Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v18n1p48

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15 March 2025

Electronic Journal of Applied Statistical Analysis
Vol. 18, Issue 01, march 2025, 48-70
DOI: 10.1285/i20705948v18n1p48

Topp-Leone Teissier Distribution: Neutrosophic Approach and Applications

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15 March 2025

In this paper, we introduce a new two-parameter extension of the Teisisier distribution using the Topp-Leone distribution as a generator, namely Topp-Leone Teissier distribution. The new model exhibits increasing, decreasing and bathtub shaped hazard rate functions. Several properties of the model are derived utilizing the Lambert W, the generalized integro-exponential and the incomplete generalized integro-exponential functions. Maximum likelihood and Bayesian procedures are used to estimate the model parameters. Lindley's approximation under squared error loss function is utilized for Bayesian computations. Moreover, a simulation study is carried out to analyze the performance of these estimators on the basis of mean squared error. The applicability of the proposed model is evaluated using two real data sets. Also, we highlight the neutrosophic approach on Topp-Leone Teissier distribution as a pathway to address issues related to indeterminate, vague, or uncertain data set. This enhancement integrates neutrosophic logic into the model parameters, providing a robust framework for addressing the inherent challenges of ambiguous data and thereby broadening the model's applicability to real-world scenarios involving incomplete or imprecise information.

keywords: Bayesian estimation, Lambert W function, Neutrosophic, Teissier distribution, Topp-Leone distribution.

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ISSN: 2070-5948
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1 Introduction

The Teissier distribution (TD) was introduced by Teissier (1934) for modeling mortality data of animal species due to pure ageing. Laurent (1975) showed the Teissier model as a generalization of the negative exponential distribution and highlighted its applications in demographic areas, biometric and failure theory. Leemis and McQueston (2008) called the extended Teissier model as Muth distribution and used its pdf to show the properties and relationships between several univariate distributions via a graph. Jodra et al. (2015) derived the mathematical properties of the Muth in terms of integro-exponential and Lambert W function. Kolev et al. (2017) introduced the symmetric and asymmetric versions of bivariate Teissier model. Further extension include the Exponentiated Teissier distribution by Sharma et al. (2022) and the Unit Teissier distribution by Krishna et al. (2022), studied its various statistical properties. Recently, Poonia and Azad (2022) developed the alpha power exponentiated Teissier distribution, while Singh et al. (2022) introduced the discrete Teissier distribution. Alsadat et al. (2023) contributed to the literature by introducing the Inverse Unit Teissier distribution.
If X follows Teissier distribution with parameter θ, then its pdf and cdf are respectively given by

$$g_T(x) = \theta(e^{\theta x} - 1)e^{(\theta x - e^{\theta x} + 1)}; \quad x > 0, \ \theta > 0 \tag{1}$$

and

$$G_T(x) = 1 - e^{(\theta x - e^{\theta x} + 1)}; \quad x > 0, \ \theta > 0 \tag{2}$$

where θ is the scale parameter.

The Teissier distribution possesses a heavier tail than some of the noted life time distributions like Weibull, gamma and lognormal, see Muth (1977). This property makes it a better model for heavy-tailed data sets encountered in fields such as financial, actuarial, public health, industrial reliability, survival data etc. However, a major limitation of the Teissier distribution is its inability to model data sets with non-monotonic hazard rate functions; it is restricted to increasing hazard rate functions. This limitation significantly reduces its flexibility for real-world applications, especially in scenarios requiring the modeling of data with decreasing or bathtub-shaped hazard rate functions. Motivated by the practical challenges of the Teissier distribution in capturing complex hazard rate behaviors, we propose a novel two-parameter extension of the Teissier distribution, termed the Topp-Leone Teissier (TL-T) distribution. The new model, derived from the Topp-Leone generated (TL-G) family of distributions pioneered by Al-Shomrani et al. (2016), addresses the limitations of the original Teissier distribution by accommodating increasing, decreasing, and bathtub-shaped hazard rate functions. As a generalization of the Teissier distribution, the TL-T model is more flexible than the Teissier distribution to model heavy tailed data sets. The cdf and pdf of the TL-G family are defined by

$$F_{TL-G}(x) = [1 - [1 - G(x)]^2]^{\lambda} \tag{3}$$

and

$$f_{TL-G}(x) = 2\lambda g(x)[1 - G(x)][1 - [1 - G(x)]^2]^{\lambda - 1} \tag{4}$$

respectively, where $x \in \mathcal{R}$, $\lambda > 0$ is the shape parameter and G(x) is a baseline cdf. Beyond addressing the limitations of the Teissier distribution, the applicability of the TL-T distribution is showcased through the analysis of March precipitation and tensile strength of carbon fibres data sets. It demonstrate the efficacy of proposed distribution in modeling the variability of precipitation patterns crucial for applications in hydrology and climate science, as well as its ability to capture the distribution of material properties with precision.

Furthermore, within the conventional statistical paradigm, the representation of variability in data typically overlooks considerations of fuzziness. Neutrosophic logic, introduced by Smarandache (1998) as an extension of fuzzy logic, addresses this limitation by specifically addressing uncertainties associated with studied variables. Additionally, Smarandache (2014) introduced the concepts of neutrosophic statistics, extending classical statistical procedures to accommodate these nuanced uncertainties. In our reality, the prevalence of indeterminate data surpasses that of determinate data, necessitating a greater reliance on neutrosophic statistical methods as opposed to classical ones. In the practical realm, when faced with ambiguous scenarios, assigning a specific precise value may not be feasible, leading to inaccurate outcomes. Neutrosophic statistics, on the other hand, utilize precise numbers to depict data within intervals. For example, assessing melting points is typically a complex task, leading to indeterministic observations that are often expressed in intervals. In this context, the distribution for alloy metal melting point data relies on an interval set of values for uncertain parameters. Neutrosophic statistics emerges as a more suitable framework for addressing alloy metal melting points based on interval data (see Rao (2023)).

The utilization of conventional classical distributions is not appropriate for situations where data is frequently imprecise, uncertain, and lacks precision. Several authors recently developed neutrosophic distributions viz., Neutrosophic Uniform and Neutrosophic Poisson (Alhabib et al., 2018), Neutrosophic Weibull (Hamza Alhasan and Smarandache, 2019), Neutrosophic exponential (Duan et al., 2021), Neutrosophic Rayleigh, Neutrosphic Beta (Khan Sherwani et al., 2021), Neutrosophic Kumaraswamy (Ahsan-ul Haq, 2022), Neutrosophic Log-Logistic (Rao, 2023), Neutrosophic generalized Pareto (Eassa et al., 2023) and Neutrosophic Laplace (Thakur et al., 2023) distributions. Considering the wide applications and feasible properties of TL-T distribution, here also, we introduce a new neutrosophic distribution based on Topp-Leone Teissier distribution namely, Neutrosophic Topp-Leone Teissier (NTL-T) distribution.

The remaining part of the article is laid out into eight sections. The next section presents the development of the TL-T distribution. Section 3 discuss some of its mathematical and statistical properties. The method of maximum likelihood and Bayesian estimation to estimate the model parameters are given in Section 4. A simulation study is addressed in Section 5. Two real-life data sets are analyzed in Section 6. In the penultimate section, we introduce the Neutrosophic Topp-Leone Teissier distribution, some of its properties and applications. Finally, the article is concluded in Section 8.

2 Topp-Leone Teissier Distribution

Inserting equation (2) in equation (3) yields the cdf of the proposed two parameter TL-T distribution as

$$F(x;\theta,\lambda) = [1-e^{A(x,\theta)}]^{\lambda}; \quad x,\ \theta\ \lambda, > 0, \tag{5}$$

thereby the probability density function corresponding to equation (5) becomes

$$f(x;\theta,\lambda) = 2\lambda\theta(e^{\theta x}-1)e^{A(x,\theta)}[1-e^{A(x,\theta)}]^{\lambda-1}; \quad x,\ \theta,\ \lambda > 0 \tag{6}$$

where $A(x,\theta) = 2\theta x - 2(e^{\theta x}-1)$.

