Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v17n3p609

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15 December 2024

Electronic Journal of Applied Statistical Analysis
Vol. 17, Issue 03, December 2024, 609-620
DOI: 10.1285/i20705948v17n3p609

E-Bayesian estimation of traffic intensity in a *M/M/1* system using expected posterior risk criteria

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15 December 2024

The strategy of E-Bayesian estimation for traffic intensity in a queueing *M/M/1* system is developed under different loss functions. The Bayesian and E-Bayesian estimators are derived using a power prior density of traffic intensity and a robust prior for the hyperparameter of the prior distribution. The posterior risk of Bayesian estimators and the associated expected posterior risks of traffic intensity are computed for comparison purposes. A Monte Carlo simulation is conducted for performace analysis of the proposed E-Bayesian estimators using expected posterior criteria.

keywords: *M/M/1* system; Robust Bayesian estimation; Traffic intensity.

1 Introduction

Queueing systems are widely used in various fields such as stochastic processes, operations research, and statistical analysis due to their inherent ability to model and analyze complex queues that occur naturally in a variety of scenarios. These scenarios can range from the study of traffic flow in transportation networks to the analysis of customer service in call centers. The versatility of queueing systems stems from their ability to capture the dynamic behavior of queues, including arrival rates, service times, and queue capacities, which allows for the examination of various performance measures such as waiting times, queue lengths, and system utilization. Moreover, the study of queueing systems has been instrumental in addressing real-world problems and optimizing system

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ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

performance by providing insights into resource allocation, capacity planning, and service level agreements. By understanding the fundamental principles and mathematical models underlying queueing systems, researchers and practitioners can make informed decisions and design efficient systems that meet the demands of today's complex and dynamic environments. Therefore, queueing systems play a crucial role in the field of stochastic processes, operations research, and statistical analysis, offering invaluable tools for analyzing and optimizing the performance of various complex queues.

Several categories of queues have been identified, and a standardized notation has been developed to represent them, mostly attributed to Kendall (1953), utilizing a sequence of symbols and slashes like $A/B/X$, where A is the symbol denoting the interarrival-time distribution (for instance, M is utilized to represent exponential or Markovian distribution), B represents the service pattern (e.g., D is used for deterministic or constant service times, M for Markovian, Er for Erlang with r phases and G for general) and X denotes the number of parallel service servers. In classical methods of estimation, Srinivas and Kale (2016) derived the maximum likelihood estimator (*MLE*) and uniformly minimum variance unbiased estimator (*UMVUE*) of traffic intensity in a *M/D/1* queueing model. Yadavalli et al. (2017) computed UMVUE for the expected length in a *M/Er/1* system. Bayesian methods up to select prior density and loss function. Chowdhury and Maiti (2014) assumed the Bayesian estimation of traffic intensity in this model under the squared error loss (*SEL*) and precautionary loss (*PL*) functions. The classical and Bayesian estimation in the *M/D/1* queueing system is considered by Chandrasekhar et al. (2021). Moreover, Yu et al. (2023) studied *UMVUEs* and Bayesian estimators for various performance measures on a Poisson queue with discouraged arrivals.

The system *M/M/1* which represents a single-server queue with Poisson arrival rate $\mu > 0$ and exponential service rate $\lambda > 0$, is the simplest and one of the most applicable queueing models in numerous practical applications. Clarke (1957) computed the *MLE* of the arrival rate and service time. Srinivas et al. (2011) considered *MLE* and *UMVUE* for some measures of this queue system. Armero and Bayarri (1994) have comupted the Bayesian estimator of the traffic intensity. Sharma and Kumar (1999) and Mukherjee and Chowdhury (2005) considered the Bayesian estimation of the traffic intensity under the SEL function and linear-exponential (*LINEX*) loss function, respectively. Dey (2008) discussed the Bayesian estimation of several characteristics under the SEL function. Ren and Wang (2012) used a PL function for finding the Bayesian estimator of the traffic intensity. Shrinkage estimation of the expected length is developed by Kiapour and Naghizadeh Qomi (2019). Quinino and Cruz (2017) studied the problem of Bayesian sample size determination. Singh and Acharya (2019) founded the bound for the equivalence of Bayesian and MLE for the arrival process.

