upside down bathtub among others. This additional feature enhances the distribution's utility in survival analysis and risk assessment. Overall, the introduction of the EHL-TIHT-G Fod represents a significant contribution to the field, providing statisticians, researchers, and practitioners with an advanced tool to tackle the challenges posed by heavy-tailed data. Its capabilities in capturing complex characteristics and its superior performance in various applications make it a valuable asset in statistical modeling and analysis.

The paper is structured as follows: In Section 2, we introduce the EHL-TIHT-G Fod and provide its linear representation and sub-families. Section 3 delves into specific cases of the EHL-TIHT-G Fod. In Section 4, we present the mathematical properties associated with this distribution. Section 5 examines various risk measures applicable to the EHL-TIHT-G Fod. Section 6 covers seven different estimation techniques and comparisons of these methods via Monte-Carlo simulations for this distribution. To demonstrate its practical usefulness, Section 7 presents two real data examples. Finally, in Section 8, we summarize the key conclusions of the paper and emphasize the importance of the EHL-TIHT-G Fod.

2 The New Distribution

We derive the exponentiated half logistic-type I heavy-tailed-G (EHL-TIHT-G) in this section. We use Π for G in this paper to reduce textual overlap. The cdf of the EHL-TIHT-Π Fod is

$$F_{EHL-TIHT-\Pi}(x;\theta,\alpha,\vartheta) = \left[\frac{1-K_{\theta}(x;\vartheta)}{1+K_{\theta}(x;\vartheta)}\right]^{\alpha}, \qquad (5)$$

and pdf is

$$
\begin{aligned}
f_{EHL-TIHT-\Pi}(x;\theta,\alpha,\vartheta) &= \frac{2\alpha\theta^2\pi(x;\vartheta)\left(1-\Pi(x;\vartheta)\right)^{\theta-1}}{\left[1-(1-\theta)\Pi(x;\vartheta)\right]^{\theta+1}} \\
&\times \frac{\left[1-K_\theta(x;\vartheta)\right]^{\alpha-1}}{\left[1+K_\theta(x;\vartheta)\right]^{\alpha+1}},
\end{aligned} \tag{6}
$$

where $K_\theta(x;\vartheta) = \left(\frac{1-\Pi(x;\vartheta)}{1-(1-\theta)\Pi(x;\vartheta)}\right)^\theta$ for $\theta, \alpha > 0$ and parameter vector ϑ. The hazard rate function (hrf) of the EHL-TIHT-Π Fod is given by

$$
\begin{aligned}
h(x;\theta,\alpha,\vartheta) &= \frac{2\alpha\theta^2\pi(x;\vartheta)\left(1-\Pi(x;\vartheta)\right)^{\theta-1}}{\left[1-(1-\theta)\Pi(x;\vartheta)\right]^{\theta+1}} \frac{\left[1-K_\theta(x;\vartheta)\right]^{\alpha-1}}{\left[1+K_\theta(x;\vartheta)\right]^{\alpha+1}} \\
&\times \frac{1}{1-\left[\frac{1-K_\theta(x;\vartheta)}{1+K_\theta(x;\vartheta)}\right]^\alpha},
\end{aligned} \tag{7}
$$

for $\alpha, \theta > 0$ and parameter vector ϑ.

2.1 Expansion of the Density Function

This section provides the linear representation of the EHL-TIHT-Π Fod. The pdf of the EHL-TIHT-Π Fod can be expressed as

$$
f_{EHL-TIHT-G}(x;\theta,\alpha,\vartheta) = \sum_{u=0}^{\infty} \omega_{u+1}\pi_{u+1}(x;\vartheta), \tag{8}
$$

where

$$
\begin{aligned}
\omega_{u+1} &= 2\alpha\theta^2 \sum_{p,q,r,s,t=0}^{\infty} (-1)^{p+s+r+t+u} \binom{\alpha-1}{p} \binom{-(\alpha+1)}{q} (1-\theta)^s \binom{r+s}{t} \binom{t}{u} \\
&\times \binom{(1+q+p)\theta-1}{r} \binom{1-((1+q+p)\theta)}{s} \frac{1}{(u+1)},
\end{aligned} \tag{9}
$$

and $\pi_{u+1}(x;\vartheta) = (u+1)\pi(x;\vartheta)\Pi^u(x;\vartheta)$ is an exponentiated-Π (Expo-Π) distribution with exponentiation parameter $(u+1)$. The EHL-TIHT-Π Fod is tractable to the Expo-Π distribution. The full expansion is shown in the appendix.

2.2 Sub-Families

Let X follows the EHL-TIHT-Π Fod with parameter vector $(\theta, \alpha, \vartheta)$.

- When $\alpha = 1$, we have the half logistic type I heavy-tailed Π (HL-TIHT-Π) Fod with cdf is

$$
F_{HL-TIHT-\Pi}(x;\theta,\vartheta) = \frac{1-K_\theta(x;\vartheta)}{1+K_\theta(x;\vartheta)}, \tag{10}
$$

  for $\theta > 0$ and baseline vector of parameters ϑ.

- When $\theta = 1$, the exponentiated half logistic-Π (EHL-Π) Fod is obtained (see Cordeiro et al. (2014) for details).

- When $\theta = \alpha = 1$, the half logistic-Π (HL-Π) Fod is obtained (see Cordeiro et al. (2016)).

3 Specified Cases

This section presents specified cases for the EHL-TIHT-Π Fod, particularly when Burr XII, Burr III, and Weibull distributions, respectively, serve as the parent distributions.

3.1 Exponentiated Half Logistic-Type I Heavy-Tailed-Burr XII (EHL-TIHT-BXII) Distribution

Using the Burr XII distribution as the baseline distribution with the cdf $\Pi(x;b,\beta) = 1-(1+x^b)^{-\beta}$ and pdf $\pi(x;b,\beta) = b\beta x^{\beta-1}(1+x^b)^{-\beta-1}$, for $b,\beta > 0$, the cdf and pdf of the new EHL-TIHT-BXII distribution are

$$F_{EHL-TIHT-BXII}(x;\alpha,b,\beta,\theta) = \left[\frac{1-K_\theta^*(x;b,\beta)}{1+K_\theta^*(x;b,\beta)}\right]^\alpha, \tag{11}$$

and

$$f_{EHL-TIHT-BXII}(x;\alpha,b,\beta,\theta) = \frac{2\alpha\theta^2 b\beta x^{\beta-1}(x^b+1)^{-\beta\theta-1}}{[1-(1-\theta)(1-(x^b+1)^{-\beta})]^{1+\theta}} \times \frac{[1-K_\theta^*(x;b,\beta)]^{\alpha-1}}{\left[1+K_\theta^*(x;b,\beta)\right]^{\alpha+1}}, \tag{12}$$

respectively, where $K_\theta^*(x;\beta,b) = \left(\frac{(x^b+1)^{-\beta}}{1-(1-\theta)(1-(x^b+1)^{-\beta})}\right)^\theta$, for $\theta,\beta,\alpha,b,> 0$ and $x > 0$, while the hrf is given by

$$h(x;\alpha,b,\beta,\theta) = \frac{2\alpha\theta^2 b\beta x^{\beta-1}\left(1+x^b\right)^{-\beta(\theta+1)}}{[1-(1-\theta)\left(1-(1+x^b)^{-\beta}\right)]^{\theta+1}} \frac{[1-K_\theta(x;\vartheta)]^{\alpha-1}}{[1+K_\theta(x;\vartheta)]^{\alpha+1}} \times \frac{1}{1-\left[\frac{1-K_\theta(x;\vartheta)}{1+K_\theta(x;\vartheta)}\right]^\alpha}, \tag{13}$$

for $\alpha,\theta > 0$ and parameter vector ϑ. From the EHL-TIHT-BXII distribution, we can obtain the exponentiated half logistic-type I heavy-tailed-Lomax and the exponentiated half logistic-type I heavy-tailed-log-logistic distributions by setting $\beta = 1$ and $b = 1$, respectively.

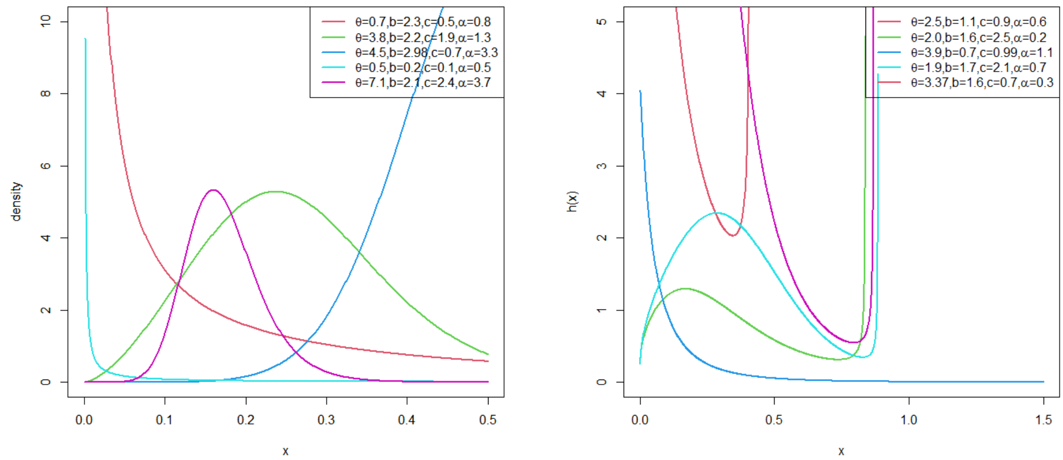


Figure 1: EHL-TIHT-BXII pdf and hrf plots

Figure 1 shows that the shape of the EHL-TIHT-BXII density can be positive or negative-skewed, reverse-J, and almost symmetrical. The hrf displays both monotonic and non-monotonic geometries.

The skewness plot indicates that skewness rises as θ and α are both increasing for

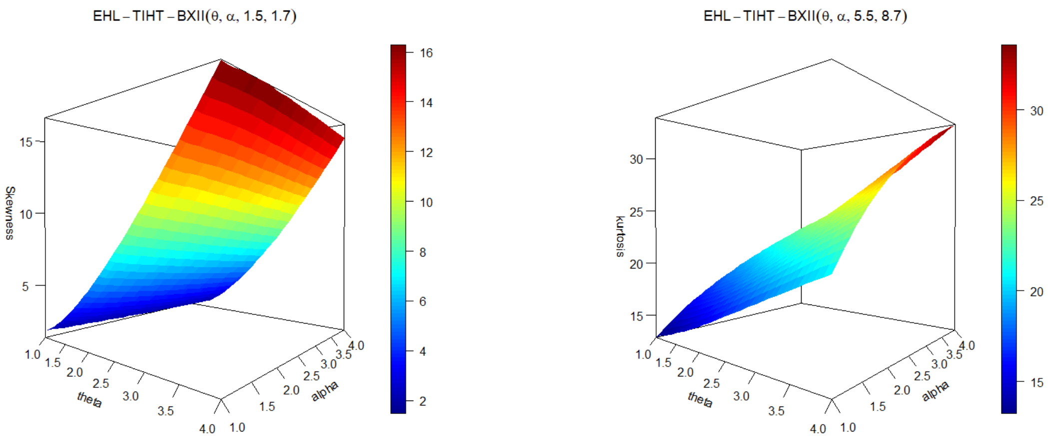


Figure 2: Skewness and kurtosis plots for EHL-TIHT-BXII distribution

$b = 1.5$ and $\beta = 1.7$. From the plot of kurtosis, we can see that kurtosis increases as α is increasing and θ is increasing for $b = 5.5$ and $\beta = 8.7$.

