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The Extended Kumaraswamy Generated Family: Properties, Inference and Applications in Applied Fields

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In this paper, a new Kumaraswamy generalized family is proposed. Four sub-models of the proposed family accommodate symmetrical, left-skewed, bimodal, right-skewed unimodal, and reverse-J densities, as well as increasing, modified bathtub, decreasing, bathtub, upside down bathtub, reverse J and J shaped hazard rates. Its fundamental properties are derived. The maximum likelihood method and seven other methods are used for estimating the model parameters. Numerical simulations are performed to explore the performance of these estimation methods. Three real-life data sets from medicine, agriculture and engineering are fitted to illustrate the flexibility of the proposed family. The proposed family is a good alternative to the Kumaraswamy-G, beta-G, and Topp-Leone-G families.

keywords: Exponential distribution; Generating functions; Maximum likelihood; Moments; Simulation; Stochastic ordering; Weibull distribution.

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1 Introduction

Recently, a meaningful concentration has been paid to the construction of generalized families and trigonometric probability distributions. These generalizations provide great flexibility in the modeling of real-life phenomena based on data. They involve some transformations that are possibly modulated by one or more shape parameters.

Many researchers proposed generalized families of and trigonometric distributions. For example, the tan-G by Ampadu Ampadu (2019), generalized raised cosine distribution by Ahsanullah et al. Ahsanullah et al. (2019), beta-G by Eugene et al. Eugene et al. (2002), Kumaraswamy-G (Kw-G) by Cordeiro and de Castro Cordeiro and de Castro (2011), McDonald-G by Alexander et al. Alexander et al. (2012), odd exponentiated generalized by Cordeiro et al. Cordeiro et al. (2013), transformed-transformer (T-X) by Alzaatreh et al. Alzaatreh et al. (2013), Weibull-G by Bourguignon et al. Bourguignon et al. (2014), complementary generalized transmuted Poisson-G by Alizadeh et al. (2018), generalized Burr XII power series by Elbatal et al. (2019), Kumaraswamy alpha-power-G by Mead et al. (2020), Marshall–Olkin Burr-III by Afify et al. (2020), Marshall–Olkin Weibull-H by Afify et al. (2022), and modified generalized-G family by Shama et al. (2023), among others.

If T is a Kumaraswamy random variable with probability density function (pdf) $f_T(x) = a\,b\,t^{b-1}\,(1-t^a)^{b-1}$ and cumulative distribution function (cdf) $F_T(x) = 1 - (1-t^a)^{b-1}$, then the Kw-G family (Cordeiro and de Castro Cordeiro and de Castro (2011)) is defined by the cdf

$$F(x) = 1 - (1 - W[G(x)]^a)^b\,, \quad x > 0,\ a, b > 0. \tag{1}$$

The corresponding pdf is

$$f(x) = a\,b\,W[G(x)]^{a-1}W'[G(x)]\,(1 - W[G(x)]^a)^{b-1}\,, \tag{2}$$

where $\frac{d}{dx}W[G(x)] = W'[G(x)]$.

In this paper, we define a new generator, called the Kumaraswamy tan generalized (KwT-G) family, using the trigonometric function $\exp\left\{-\tan\left(\frac{\pi G(x)}{2}\right)\right\}$. The KwT-G family provides more flexibility of the Kumaraswamy distribution as well as the T-X family. Its key features are derived.

We are also motivated to introduce the KwT-G family due to its flexibility which can provide unimodal, bimodal, asymmetric left, symmetrical, asymmetric right, and reverse-J pdfs as well as modified bathtub, increasing, bathtub, decreasing, upside down bathtub, reverse J and J shaped hazard rates. Therefore, the KwT-G family can model different types of failure data.

Furthermore, estimation of the sub-model KwT-exponential (KwT-E) parameters using eight classical estimation approaches including the least squares estimators (LSEs), maximum product of spacing estimators (MPSEs), weighted least-squares estimators (WLSEs), maximum likelihood estimators (MLEs), Anderson–Darling estimators (ADEs), percentiles estimators (PCEs), right-tail Anderson–Darling estimators (RADEs), and Cramér-von Mises estimators (CVMEs).

Additionally, a sub-model of the KwT-G family, called the KwT-Weibull (KwT-W) distribution, offers adequate adjustments than other modified Weibull models generated by other existing families. Hence, the KwT-G family is a competitive family to the most cited papers in the literature on distribution theory such as beta-G (Eugene et al., 2002) and Kumaraswamy-G (Cordeiro and de Castro, 2011) families, which also provide two additional shape parameters.

To this end, the KwT-G family is specified by the cdf (for $x \in \mathbb{R}$, $a > 0$, $b > 0$)

$$\begin{aligned} F_{\text{KwT-G}}(x) &= \int_0^{e^{-\left[\tan\left(\frac{\pi \overline{G}(x)}{2}\right)\right]}} a\, b\, t^{b-1}\, (1-t^a)^{b-1}\, dt \\ &= 1 - \left\{1 - e^{-a\left[\tan\left(\frac{\pi \overline{G}(x;\boldsymbol{\xi})}{2}\right)\right]}\right\}^b . \end{aligned} \quad (3)$$

where $G(x;\boldsymbol{\xi})$ is a baseline cdf depending the parameter vector $\boldsymbol{\xi}$, $\overline{G}(x;\boldsymbol{\xi}) = 1 - G(x;\boldsymbol{\xi})$ is the survival function, and a and b are two shape parameters. The corresponding pdf is

$$\begin{aligned} f_{\text{KwT-G}}(x) &= \frac{\pi}{2}\, a\, b\, g(x;\boldsymbol{\xi})\, e^{-a\left[\tan\left(\frac{\pi \overline{G}(x;\boldsymbol{\xi})}{2}\right)\right]} \left\{1 - e^{-a\left[\tan\left(\frac{\pi \overline{G}(x;\boldsymbol{\xi})}{2}\right)\right]}\right\}^{b-1} \\ &= \left\{\sec^2\left[\frac{\pi\,(\overline{G}(x;\boldsymbol{\xi}))}{2}\right]\right\}, \quad x \in \mathbb{R},\ a, b > 0, \end{aligned} \quad (4)$$

where $g(x;\boldsymbol{\xi})$ is the baseline pdf. The hazard rate function (hrf) of the KwT-G distribution is

$$\begin{aligned} h(x) &= \frac{\pi}{2}\, a\, b\, g(x;\boldsymbol{\xi})\, e^{-a\left[\tan\left(\frac{\pi \overline{G}(x;\boldsymbol{\xi})}{2}\right)\right]} \left\{1 - e^{-a\left[\tan\left(\frac{\pi \overline{G}(x;\boldsymbol{\xi})}{2}\right)\right]}\right\}^{-1} \\ &= \left\{\sec^2\left[\frac{\pi\,(\overline{G}(x;\boldsymbol{\xi}))}{2}\right]\right\}. \end{aligned} \quad (5)$$

The quantile function (qf) of the KwT-G distribution follows by inverting (3). If u has a Kumaraswamy (a, b) distribution, then

$$X = Q_G\left\{1 - \frac{2}{\pi}\tan^{-1}\left[\ln\left(1 - \{1-u\}^{\frac{1}{b}}\right)^{\frac{-1}{a}}\right]\right\} \quad (6)$$

has the pdf (4).

This paper is organized as follows. Section 2 provides some key features of the KwT-G family. Section 3 presents four sub-models of the family. Eight estimation approaches of the KwT-E parameters are explored in Section 4. The performance of these estimators are assessed using simulation results in Section 5. Three real-life data applications are used to explore the applicability of the KwT-G family in Section 6. Section 7 presents final remarks.

2 Mathematical properties

We present some key mathematical features of the KwT-G family.

2.1 Linear expansions

In this section, we provide a linear representation for the KwT-G pdf in terms of exponentiated-G (exp-G) densities.

The cdf of the KwT-G family (3) is expressed as

$$\begin{aligned} F(x) &= \sum_{k=1}^{\infty} \sum_{l,i,j=0}^{\infty} \frac{(-1)^{k+l+j+1}}{l!} (ak)^l \binom{b}{k} \binom{2i+l}{j} \\ & C_i(l) \, (\frac{\pi}{2})^{2i+l} \, [G(x)]^j. \end{aligned} \tag{7}$$

Then,

$$F(x) = \sum_{j=0}^{\infty} V_{k,l,i} \, H_j(x),$$

where

$$V_{k,l,i} = \sum_{k=1}^{\infty} \sum_{l,i=0}^{\infty} \frac{(-1)^{(k+l+j+1)}}{l!} (ak)^l \binom{b}{k} \binom{2i+l}{j} C_i(l) \, (\frac{\pi}{2})^{2i+l}. \tag{8}$$

The following power series can be calculated via Mathematica

$$\left\{ tan \left[\frac{\pi G(x)}{2} \right] \right\}^l = \sum_{i=0}^{\infty} C_i(l) \left[\frac{\pi G(x)}{2} \right]^{(2i+l)}, \tag{9}$$

where $C_0(l) = 1, C_1(l) = \frac{l}{3}, C_2(l) = \frac{l(5\,l+7)}{90}$, etc.

Hence, the pdf of the KwT-G family can be expressed as

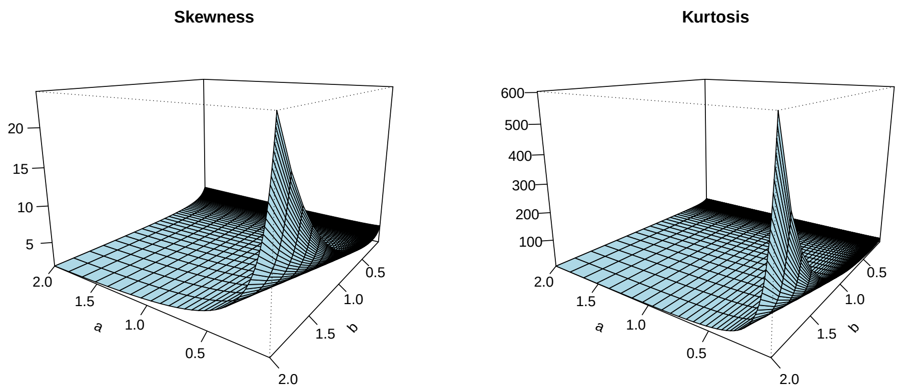
$$f(x) = \sum_{j=0}^{\infty} v_{k,l,i} \, h_j(x), \tag{10}$$

where $h_j(x)$ denotes the exp-G pdf with power parameter $j > 0$. Equation (10) shows that the KwT-G pdf is expressed as linear combination of exp-G densities. So, its mathematical features are found directly from exp-G properties.

2.2 Moments

Let Y be a random variable having the baseline cdf $G(y)$. The moments of the KwT-G distribution follow from the (r, k)th probability weighted moment (PWM) of Y defined by Greenwood et al. (1979)

$$\tau_{r,k} = \mathrm{E}[Y^r \, G(Y)^k] = \int_{-\infty}^{\infty} x^r \, G(x)^k \, g(x) dx.$$



(a) (b)

Figure 1: Plots of the skewness and kurtosis for the KwT-W distribution.

Using (10), we can write

$$\mu_r' = \mathrm{E}(X^r) = \sum_{j=0}^{\infty} v_{k,l,i}\, h_j(x)\tau_{r,j}, \tag{11}$$

where $\tau_{r,j} = \int_0^1 Q_G(u)^r\, u^j du$ can be determined numerically for any baseline qf.

The measures of skewness and kurtosis of the KT-W distribution depend on moments. They are displayed in Figure 1.

2.3 Moment generating function

The mgf of the KwT-G family follows from (10) as

$$M(s) = \sum_{j=0}^{\infty} v_{k,l,i}\, M_j(s), \tag{12}$$

where $M_j(s)$ denotes the exp-G mgf with power parameter j. (12) can also be expressed as

$$M(s) = \sum_{j=0}^{\infty} (j)\ v_{k,l,i}\, \rho_j(s), \tag{13}$$

where $\rho_j(s) = \int_0^1 \exp\left[s\, Q_G(u)\right] u^j du$ can be computed numerically.

2.4 Mean deviations

The nth incomplete moments (INM) is defined by $m_n(y) = \int_{-\infty}^{y} x^r f(x)dx$. The nth INM of the KwT-G distribution is expressed as

$$m_n(y) = \sum_{j=0}^{\infty} v_{k,l,i}\,(j) \int_{0}^{G(y;\xi)} Q_G(u)^n\, u^j\, du. \tag{14}$$

The integral in (14) can be determined numerically for most baseline distributions.

The mean deviations about the mean (say $\delta_1 = E(|X-\mu_1'|)$) and about the median (say $\delta_2 = E(|X-M|)$) of X can be expressed, respectively, as

$$\delta_1 = 2\mu_1'\,F\left(\mu_1'\right) - 2m_1\left(\mu_1'\right) \qquad \text{and} \qquad \delta_2 = \mu_1' - 2m_1(M), \tag{15}$$

where $M = Q(0.5)$, $\mu_1' = \mathrm{E}(X)$ follows from (11), $F(\mu_1')$ is calculated from (3) and $m_1(z) = \int_{-\infty}^{z} x\, f(x)dx$ is the first INM.