A problem concerning to Bayesian estimation is that a single prior distribution can not reflect personal believes about the parameter of interest. Recently, Kiapour (2022) used a new method called E-Bayesian (expected Bayesian) estimation for estimation in a $M/M/1$ queueing model. In this method, the researcher considers a class of prior distributions reflecting prior knowledge instead of single prior about model parameters of interest.

In this paper, we obtain the E-Bayesian estimators of traffic intensity in a *M/M/1* queue system and comapre them using corresponding expected posterior risks under four loss functions. To do this, the paper is arranged as follows: Section 2 is devoted to Bayesian estimation of traffic intensity. The E-Bayesian estimators of traffic intensity are calculated in Section 3. Expected posterior risks of proposed estimators are given in Section 4. A Monte Carlo simulation study is conducted in Section 5 for evaluation proposed E-Bayesian estimators. Conclusions and discussions is offered in Section 6.

2 Bayesian estimation of traffic intensity

Assume that the random variable Y represents the number of customers in a *M/M/1* system has the geometric distribution with parameter $1-\varrho$ ($Ge(1-\varrho)$), where $\varrho = \frac{\lambda}{\mu} < 1$, represents the traffic intensity. The probability mass function (p.m.f) is given as

$$P(Y = y|\varrho) = (1-\varrho)\varrho^y, \quad 0 < \varrho < 1, \quad y = 0, 1, 2, ... \tag{1}$$

Throughout the paper, let $\mathbf{y} = (y_1, ..., y_n)$ be the sample observations from a geometric distribution with p.m.f. (1). Then, the likelihood function of ϱ based on observed data is given by

$$L(\varrho) = (1-\varrho)^n \varrho^s, \tag{2}$$

where $s = \sum_{i=1}^{n} y_i$. Therefore, the *MLE* of ϱ is obtained as $\hat{\varrho}_{ml} = \frac{s}{n+s}$.

Now, Consider a power distribution as prior distribution for ϱ with probability density function (pdf)

$$\pi^a(\varrho) = a\varrho^{a-1}, \quad 0 < \varrho < 1, \ a > 0. \tag{3}$$

Posterior pdf of ϱ given $\mathbf{y}$ is given by

$$\begin{aligned} \pi^a(\varrho \mid \mathbf{y}) &= \frac{\pi^a(\varrho)L(\varrho)}{\int_0^1 \pi^a(\varrho)L(\varrho)d\varrho} = \frac{a\varrho^{a-1}(1-\varrho)^n\varrho^s}{\int_0^1 a\varrho^{a-1}(1-\varrho)^n\varrho^s d\varrho} \\ &= \frac{\varrho^{s+a-1}(1-\varrho)^n}{B(s+a, n+1)}, \end{aligned} \tag{4}$$

where $B(u, v) = \int_0^1 x^{u-1}(1-x)^{v-1}dx$ denotes the beta function.

For computing the Bayesian estimation of ϱ, we consider four loss functions:

1. Squared error loss (SEL) function: $L_1(\varrho, \hat{\varrho}) = (\hat{\varrho} - \varrho)^2$,
2. Weighted squared error loss (WSEL) function: $L_2(\varrho, \hat{\varrho}) = (\hat{\varrho} - \varrho)^2/\varrho$,
3. Precautionary loss (PL) function: $L_3(\varrho, \hat{\varrho}) = (\hat{\varrho} - \varrho)^2/\hat{\varrho}$,
4. K-loss (KL) function: $L_4(\varrho, \hat{\varrho}) = \hat{\varrho}/\varrho + \varrho/\hat{\varrho} - 2$.

The following Lemma due to Han (2020) provides the Bayesian estimation of ϱ and its posterior risk under loss functions $L_i(\varrho, \hat{\varrho}),\ i = 1, 2, 3, 4$.

Lemma 1. Let $\pi^a(\varrho)$ and $\pi^a(\varrho|\mathbf{y})$ be the prior and posterior densities of ϱ, respectively.