3.2 Exponentiated Half Logistic-Type I Heavy-Tailed-Weibull (EHL-TIHT-W) Distribution

Let the parent distribution be Weibull, with the pdf $\pi(x;\varpi) = \varpi x^{\varpi-1}e^{-x^{\varpi}}$ and cdf $\Pi(x;\varpi) = 1 - e^{-x^{\varpi}}$, for ϖ and $x > 0$, then the cdf and pdf of the EHL-TIHT-W distribution are

$$F_{EHL-TIHT-W}(x;\varpi,\alpha,\theta) = \left[\frac{1-K_{\theta}^{**}(x;\varpi)}{1+K_{\theta}^{**}(x;\varpi)}\right]^{\alpha}, \tag{14}$$

$$F_{EHL-TIHT-W}(x;\varpi,\alpha,\theta) = \left[\frac{1-\left(\frac{\exp(-x^{\varpi})}{1-(1-\theta)(1-\exp(-x^{\varpi}))}\right)^{\theta}}{1+\left(\frac{\exp(-x^{\varpi})}{1-(1-\theta)(1-\exp(-x^{\varpi}))}\right)^{\theta}}\right]^{\alpha}, \tag{15}$$

and

$$\begin{aligned} f_{EHL-TIHT-W}(x;\alpha,\theta,\varpi) &= \frac{2\alpha\theta^{2}\varpi x^{\varpi-1}\left(e^{-x^{\varpi}}\right)^{\theta}}{\left[1-(1-\theta)\left(1-\exp(-x^{\varpi})\right)\right]^{1+\theta}} \\ &\times \frac{\left[1-K_{\theta}^{**}(x;\varpi)\right]^{\alpha-1}}{\left[1+K_{\theta}^{**}(x;\varpi)\right]^{\alpha+1}}, \end{aligned} \tag{16}$$

respectively, where $K_{\theta}^{**}(x;\varpi) = \left(\frac{\exp(-x^{\varpi})}{1-(1-\theta)(1-\exp(-x^{\varpi}))}\right)^{\theta}$, for $\alpha,\varpi,\theta > 0$ and $x > 0$. The hrf is

$$\begin{aligned} h(x;\theta,\alpha,\varpi) &= \frac{2\alpha\theta^{2}\varpi x^{\varpi-1}\left(\exp(-x^{\varpi})\right)^{\theta}}{\left[1-(1-\theta)\left(1-\exp(-x^{\varpi})\right)\right]^{\theta+1}} \\ &\times \frac{\left[1-K_{\theta}^{**}(x;\varpi)\right]^{\alpha-1}}{\left[1+K_{\theta}^{**}(x;\varpi)\right]^{\alpha+1}} \frac{1}{1-\left[\frac{1-K_{\theta}^{**}(x;\varpi)}{1+K_{\theta}^{**}(x;\varpi)}\right]^{\alpha}}, \end{aligned} \tag{17}$$

for $\alpha,\varpi,\theta > 0$ and $x > 0$.

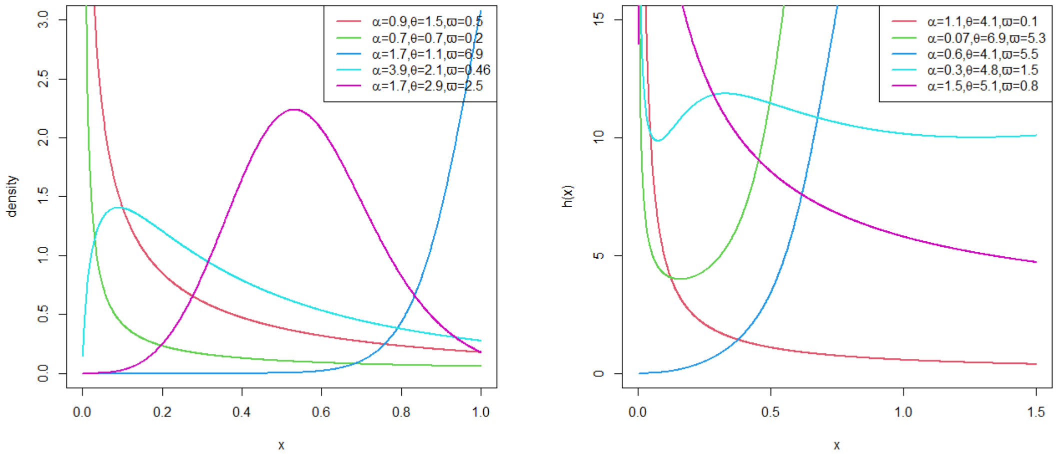


Figure 3: EHL-TIHT-W pdf and hrf plots

The shape of the distribution can be reverse-J, negative or positive-skewed. The plot of the hrf show increasing, decreasing, and bathtub, followed by an inverted bathtub geometries.

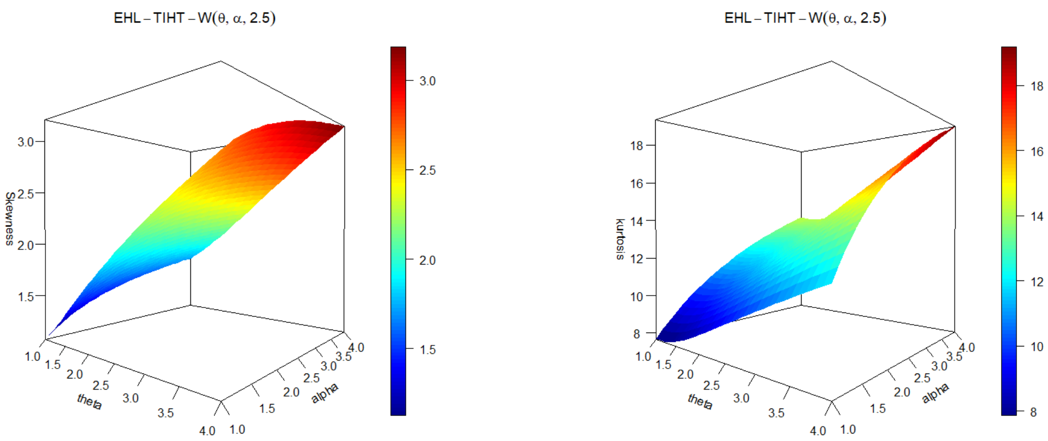


Figure 4: Skewness and kurtosis plots of the EHL-TIHT-W distribution

When α and θ increases, the skewness of the distribution also increases. Similarly, when both θ and α increase, the kurtosis of the distribution also increases, considering a fixed value of $\varpi = 2.5$.

3.3 Exponentiated Half Logistic-Type I Heavy-Tailed-Burr III (EHL-TIHT-BIII) Distribution

Using the Burr III distribution with cdf $\Pi(x;c,k) = (1+\frac{1}{x^c})^{-k}$ and pdf $\pi(x;c,k) = \frac{ckx^{-(c+1)}}{(1+\frac{1}{x^c})^{k+1}}$ for $c,k>0$ and $x>0$, as the baseline distribution, the cdf of the EHL-TIHT-BIII distribution is

$$F_{EHL-TIHT-BIII}(x;\theta,k,\alpha,c) = \left[\frac{1-B_\theta(x;k,c)}{B_\theta(x;c,k)+1}\right]^{\alpha}, \tag{18}$$

and the pdf is

$$\begin{aligned} f_{EHL-TIHT-BIII}(x;\theta,\alpha,c,k) &= \frac{2\alpha\theta^2 ckx^{-(c+1)}((1+\frac{1}{x^c})^{-(k+1)}\left(1-\left((1+\frac{1}{x^c})^{-k}\right)\right)^{\theta-1}}{\left[1-(1-\theta)\left((1+\frac{1}{x^c})^{-k}\right)\right]^{\theta+1}} \\ &\times \frac{[1-B_\theta(x;c,k)]^{\alpha-1}}{[B_\theta(x;c,k)+1]^{\alpha+1}}, \end{aligned} \tag{19}$$

where $B_\theta(x;k,c) = \left(\frac{1-\left((1+\frac{1}{x^c})^{-k}\right)}{1-(1-\theta)\left((1+\frac{1}{x^c})^{-k}\right)}\right)^{\theta}$, for $\theta,\alpha,c,k>0$.

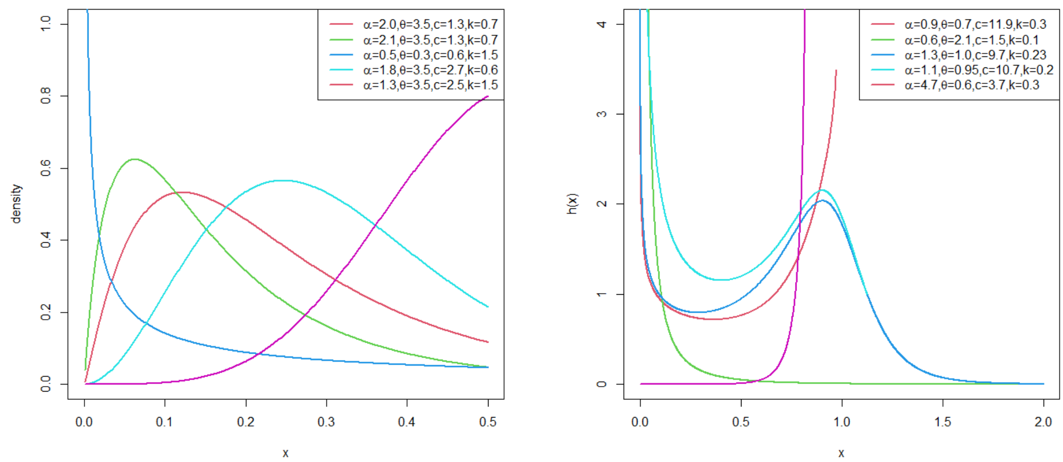


Figure 5: Plots for the EHL-TIHT-BIII distribution's pdf and hrf

Figure 5 shows that shape of the EHL-TIHT-BIII distribution density can be reverse-J, left or right-skewed and the hrf graphs display increasing, decreasing, bathtub, and bathtub followed by inverted bathtub shapes.

When α and θ increases, the skewness of the distribution also increases. Similarly, when both θ and α increase, the kurtosis of the distribution also increases, considering a fixed value of $c = 1.5$ and $k = 1.7$.

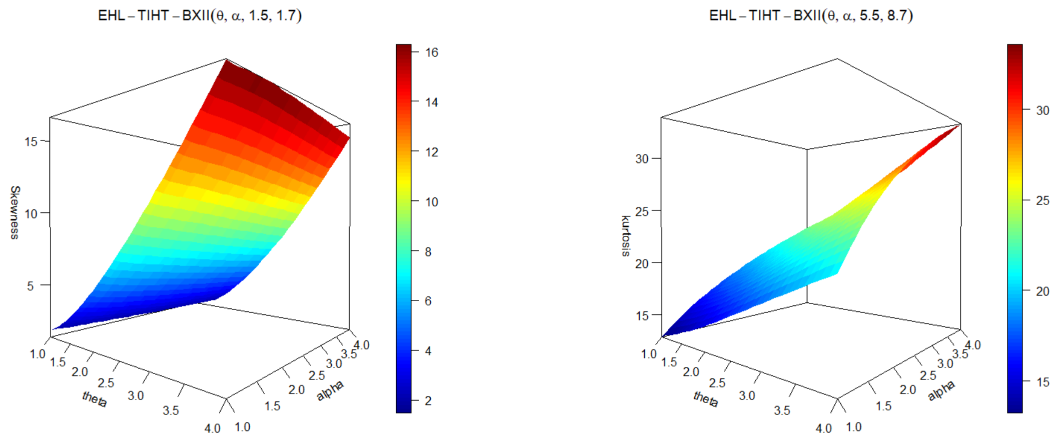


Figure 6: Skewness and kurtosis plots of the EHL-TIHT-BXII distribution

4 Some Statistical Properties

This section outlines some of the properties of the new Fod.