The reliability function, hazard rate function and reverse hazard rate function of the

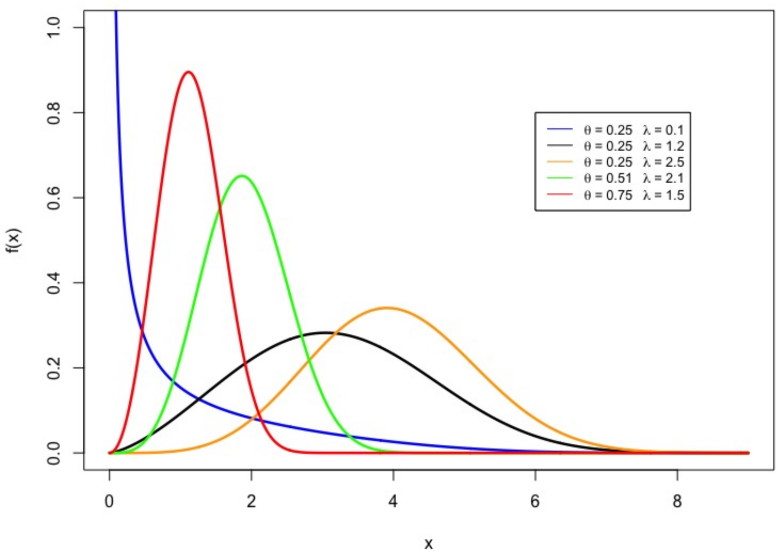


Figure 1: Density plots of TL-T distribution for certain values of θ and λ.

TL-T distribution are respectively given by

$$R(x;\theta,\lambda) = 1-(1-e^{A(x,\theta)})^{\lambda},$$

$$h(x;\theta,\lambda) = \frac{2\lambda\theta(e^{\theta x}-1)e^{A(x,\theta)}(1-e^{A(x,\theta)})^{\lambda-1}}{1-(1-e^{A(x,\theta)})^{\lambda}}$$

and

$$h_r(x;\theta,\lambda) = \frac{2\lambda\theta(e^{\theta x}-1)e^{A(x,\theta)}}{1-e^{A(x,\theta)}}.$$

Figures 1 and 2, respectively, shows the behavior of pdf and hazard rate function of TL-T distribution for various parametric values. In Figure 1, it is clear that the pdf of

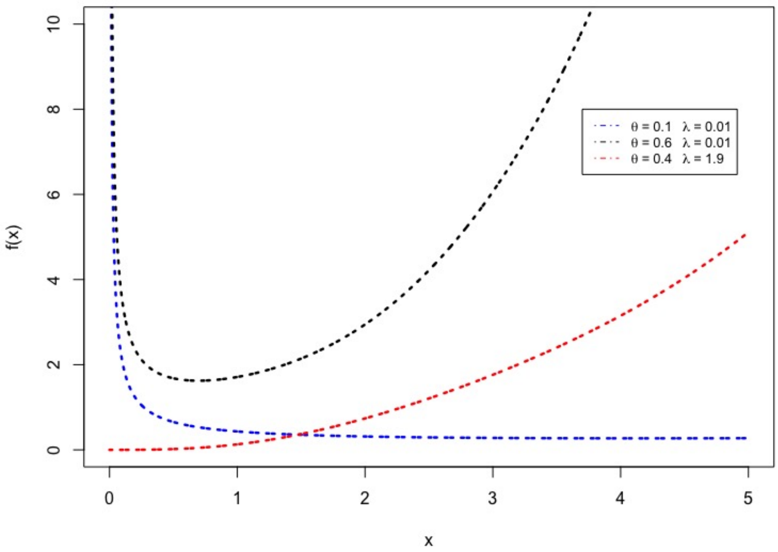


Figure 2: Hazard rate plots of TL-T distribution for certain values of θ and λ.

TL-T model can be unimodal, positively skewed, approximately symmetric and monotonically decreasing forms. Figure 2 illustrates the flexibility of the TL-T distribution in modeling a wide variety of hazard rate behaviors, including increasing, decreasing, and bathtub-shaped patterns. These characteristics enhance the model's applicability to real-world datasets encountered in fields such as reliability analysis, survival studies, and risk assessment. By accommodating diverse hazard rate shapes, the TL-T model is particularly suitable for modeling the complexities of data where hazard rates vary under different conditions.

3 Statistical Properties

Here, we discuss some statistical properties of the proposed TL-T distribution.

3.1 Linear Representation

Al-Shomrani et al. (2016) showed that the pdf of TL-G family has the following mixture representation

$$f(x) = \sum_{k=0}^{\infty} \sum_{i=0}^{2k+1} a(k,i) c_{i+1}(x)$$

where $c_{i+1}(x) = (i+1)g(x)[G(x)]^i$ is the exp-G distribution with power parameter i and

$$a(k,i) = \frac{(-1)^{k+i} 2\Gamma(\lambda+1)}{k!\Gamma(\lambda-k)(i+1)} \binom{2k+1}{i}.$$

Using the above expression, TL-T density function in equation (6) can be expressed as

$$f(x;\theta,\lambda) = 2\lambda\theta(e^{\theta x}-1)e^{\theta x - e^{\theta x}+1} \sum_{k=0}^{\infty} \sum_{i=0}^{2k+1} (-1)^{k+i} \binom{\lambda-1}{k} \binom{2k+1}{i} [1-e^{\theta x - e^{\theta x}+1}]^i.$$

3.2 Quantile Function

By inverting equation (5), the quantile function of the TL-T distribution can be obtained as

$$x = \frac{-1}{\theta} W_{-1}\left(\frac{-\sqrt{1-p^{1/\lambda}}}{e}\right) + \frac{1}{2\theta} log(1-p^{1/\lambda}) - \frac{1}{\theta} \qquad (7)$$

where $p \in (0,1)$ and W_{-1} refers the negative branch of Lambert W function, for more details, see Corless et al. (1996).

3.3 Probability Weighted Moments

In order to derive the probability weighted moments (PWMs), we rely on the generalized integro-exponential function mentioned in Milgram (1985) and it is

$$E_s^q(z) = \frac{1}{q!} \int_1^{\infty} (log\ u)^q u^{-s} e^{-zu} du,$$

where $s \in \mathbb{R}$, $q > -1$.

The (r,s)th PWMs of the TL-T model is defined by

$$\begin{aligned} M_{r,s}(x;\theta,\lambda) &= E[X^r F(X;\theta,\lambda)^s] = \int_0^{\infty} x^r F(x;\theta,\lambda)^s f(x;\theta,\lambda) dx \\ &= 2\lambda\theta \int_0^{\infty} x^r (e^{\theta x}-1) e^{A(x,\theta)} [1-e^{A(x,\theta)}]^{(s+1)\lambda-1} dx. \end{aligned}$$

By employing the binomial expansion in the above equation, we have

$$\begin{aligned} M_{r,s}(x;\theta,\lambda) &= 2\lambda\theta \sum_{j=0}^{\infty} (-1)^j \binom{(s+1)\lambda-1}{j} \int_0^{\infty} x^r (e^{\theta x}-1) e^{(j+1)A(x,\theta)} dx \\ &= 2\lambda\theta \sum_{j=0}^{\infty} (-1)^j \binom{(s+1)\lambda-1}{j} \left[\int_0^{\infty} x^r e^{\theta x + (j+1)A(x,\theta)} dx - \int_0^{\infty} x^r e^{(j+1)A(x,\theta)} dx \right]. \end{aligned}$$