δ_1 and δ_2 can be computed in two ways. A general equation for $m_1(z)$ can be derived from (10) as

$$m_1(z) = \sum_{j=0}^{\infty} v_{k,l,i}\, J_j(z), \tag{16}$$

where

$$J_j(z) = \int_{-\infty}^{z} x\, h_j(x)dx$$

and the quantity (16) can be used to obtain the mean deviations in (15).

A second formula for $m_1(z)$ can be derived by setting $u = G(x)$ in (10) as follows

$$m_1(z) = \sum_{j=0}^{\infty} (j)\, v_{k,l,i}\; T_j(z), \tag{17}$$

where $T_j(z) = \int_0^{G(z)} Q_G(u)\, u^{(j)} du$.

2.5 Stochastic ordering

We address four different stochastic orders for the KwT-G distribution. If X and Y are independent random variables with respective cdfs F_X and F_Y, then the random variable X is said to be smaller than Y in

- Stochastic order (X $\leqslant_{st}$ (Y)) if $F_X(x) \geqslant F_Y(x)$ for all x.
- Hazard rate order (X $\leqslant_{hr}$ (Y)) if $h_X(x) \geqslant h_Y(x)$ for all x.
- Mean residual life order (X $\leqslant_{mrl}$ (Y)) if $m_X(x) \geqslant m_Y(x)$ for all x.

- Likelihood ratio order (X $\leqslant_{lr}$ (Y)) if $f_X(x)/f_y(x)$ decreases in x.

The following theorem proves that the KwT-G distribution satisfies likelihood ratio ordering.
Theorem. Suppose X follows KwT-G(a_1, b_1) and Y follows KwT-G(a_2, b_2). If $a_1 = a_2 = a$ and $b_1 \geq b_2$, then $X \leqslant_{lr} (Y)$, $X \leqslant_{hr} (Y)$, $X \leqslant_{mrl} (Y)$ and $X \leqslant_{st} (Y)$.
Proof. The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{b_1}{b_2} \left\{ 1 - e^{-a \tan\left[\frac{\pi(1-G(x))}{2}\right]} \right\}^{b_1 - b_2}. \tag{18}$$

Now if $a_1 = a_2 = a$ and $b_1 > b_2$, then $\frac{d}{dx} \ln \frac{f_X(x)}{f_Y(x)} \leq 0$, which implies that $X \geqslant_{lr} (Y)$. Hence, $X \geqslant_{lr} (Y)$, $X \geqslant_{hr} (Y)$, $X \geqslant_{mrl} (Y)$ and $X \geqslant_{st} (Y)$. The proof is complete.

2.6 Order statistics

Order statistics appear in several fields of statistical theory and applications. Consider a random sample from the KwT-G distribution, say $X_1, \ldots, X_n$. The pdf of the ith order statistic, say $X_{i:n}$, can be expressed as

$$f_{i:n}(x) = K\, f(x)\, F^{i-1}(x)\, \{1 - F(x)\}^{n-i} = K \sum_{p=0}^{n-i} (-1)^p \binom{n-i}{p} f(x)\, F(x)^{p+i-1},$$

where $K = n! / [(i-1)!\, (n-i)!]$. According to Nadarajah et al. **?**, the pdf of $X_{i:n}$ can also be expressed by

$$f_{i:n}(x) = \sum_{r=0}^{\infty} d_r\, h_{r+1}(x), \tag{19}$$

where $h_{r+1}(x)$ denotes the exp-G pdf with power parameter $r + 1$, and

$$\begin{aligned} d_r \quad = \quad & \sum_{j,k,l,m,n,p,q=0}^{\infty} \frac{(-1)^{j+k+l+m+n+r} a^{n+1}\, (l+m+1)^n\, b}{n!\, (r+1)} \left(\frac{\pi}{2}\right)^{2(p+q)+n+1} \\ & \binom{2(p+q)+n}{r} \binom{n-i}{j} \binom{i+j-1}{k} \binom{bk}{l} \binom{b-1}{m} c_p(n)\, c_q(2), \end{aligned}$$

where $c_p(n)$ and $c_q(2)$ are coefficients of the power series expansions of tan and sec as obtained by Mathematica. (19) reveals that the density of KwT-G order statistics are expressed as linear combinations of exp-G densities.

3 Four sub-models

In this section, we present four special sub-models of the KwT-G family based on the Weibull, Burr XII, normal, and exponential baseline models. This section presents the

KwT-W, KwT-Burr XII (KwT-BXII), KwT-normal (KwT-N), and KwT-exponential (KwT-E) distributions. Plots of the densities of the KwT-W, KwT-BXII, KwT-N, and KwT-E distributions for different values of their parameters are displayed in Figure 2. The corresponding hazard rates are shown in Figure 3. The pdfs of the KwT-G family can be right-skewed, bimodal, unimodal, left-skewed, and reversed J shaped. The hazard rates of the KwT-G family can be flexible to exhibit all important hazard rate shapes, including monotone and non-monotone shapes.

3.1 KwT-W distribution

The pdf of the Weibull distribution is $g(x;\alpha\,,\beta) = \frac{\beta}{\alpha^{\beta}}\, x^{\beta-1} e^{-(\frac{x}{\alpha})^{\beta}},\ x>0,\ \alpha,\ \beta>0$. The pdf of the KwT-W distribution follows from (4) as

$$
\begin{aligned}
f(x;\alpha,\beta) \quad = \quad & \frac{\pi}{2}\, a\, b\, \frac{\beta}{\alpha^{\beta}}\, x^{\beta-1} e^{-(\frac{x}{\alpha})^{\beta}}\, e^{-a\left[\tan\left(\frac{\pi e^{-(\frac{x}{\alpha})^{\beta}}}{2}\right)\right]} \\
& \left\{1-e^{-a\left[\tan\left(\frac{\pi\, e^{-(\frac{x}{\alpha})^{\beta}}}{2}\right)\right]}\right\}^{b-1} \left\{\sec^2\left[\frac{\pi\, e^{-(\frac{x}{\alpha})^{\beta}}}{2}\right]\right\}, \\
& x>0,\ \ a,\ b,\ \alpha,\ \beta>0. \qquad (20)
\end{aligned}
$$

3.2 KwTBXII distribution

The pdf of the Burr XII distribution is $g(x;c,k) = c\,k\,x^{c-1}\,[1-(1+x^{c})^{(-k-1)}]$, $x>0$, c, $k>0$. Based on (4), the pdf of the KwT-BXII distribution is (for $x>0$)

$$
\begin{aligned}
f(x;a,b,c,k) \quad = \quad & \frac{\pi}{2} abckx^{c-1}\left\{[1+x^{c}]^{(-k-1)}\right\} e^{-a\left[\tan\left(\frac{\pi(1+x^{c})^{-k}}{2}\right)\right]} \\
\times \quad & \left\{1-e^{-a\left[\tan\left(\frac{\pi(1+x^{c})^{-k}}{2}\right)\right]}\right\}^{b-1} \left\{\sec^2\left[\frac{\pi\,(1+x^{c})^{-k}}{2}\right]\right\}, \\
& a,b,c,k>0. \qquad (21)
\end{aligned}
$$

3.3 KwT-N distribution

Using (4) and the normal pdf, say $g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}$, $x \in \mathbb{R}$, $\sigma > 0$, we obtain the pdf of the KwT-N distribution as

$$f_{\text{KwT-G}}(x) = \frac{\pi}{2} a\, b \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} e^{-a\left[\tan\left(\frac{\pi\left(1-\frac{1}{2}\left(1+erf(\frac{x}{\sigma\sqrt{2}})\right)\right)}{2}\right)\right]} \tag{22}$$

$$\left\{1 - e^{-a\left[\tan\left(\frac{\pi\left(1-\frac{1}{2}\left(1+erf(\frac{x}{\sigma\sqrt{2}})\right)\right)}{2}\right)\right]}\right\}^{b-1}$$

$$\left\{\sec^2\left[\frac{\pi\left(1-\frac{1}{2}\left(1+erf(\frac{x}{\sigma\sqrt{2}})\right)\right)}{2}\right]\right\}, \; x \in \mathbb{R}, \; a, \, b, \, \sigma > 0. \tag{23}$$

3.4 KwT-E distribution

The pdf the exponential distribution with scale parameter $\beta > 0$ is $g(x;\beta) = \beta\, e^{-\beta x}$, $x, \beta > 0$. The pdf of the KwT-E distribution follows from (4) as

$$f_{\text{KwT-G}}(x) = \frac{\pi}{2} a\, b\, \beta\, e^{-\beta x} e^{-a\left[\tan\left(\frac{\pi e^{-\beta x}}{2}\right)\right]} \left\{1 - e^{-a\left[\tan\left(\frac{\pi e^{-\beta x}}{2}\right)\right]}\right\}^{b-1}$$

$$\left\{\sec^2\left[\frac{\pi\,(e^{-\beta x})}{2}\right]\right\}. \tag{24}$$

4 Estimation methods for the KwT-E distribution

This section addresses eight estimation methods of the KwT-E parameters. Consider a random sample of size n from the KwT-E distribution, say $x_1, \ldots, x_n$, and its corresponding order statistics, say $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$.

The log-likelihood function of the KwT-E distribution (for $\boldsymbol{\varphi} = (a, b, \beta)^T$) takes the form

$$\ell_n(\boldsymbol{\varphi}) = n\ln(\pi) - n\ln(2) - (b-1)\sum_{i=1}^{n} \ln\left\{1 - \exp\left[-a\tan K_i\right]\right\}$$

$$-a\sum_{i=1}^{n} K_i + \sum_{i=1}^{n} \ln\left[\sec^2 K_i\right] - \sum_{i=1}^{n} \exp(-\beta\, x_i) + n\ln(a\, b\, \beta), \tag{25}$$

where $(\frac{\pi e^{-\beta x_i}}{2}) = K_i$.

The MLEs of the KwT-E parameters can be found by maximizing (25) with respect to (a, b, β).

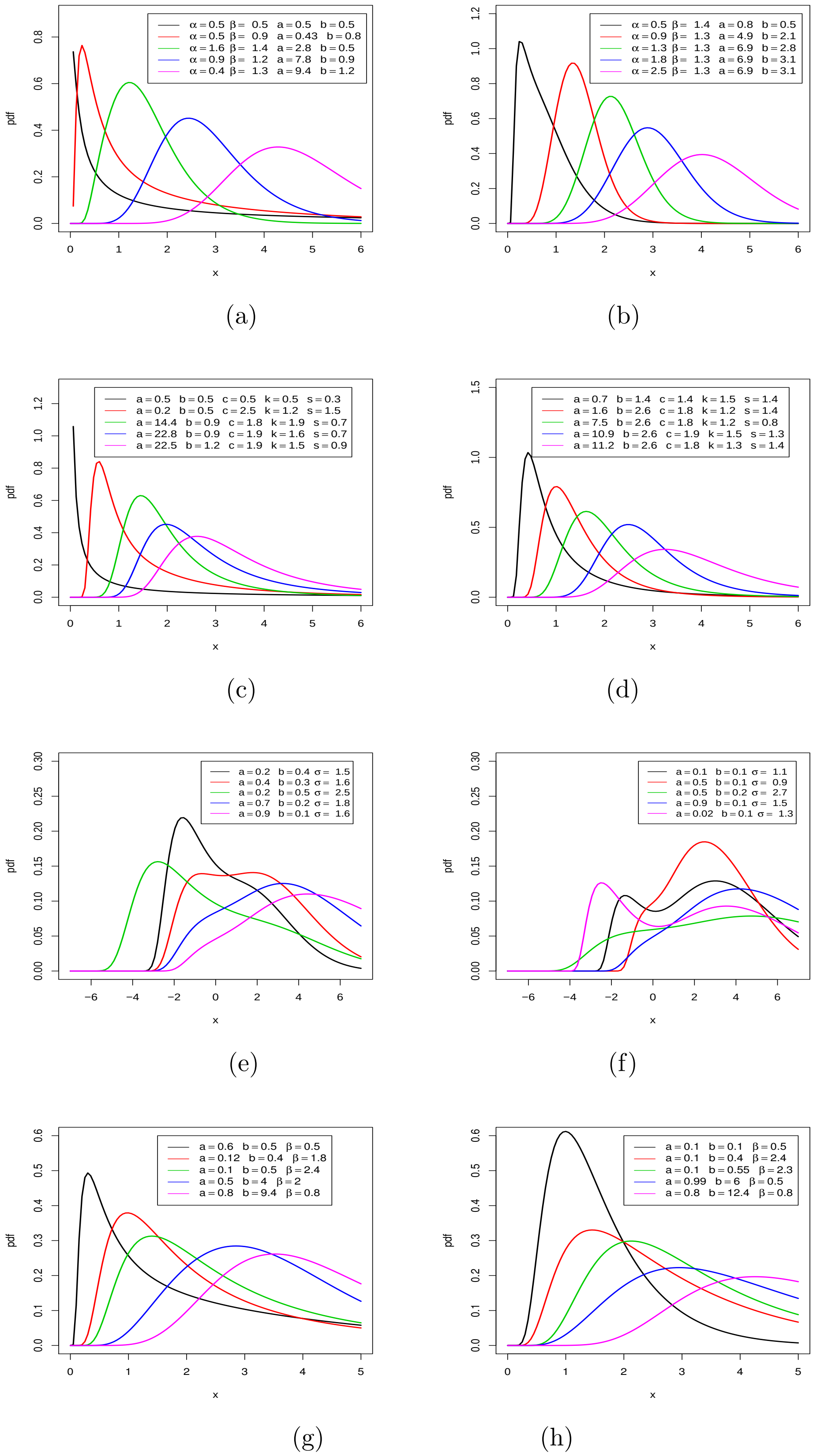


Figure 2: Plots of the pdfs of the KwT-W (panels a and b), KwT-BXII (panels c and d), KwT-N (panels e and f), and KwT-E (panels g and h) distributions.