Then, we have the following results:
(i) Under SEL function, the Bayesian estimation of ϱ is

$$\hat{\varrho}_{B1}(\mathbf{y}) = E(\varrho|\mathbf{y}),$$

and the corresponding expected posterior risk is as

$$PR_1(\hat{\varrho}) = Var(\varrho|\mathbf{y}).$$

(ii) Under WSEL function, the Bayesian estimation of ϱ and its posterior risks are respectively as

$$\hat{\varrho}_{B2}(\mathbf{y}) = E[\varrho^{-1}|\mathbf{y}]^{-1},$$

and

$$PR_2(\hat{\varrho}) = E(\varrho|\mathbf{y}) - E[\varrho^{-1}|\mathbf{y}]^{-1}.$$

(iii) Under PL function, the Bayesian estimation of ϱ is computed as

$$\hat{\varrho}_{B3}(\mathbf{y}) = \sqrt{E[\varrho^2|\mathbf{y}]},$$

and the associated expected posterior risk is given by

$$PR_3(\hat{\varrho}) = 2[\sqrt{E(\varrho^2|\mathbf{y})} - E(\varrho|\mathbf{y})].$$

(iv) Under KL function, the Bayesian estimation of ϱ is given by

$$\hat{\varrho}_{B4}(\mathbf{y}) = \sqrt{\frac{E(\varrho|\mathbf{y})}{E(\varrho^{-1}|\mathbf{y})}}$$

and its expected posterior risk is of the form

$$PR_4(\hat{\varrho}) = 2[\sqrt{E(\varrho|\mathbf{y})E(\varrho^{-1}|\mathbf{y})} - 1].$$

3 E-Bayesian estimation of traffic intensity

Consider the power prior distribution presented in (3). For $0 < a < 1$, we have $\frac{d\pi^a(\varrho)}{d\varrho} = a(a-1)\varrho^{a-2} < 0$ and $\pi^a(\varrho)$ is a decreasing function of ϱ, so this prior distribution meets the structure of hierarchical prior distribution proposed by Han (1997). Following Han (1997), the expected Bayesian (E-Bayesian) estimation of ϱ is defined as

$$\hat{\varrho}_{EB} = \int_{\mathcal{D}} \hat{\varrho}_B(a)\pi(a)da = E[\hat{\varrho}_B(a)], \tag{5}$$

where $\mathcal{D}$ is the domain of hyperparameter a and $\pi(a)$ is the prior density of a over $\mathcal{D}$.

In the following Lemma, we compute the E-Bayesian estimators of ϱ under the proposed loss functions.

Lemma 2. Assuming a uniform (0,1) distribution for hyperparameter a, the Bayesian and E-Bayesian estimators of ϱ under different loss functions are obtained as follows:
(i) Under the SEL function, the Bayesian estimator of ϱ is given by

$$\hat{\varrho}_{B1}(a) = \frac{s+a}{s+a+n+1}, \tag{6}$$

and the corresponding E-Bayesian estimator is

$$\hat{\varrho}_{EB1} = 1-(n+1)\ln\left(\frac{s+n+2}{s+n+1}\right). \tag{7}$$

(ii) Using the WSEL function, the Bayesian estimator of ϱ is computed as

$$\hat{\varrho}_{B2} = \frac{s+a-1}{s+a+n}, \tag{8}$$

and the corresponding E-Bayesian estimator is

$$\hat{\varrho}_{EB2} = 1-(n+1)\ln\left(\frac{s+n+1}{s+n}\right). \tag{9}$$

(iii) For the PL function, the Bayesian estimator of ϱ is

$$\hat{\varrho}_{B3} = \sqrt{\frac{(s+a+1)(s+a)}{(s+a+n+2)(s+a+n+1)}}, \tag{10}$$

and the associated E-Bayesian estimator is obtained as

$$\hat{\varrho}_{EB3} = \int_0^1 \sqrt{\frac{(s+a+1)(s+a)}{(s+a+n+2)(s+a+n+1)}}\, da. \tag{11}$$