Assuming $v = e^{\theta x}$ and using some algebra, we obtain the $(r,s)^{th}$ PWMs of the TL-T model as

$$M_{r,s}(x;\theta,\lambda) = 2\lambda \sum_{j=0}^{\infty} (-1)^j \binom{(s+1)\lambda-1}{j} \frac{e^{2(j+1)}}{\theta^r} r! \left[E^r_{-2j-2}(2(j+1)) - E^r_{-2j-1}(2(j+1)) \right]. \qquad (8)$$

Putting $s = 0$ in equation (8), we get the rth moment of the TL-T distribution and it is

$$E[X^r] = 2\lambda \sum_{j=0}^{\infty} (-1)^j \binom{\lambda - 1}{j} \frac{e^{2(j+1)}}{\theta^r} r! \left[E^r_{-2j-2}(2(j+1)) - E^r_{-2j-1}(2(j+1)) \right]. \quad (9)$$

The mean, variance and higher order moments of the TL-T distribution are obtained from equation (9) by putting $r = 1, 2, 3, ...$

3.4 Moment Generating Function

The moment generating function is formally defined as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E[X^r]. \quad (10)$$

Therefore, the moment generating function of TL-T distribution easily follows from equations (9) and (10) as

$$M_X(t) = 2\lambda \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{t}{\theta}\right)^r (-1)^j \binom{\lambda - 1}{j} e^{2(j+1)} \left[E^r_{-2j-2}(2(j+1)) - E^r_{-2j-1}(2(j+1)) \right]. \quad (11)$$

3.5 Incomplete Moments

The rth incomplete moment of the TL-T distribution is defined by

$$\begin{aligned} \phi_r(t) &= \int_0^t x^r f(x; \theta, \lambda) dx \\ &= 2\lambda\theta \int_0^t x^r (e^{\theta x} - 1) e^{A(x,\theta)} (1 - e^{A(x,\theta)})^{\lambda - 1} dx. \end{aligned}$$

On applying the binomial expansion and the transformation $v = e^{\theta x}$ in the above equation, we obtain

$$\phi_r(t) = \frac{2\lambda}{\theta^r} \sum_{j=0}^{\infty} (-1)^j \binom{\lambda - 1}{j} \times$$
$$\left[\int_1^{e^{\theta t}} (log\ v)^r v^{2(j+1)} e^{2(j+1)(1-v)} dv - \int_1^{e^{\theta t}} (log\ v)^r v^{2(j+1)-1} e^{2(j+1)(1-v)} dv \right].$$

Using the incomplete generalized integro-exponential function

$$E^q_s(t; z) = \frac{1}{q!} \int_1^z (log\ u)^q u^{-s} e^{-tu} du, \quad t \in (-\infty, \infty)$$

where $\lim_{z\to\infty} E_s^q(t;z) = E_s^q(t)$, we can express the rth incomplete moment of the TL-T distribution as

$$\phi_r(t) = \frac{2\lambda}{\theta^r}\sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}e^{2(j+1)}r!\left[E^r_{-2j-2}(2(j+1);e^{\theta t}) - E^r_{-2j-1}(2(j+1);e^{\theta t})\right]. \tag{12}$$

The first incomplete moment, $\phi_1(t)$ is determined by inserting $r = 1$ in equation (12) and it is

$$\phi_1(t) = \frac{2\lambda}{\theta}\sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}e^{2(j+1)}\left[E^1_{-2j-2}(2(j+1);e^{\theta t}) - E^1_{-2j-1}(2(j+1);e^{\theta t})\right]. \tag{13}$$

3.6 Mean Residual Life Function

One of the important application of the $\phi_1(t)$ is related to the mean residual life function which is the expected remaining life given the survival to a fixed time t. The MRL is defined as

$$m(t) = \frac{E[X] - \phi_1(t)}{1 - F(t;\theta,\lambda)} - t. \tag{14}$$

Using equations (5), (9) and (13), the MRL of the TL-T distribution is obtained as follows

$$m(t) =$$
$$\sum_{j=0}^{\infty} D\left[E^1_{-2j-2}(2(j+1)) - E^1_{-2j-1}(2(j+1)) - E^1_{-2j-2}(2(j+1);v) + E^1_{-2j-1}(2(j+1);v)\right] - t,$$

$$\text{where } D = \frac{2\lambda(-1)^j\binom{\lambda-1}{j}e^{2(j+1)}}{\theta[1-(1-e^{A(t,\theta)})^\lambda]}.$$

3.7 Entropies

Rényi entropy and Tsallis entropy of the TL-T distribution are respectively defined as

$$I_{entropy}(\xi) = \frac{1}{1-\xi}log[I(\xi)] \quad \text{and} \tag{15}$$

$$T_{entropy}(\xi) = \frac{1}{\xi-1}[1 - I(\xi)] \tag{16}$$

where $I(\xi) = \int_0^\infty f^\xi(x;\theta,\lambda)dx,\ \xi > 0$ and $\xi \neq 1$.

Based on equation (6), we can write

$$I(\xi) = \int_0^\infty (2\lambda\theta)^\xi (e^{\theta x} - 1)^\xi e^{\xi A(x,\theta)} (1 - e^{A(x,\theta)})^{(\lambda-1)\xi} dx$$
$$= (2\lambda\theta)^\xi \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} (-1)^{u_1+u_2+\xi} \binom{(\lambda-1)\xi}{u_1} \binom{\xi}{u_2} \int_0^\infty e^{\theta u_2 x} e^{(u_1+\xi)A(x,\theta)}\ dx.$$

Let $v = e^{\theta x}$ and integrating, $I(\xi)$ becomes

$$I(\xi) = (2\lambda)^\xi \theta^{\xi-1} \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} (-1)^{u_1+u_2+\xi} \binom{(\lambda-1)\xi}{u_1} \binom{\xi}{u_2} e^{2(u_1+\xi)} E^0_{2\xi-2u_1-u_2+1}(2(u_1+\xi)). \tag{17}$$

Hence, the Rényi entropy and Tsallis entropy of TL-T model are immediately derived from equations (15) and (16) respectively.

3.8 Order Statistics

Suppose $X_1, X_2, ..., X_n$ be a random sample from TL-T distribution and $X_{(1)}, X_{(2)}, ...X(n)$ be the corresponding order statistics. The pdf $f_{d:n}(x; \theta, \lambda)$ of the dth order statistic $X_{d:n}$ is

$$f_{d:n}(x; \theta, \lambda) = \frac{n!}{(d-1)!(n-d)!} f(x; \theta, \lambda) F(x; \theta, \lambda)^{d-1} [1 - F(x; \theta, \lambda)]^{n-d}. \tag{18}$$

Substituting equations (5) and (6) in equation (18) and using binomial expansion, we get

$$f_{d:n}(x; \theta, \lambda) = \frac{2\lambda\theta n!}{(d-1)!(n-d)!} \sum_{u_1=0}^{\infty} \sum_{u_2}^{\infty} (-1)^{u_1+u_2} \binom{n-d}{u_1} \binom{\lambda(u_1+d)-1}{u_2} (e^{\theta x} - 1) e^{(1+u_2)A(x,\theta)}. \tag{19}$$

and the corresponding cdf is

$$F_{d:n}(x; \theta, \lambda) = \sum_{l=u_1}^{n} \sum_{u_2=0}^{l} (-1)^{u_2} \binom{n}{l} \binom{l}{u_2} [1 - (1 - e^{A(x,\theta)})^\lambda]^{n-l+u_2}. \tag{20}$$

The pdf of the smallest and largest order statistics are obtained, respectively, by putting $d = 1$ and $d = n$ in equation (19).