4.1 Quantile Function

If $Q(u)$ denotes the quantile function of the EHL-TIHT-G Fod, then it is given by

$$Q(u) = \Pi^{-1}\left(\frac{1-\left(\frac{1-u^{\frac{1}{\alpha}}}{u^{\frac{1}{\alpha}}+1}\right)^{\frac{1}{\theta}}}{\left(1-(1-\theta)\left(\frac{1-u^{\frac{1}{\alpha}}}{u^{\frac{1}{\alpha}}+1}\right)^{\frac{1}{\theta}}\right)}\right), \tag{20}$$

for specified baseline cdf $\Pi(x;\vartheta)$ and $0 < u < 1$. See appendix for details.

4.2 Moments

Consider a random variable X which follows a EHL-TIHT-Π Fod, then its r^{th} non-central moment, $E(X^r)$ is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r f_{EHL-TIHT-\Pi}(x;\theta,\alpha,\vartheta)dx. \tag{21}$$

Using Equation (8), the r^{th} moment is

$$E(X^r) = \sum_{u=0}^{\infty}\omega_{u+1}\int_{-\infty}^{\infty} x^r \pi_{u+1}(x;\vartheta)dx = \sum_{u=0}^{\infty}\omega_{u+1}E(Y_{u+1}^r), \tag{22}$$

where ω_{u+1} is defined by Equation (9) and $E(Y_{u+1}^r)$ is the r^{th} moment of the Expo-Π distribution with exponentiation parameter (u+1).

4.3 Probability Weighted Moments

Probability Weighted Moments (PWMs) is a statistical method used to estimate parameters of a probability distribution based on weighted moments. It is particularly useful when dealing with heavy-tailed or skewed distributions (see Hosking (1990)). The $(m,n)^{th}$ PWMs, say $\eta_{m,n}$ of $X\sim$ EHL-TIHT-Π Fod is

$$\eta_{m,n} = E(X^m\,(F(X))^n) = \int_{-\infty}^{\infty} x^m f(x;\theta,\alpha,\vartheta)\,(F(x;\theta,\alpha,\vartheta))^n\,dx. \tag{23}$$

Note that $f(x;\theta,\alpha,\vartheta)(F(x;\theta,\alpha,\vartheta))^n$ can be written as

$$f(x;\theta,\alpha,\vartheta)(F(x;\theta,\alpha,\vartheta))^n = \sum_{u=0}^{\infty}\omega^*_{u+1}\pi_{u+1}(x;\vartheta), \tag{24}$$

where

$$\begin{aligned}\omega^*_{u+1} &= 2\alpha\theta^2\sum_{p,r,q,s,t=0}^{\infty}(-1)^{p+r+s+t+u}\binom{\alpha(1+n)-1}{p}\binom{-(\alpha(1+n)+1)}{q}\\ &\times \binom{r+s}{t}\binom{t}{u}\binom{(1+q+p)\theta+1}{r}\binom{-((1+q+p)\theta-1)}{s}\frac{(1-\theta)^s}{u+1},\end{aligned} \tag{25}$$

and $\pi_{u+1}(x;\vartheta)$ is an Expo-Π density with (u+1) as the power parameter. Therefore, the PWMs is given by

$$\eta_{m,n} = \sum_{t=0}^{\infty}\omega^*_{u+1}\int_{-\infty}^{\infty}x^m\pi_{u+1}(x;\vartheta)dx = \sum_{u=0}^{\infty}\omega^*_{u+1}E(Y^m_{u+1}), \tag{26}$$

where $E(Y^m_{u+1})$ is m^{th} moment of an Expo-Π distributed random variable with exponentiation parameter (u+1).

Details of the derivatives are provided in the appendix.

4.4 Distribution of Order Statistics

Order statistics are crucial for summarizing data, estimating parameters, conducting hypothesis tests, and analyzing extreme events. They provide insights into the range, central tendency, and spread of data, aiding in reliability analysis and model selection. Their importance extends to fields such as finance, insurance, and environmental sciences. The pdf of the i^{th} order statistic from the EHL-TIHT-Π Fod is

$$\begin{aligned}f_{i:n}(x;\theta,\alpha,\vartheta) &= \frac{f(x;\theta,\alpha,\vartheta)}{B(i,n-i+1)}\sum_{j=0}^{n-i}\binom{n-i}{j}(F(x;\theta,\alpha,\vartheta))^{j+i-1}\\ &= \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}\sum_{u=0}^{\infty}\binom{n-i}{j}\phi_{u+1}\pi_{u+1}(x;\vartheta),\end{aligned} \tag{27}$$

where $B(.,.)$ is the beta function, $\pi_{u+1}(x;\vartheta) = (u+1)\pi(x;\vartheta)\Pi^{u}(x;\vartheta)$ is an expo-Π distribution whose power parameter is $(u+1)$, and

$$\begin{aligned}
\phi_{u+1} &= 2\alpha\theta^2 \sum_{j=0}^{n-i} \sum_{p,q,r,s,t=0}^{\infty} (-1)^{p+r+s+t+u} \binom{\alpha(j+1+i)-1}{p} \\
&\times \binom{-(\alpha(j+1+i)+1)}{q} (1-\theta)^s \binom{r+s}{t} \binom{t}{u} \binom{(1+q+p)\theta-1}{r} \\
&\times \binom{-((1+q+p)\theta+1)}{s} \frac{1}{u+1}. \qquad (28)
\end{aligned}$$

The full expansion is found in the appendix.

4.5 Entropy

Rényi entropy Rényi (1961) is an extension of the Shannon entropy Shannon (1951). These measures offer insights into the information content and uncertainty within the EHL-TIHT-Π Fod. Rényi entropy for the new Fod is given as

$$\begin{aligned}
I_R(\varsigma) &= (1-\varsigma)^{-1} \log \left[\int_0^{\infty} f^{\varsigma}(x) dx \right] \\
&= (1-\varsigma)^{-1} \log \left[\sum_{u=0}^{\infty} v_u^* e^{(1-\varsigma) I_{RE\Pi}} \right], \varsigma \neq 1, \varsigma > 0, \qquad (29)
\end{aligned}$$

where

$$\begin{aligned}
v_u^* &= \sum_{p,q,r,s,t=0}^{\infty} (-1)^{p+r+s} \binom{\varsigma(\alpha-1)}{p} \binom{-\varsigma(1+\alpha)}{q} \binom{(\theta(\varsigma+q+p)-\varsigma)}{r} \\
&\times \binom{-(\theta(\varsigma+q+p)+\varsigma)}{s} (1-\theta)^s \binom{r+s}{t} \binom{t}{u} 2\alpha^{\varsigma}\theta^{2\varsigma} \left(1+\frac{u}{\varsigma}\right)^{\varsigma} \qquad (30)
\end{aligned}$$

and

$$I_{RE\Pi} = (1-\varsigma)^{-1} \log \int_0^{\infty} \left[\left(\frac{u}{\varsigma}+1\right) \pi(x;\vartheta) \Pi^{\frac{u}{\varsigma}}(x;\vartheta) \right]^{\varsigma} dx \qquad (31)$$

is Rényi entropy of the Expo-Π distribution whose power parameter is $\left(\frac{u}{\varsigma}+1\right)$. See the appendix for details.

4.6 Stochastic Ordering

We present stochastic ordering of the EHL-TIHT-Π Fod in this subsection. Yu (2009) describes stochastic ordering as an ordering of how big random variables or random vectors are. It is useful in fields such as life sciences, economics, and insurance. Stochastic ordering is crucial in determining how well a model performs. There are several stochastic orders, namely, hazard, likelihood ratio, variability, convex, Laplace transform and realizable monotonicity orders to mention a few. See Yu (2009) and references there in. In this paper, we present the likelihood ratio order.

If X_1 and X_2 are continuous random variables whose pdfs are defined by $f_1(x)$ and $f_2(x)$, respectively, then X_2 is greater than X_1 according to the likelihood-ratio ordering if $\frac{f_1(x)}{f_2(x)}$ is non-decreasing in x.

Let $f_1(x)$ and $f_2(x)$ be pdfs given by

$$\begin{aligned} f_1(x) = f_{EHL-TIHT-\Pi}(x;\theta,\alpha_1,\vartheta) &= \frac{2\alpha_1\theta^2\pi(x;\vartheta)\left(1-\Pi(x;\vartheta)\right)^{\theta-1}}{\left(1-(1-\theta)\Pi(x;\vartheta)\right)^{1+\theta}} \\ &\times \frac{[1-K_\theta(x;\vartheta)]^{\alpha_1-1}}{[1+K_\theta(x;\vartheta)]^{1+\alpha_1}}, \end{aligned} \tag{32}$$

and

$$\begin{aligned} f_2(x) = f_{EHL-TIHT-\Pi}(x;\theta,\alpha_2,\vartheta) &= \frac{2\alpha_2\theta^2\pi(x;\vartheta)\left(1-\Pi(x;\vartheta)\right)^{\theta-1}}{\left(1-(1-\theta)\Pi(x;\vartheta)\right)^{1+\theta}} \\ &\times \frac{[1-K_\theta(x;\vartheta)]^{\alpha_2-1}}{[1+K_\theta(x;\vartheta)]^{1+\alpha_2}}, \end{aligned} \tag{33}$$

respectively for $\alpha_1,\alpha_2,\theta > 0$, $x > 0$, and parameter vector ϑ, where $K_\theta(x;\vartheta) = \left(\frac{1-\Pi(x;\vartheta)}{1-(1-\theta)\Pi(x;\vartheta)}\right)^\theta$. Then the ratio of $\frac{f_1(x)}{f_2(x)}$ is given by

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha_1}{\alpha_2}\frac{[1+K_\theta(x;\vartheta)]^{\alpha_1-\alpha_2}}{[1-K_\theta(x;\vartheta)]^{\alpha_1-\alpha_2}}. \tag{34}$$

Differentiating Equation (34) with respect to x, we have

$$\frac{d}{dx}\left[\frac{\alpha_1}{\alpha_2}\frac{[1-K_\theta(x;\vartheta)]^{\alpha_1-\alpha_2}}{[1+K_\theta(x;\vartheta)]^{\alpha_1-\alpha_2}}\right] = \frac{2\alpha_1(\alpha_1-\alpha_2)K_\theta'(x;\vartheta)\,[1+K_\theta(x;\vartheta)]^{\alpha_1-\alpha_2-1}}{\alpha_2\left(1-K_\theta(x;\vartheta)\right)^{\alpha_1-\alpha_2+1}}. \tag{35}$$

If $\alpha_1 < \alpha_2$, then $\frac{\partial}{\partial x}\left(\frac{f_1(x)}{f_2(x)}\right) < 0$. Therefore, likelihood ratio order exists between X_1 and X_2, which implies that X_1 and X_2 are stochastically ordered.

5 Actuarial Measures

We address certain risk metrics, including the value at risk (VaR), tail value at risk (TVaR), tail variance (TV), and tail variance premium (TVP) for the proposed Fod in this section.

5.1 Value at Risk

VaR provides an estimate of the maximum potential loss that could be incurred with a certain level of confidence, indicating the risk exposure of a portfolio or investment. The VaR of X for the proposed Fod, denoted by $VaR_q(X)$, is

$$VaR_q = \Pi^{-1}\left(\frac{1-\left(\frac{1-q^{\frac{1}{\alpha}}}{1+q^{\frac{1}{\alpha}}}\right)^{\frac{1}{\theta}}}{\left((1-\theta)\left(\frac{1-q^{\frac{1}{\alpha}}}{1+q^{\frac{1}{\alpha}}}\right)^{\frac{1}{\theta}}-1\right)}\right), \tag{36}$$

for $0 < q < 1$, and $\alpha, \theta > 0$.