(iv) For the PL function, the Bayesian estimator of ϱ is

$$\hat{\varrho}_{B4} = \sqrt{\frac{(s+a)(s+a-1)}{(s+a+n)(s+a+n+1)}}, \tag{12}$$

and the associated E-Bayesian estimator is obtained as

$$\hat{\varrho}_{EB4} = \int_0^1 \sqrt{\frac{(s+a)(s+a-1)}{(s+a+n)(s+a+n+1)}}\, da. \tag{13}$$

Proof. (i) Using the posterior density of ϱ given in (4), the Bayesian estimation of ϱ under the SEL function is calculated as

$$\hat{\varrho}_{B1} \quad = \quad E(\varrho|\mathbf{y}) = \int_0^1 \varrho\pi^a(\varrho \mid \mathbf{y})d\varrho = \frac{\int_0^1 \varrho^{s+a}(1-\varrho)^n}{B(s+a,n+1)}$$

$$= \frac{B(s+a+1,n+1)}{B(s+a,n+1)} = \frac{s+a}{s+a+n+1}.$$

Under SEL function and the prior density of a, the E-Bayesian estimator of ϱ will be

$$\begin{aligned}
\hat{\varrho}_{EB1} &= \int \hat{\varrho}_{B1}(a)\pi(a)da = \int_0^1 \frac{s+a}{s+a+n+1} \times 1 \quad da \\
&= 1-(n+1)\int_0^1 \frac{1}{s+a+n+1} \quad da \\
&= 1-(n+1)\ln\left(\frac{s+n+2}{s+n+1}\right).
\end{aligned}$$

(ii). We have

$$\begin{aligned}
E(\varrho^{-1} \mid \mathbf{y}) &= \int_0^1 \varrho^{-1}\pi^a(\varrho \mid \mathbf{y})d\varrho = \int_0^1 \frac{\varrho^{-1}\varrho^{s+a-1}(1-\varrho)^n}{B(s+a,n+1)}d\varrho \\
&= \frac{B(s+a-1,n+1)}{B(s+a,n+1)} = \frac{s+a+n}{s+a-1}.
\end{aligned}$$

Therefore, the Bayesian estimation of ϱ under the WSEL function is given by

$$\hat{\varrho}_{B2} = [E(\varrho^{-1} \mid \mathbf{y})]^{-1} = \frac{s+a-1}{s+a+n}.$$

The E-Bayesian estimator of ϱ can be obtained as

$$\begin{aligned}
\hat{\varrho}_{EB2} &= \int \hat{\varrho}_{B2}\pi(a)da = \int_0^1 \frac{s+a-1}{s+a+n} \quad da \\
&= 1-(n+1)\int_0^1 \frac{1}{s+n+a}\, da \\
&= 1-(n+1)\ln\left(\frac{s+n+1}{s+n}\right).
\end{aligned}$$

(iii). Using the fact

$$\begin{aligned}
E(\varrho^2 \mid \mathbf{y}) &= \int \varrho^2\pi^a(\varrho \mid \mathbf{y})d\varrho = \frac{\int_0^1 \varrho^2\varrho^{s+a-1}(1-\varrho)^n}{B(s+a,n+1)}d\varrho \\
&= \frac{(s+a+1)(s+a)}{(s+a+n+2)(s+a+n+1)},
\end{aligned}$$

the Bayesian estimation of ϱ under the PL function is given by

$$\hat{\varrho}_{B3} = \sqrt{E(\varrho^2 \mid \mathbf{y})} = \sqrt{\frac{(s+a+1)(t+a)}{(s+a+n+2)(s+a+n+1)}}.$$

Therefore, the E-Bayesian estimator of ϱ is as

$$\hat{\varrho}_{EB3} = \int \hat{\varrho}_{B3}\pi(a)da = \int_0^1 \sqrt{\frac{(s+a+1)(s+a)}{(s+a+n+2)(s+a+n+1)}}\, da.$$

(iv). Using the relations $E(\varrho|\mathbf{y})$ and $E(\varrho^{-1}|\mathbf{y})$ computed in sections (i) and (ii) of proof, the the Bayesian estimation of ϱ under the KL function is given by

$$\hat{\varrho}_{B4} = \sqrt{\frac{E(\varrho \mid \mathbf{y})}{E(\varrho^{-1} \mid \mathbf{y})}} = \sqrt{\frac{(s+a)(s+a-1)}{(s+a+n)(s+a+n+1)}}.$$