The rth moment of the dth order statistic $X_{d:n}$ is computed using equation(19) as

$$E[X_d^r] = \frac{2\lambda\theta n!}{(d-1)!(n-d)!} \sum_{u_1=0}^{\infty} \sum_{u_2}^{\infty} (-1)^{u_1+u_2} \binom{n-d}{u_1} \binom{\lambda(u_1+d)-1}{u_2} \int_0^\infty x_d^r (e^{\theta x_d} - 1) e^{(1+u_2)A(x_d,\theta)} dx_d. \tag{21}$$

After some simplifications and also by applying generalised integro-exponential function, we have

$$E[X_d^r] = \sum_{u_1=0}^{\infty}\sum_{u_2=0}^{\infty} D_1 e^{2(1+u_2)}[E^r_{-(2u_2+2)}(2(u_2+1)) - E^r_{-(2u_2+1)}(2(u_2+1))]$$

where $D_1 = (-1)^{u_1+u_2}\frac{2\lambda n!}{(d-1)!(n-d)!}\frac{r!}{\theta^r}\binom{n-d}{u_1}\binom{\lambda(u_1+d)-1}{u_2}$.

4 Parameter Estimation

Here, by using the method of maximum likelihood and Bayesian estimation method, we estimate the parameter vector $\gamma = (\theta, \lambda)^T$ of Topp-Leone Teissier distribution.

4.1 Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ be a random sample taken from the TL-T distribution with pdf equation (6). The likelihood function is

$$L(\gamma|x) = (2\lambda\theta)^n \prod_{i=1}^{n}(e^{\theta x_i}-1)e^{\sum_{i=1}^{n}(2\theta x_i - 2(e^{\theta x_i}-1))}\prod_{i=1}^{n}(1-e^{2\theta x_i}-2(e^{\theta x_i}-1))^{\lambda-1} \quad (22)$$

and the corresponding log-likelihood function is

$$l(\gamma) = nlog(2\lambda\theta) + \sum_{i=1}^{n} log(e^{\theta x_i}-1) + 2\theta\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n}(e^{\theta x_i}-1) + (\lambda-1)\sum_{i=1}^{n} log(1-e^{A(x,\theta)}). \quad (23)$$

The first partial derivatives of $l(\gamma)$ with respect to θ and λ are:

$$\frac{\partial l(\gamma)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{x_i e^{\theta x_i}}{(e^{\theta x_i}-1)} + 2\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i e^{\theta x_i} + (\lambda-1)\sum_{i=1}^{n}\frac{2x_i e^{2\theta x_i+2}(e^{\theta x_i}-1)}{e^{2e^{\theta x_i}} - e^{2\theta x_i+2}} \quad (24)$$

and

$$\frac{\partial l(\gamma)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} log(1-e^{A(x_i,\theta)}). \quad (25)$$

The ML estimates of the model parameters are obtained by equating both the above equations to zero and solving them simultaneously. For that, we use the maxLik package in the R software.

4.2 Bayesian Estimation

In this section, our main objective is to obtain the Bayes estimates for the model parameter θ and λ of the TL-T distribution under squared error loss function (SELF).

Let $X_1, X_2, ..., X_n$ be a random sample drawn from TL-T distribution. Assume that the prior distributions for θ and λ follow independent Gamma(a_i, b_i), i=1, 2 distributions respectively. The joint prior distribution of θ and λ is

$$\pi(\gamma) = \pi(\theta)\pi(\lambda) \propto \theta^{a_1-1}e^{-b_1\theta}\lambda^{a_2-1}e^{-b_2\lambda}, \tag{26}$$

where $a_i, b_i > 0$; i=1, 2 denote the hyper parameters. In addition, the non-informative priors are obtained by setting $a_i = b_i = 0$ in equation (26).

Now, using the equations (23) and (26), we have the posterior distribution of θ and λ as

$$\pi(\gamma|x) = \frac{1}{K}(2\lambda\theta)^n\theta^{a_1-1}\lambda^{a_2-1}e^{-(b_1\theta+b_2\lambda)}\prod_{i=1}^{n}(e^{\theta x_i}-1)e^{\sum_{i=1}^{n}(2\theta x_i-2(e^{\theta x_i}-1))} \prod_{i=1}^{n}\left(1-e^{2\theta x_i-2(e^{\theta x_i}-1)}\right)^{\lambda-1} \tag{27}$$

where

$$K = \int_0^\infty\int_0^\infty (2\lambda\theta)^n\theta^{a_1-1}\lambda^{a_2-1}e^{-(b_1\theta+b_2\lambda)}\prod_{i=1}^{n}(e^{\theta x_i}-1)e^{\sum_{i=1}^{n}(2\theta x_i-2(e^{\theta x_i}-1))} \prod_{i=1}^{n}\left(1-e^{2\theta x_i-2(e^{\theta x_i}-1)}\right)^{\lambda-1} d\theta d\lambda.$$

The corresponding Bayes estimate of θ under SELF is obtained as

$$\hat{\theta}_B = \frac{1}{K}\int_0^\infty\int_0^\infty (2\lambda)^n\theta^{n+a_1}\lambda^{a_2-1}e^{-(b_1\theta+b_2\lambda)}\prod_{i=1}^{n}(e^{\theta x_i}-1)e^{\sum_{i=1}^{n}(2\theta x_i-2(e^{\theta x_i}-1))} \prod_{i=1}^{n}\left(1-e^{2\theta x_i-2(e^{\theta x_i}-1)}\right)^{\lambda-1} d\theta d\lambda \tag{28}$$

and that of λ as

$$\hat{\lambda}_B = \frac{1}{K}\int_0^\infty\int_0^\infty (2\theta)^n\theta^{a_1-1}\lambda^{n+a_2}e^{-(b_1\theta+b_2\lambda)}\prod_{i=1}^{n}(e^{\theta x_i}-1)e^{\sum_{i=1}^{n}(2\theta x_i-2(e^{\theta x_i}-1))} \prod_{i=1}^{n}\left(1-e^{2\theta x_i-2(e^{\theta x_i}-1)}\right)^{\lambda-1} d\theta d\lambda. \tag{29}$$

From equations (28) and (29), it is apparent that the Bayes estimators are in the form of the ratio of two integrals and hence no analytical solutions are available. Therefore, we use Lindley approximation method proposed by Lindley (1980) for the computation of Bayes estimators.

4.2.1 Lindley approximation method

For sufficiently large n, as stated in Lindley (1980), if the ratio of the integrals is in the form

$$I(x) = E[\psi(\theta,\lambda|x)] = \frac{\int_{\boldsymbol{\gamma}} \psi(\boldsymbol{\gamma}) e^{l(\boldsymbol{\gamma}|x)+\rho(\boldsymbol{\gamma})}\ d(\boldsymbol{\gamma})}{\int_{\boldsymbol{\gamma}} e^{l(\boldsymbol{\gamma}|x)+\rho(\boldsymbol{\gamma})}\ d(\boldsymbol{\gamma})} \tag{30}$$

where $\psi(\boldsymbol{\gamma})$= a function of θ and λ only, $l(\boldsymbol{\gamma}|x)$ is the log-likelihood function and $\rho(\boldsymbol{\gamma})$ is the log of joint prior of $\pi(\boldsymbol{\gamma}|x)$, using Lindley's method it can be approximated to