5.2 Tail Value at Risk

If X follows the EHL-TIHT-Π Fod with parameter vector $(\theta,\ \alpha,\ \vartheta)$, then TVaR of X can be derived as

$$\begin{aligned} TVaR_q(X) &= E(X|X > VaR_q) \\ &= \frac{1}{1-q}\int_{VaR_q}^{\infty} x f_{EHL-TIHT-\Pi}(x;\theta,\alpha,\vartheta)dx \\ &= \sum_{u=0}^{\infty}\frac{\omega_{u+1}}{1-q}\int_{VaR_q}^{\infty} x\pi_{u+1}(x;\vartheta)dx, \end{aligned} \tag{37}$$

where ω_{u+1} is given by Equation(9) and $0 < q < 1$.

5.3 Tail Variance

In finance and risk management, TV is often used as a risk measure to assess the potential losses associated with outliers. It is given by

$$TV_q = E(X^2|X > x_q) - (TVaR_q)^2. \tag{38}$$

Note that

$$E(X^2|X > x_q) = \frac{1}{1-q}\int_{VaR_q}^{\infty} x^2 f_{EHL-T1HT-\Pi}(x;\theta,\alpha,\vartheta)dx, \tag{39}$$

but $f_{EHL-T1HT-\Pi}(x;\theta,\alpha,\vartheta) = \sum_{u=0}^{\infty}\omega_{u+1}\pi_{u+1}(x;\vartheta)$, so that

$$E(X^2|X > x_q) = \sum_{u=0}^{\infty}\frac{\omega_{u+1}}{1-q}\int_{VaR_q}^{\infty} x^2\pi_{u+1}(x;\vartheta)dx, \tag{40}$$

where ω_{u+1} is given by Equation(9), and $0 < q < 1$, so that

$$TV_q = \sum_{u=0}^{\infty}\frac{\omega_{u+1}}{1-q}\int_{VaR_q}^{\infty} x^2\pi_{u+1}(x;\vartheta)dx - (TVaR_q)^2\,. \tag{41}$$

5.4 Tail Variance Premium

TVP represents the extra amount that individuals or organizations need to pay to obtain protection against losses resulting from rare and severe events. TVP builds on TVaR, and also depends on TV. TVP is given by

$$TVP_q = TVaR_q + \delta(TV_q), \tag{42}$$

where $0 < \delta < 1$.

5.5 Investigation of Risk Measures

We seek to compare the values of the VaR, TVaR, TV, and TVP of the EHL-TIHT-W distribution to those of the TIHT-W distribution, and the alpha power Weibull (APW) distribution in this section. The pdfs of the TIHT-W anf APW distributions are given in the appendix. To obtain the results in Table 1, we did the following:

- Samples of size n = 100 from the EHL-TIHT-W, TIHT-W, EHL-W, and APW distributions were generated.
- The calculation of the risk metrics for these distributions was done 1000 times.
- Results are shown in Table 1.

The distribution with largest values of the risk metrics is said to have the heaviest tail (Ahmad et al. (2018)). The results from Table 1 shows that the EHL-TIHT-W distribution has heavier tail as compared to TIHT-W and APW distributions. Table 1 shows that the EHL-TIHT-W distribution has the highest risk measures than the TIHT-W and APW distributions when evaluated at similar parameter settings. This indicates that the EHL-TIHT-W distribution is more capable of capturing extreme risk, making it particularly suitable for scenarios in which tail risks are critical. It also out performs the two nested models: type I heavy-tailed-Weibull (TIHT-W) by Zhao et al. (2020) and the exponentiated half logistic-Weibull by Cordeiro et al. (2014).

6 Estimation Techniques

The section aims to provide several statistical methods for estimating the parameters of the EHL-TIHT-Π Fod. These techniques play a crucial role in estimating unknown parameters from observed data, enabling us to make informed inferences and predictions. The estimation techniques considered include maximum likelihood estimates (MLE), ordinary least squares estimates (OLSE), Cramér-von Mises estimates (CVME), Anderson-Darling estimates (ADE), weighted least squares estimates (WLSE), right-tailed Anderson-Darling estimates (RADE), and maximum product of spacing estimates (MPSE).

Table 1: Simulation Study of the Risk Metrics for EHL-TIHT-W Distribution

Distribution	Parameters	Level of Significance	VaR	TVaR	TV	TVP
		0.60	13.5332	26.6657	182.7354	136.3070
	$\theta = 0.2$	0.65	15.2703	28.4204	184.1739	148.1334
		0.70	17.2678	30.4498	185.9850	160.6393
		0.75	19.6241	32.8576	188.3032	174.0850
EHL-TIHT-W	$\alpha = 0.9$	0.80	22.5058	35.8192	191.3517	188.9006
		0.85	26.2293	39.6666	195.5434	205.8785
		0.90	31.5155	45.1551	201.7902	226.7662
		0.95	40.7166	54.7408	212.9284	257.0228
	$\omega = 0.8$	0.99	63.1156	78.0524	238.8445	314.5085
		0.60	11.2409	21.3368	106.7655	85.3961
		0.65	12.5921	22.6846	107.4645	92.5365
		0.70	14.1388	24.2409	108.3869	100.1118
		0.75	15.9566	26.0849	109.6073	108.2904
TIHT-W	$\theta = 0.2$	0.80	18.1728	28.3501	111.2505	117.3506
		0.85	21.0290	31.2895	113.5495	127.8066
		0.90	25.0748	35.4779	117.0201	140.7960
		0.95	32.1014	42.7840	123.2699	159.8904
	$\omega = 0.8$	0.99	49.1657	60.5242	137.8915	197.0368
		0.60	1.3705	3.0516	3.1746	4.9564
	$\theta = 0.2$	0.65	1.5887	3.2765	3.2130	5.3650
		0.70	1.8411	3.5374	3.2591	5.8187
		0.75	2.1404	3.8476	3.3157	6.3344
EHL-W		0.80	2.5082	4.2302	3.3876	6.9403
		0.85	2.9856	4.7285	3.4838	7.6897
		0.90	3.6667	5.4417	3.6239	8.7032
		0.95	4.8589	6.6920	3.8698	10.3683
	$\omega = 0.8$	0.99	7.7844	9.7511	4.4381	14.1449
		0.60	9.0679	12.2917	157.7922	106.9670
	$\theta = 0.2$	0.65	10.1689	13.4758	165.9039	121.3133
		0.70	11.4509	14.9070	174.3883	136.9787
		0.75	12.9833	16.6779	182.9818	153.9142
TIHT-W	$\alpha = 0.9$	0.80	14.8820	18.9410	191.0944	171.8165
		0.85	17.3650	21.9741	197.4082	189.7711
		0.90	20.9247	26.3782	198.6587	205.2796
		0.95	20.9247	26.3782	198.7797	205.2799
	$\omega = 0.8$	0.99	42.1307	51.0171	198.7797	205.2799
		0.60	1.0398	1.4215	0.1014	1.4823
	$\beta = 2.2$	0.65	1.1054	1.4714	0.0957	1.5336
		0.70	1.1760	1.5266	0.0901	1.5896
		0.75	1.2540	1.5890	0.0843	1.6522
APW	$\alpha = 0.9$	0.80	1.3430	1.6620	0.0783	1.7246
		0.85	1.4495	1.7511	0.0717	1.8121
		0.90	1.5876	1.8692	0.0643	1.9271
		0.95	1.7997	2.0548	0.0545	2.1066
	$\lambda = 0.8$	0.99	2.2158	2.4040	0.0914	2.4945

6.1 Maximum Likelihood Estimation

Maximum likelihood estimation is a widely used technique that maximizes the likelihood of observing the given data to determine the optimal parameter values.

The log-likelihood function, $\ell = \ell(\Delta)$ based on a random sample of size n is:

$$\begin{aligned}
\ell &= 2n\log(\theta) + n\log(\alpha) + \sum_{i=1}^{n}\log[\pi(x_i;\vartheta)] + (\theta-1)\sum_{i=1}^{n}\log[1-\Pi(x_i;\vartheta)] \\
&- (\theta+1)\sum_{i=1}^{n}\log\left[1-(1-\theta)\Pi(x_i;\vartheta)\right] \\
&+ (\alpha-1)\sum_{i=1}^{n}\log\left[1-\left(\frac{1-\Pi(x_i;\vartheta)}{1-(1-\theta)\Pi(x_i;\vartheta)}\right)^{\theta}\right] \\
&- (\alpha+1)\sum_{i=1}^{n}\log\left[1+\left(\frac{1-\Pi(x_i;\vartheta)}{1-(1-\theta)\Pi(x_i;\vartheta)}\right)^{\theta}\right]. \qquad (43)
\end{aligned}$$

Solving the following non-linear equations: $\frac{\partial\ell}{\partial\alpha}=0$, $\frac{\partial\ell}{\partial\theta}=0$, and $\frac{\partial\ell}{\partial\vartheta_k}=0$, where $\frac{\partial\ell}{\partial\alpha}$, $\frac{\partial\ell}{\partial\theta}$, and $\frac{\partial\ell}{\partial\vartheta_k}$ are the partial derivatives of ℓ, results in the MLE of the model parameters. Elements of the score vector are given in the appendix.

6.2 Maximum Product of Spacing Estimation

Let $x_{1:n} \leq x_{2:n} \leq x_{3:n} \leq x_{4:n}\ldots\ldots \leq x_{n:n}$ be order statistics of a random sample of size n from the EHL-TIHT-G distribution. Then the MPSE of the parameters of the EHL-TIHT-Π Fod is obtained by maximising

$$MPS(\alpha,\theta,\vartheta) = \frac{1}{1+n}\sum_{i=1}^{1+n}\ln\left[F\left(x_i;\alpha,\theta,\vartheta\right) - F\left(x_{i-1};\alpha,\theta,\vartheta\right)(1+n)\right]. \qquad (44)$$

We find the MPSE by solving the non-linear equations

$$\left[\frac{\partial(MPS(\alpha,\theta,\vartheta))}{\partial\alpha}, \frac{\partial(MPS(\alpha,\theta,\vartheta))}{\partial\theta}, \frac{\partial(MPS(\alpha,\theta,\vartheta))}{\partial\vartheta_k}\right]^T = \mathbf{0}, \qquad (45)$$

where $\frac{\partial(MPS(\alpha,\theta,\vartheta))}{\partial\alpha}$, $\frac{\partial(MPS(\alpha,\theta,\vartheta))}{\partial\theta}$, and $\frac{\partial(MPS(\alpha,\theta,\vartheta))}{\partial\vartheta_k}$ are the partial derivatives of $MPS(\alpha,\theta,\vartheta)$ with respect to the parameters α, θ, and ϑ_k.