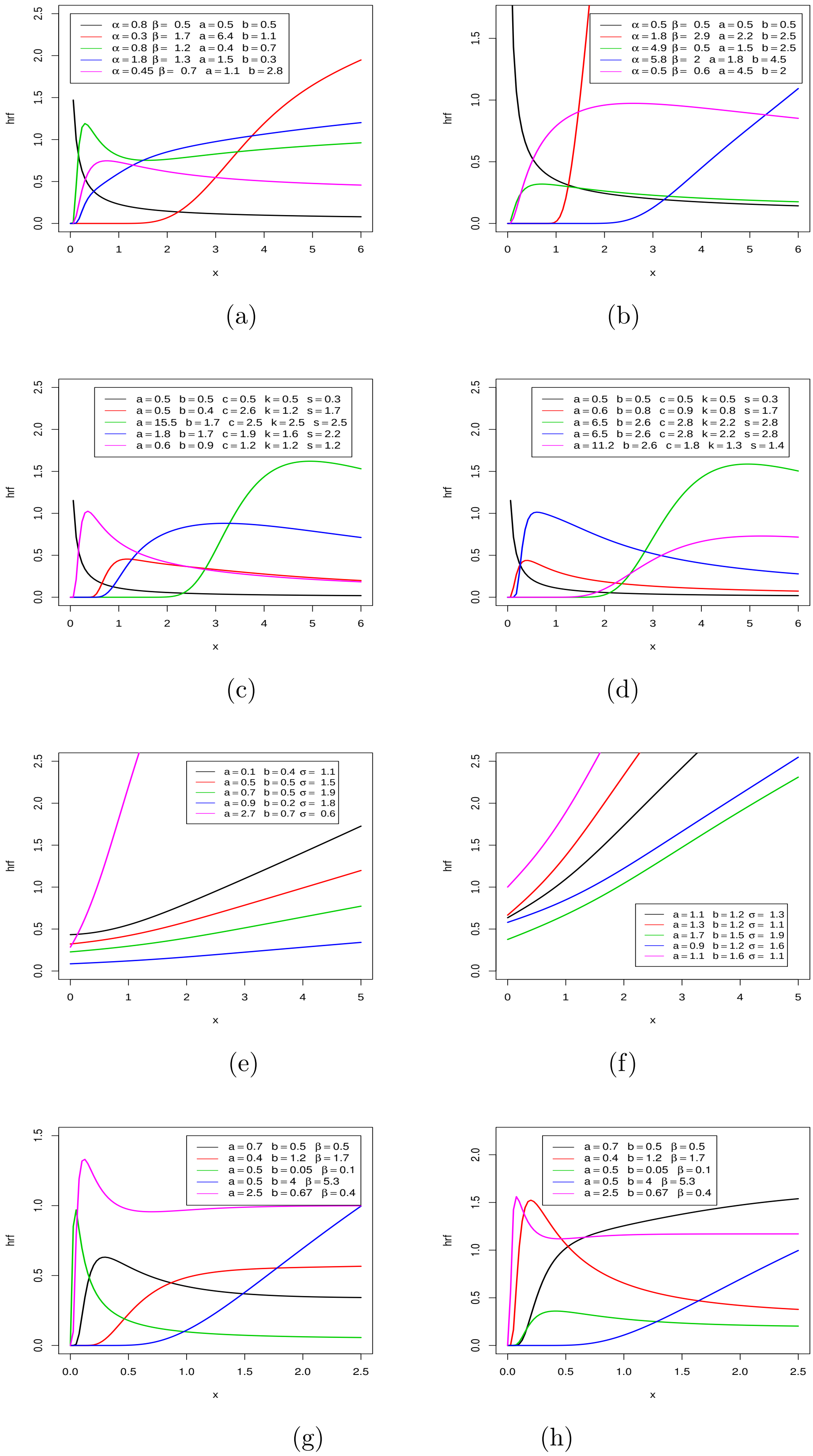


Figure 3: Plots of the hazard rate functions of the KwT-W (panels a and b), KwT-BXII (panels c and d), KwT-N (panels e and f), and KwT-E (panels g and h) distributions.

The LSEs and WLSEs of the KwT-E parameters can be found by minimizing

$$V\left(a,\, b,\, \beta\right)=\sum_{i=1}^{n}v_i\left(1-\left\{1-e^{-a\left[\tan\left(\frac{\pi e^{-\beta x_{(i)}}}{2}\right)\right]}\right\}^{b}-\frac{i}{n+1}\right)^{2},$$

where $v_i = 1$ for the LSEs and $v_i = (n+1)^2(n+2)/\left[i(n-i+1)\right]$ for the WLSEs. The LSEs and WLSEs are also found by solving

$$\sum_{i=1}^{n}v_i\left(1-\left\{1-e^{-a\left[\tan\left(\frac{\pi e^{-\beta x_{(i)}}}{2}\right)\right]}\right\}^{b}-\frac{i}{n+1}\right)\Delta_l\left(x_{(i)}|a,\, b,\, \beta\right)=0,$$

where

$$\begin{aligned}\Delta_1\left(x_{(i)}|a,\, b,\, \beta\right) &= \frac{\partial}{\partial a}F\left(x_{(i)}|a,\, b,\, \beta\right),\ \Delta_2\left(x_{(i)}|a,\, b,\, \beta\right)=\frac{\partial}{\partial b}F\left(x_{(i)}|a,\, b,\, \beta\right),\\ \Delta_3\left(x_{(i)}|a,\, b,\, \beta\right) &= \frac{\partial}{\partial \beta}F\left(x_{(i)}|a,\, b,\, \beta\right). \end{aligned} \tag{26}$$

Uniform spacings are defined by

$$D_i = F(x_{(i)}|a,\, b,\, \beta) - F(x_{(i-1)}|a,\, b,\, \beta),$$

where D_i denotes to the uniform spacings, $F(x_{(0)}|a,\, b,\, \beta) = 0$, $F(x_{(n+1)}|a,\, b,\, \beta) = 1$ and $D_0(a,\, b,\, \beta) + D_1(a,\, b,\, \beta) + \cdots + D_{n+1}(a,\, b,\, \beta) = 1$.

The MPSEs of the KwT-E parameters can be calculated by maximizing

$$G(a,\, b,\, \beta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(a,\, b,\, \beta).$$

The MPSEs can also be found by solving

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(a,\, b,\, \beta)}\left[\Delta_s(x_{(i)}|a,\, b,\, \beta)-\Delta_s(x_{(i-1)}|a,\, b,\, \beta)\right]=0,\quad s=1,2,3.$$

Suppose $u_i = i/\left(n+1\right)$ is an unbiased estimator for $F\left(x_{(i)}|a, b, \lambda\right)$. The PCEs of the KwT-E parameters are found by minimizing

$$P(a,\, b,\, \beta)=\sum_{i=1}^{n}\left(x_{(i)}-\ln\left\{\frac{2}{\pi}\tan^{-1}\left[\ln\left(1-\{1-u_i\}^{\frac{1}{b}}\right)^{\frac{-1}{a}}\right]\right\}^{\frac{-1}{\beta}}\right)^{2}.$$

The CVMEs of the KwT-E parameters are obtained by minimizing

$$C(a,\, b,\, \beta)=\frac{1}{12n}+\sum_{i=1}^{n}\left(1-\left\{1-e^{-a\left[\tan\left(\frac{\pi e^{-\beta x}}{2}\right)\right]}\right\}^{b}-\frac{2i-1}{2n}\right)^{2}.$$

These estimators can also be obtained by solving

$$\sum_{i=1}^{n}\left(1-\left\{1-e^{-a\left[\tan\left(\frac{\pi e^{-\beta x_{(i)}}}{2}\right)\right]}\right\}^{b}-\frac{2i-1}{2n}\right)\Delta_{l}\left(x_{(i)}|a,\,b,\,\beta\right)=0.$$

The ADEs of the KwT-E parameters are obtained by minimizing

$$A(a,\,b,\,\beta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F\left(x_{(i)}|a,b,\beta\right)+\ln\overline{F}\left(x_{(n+1-i)}|a,\,b,\,\beta\right)\right].$$

The ADEs are also found by solving

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_{s}\left(x_{(i)}|a,\,b,\,\beta\right)}{F\left(x_{(i)}|a,\,b,\,\beta\right)}-\frac{\Delta_{j}\left(x_{(n+1-i)}|a,\,b,\,\beta\right)}{S\left(x_{(n+1-i)}|a,\,b,\,\beta\right)}\right]=0.$$

The RADEs of the KwT-E parameters are found by minimizing

$$R(a,\,b,\,\beta)=\frac{n}{2}-2\sum_{i=1}^{n}F\left(x_{(i)}|a,\,b,\,\beta\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\,\ln\overline{F}\left(x_{(n+1-i)}|a,\,b,\,\beta\right).$$

5 Simulation results

In this section, the performance of the eight estimators of the KwT-E parameters are explored by simulation results. For this purpose, we generate $N = 10000$ samples from the KwT-E distribution each of size $n = (20, 50, 100, 200, 400)$ for $a = (0.25, 0.50, 0.80, 2.50)$, $b = (0.60, 0.80, 1.20, 3.50)$ and $\beta = (0.70, 1.20, 0.20, 1.50)$.

The averages of the following quantities are obtained over the 10000 samples:
mean square error (MSE), MSE $= \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\varphi}}-\boldsymbol{\varphi})^2$, absolute bias ($|\text{Bias}(\widehat{\boldsymbol{\varphi}})|$), $|\text{Bias}(\widehat{\boldsymbol{\varphi}})| = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\varphi}}-\boldsymbol{\varphi}|$ and mean relative error (MRE), MRE $= \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\varphi}}-\boldsymbol{\varphi}|/\boldsymbol{\varphi}$.

All computations were performed using `R` (`R` Core Team, 2020, version 4.0.3). The `nlminb` function in the `stats` package was used for maximization / minimization. The simulation results are presented in Tables 1–4. The superscripts in each row illustrate the ranks of the estimates among all methods. The partial sum of the ranks is denoted by $\sum Ranks$.

The values in these tables show that the consistency property holds for all estimation methods, which means the MSE and MRE decrease when n increases for all considered cases implying that the eight estimators are consistent for the KwT-E parameters. Additionally, these estimates are quite reliable since all estimation approaches provide small biases and MSE for all studied cases.

The maximum likelihood (ML) outperforms all other estimation methods with an overall score rank of 39. The maximum product of spacing with overall score rank of 42 is also considered a competitive method for the ML.