Thus, the associated E-Bayesian estimator of ϱ is obtained to be

$$\hat{\varrho}_{EB4} = \int \hat{\varrho}_{B4}\pi(a)da = \int_0^1 \sqrt{\frac{(s+a)(s+a-1)}{(s+a+n)(s+a+n+1)}}\, da.$$

4 Expected posterior risk of E-Bayesian estimators

In this section, we compute the expected posterior risk (E-posterior risk) of E-Bayesian estimators for comparison purposes. Following Han (2021), the E-posterior risk is defined as

$$ER(\hat{\varrho}_{EB}) = \int_{\mathcal{D}} PR(\hat{\varrho}_B)\pi(a)da,$$

where $PR(\hat{\varrho}_B)$ is the posterior risk of Bayesian estimator $\hat{\varrho}_B$.

In the following Lemma, the E-posterior risk of E-Bayesian estimators are computed under different loss functions.

Lemma 3. The posterior risk of Bayesian estimators and the corresponding E-posterior risks under the proposed loss functions are given as foloows:
(i). Under the SEL function, the posterior risk of Bayesian estimator $\hat{\varrho}_{B1}$ is

$$PR(\hat{\varrho}_{B1}) = \frac{(s+a)(n+1)}{(s+a+n+2)(s+a+n+1)},$$

and the corresponding E-posterior risk is given by

$$ER(\hat{\varrho}_{EB1}) = \int_0^1 \frac{(s+a)(n+1)}{(s+a+n+2)(s+a+n+1)}da.$$

(ii). The posterior risk of Bayesian estimator $\hat{\varrho}_{B2}$ using the WSEL function is

$$PR(\hat{\varrho}_{B2}) = \frac{n+1}{(s+a+n+1)(s+a+n)},$$

and the corresponding E-posterior risk is given by

$$ER(\hat{\varrho}_{EB2}) = \int_0^1 \frac{n+1}{(s+a+n+1)(s+a+n)}da.$$

(iii). The posterior risk of Bayesian estimator $\hat{\varrho}_{B3}$ using the PL function is

$$PR(\hat{\varrho}_{B3}) = 2\left[\sqrt{\frac{(s+a+1)(s+a)}{(s+a+n+2)(s+a+1)}} - \frac{s+a}{s+a+n+1}\right]$$

and its E-posterior risk is given by

$$ER(\hat{\varrho}_{EB3}) = 2\int_0^1 \left[\sqrt{\frac{(s+a+1)(s+a)}{(s+a+n+2)(s+a+n+1)}} - \frac{s+a}{s+a+n+1}\right] da.$$

(iv). The posterior risk of Bayesian estimator $\hat{\varrho}_{B3}$ under the KL function is

$$PR(\hat{\varrho}_{B4}) = 2\left[\sqrt{\frac{(s+a)(s+a+n)}{(n+a+1)(s+a-1)}} - 1\right],$$

and its E-posterior risk is given by

$$ER(\hat{\varrho}_{EB4}) = 2\int_0^1 \left[\sqrt{\frac{(s+a)(s+a+n)}{(n+a+1)(s+a-1)}} - 1\right] da.$$

Proof. (i). Under SEL function we have,

$$\begin{aligned} PR(\varrho_{B1}) = Var(\varrho \mid \mathbf{y}) &= E(\varrho^2 \mid \mathbf{y}) - (E(\varrho \mid \mathbf{y}))^2 \\ &= \frac{(s+a+1)(t+a)}{(n+s+a+2)(n+s+a+1)} - \left(\frac{s+a}{s+a+n+1}\right)^2 \\ &= \frac{(s+a)(n+1)}{(s+a+n+2)(s+a+n+1)^2}. \end{aligned}$$

Therefore, the E-posterior risk is given by

$$ER(\hat{\varrho}_{EB1}) = \int_0^1 PR(\hat{\varrho}_{B1})\pi(a)da = \int_0^1 \frac{(s+a)(n+1)}{(s+a+n+2)(s+a+n+1)^2} da.$$