6.3 Anderson-Darling Estimation and Right-Tailed Anderson-Darling Estimation

The Anderson-Darling estimates (ADE), (see Raschke (2017)) are defined by

$$\begin{aligned} ADE(\alpha,\theta,\vartheta) &= -n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(F(x_{i:n};\alpha,\theta,\vartheta)\right)+\log\left(1-F(x_{i:n};\alpha,\theta,\vartheta)\right)\right] \\ &= -n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(\left[\frac{1-K_\theta(x_{i:n};\vartheta)}{1+K_\theta(x_{i:n};\vartheta)}\right]^{\alpha}\right)\right] \\ &- \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(1-\left(\left[\frac{1-K_\theta(x_{i:n};\vartheta)}{1+K_\theta(x_{i:n};\vartheta)}\right]^{\alpha}\right)\right)\right]. \end{aligned} \quad (46)$$

We obtain the ADE by solving the non-linear equation

$$\left[\frac{\partial(ADE(\alpha,\theta,\vartheta))}{\partial\alpha},\frac{\partial(ADE(\alpha,\theta,\vartheta))}{\partial\theta},\frac{\partial(ADE(\alpha,\theta,\vartheta))}{\partial\vartheta_k}\right]^T = \mathbf{0}, \quad (47)$$

where $\frac{\partial(ADE(\alpha,\theta,\vartheta))}{\partial\alpha}$, $\frac{\partial(ADE)(\alpha,\theta,\vartheta)}{\partial\theta}$, and $\frac{\partial(ADE(\alpha,\theta,\vartheta))}{\partial\vartheta_k}$ are the partial derivatives of $ADE(\alpha,\theta,\vartheta)$ with respect to the parameters α, θ, and ϑ_k. The RADE are obtained by minimizing the function:

$$\begin{aligned} RADE(\alpha,\theta,\vartheta) &= \frac{n}{2}-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(1-F(x_{i:n};\alpha,\theta,\vartheta)\right)\right] \\ &- 2\sum_{i=1}^{n}\log\left(1-F(x_{i:n};\alpha,\theta,\vartheta)\right), \end{aligned} \quad (48)$$

with respect to parameters α, θ, and ϑ_k.

6.4 Ordinary Least Squares Estimation

The ordinary least squares estimation (OLSE) of the parameters of the EHL-TIHT-Π Fod is obtained by minimizing the function:

$$OLSE(\alpha,\theta,\vartheta)=\sum_{i=1}^{n}\left[F\left(x_{i:n};\alpha,\theta,\vartheta\right)-\frac{i}{n+1}\right]^2, \quad (49)$$

with respect to parameters α, θ, and ϑ_k.

6.5 Weighted Least Squares Estimation

The weighted least squares estimation (WLSE) of the parameters of the EHL-TIHT-Π Fod is obtained by minimizing following function:

$$WLSE(\alpha,\theta,\vartheta)=\sum_{i=1}^{n}\frac{(1+n)^2(n+2)}{i(n-1+1)}\left[F\left(x_{i:n};\alpha,\theta,\vartheta\right)-\frac{i}{1+n}\right]^2. \quad (50)$$

with respect to α,θ, and ϑ.

6.6 Cramér-von Mises Estimation

Cramér-von Mises estimation (CVME) of the parameters for the EHL-TIHT-Π Fod can be obtained through the minimization of the function:

$$CVME\left(\alpha,\theta,\vartheta\right) = -\frac{1}{12n}\sum_{i=1}^{n}\left[F\left(x_{i:n};\alpha,\theta,\vartheta\right) - \frac{2i-1}{2n}\right]^2, \tag{51}$$

with respect to the parameters α, θ, and ϑ_k.

6.7 Simulation Study

This subsection presents results of the simulation study for various sample sizes. Tables 2, 3, 4, and 5 show the simulation results. The tables give the average bias (AvBias) and root mean squared errors (RtMse) for the different estimation methods. The ranks are given in the brackets. The AvBias and RtMse for the estimated parameter, say, $\hat{\varpi}$, are given by:

$$AvBias(\hat{\varpi}) = \frac{\sum_{i=1}^{N}\hat{\varpi}_i}{N} - \varpi, \quad \text{and} \quad RtMse(\hat{\varpi}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\varpi}_i - \varpi)^2}{N}}, \tag{52}$$

respectively.

Table 2: Simulation Results for $(\alpha,\theta,\omega) = (0.4, 1.2, 1.6)$

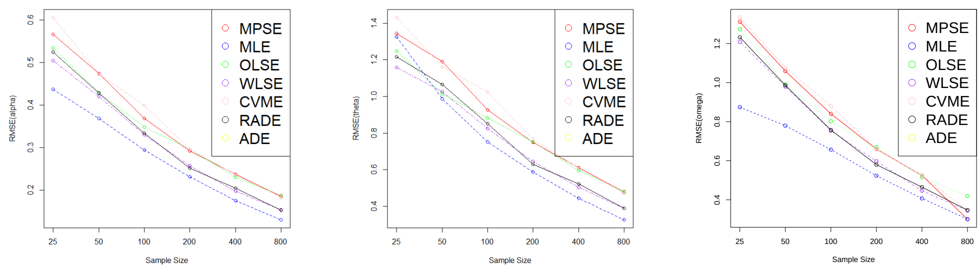
		MPSE		MLE		OLSE		WLSE	
Parameter		RtMse	AvBias	RtMse	AvBias	RtMse	AvBias	RtMse	AvBias
α	25	8.1941$_{(5)}$	1.6502$_{(6)}$	0.1712$_{(1)}$	0.0252$_{(1)}$	6.1794$_{(2)}$	1.2977$_{(4)}$	6.8158$_{(4)}$	1.1312$_{(3)}$
θ	25	0.5096$_{(5)}$	0.1811$_{(2)}$	0.3792$_{(1)}$	0.1592$_{(1)}$	0.5093$_{(4)}$	0.2395$_{(5)}$	0.4707$_{(3)}$	0.2723$_{(7)}$
ω	25	3.1462$_{(7)}$	2.9393$_{(7)}$	0.8668$_{(1)}$	0.0313$_{(1)}$	2.8693$_{(6)}$	2.7028$_{(3)}$	3.0032$_{(4)}$	2.8585$_{(6)}$
$\sum$ ranks			32		6		24		27
α	50	0.8195$_{(2)}$	0.4334$_{(3)}$	0.1549$_{(1)}$	0.0190$_{(1)}$	5.5171$_{(7)}$	0.8075$_{(7)}$	1.4757$_{(4)}$	0.4691$_{(4)}$
θ	50	0.3059$_{(1)}$	0.3067$_{(5)}$	0.3170$_{(2)}$	0.0936$_{(1)}$	0.4043$_{(6)}$	0.2700$_{(3)}$	0.3419$_{(4)}$	0.3120$_{(6)}$
ω	50	2.9926$_{(6)}$	2.9659$_{(5)}$	0.7836$_{(2)}$	0.0055$_{(1)}$	2.7297$_{(5)}$	2.6745$_{(2)}$	2.8046$_{(3)}$	2.7995$_{(6)}$
$\sum ranks$			22		8		30		27
α	100	0.3836$_{(2)}$	0.2563$_{(2)}$	0.1340$_{(1)}$	0.0153$_{(1)}$	0.5235$_{(5)}$	0.3217$_{(5)}$	0.4317$_{(4)}$	0.2712$_{(4)}$
θ	100	0.2066$_{(1)}$	0.3813$_{(7)}$	0.2606$_{(5)}$	0.0547$_{(1)}$	0.2685$_{(6)}$	0.3361$_{(3)}$	0.2292$_{(3)}$	0.3621$_{(6)}$
ω	100	2.9064$_{(5)}$	3.0245$_{(7)}$	0.6630$_{(3)}$	0.0128$_{(1)}$	2.6842$_{(4)}$	2.7511$_{(3)}$	2.6969$_{(1)}$	2.8212$_{(6)}$
$\sum ranks$			24		12		26		25
α	200	0.2799$_{(2)}$	0.2155$_{(2)}$	0.1107$_{(1)}$	0.0187$_{(1)}$	0.3648$_{(6)}$	0.2652$_{(6)}$	0.3192$_{(4)}$	0.2394$_{(5)}$
θ	200	0.1478$_{(1)}$	0.3960$_{(7)}$	0.2170$_{(7)}$	0.0500$_{(1)}$	0.1974$_{(5)}$	0.3513$_{(3)}$	0.1663$_{(3)}$	0.3690$_{(4)}$
ω	200	2.7170$_{(7)}$	2.9238$_{(7)}$	0.5480$_{(1)}$	0.0378$_{(1)}$	2.5284$_{(3)}$	2.6891$_{(3)}$	2.5346$_{(5)}$	2.7421$_{(4)}$
$\sum ranks$			26		12		26		25
α	400	0.2095$_{(2)}$	0.1743$_{(2)}$	0.0954$_{(1)}$	0.0179$_{(1)}$	0.3111$_{(7)}$	0.2550$_{(7)}$	0.2460$_{(3)}$	0.2100$_{(4)}$
θ	400	0.1103$_{(1)}$	0.4215$_{(7)}$	0.1953$_{(7)}$	0.0424$_{(1)}$	0.1501$_{(6)}$	0.3526$_{(2)}$	0.1106$_{(2)}$	0.3858$_{(5)}$
ω	400	2.6800$_{(7)}$	2.9338$_{(7)}$	0.4627$_{(1)}$	0.0473$_{(1)}$	2.3979$_{(2)}$	2.6279$_{(2)}$	2.4771$_{(4)}$	2.7373$_{(4)}$
$\sum ranks$			26		12		26		22
α	800	0.1850$_{(2)}$	0.1668$_{(2)}$	0.0743$_{(1)}$	0.0127$_{(1)}$	0.2736$_{(7)}$	0.2442$_{(6)}$	0.2226$_{(4)}$	0.2030$_{(4)}$
θ	800	0.0783$_{(1)}$	0.4241$_{(7)}$	0.1457$_{(7)}$	0.0243$_{(1)}$	0.1125$_{(6)}$	0.3543$_{(2)}$	0.0817$_{(2)}$	0.3888$_{(5)}$
ω	800	2.6182$_{(7)}$	2.8980$_{(7)}$	0.3542$_{(1)}$	0.0320$_{(1)}$	2.3255$_{(2)}$	2.5914$_{(2)}$	2.4461$_{(5)}$	2.7255$_{(5)}$
$\sum ranks$			26		12		25		25

The numbers in brackets represent the ranks.