Table 1: Simulation results for $\boldsymbol{\varphi} = (a = 0.25, b = 0.60, \beta = 0.70)^\intercal$.

n	Est.	Est. Par.	WLSE	LSE	MLE	MPSE	CVME	ADE	RADE	PCE
20	$\lvert BIAS \rvert$	$\widehat{a}$	$0.31306^{\{8\}}$	$0.11393^{\{5\}}$	$0.09144^{\{1\}}$	$0.10323^{\{4\}}$	$0.11689^{\{6\}}$	$0.09897^{\{2\}}$	$0.09928^{\{3\}}$	$0.14153^{\{7\}}$
		$\widehat{b}$	$0.25108^{\{7\}}$	$0.18547^{\{4\}}$	$0.17451^{\{2\}}$	$0.17010^{\{1\}}$	$0.20250^{\{5\}}$	$0.17604^{\{3\}}$	$0.21063^{\{6\}}$	$0.26801^{\{8\}}$
		$\widehat{\beta}$	$0.91443^{\{5\}}$	$1.02252^{\{7\}}$	$0.70426^{\{1\}}$	$0.88376^{\{3\}}$	$1.01131^{\{6\}}$	$0.84437^{\{2\}}$	$0.90068^{\{4\}}$	$1.26004^{\{8\}}$
	MSE	$\widehat{a}$	$0.09801^{\{8\}}$	$0.01298^{\{5\}}$	$0.00836^{\{1\}}$	$0.01066^{\{4\}}$	$0.01366^{\{6\}}$	$0.01028^{\{3\}}$	$0.00986^{\{2\}}$	$0.02003^{\{7\}}$
		$\widehat{b}$	$0.06304^{\{7\}}$	$0.03440^{\{4\}}$	$0.03045^{\{2\}}$	$0.02894^{\{1\}}$	$0.04101^{\{5\}}$	$0.03372^{\{3\}}$	$0.04437^{\{6\}}$	$0.07183^{\{8\}}$
		$\widehat{\beta}$	$0.83618^{\{5\}}$	$1.04555^{\{7\}}$	$0.49598^{\{1\}}$	$0.78103^{\{3\}}$	$1.02275^{\{6\}}$	$0.75659^{\{2\}}$	$0.81123^{\{4\}}$	$1.58769^{\{8\}}$
	MRE	$\widehat{a}$	$0.41742^{\{4\}}$	$0.45573^{\{6\}}$	$0.36575^{\{1\}}$	$0.41290^{\{4\}}$	$0.46758^{\{7\}}$	$0.39589^{\{2\}}$	$0.39711^{\{3\}}$	$0.56611^{\{8\}}$
		$\widehat{b}$	$0.50216^{\{8\}}$	$0.30911^{\{4\}}$	$0.29084^{\{2\}}$	$0.28351^{\{1\}}$	$0.33750^{\{5\}}$	$0.29340^{\{3\}}$	$0.35105^{\{6\}}$	$0.44669^{\{7\}}$
		$\widehat{\beta}$	$0.60962^{\{7\}}$	$0.58430^{\{6\}}$	$0.40244^{\{1\}}$	$0.50501^{\{3\}}$	$0.57789^{\{5\}}$	$0.48250^{\{2\}}$	$0.51467^{\{4\}}$	$0.72002^{\{8\}}$
	$\sum Ranks$		$59^{\{7\}}$	$48^{\{5\}}$	$12^{\{1\}}$	$24^{\{3\}}$	$51^{\{6\}}$	$22^{\{2\}}$	$38^{\{4\}}$	$69^{\{8\}}$
50	$\lvert BIAS \rvert$	$\widehat{a}$	$0.20056^{\{8\}}$	$0.07509^{\{6\}}$	$0.06027^{\{1\}}$	$0.06405^{\{4\}}$	$0.07256^{\{5\}}$	$0.06278^{\{3\}}$	$0.06210^{\{2\}}$	$0.08201^{\{7\}}$
		$\widehat{b}$	$0.19365^{\{7\}}$	$0.11968^{\{4\}}$	$0.10919^{\{1\}}$	$0.11217^{\{3\}}$	$0.12086^{\{5\}}$	$0.11110^{\{2\}}$	$0.12965^{\{6\}}$	$0.23959^{\{8\}}$
		$\widehat{\beta}$	$0.65506^{\{7\}}$	$0.62544^{\{5\}}$	$0.48420^{\{1\}}$	$0.53588^{\{3\}}$	$0.62600^{\{6\}}$	$0.52750^{\{2\}}$	$0.56325^{\{4\}}$	$1.02853^{\{8\}}$
	MSE	$\widehat{a}$	$0.04023^{\{8\}}$	$0.00564^{\{6\}}$	$0.00363^{\{1\}}$	$0.00410^{\{4\}}$	$0.00527^{\{5\}}$	$0.00394^{\{2\}}$	$0.00386^{\{3\}}$	$0.00673^{\{7\}}$
		$\widehat{b}$	$0.03750^{\{7\}}$	$0.01432^{\{4\}}$	$0.01192^{\{1\}}$	$0.01258^{\{3\}}$	$0.01461^{\{5\}}$	$0.01234^{\{2\}}$	$0.01681^{\{6\}}$	$0.05740^{\{8\}}$
		$\widehat{\beta}$	$0.42910^{\{7\}}$	$0.39117^{\{5\}}$	$0.23445^{\{1\}}$	$0.28717^{\{3\}}$	$0.39188^{\{6\}}$	$0.27825^{\{2\}}$	$0.31725^{\{4\}}$	$1.05788^{\{8\}}$
	MRE	$\widehat{a}$	$0.26742^{\{5\}}$	$0.30035^{\{7\}}$	$0.24108^{\{1\}}$	$0.25621^{\{4\}}$	$0.29026^{\{6\}}$	$0.25114^{\{3\}}$	$0.24842^{\{2\}}$	$0.32803^{\{8\}}$
		$\widehat{b}$	$0.38731^{\{7\}}$	$0.19947^{\{4\}}$	$0.18198^{\{1\}}$	$0.18695^{\{3\}}$	$0.20143^{\{5\}}$	$0.18517^{\{2\}}$	$0.21608^{\{6\}}$	$0.39931^{\{8\}}$
		$\widehat{\beta}$	$0.43671^{\{7\}}$	$0.35739^{\{5\}}$	$0.27668^{\{1\}}$	$0.30622^{\{3\}}$	$0.35772^{\{6\}}$	$0.30143^{\{2\}}$	$0.32186^{\{4\}}$	$0.58773^{\{8\}}$
	$\sum Ranks$		$63^{\{7\}}$	$46^{\{5\}}$	$09^{\{1\}}$	$29^{\{3\}}$	$49^{\{6\}}$	$20^{\{2\}}$	$37^{\{4\}}$	$70^{\{8\}}$
100	$\lvert BIAS \rvert$	$\widehat{a}$	$0.13394^{\{8\}}$	$0.05131^{\{6\}}$	$0.04174^{\{1\}}$	$0.04473^{\{4\}}$	$0.05038^{\{5\}}$	$0.04367^{\{3\}}$	$0.04142^{\{2\}}$	$0.05325^{\{7\}}$
		$\widehat{b}$	$0.14115^{\{7\}}$	$0.08715^{\{5\}}$	$0.07568^{\{1\}}$	$0.07961^{\{2\}}$	$0.08578^{\{4\}}$	$0.07979^{\{3\}}$	$0.09005^{\{6\}}$	$0.20492^{\{8\}}$
		$\widehat{\beta}$	$0.45879^{\{7\}}$	$0.44188^{\{5\}}$	$0.35596^{\{1\}}$	$0.37817^{\{2\}}$	$0.45063^{\{6\}}$	$0.37994^{\{3\}}$	$0.39223^{\{4\}}$	$0.81413^{\{8\}}$
	MSE	$\widehat{a}$	$0.01794^{\{8\}}$	$0.00263^{\{6\}}$	$0.00174^{\{1\}}$	$0.00200^{\{4\}}$	$0.00254^{\{5\}}$	$0.00191^{\{2\}}$	$0.00172^{\{3\}}$	$0.00284^{\{7\}}$
		$\widehat{b}$	$0.01992^{\{7\}}$	$0.00760^{\{5\}}$	$0.00573^{\{1\}}$	$0.00634^{\{2\}}$	$0.00736^{\{4\}}$	$0.00637^{\{3\}}$	$0.00811^{\{6\}}$	$0.04199^{\{8\}}$
		$\widehat{\beta}$	$0.21048^{\{7\}}$	$0.19526^{\{6\}}$	$0.12671^{\{1\}}$	$0.14301^{\{2\}}$	$0.20307^{\{6\}}$	$0.14436^{\{3\}}$	$0.15385^{\{4\}}$	$0.66280^{\{8\}}$
	MRE	$\widehat{a}$	$0.17859^{\{4\}}$	$0.20524^{\{7\}}$	$0.16697^{\{2\}}$	$0.17891^{\{5\}}$	$0.20153^{\{6\}}$	$0.17466^{\{3\}}$	$0.16569^{\{1\}}$	$0.21301^{\{8\}}$
		$\widehat{b}$	$0.28230^{\{7\}}$	$0.14525^{\{5\}}$	$0.12613^{\{1\}}$	$0.13268^{\{2\}}$	$0.14296^{\{4\}}$	$0.13299^{\{3\}}$	$0.15009^{\{6\}}$	$0.34154^{\{8\}}$
		$\widehat{\beta}$	$0.30586^{\{7\}}$	$0.25250^{\{5\}}$	$0.20341^{\{1\}}$	$0.21610^{\{2\}}$	$0.25750^{\{6\}}$	$0.21711^{\{3\}}$	$0.22413^{\{4\}}$	$0.46522^{\{8\}}$
	$\sum Ranks$		$62^{\{7\}}$	$50^{\{6\}}$	$10^{\{1\}}$	$29^{\{3\}}$	$46^{\{5\}}$	$26^{\{2\}}$	$36^{\{4\}}$	$70^{\{8\}}$
200	$\lvert BIAS \rvert$	$\widehat{a}$	$0.09581^{\{8\}}$	$0.03629^{\{6\}}$	$0.02964^{\{1\}}$	$0.03030^{\{2\}}$	$0.03736^{\{7\}}$	$0.03042^{\{3\}}$	$0.03087^{\{4\}}$	$0.03441^{\{5\}}$
		$\widehat{b}$	$0.10146^{\{7\}}$	$0.06094^{\{5\}}$	$0.05317^{\{1\}}$	$0.05426^{\{2\}}$	$0.06093^{\{4\}}$	$0.05606^{\{3\}}$	$0.06294^{\{6\}}$	$0.16458^{\{8\}}$
		$\widehat{\beta}$	$0.32650^{\{7\}}$	$0.31370^{\{5\}}$	$0.24743^{\{1\}}$	$0.25865^{\{2\}}$	$0.31384^{\{6\}}$	$0.26028^{\{3\}}$	$0.28054^{\{4\}}$	$0.63369^{\{8\}}$
	MSE	$\widehat{a}$	$0.00918^{\{8\}}$	$0.00132^{\{6\}}$	$0.00088^{\{1\}}$	$0.00092^{\{2\}}$	$0.00140^{\{7\}}$	$0.00093^{\{3\}}$	$0.00095^{\{4\}}$	$0.00118^{\{5\}}$
		$\widehat{b}$	$0.01030^{\{7\}}$	$0.00371^{\{4.5\}}$	$0.00283^{\{1\}}$	$0.00294^{\{2\}}$	$0.00371^{\{4.5\}}$	$0.00314^{\{3\}}$	$0.00396^{\{6\}}$	$0.02709^{\{8\}}$
		$\widehat{\beta}$	$0.10661^{\{7\}}$	$0.09841^{\{5\}}$	$0.06122^{\{1\}}$	$0.06690^{\{2\}}$	$0.09850^{\{6\}}$	$0.06775^{\{3\}}$	$0.07870^{\{4\}}$	$0.40156^{\{8\}}$
	MRE	$\widehat{a}$	$0.12774^{\{5\}}$	$0.14516^{\{7\}}$	$0.11856^{\{1\}}$	$0.12120^{\{2\}}$	$0.14943^{\{8\}}$	$0.12167^{\{3\}}$	$0.12349^{\{4\}}$	$0.13765^{\{6\}}$
		$\widehat{b}$	$0.20293^{\{7\}}$	$0.10157^{\{5\}}$	$0.08862^{\{1\}}$	$0.09043^{\{2\}}$	$0.10155^{\{4\}}$	$0.09343^{\{3\}}$	$0.10490^{\{6\}}$	$0.27430^{\{8\}}$
		$\widehat{\beta}$	$0.21767^{\{7\}}$	$0.17926^{\{5\}}$	$0.14139^{\{1\}}$	$0.14780^{\{2\}}$	$0.17934^{\{6\}}$	$0.14873^{\{3\}}$	$0.16031^{\{4\}}$	$0.36211^{\{8\}}$
	$\sum Ranks$		$53^{\{7\}}$	$48.8^{\{5\}}$	$09^{\{1\}}$	$18^{\{2\}}$	$52.5^{\{6\}}$	$27^{\{3\}}$	$42^{\{4\}}$	$64^{\{8\}}$
400	$\lvert BIAS \rvert$	$\widehat{a}$	$0.06487^{\{8\}}$	$0.02561^{\{7\}}$	$0.02088^{\{1\}}$	$0.02168^{\{3\}}$	$0.02492^{\{6\}}$	$0.02205^{\{4\}}$	$0.02148^{\{2\}}$	$0.02277^{\{5\}}$
		$\widehat{b}$	$0.06805^{\{7\}}$	$0.04247^{\{5\}}$	$0.03822^{\{1\}}$	$0.03837^{\{2\}}$	$0.04163^{\{4\}}$	$0.03916^{\{3\}}$	$0.04535^{\{6\}}$	$0.12089^{\{8\}}$
		$\widehat{\beta}$	$0.22477^{\{7\}}$	$0.22011^{\{5\}}$	$0.17855^{\{1\}}$	$0.18315^{\{2\}}$	$0.21292^{\{6\}}$	$0.18552^{\{3\}}$	$0.19527^{\{4\}}$	$0.46764^{\{8\}}$
	MSE	$\widehat{a}$	$0.00421^{\{8\}}$	$0.00066^{\{7\}}$	$0.00044^{\{1\}}$	$0.00047^{\{2\}}$	$0.00062^{\{6\}}$	$0.00049^{\{4\}}$	$0.00046^{\{3\}}$	$0.00052^{\{5\}}$
		$\widehat{b}$	$0.00463^{\{7\}}$	$0.00180^{\{5\}}$	$0.00146^{\{1\}}$	$0.00147^{\{2\}}$	$0.00173^{\{4\}}$	$0.00153^{\{3\}}$	$0.00206^{\{6\}}$	$0.01461^{\{8\}}$
		$\widehat{\beta}$	$0.05052^{\{7\}}$	$0.04845^{\{6\}}$	$0.03188^{\{1\}}$	$0.03354^{\{2\}}$	$0.04534^{\{5\}}$	$0.03442^{\{3\}}$	$0.03813^{\{4\}}$	$0.21869^{\{8\}}$
	MRE	$\widehat{a}$	$0.08649^{\{3\}}$	$0.10245^{\{8\}}$	$0.08353^{\{1\}}$	$0.08673^{\{4\}}$	$0.09966^{\{7\}}$	$0.08819^{\{5\}}$	$0.08593^{\{2\}}$	$0.09107^{\{6\}}$
		$\widehat{b}$	$0.13610^{\{7\}}$	$0.07079^{\{5\}}$	$0.06370^{\{1\}}$	$0.06396^{\{2\}}$	$0.06938^{\{4\}}$	$0.06527^{\{3\}}$	$0.07559^{\{6\}}$	$0.20148^{\{8\}}$
		$\widehat{\beta}$	$0.14985^{\{7\}}$	$0.12578^{\{6\}}$	$0.10203^{\{1\}}$	$0.10466^{\{2\}}$	$0.12167^{\{5\}}$	$0.10601^{\{3\}}$	$0.11158^{\{4\}}$	$0.26723^{\{8\}}$
	$\sum Ranks$		$61^{\{7\}}$	$54^{\{5\}}$	$09^{\{1\}}$	$21^{\{2\}}$	$57^{\{6\}}$	$31^{\{3\}}$	$37^{\{4\}}$	$64^{\{8\}}$