(ii). The E-posterior risk of E-Bayesian estimator $\hat{\varrho}_{EB2}$ under WSEL function is given by

$$\begin{aligned} R(\hat{\varrho}_{B2}) &= E(\varrho \mid \mathbf{y}) - (E(\varrho \mid \mathbf{y}))^{-1} \\ &= \frac{s+a}{s+n+a+1} - \frac{s+a-1}{s+a+n} = \frac{n+1}{(s+a+n+1)(s+a+n)}. \end{aligned}$$

Thus, the E-posterior risk is obtained to be

$$ER(\hat{\varrho}_{EB2}) = \int_0^1 PR(\hat{\varrho}_{B2})\pi(a)da = \int_0^1 \frac{n+1}{(s+a+n+1)(s+a+n)}.da.$$

(iii). Using the PL function, the E-posterior risk of E-Bayesian estimator $\hat{\varrho}_{EB3}$ is given by

$$\begin{aligned} PR(\hat{\varrho}_{B3}) &= 2[\sqrt{E(\varrho^2 \mid \mathbf{y})} - E(\varrho \mid \mathbf{y})] \\ &= = 2\left[\sqrt{\frac{(s+a+1)(s+a)}{(s+a+n+2)(s+a+1)}} - \frac{s+a}{s+a+n+1}\right] \end{aligned}$$

Thus, the E-posterior risk is obtained to be

$$ER(\hat{\varrho}_{EB3}) = 2\int_0^1 \left[\sqrt{\frac{(s+a+1)(s+a)}{(s+a+n+2)(s+a+n+1)}} - \frac{s+a}{s+a+n+1}\right] da.$$

(iv). Similar to the above, under KL function, we get

$$\begin{aligned} PR(\hat{\varrho}_{B4}) &= 2\left[\sqrt{E(\varrho \mid \mathbf{y})E(\varrho^{-1} \mid \mathbf{y})} - 1\right] \\ &= 2\left[\sqrt{\frac{s+a}{s+a+n+1} \times \frac{s+a+n}{s+a-1}} - 1\right]. \end{aligned}$$

The associated E-posterior risk can be calculated as

$$ER(\hat{\varrho}_{EB4}) = 2\int_0^1 \left[\sqrt{\frac{(s+a)(s+a+n)}{(s+a+n+1)(s+a-1)}} - 1\right] da.$$

5 Comparison of proposed estimators

In this section, we conduct a simulation study to campare the E-Bayesian estimators of ϱ in terms of the expected posterior measure. To do this, the following steps are considered:

(1) Generate random observations $y_1, ..., y_n$ from the $Ge(1-\varrho)$ distribution with p.m.f. given in (1) with $\varrho = 0.4, 0.6, 0.8$.

(2) Compute the E-Bayesian estimates $\hat{\varrho}_{EBi},\ i = 1, 2, 3, 4$ and corresponding expected posterior risks for $n = 5, 15, 40, 80, 100$.

(3) Repeat the above steps $M = 10^4$ times and compute the E-Bayesian estimates (EB) and associated estimated expected posterior risks (EER) respectively as

$$EB(\hat{\varrho}_{EBk}) = \frac{1}{M}\sum_{i=1}^{M} \hat{\varrho}_{EBk}, \quad EER(k) = \frac{1}{M}\sum_{i=1}^{M} ER(\hat{\varrho}_{EBk}).$$

Tables 1-3 present the values of $\hat{\varrho}_{EBi},\ i = 1, 2, 3, 4$ and the associated estimated expected posterior risks for different values of traffic intensity. In view of these tables, we conclude the following observations:

Table 1: The E-Bayesian estimates (EB) and associated estimated expected posterior risks (EER) based on true value of $\varrho = 0.4$.