Table 3: Simulation Results for $(\alpha, \theta, \omega) = (0.4, 1.2, 1.6)$

		CVME		RTADE		ADE	
parameter	n	RtMse	AvBias	RtMse	AvBias	RtMse	AvBias
α	25	$8.4745_{(7)}$	$1.6001_{(5)}$	$8.3113_{(6)}$	$1.8082_{(7)}$	$6.7287_{(3)}$	$1.0484_{(2)}$
θ	25	$0.5213_{(2)}$	$0.2205_{(4)}$	$0.5302_{(7)}$	$0.1886_{(3)}$	$0.4432_{(2)}$	$0.2595_{(6)}$
ω	25	$2.8887_{(5)}$	$2.7551_{(4)}$	$2.7756_{(2)}$	$2.6720_{(2)}$	$2.8695_{(3)}$	$2.7907_{(5)}$
$\sum ranks$			31		27		21
α	50	$2.7480_{(6)}$	$0.7536_{(6)}$	$1.4911_{(5)}$	$0.5632_{(5)}$	$1.0720_{(3)}$	$0.4259_{(2)}$
θ	50	$0.4153_{(7)}$	$0.2588_{(2)}$	$0.3656_{(5)}$	$0.3010_{(4)}$	$0.3214_{(3)}$	$0.3155_{(7)}$
ω	50	$2.7575_{(4)}$	$2.7084_{(3)}$	$2.8077_{(6)}$	$2.8285_{(7)}$	$2.7496_{(1)}$	$2.7910_{(4)}$
$\sum ranks$			28		32		20
α	100	$0.5970_{(7)}$	$0.3564_{(7)}$	$0.5541_{(6)}$	$0.3227_{(6)}$	$0.3943_{(3)}$	$0.2703_{(3)}$
θ	100	$0.2758_{(7)}$	$0.3228_{(2)}$	$0.2597_{(4)}$	$0.3471_{(4)}$	$0.2126_{(2)}$	$0.3595_{(5)}$
ω	100	$2.6483_{(2)}$	$2.7285_{(2)}$	$2.7045_{(5)}$	$2.8202_{(5)}$	$2.6638_{(6)}$	$2.7996_{(4)}$
$\sum ranks$			28		30		23
α	200	$0.4036_{(7)}$	$0.2935_{(7)}$	$0.3366_{(5)}$	$0.2243_{(3)}$	$0.2974_{(3)}$	$0.2302_{(4)}$
θ	200	$0.2042_{(6)}$	$0.3362_{(2)}$	$0.1853_{(4)}$	$0.3926_{(6)}$	$0.1577_{(2)}$	$0.3724_{(5)}$
ω	200	$2.4964_{(4)}$	$2.660_{(2)}$	$2.7097_{(6)}$	$2.8890_{(6)}$	$2.5302_{(5)}$	$2.7431_{(5)}$
$\sum ranks$			28		30		24
α	400	$0.3076_{(6)}$	$0.2529_{(6)}$	$0.2588_{(5)}$	$0.1988_{(3)}$	$0.2478_{(4)}$	$0.2104_{(5)}$
θ	400	$0.1460_{(5)}$	$0.3542_{(3)}$	$0.1376_{(4)}$	$0.3993_{(6)}$	$0.1141_{(3)}$	$0.3853_{(4)}$
ω	400	$2.4097_{(3)}$	$2.6406_{(3)}$	$2.5977_{(6)}$	$2.8315_{(6)}$	$2.4801_{(5)}$	$2.7376_{(5)}$
$\sum ranks$			26		29		26
α	800	$0.2688_{(6)}$	$0.2417_{(1)}$	$0.2037_{(3)}$	$0.1710_{(3)}$	$0.2328_{(5)}$	$0.2120_{(5)}$
θ	800	$0.1069_{(5)}$	$0.3571_{(3)}$	$0.0993_{(4)}$	$0.4181_{(6)}$	$0.0865_{(3)}$	$0.3806_{(4)}$
ω	800	$2.3349_{(3)}$	$2.6051_{(3)}$	$2.5908_{(6)}$	$2.8566_{(6)}$	$2.4192_{(4)}$	$2.6949_{(4)}$
$\sum ranks$			21		28		25

The numbers in brackets represent the ranks.



(i) **(ii)** **(iii)**

Figure 7: Simulations Results for $(\alpha, \theta, \omega) = (0.8, 0.4, 3.2)$

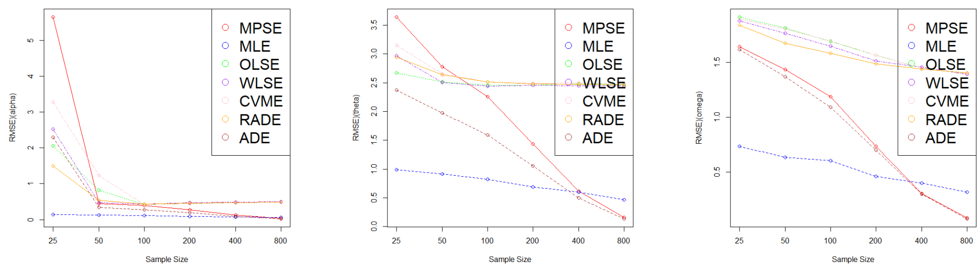
In Table 2, Table 3, Table 4, and Table 5, the row indicating $\sum$ ranks corresponds to the partial sum of the ranks. Among all the estimators for a given metric, the subscript indicates the rank. Table 1 presents, for example, the RtMse of $\hat{\alpha}$, obtained via the MPSE method, as $8.1941_{(5)}$ for $n = 25$. This indicates that the RtMse of $\hat{\alpha}$ obtained using the MPSE method ranks fifth among all other estimators.

Table 4: Simulations Results for $(\alpha, \theta, \omega) = (0.8, 0.4, 3.2)$

		MPSE		MLE		OLSE		WLSE	
parameter	n	RtMse	AvBias	RtMse	AvBias	RtMse	AvBias	RtMse	AvBias
α	25	$5.6513_{(7)}$	$1.1265_{(7)}$	$0.1389_{(1)}$	$0.0072_{(4)}$	$2.0629_{(3)}$	$0.1714_{(5)}$	$2.5294_{(5)}$	$0.0686_{(3)}$
θ	25	$3.6404_{(7)}$	$3.3278_{(7)}$	$0.9853_{(1)}$	$0.2912_{(1)}$	$2.6704_{(3)}$	$2.5185_{(4)}$	$2.9729_{(5)}$	$2.5101_{(3)}$
ω	25	$1.6432_{(3)}$	$1.2862_{(2)}$	$0.7352_{(1)}$	$0.0876_{(1)}$	$1.9109_{(7)}$	$1.6962_{(7)}$	$1.8746_{(4)}$	$1.6634_{(6)}$
$\sum ranks$			33		9		29		26
α	50	$0.4492_{(3)}$	$0.3275_{(7)}$	$0.1214_{(1)}$	$0.0045_{(1)}$	$0.8202_{(6)}$	$0.2006_{(4)}$	$0.4788_{(4)}$	$0.2999_{(5)}$
θ	50	$2.7821_{(7)}$	$2.5819_{(7)}$	$0.9157_{(1)}$	$0.2390_{(1)}$	$2.5142_{(4)}$	$2.4539_{(4)}$	$2.5023_{(3)}$	$2.4500_{(3)}$
ω	50	$1.4363_{(3)}$	$1.2017_{(3)}$	$0.6343_{(1)}$	$0.0577_{(1)}$	$1.8134_{(7)}$	$1.6692_{(7)}$	$1.7635_{(5)}$	$1.6383_{(5)}$
$\sum ranks$			30		6		32		25
α	100	$0.4008_{(3)}$	$0.3237_{(2)}$	$0.1098_{(1)}$	$0.0148_{(1)}$	$0.4225_{(4)}$	$0.3660_{(5)}$	$0.4314_{(6)}$	$0.4056_{(7)}$
θ	100	$2.2551_{(3)}$	$1.8737_{(3)}$	$0.8212_{(1)}$	$0.1222_{(1)}$	$2.4590_{(5)}$	$2.4216_{(5)}$	$2.4362_{(4)}$	$2.4090_{(4)}$
ω	100	$1.1877_{(3)}$	$0.9172_{(4)}$	$0.6079_{(1)}$	$0.1018_{(1)}$	$1.6890_{(6)}$	$1.6004_{(6)}$	$1.6474_{(5)}$	$1.5694_{(5)}$
$\sum ranks$			18		6		32		25
α	200	$0.2801_{(3)}$	$0.1537_{(3)}$	$0.0846_{(1)}$	$0.0041_{(1)}$	$0.4490_{(5)}$	$0.4333_{(5)}$	$0.4689_{(6)}$	$0.4609_{(7)}$
θ	200	$1.4332_{(3)}$	$0.7875_{(3)}$	$0.6897_{(1)}$	$0.1415_{(1)}$	$2.4523_{(4)}$	$2.4346_{(4)}$	$2.4535_{(5)}$	$2.4368_{(5)}$
ω	200	$0.7379_{(3)}$	$0.3785_{(3)}$	$0.4634_{(1)}$	$0.0241_{(1)}$	$1.5656_{(7)}$	$1.5129_{(7)}$	$1.5128_{(5)}$	$1.4661_{(5)}$
$\sum ranks$			22		6		32		33
α	400	$0.1337_{(3)}$	$0.0334_{(3)}$	$0.0745_{(1)}$	$0.0011_{(1)}$	$0.4783_{(6)}$	$0.4721_{(5)}$	$0.4818_{(7)}$	$0.4786_{(7)}$
θ	400	$0.6130_{(3)}$	$0.1489_{(3)}$	$0.5998_{(2)}$	$0.1061_{(2)}$	$2.4826_{(5)}$	$2.4727_{(6)}$	$2.4512_{(6)}$	$2.4431_{(4)}$
ω	400	$0.3089_{(2)}$	$0.0713_{(3)}$	$0.4037_{(3)}$	$0.0151_{(1)}$	$1.4465_{(5)}$	$1.4126_{(5)}$	$1.4539_{(7)}$	$1.4259_{(7)}$
$\sum ranks$			17		10		33		38
α	800	$0.0386_{(2)}$	$0.0027_{(3)}$	$0.0588_{(3)}$	$0.0002_{(1)}$	$0.4904_{(6)}$	$0.4874_{(4)}$	$0.4967_{(7)}$	$0.4952_{(7)}$
θ	800	$0.1580_{(2)}$	$0.0103_{(2)}$	$0.4686_{(3)}$	$0.0580_{(3)}$	$2.4739_{(6)}$	$2.4678_{(6)}$	$2.4467_{(4)}$	$2.4427_{(4)}$
ω	800	$0.0846_{(2)}$	$0.0054_{(2)}$	$0.3199_{(3)}$	$0.0132_{(3)}$	$1.4005_{(6)}$	$1.3785_{(5)}$	$1.3912_{(4)}$	$1.3775_{(4)}$
$\sum ranks$			13		16		33		30

Table 5: Simulations Results for $(\alpha, \theta, \omega) = (0.8, 0.4, 3.2)$

		CMVE		RTADE		ADE	
parameter	n	RtMse	AvBias	RtMse	AvBias	RtMse	AvBias
α	25	$3.2790_{(6)}$	$0.3312_{(6)}$	$1.5025_{(2)}$	$0.0012_{(1)}$	$2.3005_{(4)}$	$0.0276_{(2)}$
θ	25	$3.1508_{(6)}$	$2.8071_{(6)}$	$2.9424_{(4)}$	$2.7441_{(5)}$	$2.3735_{(2)}$	$2.0207_{(2)}$
ω	25	$1.8929_{(6)}$	$1.6434_{(5)}$	$1.8364_{(5)}$	$1.5469_{(4)}$	$1.6186_{(2)}$	$1.3086_{(3)}$
$\sum ranks$			35		21		15
α	50	$1.2275_{(7)}$	$0.1521_{(2)}$	$0.5378_{(5)}$	$0.3056_{(6)}$	$0.3352_{(2)}$	$0.1873_{(3)}$
θ	50	$2.6560_{(6)}$	$2.5774_{(6)}$	$2.6363_{(5)}$	$2.5762_{(5)}$	$1.9775_{(2)}$	$1.5272_{(2)}$
ω	50	$1.8026_{(6)}$	$1.6537_{(6)}$	$1.6708_{(4)}$	$1.5282_{(4)}$	$1.3692_{(2)}$	$1.0246_{(2)}$
$\sum ranks$			33		29		13
α	100	$0.4164_{(4)}$	$0.3559_{(4)}$	$0.4384_{(7)}$	$0.4038_{(6)}$	$0.2696_{(2)}$	$0.1632_{(3)}$
θ	100	$2.5102_{(6)}$	$2.4706_{(6)}$	$2.5117_{(7)}$	$2.4848_{(7)}$	$1.5961_{(2)}$	$1.0330_{(2)}$
ω	100	$1.6925_{(7)}$	$1.6050_{(7)}$	$1.5831_{(4)}$	$1.5079_{(3)}$	$1.0930_{(2)}$	$0.6906_{(2)}$
$\sum ranks$			33		29		13
α	200	$0.4479_{(4)}$	$0.4323_{(4)}$	$0.4656_{(7)}$	$0.4555_{(6)}$	$0.1929_{(2)}$	$0.0832_{(2)}$
θ	200	$2.4878_{(7)}$	$2.4683_{(6)}$	$2.4816_{(6)}$	$2.4701_{(7)}$	$1.0588_{(2)}$	$0.4633_{(2)}$
ω	200	$1.5625_{(6)}$	$1.5067_{(6)}$	$1.4853_{(4)}$	$1.4447_{(4)}$	$0.6994_{(2)}$	$0.3025_{(2)}$
$\sum ranks$			33		34		13
α	400	$0.4761_{(4)}$	$0.4696_{(4)}$	$0.4784_{(5)}$	$0.4741_{(6)}$	$0.0931_{(2)}$	$0.0187_{(2)}$
θ	400	$2.4976_{(7)}$	$2.4870_{(7)}$	$2.4724_{(4)}$	$2.4659_{(5)}$	$0.4969_{(1)}$	$0.1004_{(1)}$
ω	400	$1.4485_{(6)}$	$1.4113_{(4)}$	$1.4391_{(4)}$	$1.4161_{(6)}$	$0.3004_{(1)}$	$0.0603_{(2)}$
$\sum ranks$			32		30		9
α	800	$0.4898_{(5)}$	$0.4869_{(6)}$	$0.4878_{(4)}$	$0.4855_{(5)}$	$0.0257_{(1)}$	$0.0014_{(2)}$
θ	800	$2.4776_{(7)}$	$2.4718_{(7)}$	$2.4673_{(5)}$	$2.4640_{(5)}$	$0.1385_{(1)}$	$0.0076_{(1)}$
ω	800	$1.4002_{(5)}$	$1.3786_{(6)}$	$1.4062_{(7)}$	$1.3907_{(7)}$	$0.0783_{(1)}$	$0.0043_{(1)}$
$\sum ranks$			36		33		7



(i) (ii) (iii)

Figure 8: Line plots for RtMse performance of the EHL-TIHT-W model parameters for Tables 4 and 5.