Table 2: Simulation results for $\boldsymbol{\varphi} = (a = 0.5, b = 0.80, \beta = 1.20)^{\intercal}$.

n	Est.	Est. Par.	WLSE	LSE	MLE	MPSE	CVME	ADE	RADE	PCE
20	$\|BIAS\|$	$\widehat{a}$	$0.22272^{\{5\}}$	$0.23712^{\{6\}}$	$0.18140^{\{1\}}$	$0.21159^{\{4\}}$	$0.24240^{\{7\}}$	$0.21056^{\{3\}}$	$0.20470^{\{2\}}$	$0.27138^{\{8\}}$
		$\widehat{b}$	$0.31567^{\{6\}}$	$0.30887^{\{4\}}$	$0.27285^{\{1\}}$	$0.28372^{\{2\}}$	$0.32772^{\{8\}}$	$0.30122^{\{3\}}$	$0.32117^{\{7\}}$	$0.31414^{\{5\}}$
		$\widehat{\beta}$	$1.09802^{\{5\}}$	$1.16678^{\{7\}}$	$0.75399^{\{1\}}$	$0.99443^{\{3\}}$	$1.12464^{\{6\}}$	$0.95460^{\{2\}}$	$1.03360^{\{4\}}$	$1.21005^{\{8\}}$
	MSE	$\widehat{a}$	$0.04960^{\{5\}}$	$0.05622^{\{6\}}$	$0.03291^{\{1\}}$	$0.04477^{\{4\}}$	$0.05876^{\{7\}}$	$0.04434^{\{3\}}$	$0.04190^{\{2\}}$	$0.07365^{\{8\}}$
		$\widehat{b}$	$0.09965^{\{6\}}$	$0.09540^{\{4\}}$	$0.07445^{\{1\}}$	$0.08050^{\{2\}}$	$0.10740^{\{8\}}$	$0.09073^{\{3\}}$	$0.10315^{\{7\}}$	$0.09868^{\{5\}}$
		$\widehat{\beta}$	$1.20564^{\{5\}}$	$1.36138^{\{7\}}$	$0.56851^{\{1\}}$	$0.98888^{\{3\}}$	$1.26482^{\{6\}}$	$0.91126^{\{2\}}$	$1.06833^{\{4\}}$	$1.46423^{\{8\}}$
	MRE	$\widehat{a}$	$0.44543^{\{5\}}$	$0.47423^{\{6\}}$	$0.36280^{\{1\}}$	$0.42318^{\{4\}}$	$0.48480^{\{7\}}$	$0.42112^{\{3\}}$	$0.40940^{\{2\}}$	$0.54277^{\{8\}}$
		$\widehat{b}$	$0.39459^{\{5\}}$	$0.38609^{\{3\}}$	$0.34106^{\{1\}}$	$0.53465^{\{8\}}$	$0.40965^{\{7\}}$	$0.37652^{\{2\}}$	$0.40147^{\{6\}}$	$0.39268^{\{4\}}$
		$\widehat{\beta}$	$0.62744^{\{5\}}$	$0.66673^{\{7\}}$	$0.43085^{\{1\}}$	$0.56824^{\{3\}}$	$0.64265^{\{6\}}$	$0.54549^{\{2\}}$	$0.59063^{\{4\}}$	$0.69146^{\{8\}}$
	$\sum Ranks$		$47^{\{5\}}$	$50^{\{6\}}$	$09^{\{1\}}$	$33^{\{3\}}$	$62^{\{7.5\}}$	$23^{\{2\}}$	$38^{\{4\}}$	$62^{\{7.5\}}$
50	$\|BIAS\|$	$\widehat{a}$	$0.13527^{\{4\}}$	$0.14813^{\{7\}}$	$0.11835^{\{1\}}$	$0.13653^{\{5\}}$	$0.14690^{\{6\}}$	$0.12813^{\{3\}}$	$0.12655^{\{2\}}$	$0.17216^{\{8\}}$
		$\widehat{b}$	$0.20600^{\{4\}}$	$0.21606^{\{5\}}$	$0.18174^{\{1\}}$	$0.18887^{\{2\}}$	$0.21813^{\{6\}}$	$0.19500^{\{3\}}$	$0.22976^{\{7\}}$	$0.30964^{\{8\}}$
		$\widehat{\beta}$	$0.66144^{\{4\}}$	$0.73531^{\{7\}}$	$0.52784^{\{1\}}$	$0.63370^{\{3\}}$	$0.72210^{\{6\}}$	$0.60829^{\{2\}}$	$0.68195^{\{5\}}$	$0.97090^{\{8\}}$
	MSE	$\widehat{a}$	$0.01830^{\{4\}}$	$0.02194^{\{7\}}$	$0.01401^{\{1\}}$	$0.01864^{\{5\}}$	$0.02158^{\{6\}}$	$0.01642^{\{3\}}$	$0.01602^{\{2\}}$	$0.02964^{\{8\}}$
		$\widehat{b}$	$0.04243^{\{4\}}$	$0.04668^{\{5\}}$	$0.03303^{\{1\}}$	$0.03567^{\{2\}}$	$0.04758^{\{6\}}$	$0.03803^{\{3\}}$	$0.05279^{\{7\}}$	$0.09588^{\{8\}}$
		$\widehat{\beta}$	$0.43751^{\{4\}}$	$0.54068^{\{7\}}$	$0.27861^{\{1\}}$	$0.40157^{\{3\}}$	$0.52143^{\{6\}}$	$0.37001^{\{2\}}$	$0.46506^{\{5\}}$	$0.94265^{\{8\}}$
	MRE	$\widehat{a}$	$0.27054^{\{4\}}$	$0.29626^{\{7\}}$	$0.23670^{\{1\}}$	$0.27305^{\{5\}}$	$0.29380^{\{6\}}$	$0.25625^{\{3\}}$	$0.25310^{\{2\}}$	$0.34433^{\{8\}}$
		$\widehat{b}$	$0.25750^{\{4\}}$	$0.27007^{\{5\}}$	$0.22717^{\{1\}}$	$0.23609^{\{2\}}$	$0.27266^{\{6\}}$	$0.24375^{\{3\}}$	$0.28720^{\{7\}}$	$0.38705^{\{8\}}$
		$\widehat{\beta}$	$0.37797^{\{4\}}$	$0.42018^{\{7\}}$	$0.30162^{\{1\}}$	$0.36211^{\{3\}}$	$0.41263^{\{6\}}$	$0.34759^{\{2\}}$	$0.38969^{\{5\}}$	$0.55480^{\{8\}}$
	$\sum Ranks$		$36^{\{4\}}$	$57^{\{7\}}$	$09^{\{1\}}$	$30^{\{3\}}$	$54^{\{6\}}$	$24^{\{2\}}$	$42^{\{5\}}$	$72^{\{8\}}$
100	$\|BIAS\|$	$\widehat{a}$	$0.09316^{\{5\}}$	$0.10498^{\{6\}}$	$0.08482^{\{1\}}$	$0.09270^{\{4\}}$	$0.10778^{\{7\}}$	$0.08845^{\{3\}}$	$0.08581^{\{2\}}$	$0.12099^{\{8\}}$
		$\widehat{b}$	$0.13558^{\{3\}}$	$0.15263^{\{5\}}$	$0.12848^{\{1\}}$	$0.13514^{\{2\}}$	$0.15452^{\{6\}}$	$0.13930^{\{4\}}$	$0.16358^{\{7\}}$	$0.27093^{\{8\}}$
		$\widehat{\beta}$	$0.44195^{\{3\}}$	$0.51633^{\{7\}}$	$0.39357^{\{1\}}$	$0.44397^{\{4\}}$	$0.51166^{\{6\}}$	$0.42122^{\{2\}}$	$0.48589^{\{5\}}$	$0.81461^{\{8\}}$
	MSE	$\widehat{a}$	$0.00868^{\{5\}}$	$0.01102^{\{6\}}$	$0.00719^{\{1\}}$	$0.00859^{\{4\}}$	$0.01162^{\{7\}}$	$0.00782^{\{3\}}$	$0.00736^{\{2\}}$	$0.01464^{\{8\}}$
		$\widehat{b}$	$0.01838^{\{3\}}$	$0.02330^{\{5\}}$	$0.01651^{\{1\}}$	$0.01826^{\{2\}}$	$0.02388^{\{6\}}$	$0.01941^{\{4\}}$	$0.02676^{\{7\}}$	$0.07341^{\{8\}}$
		$\widehat{\beta}$	$0.19532^{\{3\}}$	$0.26660^{\{7\}}$	$0.15490^{\{1\}}$	$0.19711^{\{4\}}$	$0.26179^{\{6\}}$	$0.17743^{\{2\}}$	$0.23609^{\{5\}}$	$0.66359^{\{8\}}$
	MRE	$\widehat{a}$	$0.18632^{\{5\}}$	$0.20997^{\{6\}}$	$0.16964^{\{1\}}$	$0.18540^{\{4\}}$	$0.21557^{\{7\}}$	$0.17691^{\{3\}}$	$0.17162^{\{2\}}$	$0.24199^{\{8\}}$
		$\widehat{b}$	$0.16948^{\{3\}}$	$0.19079^{\{5\}}$	$0.16060^{\{1\}}$	$0.16892^{\{2\}}$	$0.19315^{\{6\}}$	$0.17413^{\{4\}}$	$0.20448^{\{7\}}$	$0.33867^{\{8\}}$
		$\widehat{\beta}$	$0.25254^{\{3\}}$	$0.29505^{\{7\}}$	$0.22490^{\{1\}}$	$0.25370^{\{4\}}$	$0.29238^{\{6\}}$	$0.24070^{\{2\}}$	$0.27765^{\{5\}}$	$0.46549^{\{8\}}$
	$\sum Ranks$		$33^{\{4\}}$	$54^{\{6\}}$	$09^{\{1\}}$	$30^{\{3\}}$	$57^{\{7\}}$	$26^{\{2\}}$	$42^{\{5\}}$	$72^{\{8\}}$
200	$\|BIAS\|$	$\widehat{a}$	$0.06248^{\{3\}}$	$0.07214^{\{6\}}$	$0.06058^{\{1\}}$	$0.06444^{\{5\}}$	$0.07273^{\{7\}}$	$0.06277^{\{4\}}$	$0.06173^{\{2\}}$	$0.08337^{\{8\}}$
		$\widehat{b}$	$0.09597^{\{4\}}$	$0.10742^{\{5\}}$	$0.09562^{\{3\}}$	$0.09268^{\{1\}}$	$0.10764^{\{6\}}$	$0.09408^{\{2\}}$	$0.11650^{\{7\}}$	$0.22133^{\{8\}}$
		$\widehat{\beta}$	$0.30276^{\{4\}}$	$0.35323^{\{7\}}$	$0.28695^{\{1\}}$	$0.29900^{\{2\}}$	$0.35182^{\{6\}}$	$0.30191^{\{3\}}$	$0.33832^{\{5\}}$	$0.63439^{\{8\}}$
	MSE	$\widehat{a}$	$0.00390^{\{3\}}$	$0.00520^{\{6\}}$	$0.00367^{\{1\}}$	$0.00415^{\{5\}}$	$0.00529^{\{7\}}$	$0.00394^{\{4\}}$	$0.00381^{\{2\}}$	$0.00695^{\{8\}}$
		$\widehat{b}$	$0.00921^{\{4\}}$	$0.01154^{\{5\}}$	$0.00914^{\{3\}}$	$0.00859^{\{1\}}$	$0.01159^{\{6\}}$	$0.00885^{\{2\}}$	$0.01357^{\{7\}}$	$0.04899^{\{8\}}$
		$\widehat{\beta}$	$0.09167^{\{4\}}$	$0.12477^{\{7\}}$	$0.08234^{\{1\}}$	$0.08940^{\{2\}}$	$0.12378^{\{6\}}$	$0.09115^{\{3\}}$	$0.11446^{\{5\}}$	$0.40244^{\{8\}}$
	MRE	$\widehat{a}$	$0.12497^{\{3\}}$	$0.14428^{\{6\}}$	$0.12117^{\{1\}}$	$0.12887^{\{5\}}$	$0.14545^{\{7\}}$	$0.12554^{\{4\}}$	$0.12347^{\{2\}}$	$0.16673^{\{8\}}$
		$\widehat{b}$	$0.11996^{\{4\}}$	$0.13428^{\{5\}}$	$0.11953^{\{3\}}$	$0.11585^{\{1\}}$	$0.13454^{\{6\}}$	$0.11761^{\{2\}}$	$0.14562^{\{7\}}$	$0.27667^{\{8\}}$
		$\widehat{\beta}$	$0.17301^{\{4\}}$	$0.20184^{\{7\}}$	$0.16397^{\{1\}}$	$0.17086^{\{2\}}$	$0.20104^{\{6\}}$	$0.17252^{\{3\}}$	$0.19333^{\{5\}}$	$0.36251^{\{8\}}$
	$\sum Ranks$		$33^{\{4\}}$	$54^{\{6\}}$	$15^{\{1\}}$	$24^{\{2\}}$	$57^{\{7\}}$	$25^{\{3\}}$	$42^{\{5\}}$	$72^{\{8\}}$
400	$\|BIAS\|$	$\widehat{a}$	$0.04513^{\{5\}}$	$0.05112^{\{6\}}$	$0.04285^{\{1\}}$	$0.04339^{\{2\}}$	$0.05246^{\{7\}}$	$0.04466^{\{4\}}$	$0.04406^{\{3\}}$	$0.05602^{\{8\}}$
		$\widehat{b}$	$0.06743^{\{3\}}$	$0.07678^{\{6\}}$	$0.06431^{\{1\}}$	$0.06442^{\{2\}}$	$0.07247^{\{5\}}$	$0.06870^{\{4\}}$	$0.08180^{\{7\}}$	$0.16392^{\{8\}}$
		$\widehat{\beta}$	$0.21434^{\{3\}}$	$0.25359^{\{7\}}$	$0.19538^{\{1\}}$	$0.21116^{\{2\}}$	$0.24528^{\{6\}}$	$0.21446^{\{4\}}$	$0.24173^{\{5\}}$	$0.46593^{\{8\}}$
	MSE	$\widehat{a}$	$0.00204^{\{5\}}$	$0.00261^{\{6\}}$	$0.00184^{\{1\}}$	$0.00188^{\{2\}}$	$0.00275^{\{7\}}$	$0.00199^{\{4\}}$	$0.00194^{\{3\}}$	$0.00314^{\{8\}}$
		$\widehat{b}$	$0.00455^{\{3\}}$	$0.00590^{\{6\}}$	$0.00414^{\{1\}}$	$0.00415^{\{2\}}$	$0.00525^{\{5\}}$	$0.00472^{\{4\}}$	$0.00669^{\{7\}}$	$0.02687^{\{8\}}$
		$\widehat{\beta}$	$0.04594^{\{3\}}$	$0.06431^{\{7\}}$	$0.03817^{\{1\}}$	$0.04459^{\{2\}}$	$0.06061^{\{6\}}$	$0.04599^{\{4\}}$	$0.05843^{\{5\}}$	$0.21709^{\{8\}}$
	MRE	$\widehat{a}$	$0.09026^{\{5\}}$	$0.10225^{\{6\}}$	$0.08571^{\{1\}}$	$0.08677^{\{2\}}$	$0.10491^{\{7\}}$	$0.08933^{\{4\}}$	$0.08812^{\{3\}}$	$0.11204^{\{8\}}$
		$\widehat{b}$	$0.08428^{\{3\}}$	$0.09598^{\{6\}}$	$0.08039^{\{1\}}$	$0.08052^{\{2\}}$	$0.09050^{\{5\}}$	$0.08588^{\{4\}}$	$0.10225^{\{7\}}$	$0.20489^{\{8\}}$
		$\widehat{\beta}$	$0.12248^{\{3\}}$	$0.14491^{\{7\}}$	$0.11164^{\{1\}}$	$0.12066^{\{2\}}$	$0.14016^{\{6\}}$	$0.12255^{\{4\}}$	$0.13813^{\{5\}}$	$0.26624^{\{8\}}$
	$\sum Ranks$		$33^{\{3\}}$	$57^{\{7\}}$	$09^{\{1\}}$	$18^{\{2\}}$	$54^{\{6\}}$	$36^{\{4\}}$	$45^{\{5\}}$	$72^{\{8\}}$