$$\begin{aligned} I(x) \approx\ & \psi(\hat{\theta},\hat{\lambda}) \\ & + \frac{1}{2}[(\hat{\psi}_{\theta\theta} + 2\hat{\psi}_{\theta}\hat{\rho}_{\theta})\hat{\sigma}_{\theta\theta} + (\hat{\psi}_{\lambda\theta} + 2\hat{\psi}_{\lambda}\hat{\rho}_{\theta})\hat{\sigma}_{\lambda\theta} + (\hat{\psi}_{\theta\lambda} + 2\hat{\psi}_{\theta}\hat{\rho}_{\lambda})\hat{\sigma}_{\theta\lambda} + (\hat{\psi}_{\lambda\lambda} + 2\hat{\psi}_{\lambda}\hat{\rho}_{\lambda})\hat{\sigma}_{\lambda\lambda}] \\ & + \frac{1}{2}[(\hat{\psi}_{\theta}\hat{\sigma}_{\theta\theta} + \hat{\psi}_{\lambda}\hat{\sigma}_{\theta\lambda})(\hat{l}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta} + \hat{l}_{\theta\lambda\theta}\hat{\sigma}_{\theta\lambda} + \hat{l}_{\lambda\theta\theta}\hat{\sigma}_{\lambda\theta} + \hat{l}_{\lambda\lambda\theta}\hat{\sigma}_{\lambda\lambda}) \\ & + (\hat{\psi}_{\theta}\hat{\sigma}_{\lambda\theta} + \hat{\psi}_{\lambda}\hat{\sigma}_{\lambda\lambda})(\hat{l}_{\lambda\theta\theta}\hat{\sigma}_{\theta\theta} + \hat{l}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\lambda} + \hat{l}_{\lambda\theta\lambda}\hat{\sigma}_{\lambda\theta} + \hat{l}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda})], \end{aligned} \tag{31}$$

where $\hat{\theta}$ and $\hat{\lambda}$ are the MLE of θ and λ respectively. The details of the terms within I(x) are as follows:

$$\begin{aligned} \rho(\gamma) &= (a_1 - 1)log(\theta) - b_1\theta + (a_2 - 1)log(\lambda) - b_2\lambda, \\ \hat{\rho}_{\theta} &= \frac{\partial\rho(\gamma)}{\partial\theta} = \frac{(a_1 - 1)}{\hat{\theta}} - b_1 \quad \text{and} \\ \hat{\rho}_{\lambda} &= \frac{\partial\rho(\gamma)}{\partial\lambda} = \frac{(a_2 - 1)}{\hat{\lambda}} - b_2. \end{aligned}$$

Also,

$$\hat{l}_{\theta\theta} = \frac{\partial^2 l(\gamma)}{\partial\theta^2} = -\frac{n}{\hat{\theta}^2} - \sum_{i=1}^{n} \frac{x_i^2 e^{\hat{\theta}x_i}}{(e^{\hat{\theta}x_i}-1)^2} - 2\sum_{i=1}^{n} x_i^2 e^{\hat{\theta}x_i}$$
$$- 2(\hat{\lambda}-1)\sum_{i=1}^{n} \frac{x_i^2 e2\hat{\theta}x_i + 2((2e^{2\hat{\theta}x_i} - 5e^{\hat{\theta}x_i} + 2)e^{2e^{\hat{\theta}x_i}} + e^{3\hat{\theta}x_i+2})}{(e^{2e^{\hat{\theta}x_i}} - e^{2\hat{\theta}x_i+2})^2},$$

$$\hat{l}_{\theta\theta\theta} = \frac{\partial^3 l(\gamma)}{\partial\theta^3} = \frac{2n}{\hat{\theta}^3} + \sum_{i=1}^{n} \frac{x_i^3 e^{\hat{\theta}x_i}(e^{\hat{\theta}x_i}+1)}{(e^{\hat{\theta}x_i}-1)^3} - \sum_{i=1}^{n} 2x_i^3 e^{\hat{\theta}x_i} + 2(\hat{\lambda}-1)\sum_{i=1}^{n} \frac{2x_i^3 e^{2\hat{\theta}x_i+2}}{(e^{2e^{\hat{\theta}x_i}} - e^{2\hat{\theta}x_i+2})^3}$$
$$[(4e^{3\hat{\theta}x_i} - 18e^{2\hat{\theta}x_i} + 19e^{\hat{\theta}x_i} - 4)e^{4e^{\hat{\theta}x_i}} + (4e^{5\hat{\theta}x_i+2} - 6e^{4\hat{\theta}x_i+2} + 4e^{3\hat{\theta}x_i+2} - 4e^{2\hat{\theta}x_i+2})e^{2e^{\hat{\theta}x_i}}],$$

$$\hat{l}_{\lambda\lambda} = \frac{\partial^2 l(\gamma)}{\partial\lambda^2} = \frac{-n}{\hat{\lambda}^2},$$

$$\hat{l}_{\lambda\lambda\lambda} = \frac{\partial^3 l(\gamma)}{\partial\lambda^3} = \frac{2n}{\hat{\lambda}^3},$$

$$\hat{l}_{\theta\lambda} = \frac{\partial^2 l(\gamma)}{\partial\theta\partial\lambda} = 2\sum_{i=1}^{n} \frac{x_i(1-e^{\hat{\theta}x_i})e^{2\hat{\theta}x_i+2}}{e^{2\hat{\theta}x_i+2} - e^{2e^{\hat{\theta}x_i}}} = \hat{l}_{\lambda\theta},$$

$$\hat{l}_{\theta\theta\lambda} = \frac{\partial^3 l(\gamma)}{\partial\theta\theta\lambda} = 2\lambda\sum_{i=1}^{n} \frac{x_i^2 e^{2\hat{\theta}x_i+2}((2e^{2\hat{\theta}x_i} - 5e^{\hat{\theta}x_i} + 2)e^{2e^{\hat{\theta}x_i}} + e^{3\hat{\theta}x_i+2})}{(e^{2\hat{\theta}x_i+2} - e^{2e^{\hat{\theta}x_i}})^2} = \hat{l}_{\theta\lambda\theta} = \hat{l}_{\lambda\theta\theta} \quad \text{and}$$

$$\hat{l}_{\lambda\lambda\theta} = \frac{\partial^3 l(\gamma)}{\partial\lambda\lambda\theta} = 0 = \hat{l}_{\lambda\theta\lambda} = \hat{l}_{\theta\lambda\lambda}.$$

For estimating θ, we took $\psi(\theta,\lambda) = \theta$, thereby $\psi_\theta = 1$ and $\psi_{\theta\theta} = \psi_\lambda = \psi_{\lambda\lambda} = \psi_{\theta\lambda} = 0$. Thus, the approximate Bayes estimator of θ under SELF is obtained as

$$\hat{\theta}_B = \hat{\theta} + 0.5[2\hat{\rho}_\theta\hat{\sigma}_{\theta\theta} + 2\hat{\rho}_\lambda\hat{\sigma}_{\theta\lambda} + \hat{\sigma}^2_{\theta\theta}\hat{L}_{\theta\theta\theta} + 2\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\theta\lambda}\hat{L}_{\theta\lambda\theta} + \hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\lambda\lambda}\hat{L}_{\lambda\lambda\lambda}]. \quad (32)$$

Similarly, for estimating λ, we took $\psi(\theta,\lambda) = \lambda$, thereby $\psi_\lambda = 1$ and $\psi_{\lambda\lambda} = \psi_\theta = \psi_{\theta\theta} = \psi_{\theta\lambda} = 0$.
Hence, the approximate Bayes estimator of λ under SELF is obtained as

$$\hat{\lambda}_B = \hat{\lambda} + 0.5[2\hat{\rho}_\theta\hat{\sigma}_{\lambda\theta} + 2\hat{\rho}_\lambda\hat{\sigma}_{\lambda\lambda} + \hat{\sigma}_{\theta\theta}\hat{\sigma}_{\theta\lambda}\hat{L}_{\theta\theta\theta} + 2\hat{\sigma}^2_{\theta\lambda}\hat{\sigma}_{\theta\lambda}\hat{L}_{\theta\lambda\theta} + \hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\lambda\lambda}\hat{L}_{\lambda\lambda\lambda}]. \quad (33)$$

5 Simulation Study

Here, we conducted an extensive simulation study to illustrate the behaviour of proposed estimators in terms of their mean squared error (MSE) of the parameters θ and λ. We use Monte Carlo simulation method to obtain the average estimates for different combination of parameters and various sample sizes of n= 10, 20, 50 and 100. The experiment is repeated 10,000 times using R software. Table 1 displays the MLEs and their corresponding bias and MSE of the model parameters.
In the case of Bayesian estimation, both non-informative and informative priors under