Table 6 shows that the MLE had the smallest RtMse and AvBias followed by WLSE and ADE. The RADE performed badly as compared to other estimates.

Table 6: Overall Rankings of Estimation Methods for EHL-TIHT-W Distribution

Parameters	n	MPSE	MLE	LSE	WLSE	CVME	RADE	ADE
$(\alpha, \theta, \omega) = (0.4, 1.2, 1.6)$	25	7	1	3	4.5	6	4.5	2
	50	3	1	6	4	5	7	2
	100	3	1	5	4	6	7	2
	200	4.5	1	4.5	3	6	7	2
	400	4.5	1	4.5	2	4.5	7	4.5
	800	6	1	4	4	2	7	4
$(\alpha, \theta, \omega) = (0.8, 0.4, 3.2)$	25	6	1	5	4	7	3	2
	50	5	1	6	3	7	4	2
	100	3	1	6	3	4.5	4.5	4.5
	200	4	3	1	2	6	7	5
	400	5	3	1	2	6.5	6.5	4
	800	4	3	1	2	7	6	5
	$\sum$ ranks	56	18	47	37.5	67.5	70.5	39
	Overall rank	5	1	4	2	6	7	3

7 Applications

In this section, we compare the EHL-TIHT-W distribution to six other distributions using two datasets. The estimated parameter values are presented together with their standard errors (in brackets). We assess the usefulness of the model by evaluating its fit against several alternative distributions and employ various statistical plots and goodness-of-fit (GoF) statistics for comparison. We consider the following GoF statistics to compare the models: -2loglikelihood (-2 log(L)), Bayesian information criterion

(BIC), Consistent Akaike Information Criterion (CAIC), Kolmogorov-Smirnov (K-S), Cramér-von Mises (W*), and Anderson-Darling (A*). The best model is the one with the lowest GoF values, highest p-value for the K-S statistic and smallest sum of squares (SS) from the probability plots. Fitted density plots and probability plots are presented for each application to determine which model provides the best fit. Empirical cumulative distribution function (ECDF) plots provide visual comparisons of the empirical distribution of the data with fitted distributions. Furthermore, Kaplan-Meier (K-M) survival plots are provided to visualize the survival probabilities over time for each application. Plots for the hrf and total time on test transform (TTT) plots are presented to describe the hazard rate function geometry. In order to verify that the maximum likelihood estimates are global maximums, profile log-likelihood plots are given.

The models considered are the Zubair-Weibull (Z-W) distribution by Ahmad (2020), the type I heavy-tailed Weibull (TIHT-W) distribution by Ahmad et al. (2018), new heavy-tailed beta-power transformed (HTBPT) distribution by Zhao et al. (2021), new heavy-tailed Weibull (NEHTW) distribution by Arif et al. (2021), Weibull-Loss (W-Loss) model Ahmad et al. (2019) and the alpha power Weibull (APW) distribution by Nassar et al. (2017). The pdfs of these distributions are provided in the appendix.

7.1 Hospital Bills Data

This data set represents the hospital cost in the state of Wisconsin. It was also used by Alfaer et al. (2023). The data is provided in the appendix.

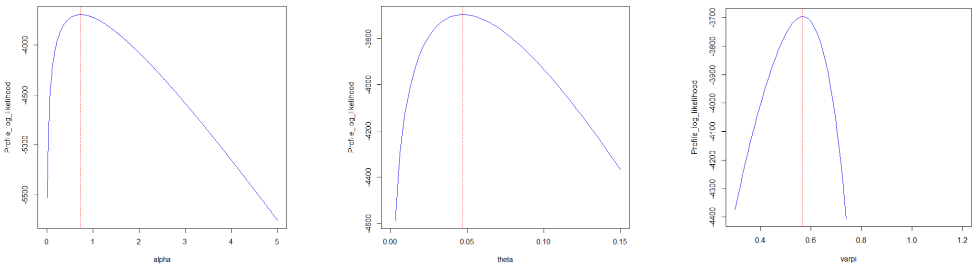
Table 7: Estimates and GoF Statistics for EHL-TIHT-W and other distributions for Hospital Bills Data

	Estimates			Goodness-of-fit Statistics								
Model	α	θ	ϖ	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	$K-S$	$p-value$	SS
$EHL-TIHT-W$	0.7313	0.0472	0.5678	7393.0900	7399.09	7399.137	7411.8810	0.0842	0.5814	0.0350	0.5407	0.0848
	(0.1023)	(0.0228)	(0.0581)									
	α	β	γ									
Z-W	0.7213	1.3906×10^{3}	1.8529×10^{-3}	7398.2820	7404.2820	7417.0720	0.1436	0.9463	0.0418	0.3176	0.0940	0.1310
	(0.0250)	(9.8667×10^{-8})	(3.0439×10^{-4})									
	θ	α	γ									
TIHTW	1.0090	0.6700	0.0186	7401.649	7407.6490	7420.6850	7420.4390	0.1147	0.9396	0.0372	0.4619	0.0976
	(0.0439)	(0.0232)	(0.0035)									
	α	β	γ									
$HTBPT$	0.3203	0.0018	0.4105	7407.6710	7413.6710	7413.7170	7426.4610	0.3078	1.7674	0.0477	0.1837	0.2937
	(0.0351)	(0.0025)	(0.1199)									
	α	γ	σ									
$NEHTW$	0.6712	0.0187	2.5082×10^{-9}	7401.708	7407.7080	7407.7540	7420.4980	0.1151	0.9468	0.0392	0.3940	0.1075
	(0.0220)	(2.8253×10^{-3})	(2.8440×10^{-3})									
	α	β	δ									
APW	0.1788	0.7608	6.4404×10^{-3}	7397.4830	7403.4830	7403.5290	7416.2730	0.1600	0.9995	0.0410	0.3393	0.7628
	(1.4708×10^{-14})	(5.7764×10^{-3})	(0.0465)									
	α	γ	σ									
$W-LOSS$	0.7782	0.4659	4.115×10^{-3}	7396.8570	7402.8570	7402.9030	7415.6470	0.1565	0.9704	0.0401	0.3683	0.1503
	(0.0569)	(0.3129)	(2.7015×10^{-3})									

For the EHL-TIHT-W distribution, the estimated variance-covariance (var-cov) matrix is provided by

$$\begin{bmatrix} 0.0105 & 0.0022 & -0.0054 \\ 0.0022 & 0.00052 & -0.0013 \\ -0.0054 & -0.0013 & 0.0034 \end{bmatrix},$$

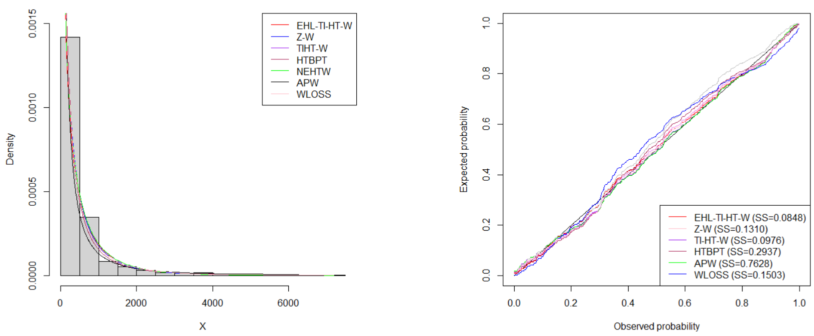
and the 95% asymptotic confidence intervals (ACI) for the parameters of the EHL-TIHT-W distribution are $\alpha \in [0.7313 \pm 0.2005]$, $\theta \in [0.0472 \pm 0.0447]$, and $\varpi \in [0.5678 \pm 0.1139]$, respectively.



(i) **(ii)** **(iii)**

Figure 9: Profiles plots of the EHL-TIHT-W distribution on the hospital bills data.

These profiles plots reveal that the log-likelihood reaches its maximum value for the estimated parameter values of α, θ, and ϖ. Thus, the figures demonstrate that the values of these three estimated parameters represent the global maximums of the log-likelihood function.



(i) **(ii)**

Figure 10: Fitted densities and probability plots for the data on hospital bills.

The fitted density plot and probability plots show that among the six specified models, the EHL-TIHT-W distribution offers the best fit for the hospital bills data.

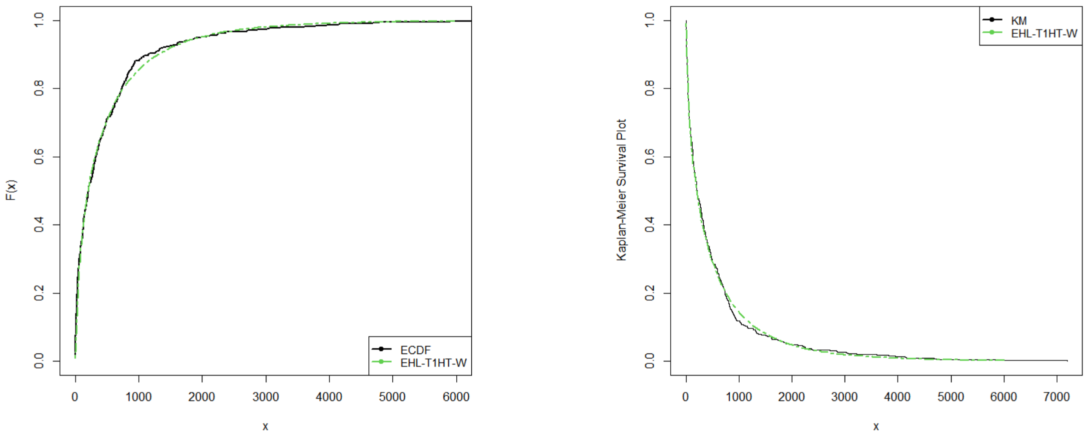


Figure 11: ECDF plot and K-M survival curve of the EHL-TIHT-W model for the hospital bills data.