Table 3: Simulation results for $\boldsymbol{\varphi} = (a = 0.8, b = 1.20, \beta = 0.20)^{\intercal}$.

n	Est.	Est. Par.	WLSE	LSE	MLE	MPSE	CVME	ADE	RADE	PCE
20	$\|BIAS\|$	$\widehat{a}$	$0.43434^{\{5\}}$	$0.45110^{\{7\}}$	$0.30494^{\{1\}}$	$0.37847^{\{2\}}$	$0.45266^{\{8\}}$	$0.38773^{\{3\}}$	$0.39215^{\{4\}}$	$0.45043^{\{6\}}$
		$\widehat{b}$	$0.53316^{\{5\}}$	$0.56038^{\{7\}}$	$0.45284^{\{2\}}$	$0.47200^{\{3\}}$	$0.56632^{\{8\}}$	$0.53217^{\{4\}}$	$0.53791^{\{6\}}$	$0.44607^{\{1\}}$
		$\widehat{\beta}$	$1.38716^{\{7\}}$	$1.48836^{\{8\}}$	$0.83438^{\{1\}}$	$1.14603^{\{2\}}$	$1.37840^{\{6\}}$	$1.16694^{\{3\}}$	$1.17221^{\{4\}}$	$1.23170^{\{5\}}$
	MSE	$\widehat{a}$	$0.18866^{\{5\}}$	$0.20349^{\{7\}}$	$0.09299^{\{1\}}$	$0.14324^{\{2\}}$	$0.20490^{\{8\}}$	$0.15033^{\{3\}}$	$0.15378^{\{4\}}$	$0.20289^{\{6\}}$
		$\widehat{b}$	$0.28426^{\{5\}}$	$0.31403^{\{7\}}$	$0.20506^{\{2\}}$	$0.22279^{\{3\}}$	$0.32072^{\{8\}}$	$0.28321^{\{4\}}$	$0.28935^{\{6\}}$	$0.19898^{\{1\}}$
		$\widehat{\beta}$	$1.92422^{\{7\}}$	$2.21523^{\{8\}}$	$0.69619^{\{1\}}$	$1.31338^{\{2\}}$	$1.87253^{\{6\}}$	$1.36175^{\{3\}}$	$1.37407^{\{4\}}$	$1.51709^{\{5\}}$
	MRE	$\widehat{a}$	$0.54293^{\{5\}}$	$0.56387^{\{7\}}$	$0.38117^{\{1\}}$	$0.47309^{\{2\}}$	$0.56582^{\{8\}}$	$0.48466^{\{3\}}$	$0.49019^{\{4\}}$	$0.56304^{\{6\}}$
		$\widehat{b}$	$0.44430^{\{5\}}$	$0.46698^{\{7\}}$	$0.37736^{\{2\}}$	$0.39334^{\{3\}}$	$0.47193^{\{8\}}$	$0.44348^{\{4\}}$	$0.44826^{\{6\}}$	$0.37173^{\{1\}}$
		$\widehat{\beta}$	$0.79266^{\{7\}}$	$0.85049^{\{8\}}$	$0.47689^{\{1\}}$	$0.65487^{\{2\}}$	$0.78194^{\{6\}}$	$0.66682^{\{3\}}$	$0.66983^{\{4\}}$	$0.70383^{\{5\}}$
	$\sum Ranks$		$51^{\{6\}}$	$66^{\{7.5\}}$	$12^{\{1\}}$	$21^{\{2\}}$	$66^{\{7.5\}}$	$30^{\{3\}}$	$42^{\{5\}}$	$34^{\{4\}}$
50	$\|BIAS\|$	$\widehat{a}$	$0.25777^{\{5\}}$	$0.28737^{\{7\}}$	$0.20435^{\{1\}}$	$0.25531^{\{4\}}$	$0.27931^{\{6\}}$	$0.23980^{\{2\}}$	$0.24347^{\{3\}}$	$0.31032^{\{8\}}$
		$\widehat{a}$	$0.38770^{\{4\}}$	$0.40652^{\{5\}}$	$0.32725^{\{1\}}$	$0.35008^{\{2\}}$	$0.42232^{\{6\}}$	$0.38190^{\{3\}}$	$0.43246^{\{7\}}$	$0.44607^{\{8\}}$
		$\widehat{\beta}$	$0.80328^{\{4\}}$	$0.91521^{\{7\}}$	$0.59415^{\{1\}}$	$0.74892^{\{2\}}$	$0.87952^{\{6\}}$	$0.75019^{\{3\}}$	$0.83197^{\{5\}}$	$1.23170^{\{8\}}$
	MSE	$\widehat{a}$	$0.06645^{\{5\}}$	$0.08258^{\{7\}}$	$0.04176^{\{1\}}$	$0.06518^{\{4\}}$	$0.07801^{\{6\}}$	$0.05750^{\{2\}}$	$0.05928^{\{3\}}$	$0.09630^{\{8\}}$
		$\widehat{b}$	$0.15031^{\{4\}}$	$0.16526^{\{5\}}$	$0.10709^{\{1\}}$	$0.12256^{\{2\}}$	$0.17836^{\{6\}}$	$0.14584^{\{3\}}$	$0.18702^{\{7\}}$	$0.19587^{\{8\}}$
		$\widehat{\beta}$	$0.64256^{\{4\}}$	$0.83760^{\{7\}}$	$0.35301^{\{1\}}$	$0.56088^{\{2\}}$	$0.77355^{\{6\}}$	$0.56278^{\{3\}}$	$0.69217^{\{5\}}$	$0.97145^{\{8\}}$
	MRE	$\widehat{a}$	$0.32222^{\{5\}}$	$0.35921^{\{7\}}$	$0.25544^{\{1\}}$	$0.31914^{\{4\}}$	$0.34914^{\{6\}}$	$0.29975^{\{2\}}$	$0.30433^{\{3\}}$	$0.38790^{\{8\}}$
		$\widehat{b}$	$0.32309^{\{4\}}$	$0.33876^{\{5\}}$	$0.27271^{\{1\}}$	$0.29173^{\{2\}}$	$0.35194^{\{6\}}$	$0.31825^{\{3\}}$	$0.36039^{\{7\}}$	$0.36881^{\{8\}}$
		$\widehat{\beta}$	$0.45902^{\{4\}}$	$0.52298^{\{7\}}$	$0.33951^{\{1\}}$	$0.42795^{\{2\}}$	$0.50258^{\{6\}}$	$0.42868^{\{3\}}$	$0.47541^{\{5\}}$	$0.56321^{\{8\}}$
	$\sum Ranks$		$39^{\{4\}}$	$57^{\{7\}}$	$09^{\{1\}}$	$24^{\{2.5\}}$	$54^{\{6\}}$	$24^{\{2.5\}}$	$45^{\{5\}}$	$72^{\{8\}}$
100	$\|BIAS\|$	$\widehat{a}$	$0.17570^{\{5\}}$	$0.19643^{\{7\}}$	$0.14970^{\{1\}}$	$0.16873^{\{2\}}$	$0.19019^{\{6\}}$	$0.17242^{\{4\}}$	$0.16947^{\{3\}}$	$0.23305^{\{8\}}$
		$\widehat{b}$	$0.26736^{\{4\}}$	$0.29675^{\{6\}}$	$0.23983^{\{1\}}$	$0.26377^{\{3\}}$	$0.29699^{\{5\}}$	$0.26326^{\{2\}}$	$0.31104^{\{7\}}$	$0.40480^{\{8\}}$
		$\widehat{\beta}$	$0.54071^{\{4\}}$	$0.63356^{\{7\}}$	$0.45934^{\{1\}}$	$0.53060^{\{3\}}$	$0.61609^{\{6\}}$	$0.52053^{\{2\}}$	$0.57777^{\{5\}}$	$0.81145^{\{8\}}$
	MSE	$\widehat{a}$	$0.03087^{\{5\}}$	$0.03858^{\{7\}}$	$0.02241^{\{1\}}$	$0.02847^{\{2\}}$	$0.03617^{\{6\}}$	$0.02973^{\{4\}}$	$0.02872^{\{3\}}$	$0.05431^{\{8\}}$
		$\widehat{a}$	$0.07148^{\{4\}}$	$0.08806^{\{5\}}$	$0.05752^{\{1\}}$	$0.06957^{\{3\}}$	$0.08820^{\{6\}}$	$0.06930^{\{2\}}$	$0.09675^{\{7\}}$	$0.16386^{\{8\}}$
		$\widehat{\beta}$	$0.29236^{\{4\}}$	$0.40139^{\{7\}}$	$0.21099^{\{1\}}$	$0.28154^{\{3\}}$	$0.37957^{\{6\}}$	$0.27095^{\{2\}}$	$0.33382^{\{5\}}$	$0.65845^{\{8\}}$
	MRE	$\widehat{a}$	$0.21963^{\{5\}}$	$0.24553^{\{7\}}$	$0.18712^{\{1\}}$	$0.21092^{\{2\}}$	$0.23774^{\{6\}}$	$0.21552^{\{4\}}$	$0.21183^{\{3\}}$	$0.29131^{\{8\}}$
		$\widehat{b}$	$0.22280^{\{4\}}$	$0.24729^{\{5\}}$	$0.19986^{\{1\}}$	$0.21981^{\{3\}}$	$0.24749^{\{6\}}$	$0.21938^{\{2\}}$	$0.25920^{\{7\}}$	$0.33733^{\{8\}}$
		$\widehat{\beta}$	$0.30897^{\{4\}}$	$0.36203^{\{7\}}$	$0.26248^{\{1\}}$	$0.30320^{\{3\}}$	$0.35205^{\{6\}}$	$0.29744^{\{2\}}$	$0.33015^{\{5\}}$	$0.46369^{\{8\}}$
	$\sum Ranks$		$39^{\{4\}}$	$58^{\{7\}}$	$09^{\{1\}}$	$24^{\{2.5\}}$	$53^{\{6\}}$	$24^{\{2.5\}}$	$45^{\{5\}}$	$72^{\{8\}}$
200	$\|BIAS\|$	$\widehat{a}$	$0.11850^{\{5\}}$	$0.14445^{\{7\}}$	$0.10781^{\{1\}}$	$0.11283^{\{2\}}$	$0.13728^{\{6\}}$	$0.11549^{\{3\}}$	$0.11831^{\{4\}}$	$0.16784^{\{8\}}$
		$\widehat{b}$	$0.18352^{\{3\}}$	$0.21040^{\{4\}}$	$0.16843^{\{1\}}$	$0.17499^{\{2\}}$	$0.21247^{\{6\}}$	$0.18501^{\{4\}}$	$0.22198^{\{7\}}$	$0.34380^{\{8\}}$
		$\widehat{\beta}$	$0.37085^{\{4\}}$	$0.44269^{\{7\}}$	$0.33359^{\{1\}}$	$0.36585^{\{3\}}$	$0.43675^{\{6\}}$	$0.36348^{\{2\}}$	$0.41616^{\{5\}}$	$0.65884^{\{8\}}$
	MSE	$\widehat{a}$	$0.01404^{\{5\}}$	$0.02087^{\{7\}}$	$0.01162^{\{1\}}$	$0.01273^{\{2\}}$	$0.01885^{\{6\}}$	$0.01334^{\{3\}}$	$0.01400^{\{4\}}$	$0.02817^{\{8\}}$
		$\widehat{b}$	$0.03368^{\{3\}}$	$0.04427^{\{5\}}$	$0.02837^{\{1\}}$	$0.03062^{\{2\}}$	$0.04514^{\{6\}}$	$0.04323^{\{4\}}$	$0.04928^{\{7\}}$	$0.11820^{\{8\}}$
		$\widehat{\beta}$	$0.13753^{\{4\}}$	$0.19598^{\{7\}}$	$0.11128^{\{1\}}$	$0.13385^{\{3\}}$	$0.19075^{\{6\}}$	$0.13211^{\{2\}}$	$0.17319^{\{5\}}$	$0.43407^{\{8\}}$
	MRE	$\widehat{a}$	$0.14812^{\{5\}}$	$0.18056^{\{7\}}$	$0.13476^{\{1\}}$	$0.14104^{\{2\}}$	$0.17160^{\{6\}}$	$0.14436^{\{3\}}$	$0.14789^{\{4\}}$	$0.20980^{\{8\}}$
		$\widehat{b}$	$0.15293^{\{3\}}$	$0.17533^{\{5\}}$	$0.14036^{\{1\}}$	$0.14582^{\{2\}}$	$0.17706^{\{6\}}$	$0.15418^{\{4\}}$	$0.18499^{\{7\}}$	$0.28650^{\{8\}}$
		$\widehat{\beta}$	$0.21191^{\{4\}}$	$0.25297^{\{7\}}$	$0.19062^{\{1\}}$	$0.20906^{\{3\}}$	$0.24957^{\{6\}}$	$0.20770^{\{2\}}$	$0.23781^{\{5\}}$	$0.37648^{\{8\}}$
	$\sum Ranks$		$36^{\{4\}}$	$56^{\{7\}}$	$09^{\{1\}}$	$21^{\{2\}}$	$54^{\{6\}}$	$27^{\{3\}}$	$48^{\{5\}}$	$72^{\{8\}}$
400	$\|BIAS\|$	$\widehat{a}$	$0.08134^{\{3\}}$	$0.09619^{\{6\}}$	$0.07665^{\{2\}}$	$0.07569^{\{1\}}$	$0.09663^{\{7\}}$	$0.08174^{\{4\}}$	$0.08297^{\{5\}}$	$0.11687^{\{8\}}$
		$\widehat{b}$	$0.12819^{\{4\}}$	$0.14271^{\{5\}}$	$0.12259^{\{2\}}$	$0.11249^{\{1\}}$	$0.14990^{\{6\}}$	$0.12628^{\{3\}}$	$0.15755^{\{7\}}$	$0.25951^{\{8\}}$
		$\widehat{\beta}$	$0.25518^{\{4\}}$	$0.29906^{\{6\}}$	$0.23403^{\{1\}}$	$0.24931^{\{2\}}$	$0.30586^{\{7\}}$	$0.25422^{\{3\}}$	$0.29081^{\{5\}}$	$0.47988^{\{8\}}$
	MSE	$\widehat{a}$	$0.00662^{\{3\}}$	$0.00925^{\{6\}}$	$0.00588^{\{2\}}$	$0.00573^{\{1\}}$	$0.00934^{\{7\}}$	$0.00668^{\{4.5\}}$	$0.00688^{\{4.5\}}$	$0.01366^{\{8\}}$
		$\widehat{b}$	$0.01643^{\{4\}}$	$0.02037^{\{5\}}$	$0.01503^{\{2\}}$	$0.01265^{\{1\}}$	$0.02247^{\{6\}}$	$0.01595^{\{3\}}$	$0.02482^{\{7\}}$	$0.06735^{\{8\}}$
		$\widehat{\beta}$	$0.06512^{\{4\}}$	$0.08944^{\{6\}}$	$0.05477^{\{1\}}$	$0.06216^{\{2\}}$	$0.09355^{\{7\}}$	$0.06463^{\{3\}}$	$0.08457^{\{5\}}$	$0.23029^{\{8\}}$
	MRE	$\widehat{a}$	$0.10167^{\{3\}}$	$0.12024^{\{6\}}$	$0.09581^{\{2\}}$	$0.09461^{\{1\}}$	$0.12078^{\{7\}}$	$0.10218^{\{4\}}$	$0.10371^{\{5\}}$	$0.14608^{\{8\}}$
		$\widehat{b}$	$0.10682^{\{4\}}$	$0.11892^{\{5\}}$	$0.10216^{\{2\}}$	$0.09374^{\{1\}}$	$0.12491^{\{6\}}$	$0.10524^{\{3\}}$	$0.13129^{\{7\}}$	$0.21626^{\{8\}}$
		$\widehat{\beta}$	$0.14582^{\{4\}}$	$0.17089^{\{6\}}$	$0.13373^{\{1\}}$	$0.14247^{\{2\}}$	$0.17478^{\{7\}}$	$0.14527^{\{3\}}$	$0.16618^{\{5\}}$	$0.27422^{\{8\}}$
	$\sum Ranks$		$33^{\{5\}}$	$51^{\{6\}}$	$15^{\{2\}}$	$12^{\{1\}}$	$60^{\{6\}}$	$30.5^{\{4\}}$	$52.5^{\{7\}}$	$72^{\{8\}}$