SELF have been considered for θ and λ. In the case of the non-informative prior (Prior 1), the hyperparameters chosen are zeroes and of informative gamma prior (Prior 2), the hyper parameters considered are ($a_1 = 0.5, b_1 = 1, a_2 = 1.5, b_2 = 1$) and ($a_1 = 2, b_1 = 0.8, a_2 = 1, b_2 = 2$) respectively for the true parameter values ($\theta = 0.5, \lambda = 1.5$) and ($\theta = 2.5, \lambda = 0.5$). Tables 2 and 3 reports the Bayes estimates and their corresponding MSEs of the model parameters.
From Tables 1, 2 and 3, it is evident that the MSE of the parameter estimates of the TL-T distribution decreases as the sample size increases, which confirms the consistency property of both MLE and Bayes estimators. But from Tables 2 and 3, it is inferred that the performance of Bayes estimators are better than MLE for both the parameters on the basis of MSEs. It is also revealed that Bayes estimators with informative prior (Prior 2) outperforms non-informative prior (Prior 1), in terms of their smaller MSEs.

6 Applications

In this section, we analyse two data sets for illustrating the applicability and potentiality of the TL-T distribution and to compare the efficacy of different estimation procedures discussed in Section 4.
The first data set, from Hinkley (1977) which consists of 30 successive values of March precipitation (in inches) in Minneapolis/St. Paul and it is
0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.
The second data set is also a secondary data obtained from Nichols and Padgett (2006) and it consists of 100 observations on tensile strength of carbon fibres (in Gba). The data set is given below.
3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.
In order to detect the shape of the hazard rate function of the data sets, we employ an empirical approach of TTT plot proposed by Barlow and Davis (1977) and they are illustrated in Figure 3. These plots show a concave upward trend, indicating increasing hazard rates for both data sets. Therefore, the TL-T distribution which exhibits increasing hazard function also, it is considered as an appropriate model to fit both the data sets.
We fit the TL-T model to the above two data sets and compare the results with respect to the Exponentiated Teissier (ET) (Sharma et al., 2022), Topp-Leone Frechet (TL-F) (Sapkota, 2021), Exponential transformed inverse Rayleigh (ETIR) (Banerjee and Bhunia, 2022) and Teissier (Teissier, 1934) distributions. The pdfs (for $x > 0$) corresponding

Table 1: The mles and their MSEs of the parameters

n	θ	λ	$\hat{\theta}_M$	Bias(θ)	MSE(θ)	$\hat{\lambda}_M$	Bias(λ)	MSE(λ)
10	0.5	0.5	0.5602	0.0602	0.0206	0.6362	0.1362	0.1173
		1.5	0.5329	0.0329	0.0079	2.2357	0.7357	0.9371
	1.5	0.5	1.6889	0.1889	0.1847	0.6377	0.1377	0.1064
		1.5	1.6027	0.1027	0.0661	2.1261	0.6261	2.6429
	2.5	0.5	2.9458	0.4458	0.5818	0.6436	0.1436	0.1366
		1.5	2.8773	0.1773	0.2052	2.1551	0.6551	3.4603
20	0.5	0.5	0.5281	0.0281	0.0079	0.5654	0.0654	0.0359
		1.5	0.5182	0.0182	0.0033	1.7330	0.2343	0.4226
	1.5	0.5	1.5941	0.0941	0.0737	0.5691	0.0691	0.0331
		1.5	1.5503	0.0503	0.0286	1.7564	0.2564	0.4889
	2.5	0.5	2.7588	0.2588	0.2144	0.5627	0.0627	0.0342
		1.5	2.5811	0.0811	0.0743	1.7195	0.2195	0.4643
50	0.5	0.5	0.5099	0.0090	0.0023	0.5193	0.0193	0.0075
		1.5	0.5056	0.0056	0.0009	1.5764	0.0764	0.0608
	1.5	0.5	1.5364	0.0363	0.0216	0.5221	0.0221	0.0077
		1.5	1.5197	0.0197	0.0094	1.5874	0.0874	0.1050
	2.5	0.5	2.6629	0.1629	0.0752	0.5265	0.0265	0.0094
		1.5	2.5313	0.0313	0.0276	1.5886	0.0886	01054
100	0.5	0.5	0.5055	0.0055	0.0011	0.5105	0.0105	0.0036
		1.5	0.5031	0.0031	0.0004	1.5332	0.0432	0.0127
	1.5	0.5	1.5193	0.0193	0.0105	0.5118	0.0118	0.0038
		1.5	1.5095	0.0095	0.0048	1.5382	0.0382	0.0453
	2.5	0.5	2.5276	0.0276	0.0362	0.5095	0.0095	0.0037
		1.5	2.5110	0.0110	0.0136	1.5348	0.0348	0.0263

to the competitive distributions are respectively, given by

$$\text{ET} : f(x;\lambda,\theta) = \lambda\theta(e^{\theta x}-1)e^{\theta x - e^{\theta x}+1}(1-e^{\theta x - e^{\theta x}+1})^{\lambda-1};$$

$$\text{TLF} : f(x;\lambda,\theta) = 2\lambda\theta x^{-(1+\theta)}e^{-x^{-\theta}}(1-e^{-x^{-\theta}})[1-(1-e^{-x^{-\theta}})^2]^{\lambda-1};$$

$$\text{ETIR} : f(x;\theta) = \frac{2\theta^2}{x^3(e-1)}e^{e^{-(\frac{\theta}{x})^2}}e^{-(\frac{\theta}{x})^2};$$

$$\text{Teissier} : f(x,\theta) = \theta(e^{\theta x}-1)e^{\theta x - e^{\theta x}+1}.$$

Table 2: Average values of estimates and MSEs (in parentheses) for $\theta = 0.5$ and $\lambda = 1.5$.

n	$\hat{\theta}_M$	$\hat{\lambda}_M$	$\hat{\theta}_{B_1}$	$\hat{\lambda}_{B_1}$	$\hat{\theta}_{B_2}$	$\hat{\lambda}_{B_2}$
10	0.5329	2.2357	0.5279	2.0614	0.5267	1.9287
	(0.0079)	(0.9371)	(0.0034)	(0.3153)	(0.0012)	(0.1839)
20	0.5182	1.7330	0.5157	1.7177	0.5141	1.6975
	(0.0033)	(0.4226)	(0.0008)	(0.0974)	(0.0005)	(0.0790)
50	0.5056	1.5764	0.5045	1.5620	0.5029	1.5596
	(0.0009)	(0.0608)	(8.0509×10^{-5})	(0.0252)	(6.0433×10^{-5})	(0.0105)
100	0.5031	1.5332	0.5021	1.5212	0.5013	1.5105
	(0.0004)	(0.0127)	(6.5162×10^{-5})	(0.0097)	(5.5068×10^{-5})	(0.0076)

Table 3: Average values of estimates and MSEs (in parentheses) for $\theta = 2.5$ and $\lambda = 0.5$.

n	$\hat{\theta}_M$	$\hat{\lambda}_M$	$\hat{\theta}_{B_1}$	$\hat{\lambda}_{B_1}$	$\hat{\theta}_{B_2}$	$\hat{\lambda}_{B_2}$
10	2.9458	0.6436	2.7728	0.6385	2.7508	0.6297
	(0.5818)	(0.1366)	(0.2452)	(0.0919)	(0.0942)	(0.0618)
20	2.7588	0.5627	2.6176	0.5551	2.6026	0.5516
	(0.2144)	(0.0342)	(0.0538)	(0.0142)	(0.0327)	(0.0093)
50	2.6629	0.5265	2.5455	0.5247	2.5347	0.5209
	(0.0752)	(0.0094)	(0.0081)	(0.0035)	(0.0039)	(0.0007)
100	2.5276	0.5095	2.5167	0.5059	2.5085	0.5036
	(0.0362)	(0.0037)	(0.0004)	(0.0003)	(0.0002)	(0.0001)