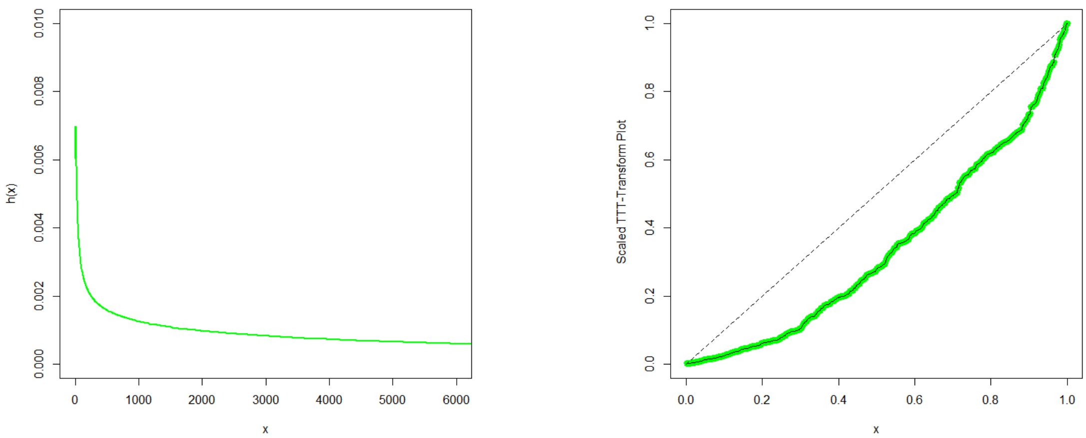


Figure 12: Plots for the hrf and TTT of the EHL-TIHT-W distribution for the hospital bills data.

The results presented in Table 7 support the assertion that the EHL-TIHT-W model is preferable because it has the lowest GoF statistics values and the K-S statistic with the largest p-value among the non-nested models investigated in this study. We conclude that the EHL-TIHT-W model better explains the hospital bills data when compared to the other models. The EHL-TIHT-W distribution outperforms the various non-nested models on the hospital bills data, as shown by the plots in Figures 10, 11 and 12.

7.2 Bladder Cancer Data

The data is about the remission times (in months) of a 128 patients suffering from bladder cancer. The data was also used by Klakattawi (2022). The data is presented in the appendix. According on data on bladder cancer, the estimated var-cov matrix for

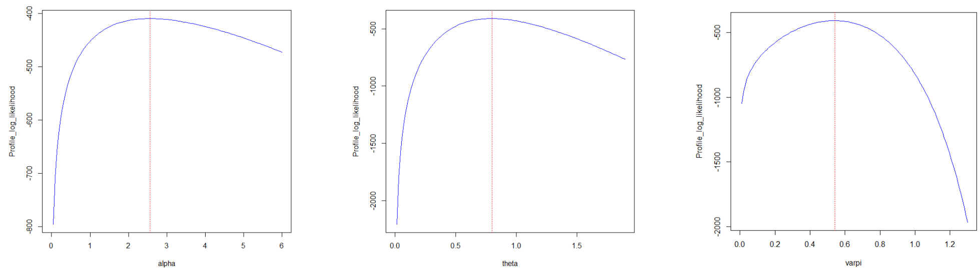
Table 8: MLEs and GoF Statistics for Bladder Cancer Data

	Estimates			GoF Statistics								
Model	α	θ	ϖ	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	$K-S$	$p-value$	SS
$EHL-TIHT-W$	2.5629	0.7987	0.5408	820.0285	826.0285	826.2221	834.5846	0.0310	0.2002	0.0393	0.9890	0.0351
	(1.0734)	(0.2161)	(0.0762)									
	α	β	γ									
ZW	1.2415	1.6158	0.0193	820.9724	826.9724	827.1659	835.5284	0.0511	0.3056	0.0484	0.9248	0.0541
	(0.0839)	(2.7932×10^{-5})	(4.3426×10^{-3})									
	θ	α	γ									
TIHTW	1.0151	1.0514	0.0909	828.1454	834.1454	834.339	842.7015	0.1304	0.7813	0.0700	0.5579	0.1774
	(0.0901)	(0.0710)	(0.0255)									
	α	β	γ									
$HTBPT$	1.0478	1.0000	0.0939	828.1738	834.1738	834.3673	842.7298	0.1314	0.7865	0.07002	0.5570	0.1779
	(0.0676)	(0.7280)	(0.0191)									
	α	γ	σ									
$NEHTW$	1.0815	0.0813	3.0091×10^{-9}	828.7594	834.7594	834.9530	843.3155	0.8214	0.1373	0.0959	0.1900	0.3796
	(0.0630)	(0.0151)	(4.8508×10^{-3})									
	α	β	δ									
APW	1.0647×10^{11}	0.3072	2.0943	843.8078	835.2691	835.4626	843.8252	0.1171	0.7922	0.0533	0.8605	0.0549
	(3.6786×10^{-14})	(0.0203)	(0.0959)									
	α	γ	σ									
$W-Loss$	1.0479	100.30	0.0929	828.1689	834.1689	834.3624	842.7250	0.1313	0.7861	0.0700	0.5568	0.2114
	(0.0676)	(2.9627×10^{-7})	(0.0189)									

the EHL-TIHT-W model is

$$\begin{bmatrix} 1.15227 & 0.2259 & -0.07512 \\ 0.2259 & 0.04671 & -0.0159 \\ -0.07512 & -0.0159 & 0.0058 \end{bmatrix},$$

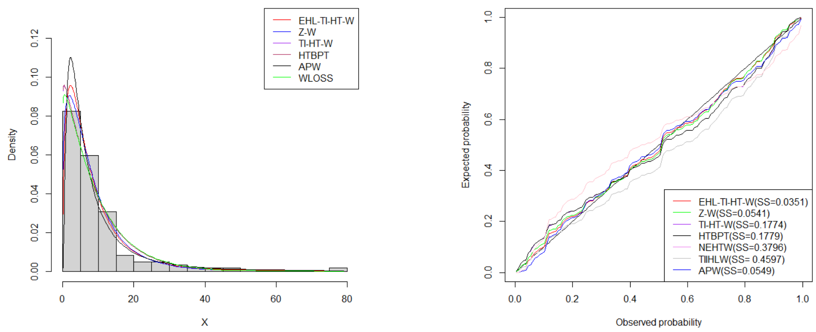
and the 95% ACI for the model parameters are given by
$\alpha \in [2.5629 \pm 2.10394]$, $\theta \in [0.7987 \pm 0.4236]$ and $\varpi \in [0.5408 \pm 0.1494]$.



(i) (ii) (iii)

Figure 13: Profiles plots against the parameters of EHL-TIHT-W distribution on the blood cancer data.

These profile plots demonstrate that the log-likelihood achieves its maximum value at specific parameter values of α, θ, and ϖ. This indicates that these estimated parameter values constitute the global maximums of the log-likelihood function for the blood cancer data.



(i) (ii)

Figure 14: Fitted densities and probability plots for the data on bladder cancer

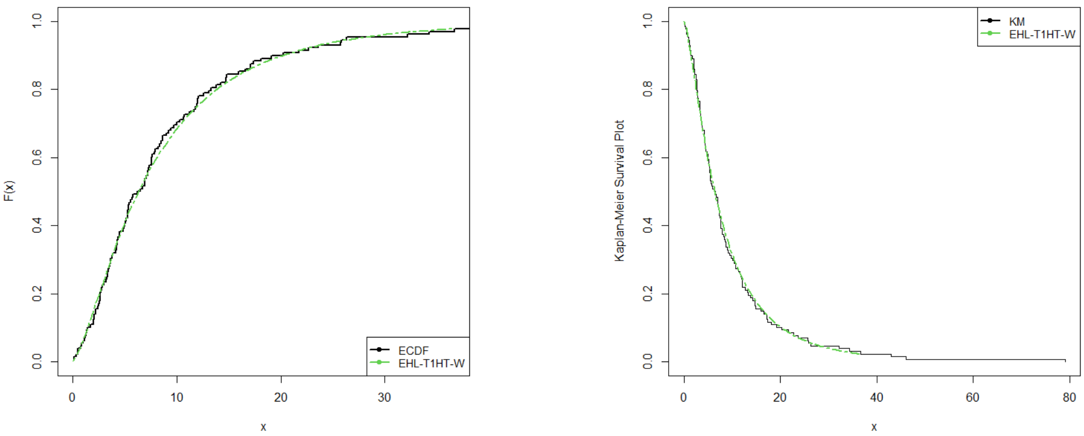


Figure 15: Plots for empirical cdf and the Kaplan–Meier survival curve of the EHL-TIHT-W model for the bladder cancer data

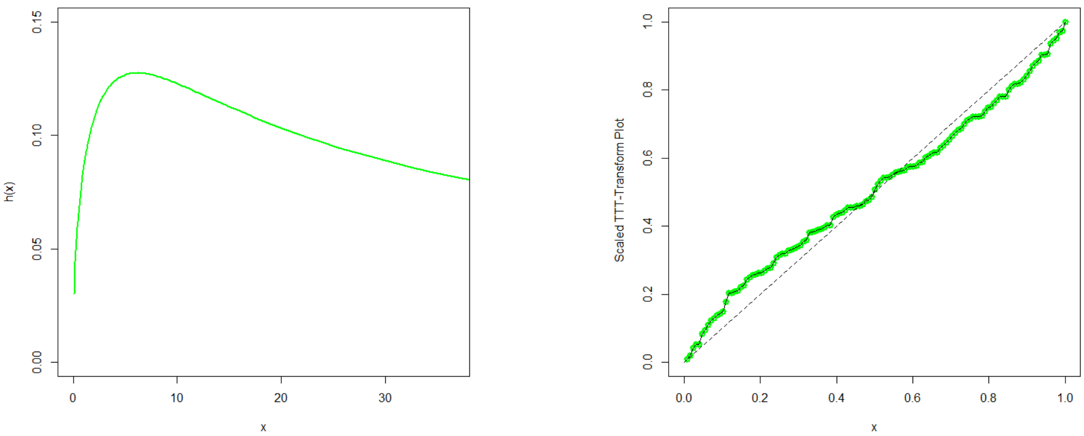


Figure 16: Plots for the hrf function and TTT plot of the EHL-TIHT-W distribution for the bladder cancer data

Table 8 shows that the EHL-TIHT-W model is preferable because it has the lowest GoF statistics values and the K-S statistic with the largest p-value among the non-nested models used in this study. We conclude that the EHL-TIHT-W distribution better explains the bladder cancer data when compared to the other no-nested models.

8 Conclusions

In our work, we provide essential mathematical properties of the EHL-TIHT-Π distribution, including a linear representation of its density function. We also derive key actuarial measures. To estimate the model parameters, we employ several estimation techniques, including the MLE, MPSE, and RADE, among others. Monte Carlo simulations are utilized to assess the accuracy of parameter estimates obtained through these methods. The research findings indicate that among the seven estimation techniques evaluated, the MLE method yielded the most accurate parameter estimates for the EHL-TIHT-W distribution, followed closely by the WLSE and ADE methods as shown in Tables 2, 3, 4, and 5. This is also supported by Figures 7 and 9 in our analysis of two real datasets. We compare the performance of the special case of the EHL-TIHT-W distribution with several other known distributions. The results in Tables 7 and 8 reveal that the EHL-TIHT-W distribution outperforms other non-nested distributions in terms of the GoF. In conclusion, the EHL-TIHT-Π distribution offers enhanced flexibility and exhibits excellent performance compared to several existing distributions. Future work could further investigate its applications across diverse fields, comparing its efficacy against a broader range of distributions to solidify its standing in statistical modeling.

To access the appendix, kindly click on the link provided hereunder: `https://drive.google.com/file/d/1_t4K01R43Bzb2gAw9SMF11T7EtJUtyrY/view?usp=drive_link`.

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