Table 4: Simulation results for $\boldsymbol{\varphi} = (a = 2.50, b = 3.50, \beta = 1.50)^{\intercal}$.

n	Est.	Est. Par.	WLSE	LSE	MLE	MPSE	CVME	ADE	RADE	PCE
20	$\|BIAS\|$	$\widehat{a}$	$1.44091^{\{8\}}$	$1.34639^{\{4\}}$	$0.97626^{\{2\}}$	$0.95168^{\{1\}}$	$1.40828^{\{7\}}$	$1.38313^{\{6\}}$	$1.37371^{\{5\}}$	$1.21318^{\{3\}}$
		$\widehat{b}$	$2.01357^{\{8\}}$	$1.75118^{\{5\}}$	$1.89372^{\{6\}}$	$1.50542^{\{2\}}$	$1.61650^{\{3\}}$	$1.94669^{\{7\}}$	$1.67678^{\{4\}}$	$1.39091^{\{1\}}$
		$\widehat{\beta}$	$1.29058^{\{5\}}$	$1.14336^{\{7\}}$	$0.77916^{\{1\}}$	$1.06628^{\{6\}}$	$0.94174^{\{4\}}$	$1.17779^{\{8\}}$	$0.86554^{\{2\}}$	$0.90342^{\{3\}}$
	MSE	$\widehat{a}$	$2.07623^{\{8\}}$	$1.81278^{\{4\}}$	$0.95308^{\{2\}}$	$0.90569^{\{1\}}$	$1.98327^{\{7\}}$	$1.91305^{\{6\}}$	$1.88707^{\{5\}}$	$1.47181^{\{3\}}$
		$\widehat{b}$	$4.05445^{\{8\}}$	$3.06663^{\{5\}}$	$3.58617^{\{6\}}$	$2.26629^{\{2\}}$	$2.61307^{\{3\}}$	$3.78960^{\{7\}}$	$2.81158^{\{4\}}$	$1.93463^{\{1\}}$
		$\widehat{\beta}$	$1.66555^{\{8\}}$	$1.30728^{\{6\}}$	$0.60710^{\{1\}}$	$1.13696^{\{5\}}$	$0.88687^{\{4\}}$	$1.38720^{\{7\}}$	$0.74916^{\{2\}}$	$0.81617^{\{3\}}$
	MRE	$\widehat{a}$	$0.57637^{\{8\}}$	$0.53856^{\{4\}}$	$0.39050^{\{2\}}$	$0.38067^{\{1\}}$	$0.56331^{\{7\}}$	$0.55325^{\{6\}}$	$0.54948^{\{5\}}$	$0.48527^{\{3\}}$
		$\widehat{b}$	$0.57530^{\{8\}}$	$0.50034^{\{5\}}$	$0.54106^{\{6\}}$	$0.43012^{\{2\}}$	$0.46186^{\{3\}}$	$0.55620^{\{7\}}$	$0.47908^{\{4\}}$	$0.39740^{\{1\}}$
		$\widehat{\beta}$	$0.73747^{\{8\}}$	$0.65335^{\{6\}}$	$0.44524^{\{1\}}$	$0.60931^{\{5\}}$	$0.53814^{\{4\}}$	$0.67302^{\{7\}}$	$0.49459^{\{2\}}$	$0.51624^{\{3\}}$
	$\sum Ranks$		$69^{\{8\}}$	$46^{\{6\}}$	$27^{\{3\}}$	$25^{\{2\}}$	$42^{\{5\}}$	$61^{\{7\}}$	$33^{\{4\}}$	$20^{\{1\}}$
50	$\|BIAS\|$	$\widehat{a}$	$1.12618^{\{8\}}$	$1.10898^{\{6\}}$	$0.78913^{\{2\}}$	$0.69123^{\{1\}}$	$1.11439^{\{7\}}$	$1.06968^{\{5\}}$	$1.03745^{\{3\}}$	$1.06007^{\{4\}}$
		$\widehat{b}$	$1.85674^{\{8\}}$	$1.75478^{\{6\}}$	$1.62018^{\{3\}}$	$1.04490^{\{1\}}$	$1.64373^{\{5\}}$	$1.83600^{\{7\}}$	$1.62170^{\{4\}}$	$1.51174^{\{2\}}$
		$\widehat{\beta}$	$1.07152^{\{7\}}$	$1.02485^{\{6\}}$	$0.72118^{\{2\}}$	$0.60224^{\{1\}}$	$0.93788^{\{4\}}$	$1.01916^{\{5\}}$	$0.80844^{\{3\}}$	$1.84786^{\{8\}}$
	MSE	$\widehat{a}$	$1.26828^{\{7\}}$	$1.81278^{\{8\}}$	$0.62272^{\{2\}}$	$0.47780^{\{1\}}$	$1.24187^{\{6\}}$	$1.14421^{\{5\}}$	$1.07630^{\{3\}}$	$1.12375^{\{4\}}$
		$\widehat{b}$	$3.44747^{\{8\}}$	$3.06663^{\{6\}}$	$2.62498^{\{3\}}$	$1.09183^{\{1\}}$	$2.70184^{\{5\}}$	$3.37091^{\{7\}}$	$2.62990^{\{4\}}$	$2.28535^{\{2\}}$
		$\widehat{\beta}$	$1.14816^{\{7\}}$	$1.30728^{\{8\}}$	$0.52010^{\{2\}}$	$0.36269^{\{1\}}$	$0.87962^{\{5\}}$	$1.03869^{\{6\}}$	$0.65358^{\{3\}}$	$0.71887^{\{4\}}$
	MRE	$\widehat{a}$	$0.45047^{\{8\}}$	$0.44360^{\{6\}}$	$0.31565^{\{2\}}$	$0.27649^{\{1\}}$	$0.44576^{\{7\}}$	$0.42787^{\{5\}}$	$0.41498^{\{3\}}$	$0.42403^{\{4\}}$
		$\widehat{b}$	$0.53050^{\{8\}}$	$0.50137^{\{6\}}$	$0.46291^{\{3\}}$	$0.29854^{\{1\}}$	$0.46964^{\{5\}}$	$0.52547^{\{7\}}$	$0.46334^{\{4\}}$	$0.43192^{\{2\}}$
		$\widehat{\beta}$	$0.61230^{\{8\}}$	$0.58563^{\{7\}}$	$0.41210^{\{2\}}$	$0.34413^{\{1\}}$	$0.53593^{\{5\}}$	$0.58238^{\{6\}}$	$0.46197^{\{3\}}$	$0.48449^{\{4\}}$
	$\sum Ranks$		$69^{\{8\}}$	$59^{\{7\}}$	$21^{\{2\}}$	$09^{\{1\}}$	$49^{\{5\}}$	$53^{\{6\}}$	$30^{\{3\}}$	$34^{\{4\}}$
100	$\|BIAS\|$	$\widehat{a}$	$0.90953^{\{7\}}$	$0.95698^{\{8\}}$	$0.69910^{\{2\}}$	$0.45533^{\{1\}}$	$0.90225^{\{6\}}$	$0.88976^{\{4\}}$	$0.84696^{\{3\}}$	$0.89086^{\{5\}}$
		$\widehat{b}$	$1.73312^{\{7\}}$	$1.73822^{\{8\}}$	$1.50438^{\{3\}}$	$0.31199^{\{1\}}$	$1.65037^{\{5\}}$	$1.71858^{\{6\}}$	$1.49575^{\{2\}}$	$1.54911^{\{4\}}$
		$\widehat{\beta}$	$0.91483^{\{7\}}$	$0.95289^{\{8\}}$	$0.67709^{\{2\}}$	$0.23095^{\{1\}}$	$0.88683^{\{5\}}$	$0.89474^{\{6\}}$	$0.74858^{\{3\}}$	$0.76716^{\{4\}}$
	MSE	$\widehat{a}$	$0.82724^{\{7\}}$	$0.91581^{\{8\}}$	$0.48873^{\{2\}}$	$0.20733^{\{1\}}$	$0.81406^{\{6\}}$	$0.79168^{\{4\}}$	$0.71735^{\{3\}}$	$0.79363^{\{5\}}$
		$\widehat{b}$	$3.00370^{\{8\}}$	$3.02140^{\{7\}}$	$2.26315^{\{3\}}$	$0.09734^{\{1\}}$	$2.72374^{\{5\}}$	$2.95353^{\{6\}}$	$2.23727^{\{2\}}$	$2.39976^{\{4\}}$
		$\widehat{\beta}$	$0.83691^{\{7\}}$	$0.90800^{\{8\}}$	$0.45845^{\{2\}}$	$0.05334^{\{1\}}$	$0.78647^{\{5\}}$	$0.80056^{\{6\}}$	$0.56037^{\{3\}}$	$0.58854^{\{4\}}$
	MRE	$\widehat{a}$	$0.36381^{\{6\}}$	$0.38279^{\{7\}}$	$0.27964^{\{2\}}$	$0.18213^{\{1\}}$	$0.36090^{\{5\}}$	$0.35590^{\{3\}}$	$0.38879^{\{8\}}$	$0.35634^{\{4\}}$
		$\widehat{b}$	$0.49518^{\{7\}}$	$0.49663^{\{8\}}$	$0.42982^{\{3\}}$	$0.08914^{\{1\}}$	$0.47154^{\{5\}}$	$0.49102^{\{6\}}$	$0.42736^{\{2\}}$	$0.44260^{\{4\}}$