For comparing the models, we compute the values of four goodness of fit measures: AIC (Akaike information criterion), BIC (Bayesian information criterion), Hannan-Quinn Information Criterion (HQIC), Consistent Akaike Information Criterion (CAIC), Kolmogorov-Smirnov (KS) statistic and its associated p-value via MLEs. The model with the smallest values of AIC, BIC, HQIC, CAIC, and KS, alongside the largest p-value indicate a best fit for the given data sets. Table 4 provides the MLEs, AIC, BIC, HQIC, CAIC, KS and its respective p-values of the fitted models for the data sets 1 and 2. Those results are achieved using R software. Furthermore, the fitted pdfs along with the estimated cdfs (ecdf) plots of the TL-T and other competitive models for data sets 1 and 2 are displayed in Figures 4 and 5 respectively. Table 4 and those graphical tools reveal that the TL-T distribution provides a better fit than other distributions to both the data sets.

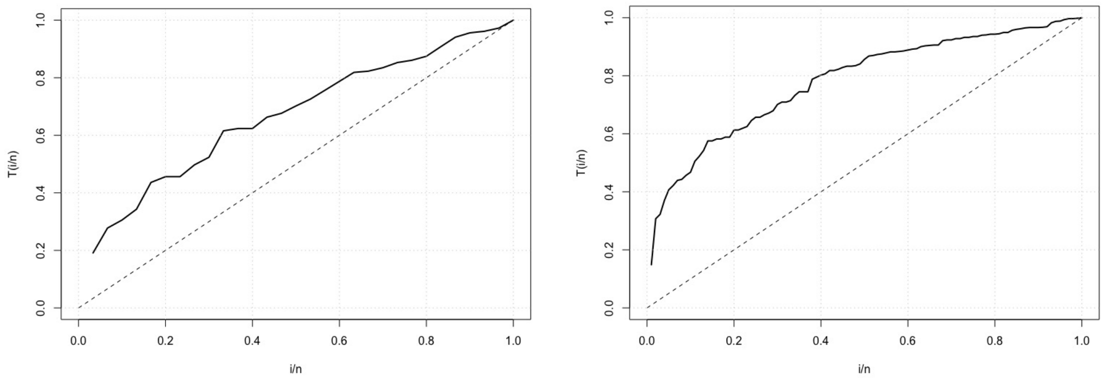


Figure 3: TTT plot for (a) March precipitation data and (b) Glass fibre data.

Table 4: Modal comparison for data sets 1 and 2.

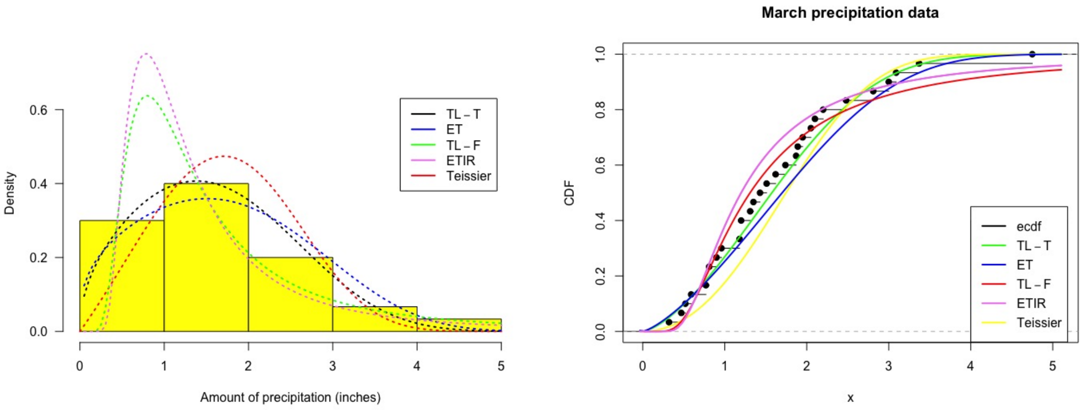
Data set	Models	Estimates	AIC	BIC	HQIC	CAIC	KS	p-value
Data 1	TL-T	θ =0.3984, λ =0.7316	84.6082	87.4106	85.5047	85.0527	0.0697	0.9986
	ET	θ =0.4708, λ =0.6548	84.8408	87.6432	85.7373	85.2852	0.1302	0.6896
	TL-F	θ =1.0545, λ =2.1104	85.1979	88.0004	86.0945	85.6424	0.1392	0.6065
	ETIR	θ =0.8293	86.0526	87.4538	86.5009	86.1955	0.1894	0.2323
	Teissier	θ =0.5643	84.9402	90.3414	89.3885	89.0831	0.1503	0.5069
Data 2	TL-T	θ =0.3222, λ =1.4002	287.1496	292.3599	289.2583	287.2733	0.0610	0.8502
	ET	θ =0.4143, λ =1.2813	287.9556	293.1659	290.0643	288.0793	0.0647	0.7959
	TL-F	θ =1.1460, λ =6.5325	335.8710	341.0813	337.9797	335.9947	0.1991	0.0007
	ETIR	θ =1.5771	333.4762	336.4762	334.9254	333.9118	0.1568	0.0147
	Teissier	θ =0.3927	289.5540	292.1592	290.6084	289.5948	0.0805	0.5359

We further compare the maximum likelihood and Bayesian estimation method utilizing the concept of Pradhan and Kundu (2011) on the basis of KS statistic and its p-value. Due to the absence of prior information, here we consider only Prior 1 for both the parameters. Table 5 presents the MLE, Bayes estimates, KS statistics and the corresponding p-values for both the data sets. It is noted that for considered data sets, Bayes estimates outperforms MLEs in terms of smaller KS and larger p-values.

7 Neutrosophic Topp-Leone Teissier (NTL-T) Distribution

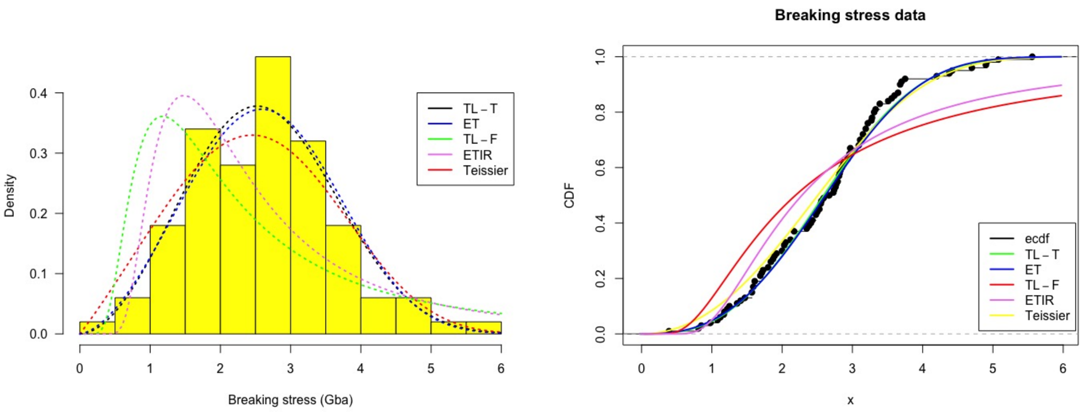
A neutrosophic random variable $Y_N = Y_L + Y_U I_N$ is considered to be NTL-T distributed if its probability density function is

$$f(y_N;\theta_N,\lambda_N) = 2\lambda_N\theta_N(e^{\theta_N y_N}-1)e^{A(y_N,\theta_N)}[1-e^{A(y_N,\theta_N)}]^{\lambda_N-1};\quad y_N,\theta_N,\lambda_N>0 \quad (34)$$



(a) (b)

Figure 4: (a) Fitted densities and (b) empirical and theoretical cdfs for the data set 1.