n	$\hat{\varrho}_{EB1}$	$\hat{\varrho}_{EB2}$	$\hat{\varrho}_{EB3}$	$\hat{\varrho}_{EB4}$	$EER(1)$	$EER(2)$	$EER(3)$	$EER(4)$
10	0.372754	0.333505	0.389552	0.352042	0.011674	0.039249	0.033598	0.201597
20	0.386834	0.368021	0.395559	0.377267	0.006444	0.018814	0.017449	0.059349
40	0.391998	0.382753	0.396453	0.387344	0.003410	0.009244	0.008911	0.025609
60	0.393607	0.387477	0.396599	0.390529	0.002316	0.006130	0.005983	0.016413
80	0.396041	0.391481	0.398280	0.393754	0.001752	0.004560	0.004478	0.011962
100	0.396098	0.392450	0.397895	0.394270	0.001410	0.003647	0.003595	0.009491

Table 2: The E-Bayesian estimates (EB) and associated estimated expected posterior risks (EER) based on true value of $\varrho = 0.6$.

n	$\hat{\varrho}_{EB1}$	$\hat{\varrho}_{EB2}$	$\hat{\varrho}_{EB3}$	$\hat{\varrho}_{EB4}$	$EER(1)$	$EER(2)$	$EER(3)$	$EER(4)$
10	0.560471	0.541036	0.569243	0.550584	0.009062	0.019435	0.017543	0.047515
20	0.581497	0.572698	0.585692	0.577075	0.004666	0.008798	0.008391	0.016672
40	0.589563	0.585351	0.591622	0.587453	0.002383	0.004212	0.004118	0.007406
60	0.591822	0.589044	0.593190	0.590431	0.001599	0.002778	0.002737	0.004804
80	0.594188	0.592127	0.595206	0.593157	0.001198	0.002060	0.002037	0.003532
100	0.594775	0.593131	0.595590	0.593953	0.000961	0.001644	0.001629	0.002806

(1) For each sample size n, the values $\hat{\varrho}_{EBi},\ i = 1, 2, 3, 4$ are close to each other. Moreover, the estimated values of E-Bayesian estimates approaches the actual value of the traffic intensity when n increases.
(2) We get the following relation:

$$\hat{\varrho}_{EB2} < \hat{\varrho}_{EB4} < \hat{\varrho}_{EB1} < \hat{\varrho}_{EB3}.$$

(3) The values of EER decreases when the sample size increases.
(4) For fixed n and ϱ, we have the following relation for EER values:

$$EER(1) < EER(3) < EER(2) < EER(4).$$

Thus, the E-Bayesian estimate $\hat{\varrho}_{EB1}$ is better than other E-Bayesian estimates in terms of expected posterior risk measure.

6 Concluding remarks

Th Bayesian, E-Bayesian and corresponding E-posterior risks of traffic intensity in a *M/M/1* system was presented in this paper. The Bayesian estimations are derived using a power prior distribution under four loss functions and the related E-Bayesian estimations are obtained using a robust prior of hyperparameter of considered prior distribution. The E-posterior risks of E-Bayesian estimators are provided. A comparison

Table 3: The E-Bayesian estimates (EB) and associated estimated expected posterior risks (EER) based on true value of $\varrho = 0.8$.

n	$\hat{\varrho}_{EB1}$	$\hat{\varrho}_{EB2}$	$\hat{\varrho}_{EB3}$	$\hat{\varrho}_{EB4}$	$EER(1)$	$EER(2)$	$EER(3)$	$EER(4)$
10	0.769207	0.763789	0.771776	0.766490	0.003733	0.005418	0.005138	0.007694
20	0.785087	0.782762	0.786223	0.783923	0.001739	0.002325	0.002272	0.003066
40	0.792321	0.791239	0.792856	0.791780	0.000838	0.001082	0.001070	0.001387
60	0.794777	0.794074	0.795127	0.794426	0.000551	0.000703	0.000698	0.000894
80	0.796602	0.796085	0.796860	0.796343	0.000408	0.000518	0.000515	0.000655
100	0.797358	0.796947	0.797562	0.797152	0.000325	0.000411	0.000409	0.000518

of E-Bayesian estimations in terms of E-posterior risks is performed. The results show that the E-Bayesian estimation associated with SEL function has good performance and can be preferd.

Acknowledgements

The authors thank the Editor and Reviewers for providing valuable comments and suggestions.

Disclosure statement

The authors declare no conflict of interest.

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