(a) (b)

Figure 5: (a) Fitted densities and (b) empirical and theoretical cdfs for the data set 2.

Table 5: Parameter estimates, KS and p-values for both data sets

Data set	Method	λ	θ	KS	p-value
Data 1 (n=30)	MLE	0.7316	0.3984	0.0697	0.9986
	Bayes	0.7212	0.3973	0.0684	0.9990
Data 2 (n=100)	MLE	1.4002	0.3222	0.0610	0.8502
	Bayes	1.3851	0.3219	0.0603	0.8605

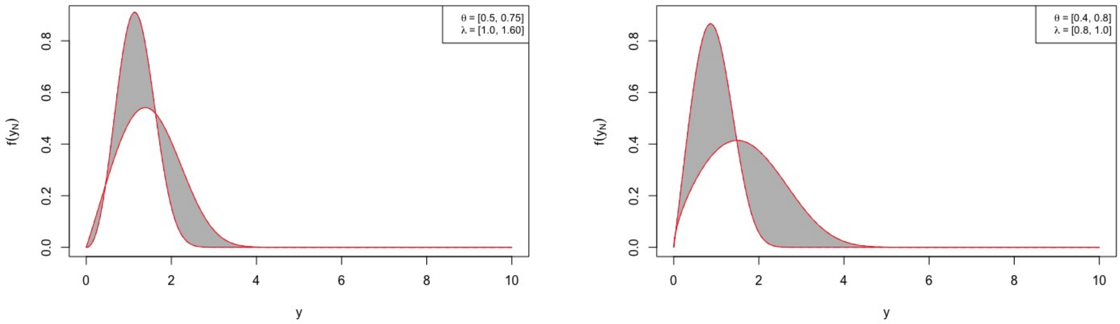


Figure 6: Pdf plot of NTL-T distribution

where $A(y_N, \theta_N) = 2\theta_N y_N - 2(e^{\theta_N y_N} - 1)$.

Here, $\theta_N = \theta_L + \theta_U I_N$ is the neutrosophic scale parameter, $\lambda_N = \lambda_L + \lambda_U I_N$ is the neutrosophic shape parameter, and L and U denote the lower and upper values of the indeterminate parameters respectively.

Also, the cdf of NTL-T distribution is

$$F(y_N; \theta_N, \lambda_N) = [1 - e^{A(y_N, \theta_N)}]^{\lambda_N}; \; y_N, \theta_N, \lambda_N > 0. \tag{35}$$

Sketches of the pdf of NTL-T distribution for selected values of θ_N and λ_N are given in Figure 6.

7.1 Reliability and Statistical Properties

Some reliability and statistical properties of NTL-T distribution are discussed below.

1. **Survival Function**
   Survival function of NTL-T distribution is given by

$$S(y_N; \theta_N, \lambda_N) = 1 - F(y_N; \theta_N, \lambda_N) = 1 - [1 - e^{A(y_N, \theta_N)}]^{\lambda_N}.$$

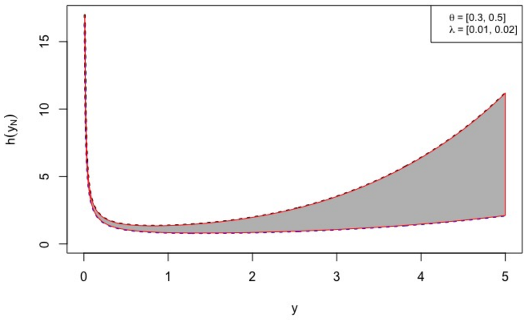


Figure 7: Hazard rate curve for NTL-T distribution.

2. **Hazard rate function**
   Hazard rate function of NTL-T distribution is expressed as

$$h(y_N;\theta_N,\lambda_N)=\frac{2\lambda_N\theta_N(e^{\theta_N y_N}-1)e^{A(y_N,\theta_N)}(1-e^{A(y_N,\theta_N)})^{\lambda_N-1}}{1-(1-e^{A(y_N,\theta_N)})^{\lambda_N}}.$$

   The hazard rate plot of NTL-T distribution is displayed in Figure 7. and it is notable that the hazard rate of NTL-T distribution has a bathtub shape, which is useful to analyze data sets possess the pattern of bathtub hazard rate.

3. **Quantile function**
   The quantile function of NTL-T distribution is given by

$$Q_N(p)=\frac{-1}{\theta}W_{-1}\left(\frac{-\sqrt{1-p^{1/\lambda_N}}}{e}\right)+\frac{1}{2\theta_N}log(1-p^{1/\lambda_N})-\frac{1}{\theta_N}.$$

4. **Moment**
   The r^{th} moment of the NTL-T distribution is given by

$$E_(y_N^r)=2\lambda_N\sum_{j=0}^{\infty}(-1)^j\binom{\lambda_N-1}{j}\frac{e^{2(j+1)}}{\theta_N^r}r!\left[E^r_{-2j-2}(2(j+1))-E^r_{-2j-1}(2(j+1))\right].$$

7.2 Application

Though a lot many continuous distributions have been introduced in the literature so far. They cannot accommodate data from various circumstances, especially with some degree of inexactness. Moreover, there are scenarios where accurate measurements are not available due to the irregular nature of the considered variables. For example, here our newly proposed TL-T distribution is illustrated as a well-suited model for the March precipitation data. But we know that there is a direct relationship between rainfall density and precipitation, viz. the higher the rainfall density, the more precipitation. Besides rainfall density, there are several factors like air temperature, humidity, the presence of mountains, and other topographic features that affect the precipitation. i.e., when the air temperature is warm, the water droplets in the clouds will evaporate more quickly, which will result in a reduced amount of precipitation, and similar is the case with high humidity. So, the precipitation patterns are considered unpredictable, and it is impossible to predict correctly how much precipitation will fall in a specified area over a particular period. So, the precipitation is considered as uncertain data and it is to be treated as such.

If we consider the precipitation data as interval data, the study can be extended to identify the track changes of precipitation patterns concerning time and hence to identify the drought period. Also, it is to be noted that precipitation data involving uncertainties cannot be modeled and analyzed using the TL-T distribution. However, our newly proposed NTL-T distribution will be a suitable model for such precipitation data.

8 Conclusion

In this paper, we proposed a two-parameter lifetime distribution called the Topp-Leone Teissier distribution which is an extension of the Teissier distribution. Being a distribution exhibiting increasing, decreasing and bathtub hazard rate functions, it is capable to model data possessing such hazard rate functions. The explicit expressions for quantile, probability weighted moments, incomplete moments, mean residual life functions, Rényi and Tsallis entropies and order statistics are obtained. The model parameters are estimated via maximum likelihood and Bayesian estimation methods and their performances were validated and compared through a Monte Carlo simulation study. The Bayes estimators are computed under non-informative and informative priors using Lindley approximation technique. The results from simulation study revealed that for both the parameters Bayes estimators perform better than MLEs. The applicability of the TL-T distribution is showcased through its superior performance on two real datasets, outshining other competitive models, with promising applications in fields such as climate studies (e.g., precipitation modeling) and materials science (e.g., tensile strength data modeling). Furthermore, a neutrosophic approach to the TL-T distribution, known as the Neutrosophic Topp-Leone Teissier Distribution is developed and studied its chief statistical properties. This distribution takes into account the indeterminancy and ambiguity of precipitation data. We encourage further exploration of the TL-T distribution to fully uncover its versatility and expand its application across different domains, unlocking new possibilities for data analysis and modeling in diverse disciplines, due to its flexibility in modeling diverse hazard rate functions.

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