		$\widehat{\beta}$	$0.52276^{\{7\}}$	$0.54451^{\{8\}}$	$0.38691^{\{2\}}$	$0.13197^{\{1\}}$	$0.50676^{\{5\}}$	$0.51128^{\{6\}}$	$0.42776^{\{3\}}$	$0.43838^{\{4\}}$
	$\sum Ranks$		$63^{\{7\}}$	$70^{\{8\}}$	$21^{\{2\}}$	$09^{\{1\}}$	$47^{\{5.5\}}$	$47^{\{5.5\}}$	$26^{\{3\}}$	$38^{\{4\}}$
200	$\|BIAS\|$	$\widehat{a}$	$0.73377^{\{5\}}$	$0.79777^{\{8\}}$	$0.58947^{\{2\}}$	$0.23916^{\{1\}}$	$0.75711^{\{7\}}$	$0.70499^{\{4\}}$	$0.68617^{\{3\}}$	$0.74947^{\{6\}}$
		$\widehat{b}$	$1.50757^{\{5\}}$	$1.64400^{\{8\}}$	$1.34236^{\{2\}}$	$0.14357^{\{1\}}$	$1.58103^{\{7\}}$	$1.53191^{\{6\}}$	$1.36990^{\{3\}}$	$1.50475^{\{4\}}$
		$\widehat{\beta}$	$0.74488^{\{6\}}$	$0.84104^{\{8\}}$	$0.57881^{\{2\}}$	$0.10610^{\{1\}}$	$0.80726^{\{7\}}$	$0.72802^{\{5\}}$	$0.66236^{\{3\}}$	$0.69454^{\{4\}}$
	MSE	$\widehat{a}$	$0.53842^{\{5\}}$	$0.63644^{\{8\}}$	$0.34747^{\{2\}}$	$0.05720^{\{1\}}$	$0.57413^{\{7\}}$	$0.49701^{\{4\}}$	$0.47083^{\{3\}}$	$0.56170^{\{6\}}$
		$\widehat{b}$	$2.27278^{\{5\}}$	$2.70272^{\{8\}}$	$1.80193^{\{2\}}$	$0.02061^{\{1\}}$	$2.49965^{\{7\}}$	$2.34675^{\{6\}}$	$1.87663^{\{3\}}$	$2.26428^{\{4\}}$
		$\widehat{\beta}$	$0.55485^{\{6\}}$	$0.70734^{\{8\}}$	$0.33502^{\{2\}}$	$0.01260^{\{1\}}$	$0.65166^{\{7\}}$	$0.53001^{\{5\}}$	$0.43872^{\{3\}}$	$0.48239^{\{4\}}$
	MRE	$\widehat{a}$	$0.29351^{\{5\}}$	$0.31911^{\{8\}}$	$0.23579^{\{2\}}$	$0.09566^{\{1\}}$	$0.30309^{\{7\}}$	$0.28200^{\{4\}}$	$0.27447^{\{3\}}$	$0.29979^{\{6\}}$
		$\widehat{b}$	$0.43074^{\{5\}}$	$0.46971^{\{8\}}$	$0.38353^{\{2\}}$	$0.04102^{\{1\}}$	$0.45172^{\{7\}}$	$0.43769^{\{6\}}$	$0.39140^{\{3\}}$	$0.42993^{\{4\}}$
		$\widehat{\beta}$	$0.42565^{\{6\}}$	$0.48059^{\{8\}}$	$0.33075^{\{2\}}$	$0.06063^{\{1\}}$	$0.46129^{\{7\}}$	$0.41601^{\{5\}}$	$0.37849^{\{3\}}$	$0.39688^{\{4\}}$
	$\sum Ranks$		$48^{\{6\}}$	$72^{\{8\}}$	$18^{\{2\}}$	$09^{\{1\}}$	$63^{\{7\}}$	$45^{\{5\}}$	$27^{\{3\}}$	$42^{\{4\}}$
400	$\|BIAS\|$	$\widehat{a}$	$0.55077^{\{5\}}$	$0.64682^{\{8\}}$	$0.46268^{\{2\}}$	$0.14397^{\{1\}}$	$0.63297^{\{6\}}$	$0.54530^{\{3\}}$	$0.54931^{\{4\}}$	$0.64264^{\{7\}}$
		$\widehat{b}$	$1.30198^{\{4\}}$	$1.47261^{\{8\}}$	$1.10089^{\{2\}}$	$0.08511^{\{1\}}$	$1.43555^{\{7\}}$	$1.30976^{\{5\}}$	$1.22018^{\{3\}}$	$1.38862^{\{6\}}$
		$\widehat{\beta}$	$0.56630^{\{4\}}$	$0.69377^{\{8\}}$	$0.46717^{\{2\}}$	$0.06300^{\{1\}}$	$0.67293^{\{7\}}$	$0.56856^{\{5\}}$	$0.56352^{\{3\}}$	$0.60144^{\{6\}}$
	MSE	$\widehat{a}$	$0.30334^{\{5\}}$	$0.41837^{\{8\}}$	$0.21407^{\{2\}}$	$0.02073^{\{1\}}$	$0.40065^{\{6\}}$	$0.29735^{\{3\}}$	$0.30174^{\{4\}}$	$0.41298^{\{7\}}$
		$\widehat{b}$	$1.69516^{\{4\}}$	$2.16859^{\{8\}}$	$1.21195^{\{2\}}$	$0.00724^{\{1\}}$	$2.06081^{\{7\}}$	$1.71548^{\{5\}}$	$1.48884^{\{3\}}$	$1.92826^{\{6\}}$
		$\widehat{\beta}$	$0.32107^{\{4\}}$	$0.48131^{\{8\}}$	$0.21825^{\{2\}}$	$0.00397^{\{1\}}$	$0.45283^{\{7\}}$	$0.32326^{\{5\}}$	$0.31756^{\{3\}}$	$0.36173^{\{6\}}$
	MRE	$\widehat{a}$	$0.22031^{\{6\}}$	$0.21972^{\{4.5\}}$	$0.18507^{\{2\}}$	$0.05759^{\{1\}}$	$0.25319^{\{7\}}$	$0.21812^{\{3\}}$	$0.21972^{\{4.5\}}$	$0.25705^{\{8\}}$
		$\widehat{b}$	$0.37199^{\{5\}}$	$0.34862^{\{3.5\}}$	$0.31454^{\{2\}}$	$0.02432^{\{1\}}$	$0.41016^{\{8\}}$	$0.37422^{\{6\}}$	$0.34862^{\{3.5\}}$	$0.39675^{\{7\}}$
		$\widehat{\beta}$	$0.32379^{\{4\}}$	$0.39644^{\{8\}}$	$0.26696^{\{2\}}$	$0.03600^{\{1\}}$	$0.38453^{\{7\}}$	$0.32489^{\{5\}}$	$0.32201^{\{3\}}$	$0.34368^{\{6\}}$
	$\sum Ranks$		$41^{\{4.5\}}$	$64^{\{8\}}$	$18^{\{2\}}$	$09^{\{1\}}$	$62^{\{7\}}$	$41^{\{4.5\}}$	$35^{\{3\}}$	$59^{\{6\}}$

Table 5: Partial and overall ranks of all estimation methods for various combinations of $\boldsymbol{\varphi}$.

$\boldsymbol{\phi}^{\intercal}$	n	**WLSE**	**LSE**	**MLE**	**MPSE**	**CVME**	**ADE**	**RADE**	**PCE**
	20	7	5	1	3	6	2	4	8
	50	7	5	1	3	6	2	4	8
$(a = 0.25, b = 0.60, \beta = 0.70)$	100	7	6	1	3	5	2	4	8
	200	7	5	1	2	6	3	4	8
	400	7	5	1	2	6	3	4	8
	20	5	6	1	3	7.5	2	4	7.5
	50	4	7	1	3	6	2	5	8
$(a = 0.5, b = 0.80, \beta = 1.20)$	100	4	6	1	3	7	2	5	8
	200	4	6	1	2	7	3	5	8
	400	3	7	1	2	6	4	5	8
	20	6	7.5	1	2	7.5	3	5	8
	50	4	7	1	2.5	6	2.5	5	8
$(a = 0.80, b = 1.20, \beta = 0.20)$	100	4	7	1	2.5	6	2.5	5	8
	200	4	7	1	2	6	3	5	8
	400	5	6	2	1	6	4	7	8
	20	8	6	3	2	5	7	4	1
	50	8	7	2	1	5	6	3	4
$(a = 2.50, b = 3.50, \beta = 1.50)$	100	7	8	2	1	5.5	5.5	3	4
	200	6	8	2	1	7	5	3	4
	400	4.5	8	2	1	7	4.5	3	6
	$\sum$ *Ranks*	91.5	129.5	39	42	123.5	58	88	138.5
	Overall Ranks	5	6	1	2	7	3	4	8

6 Applications

In this section, the flexibility of the KwT-W distribution are explored by analyzing three real-life data sets from medicine, agriculture, and engineering sciences. The first data about survival times in years of a group of patients given chemotherapy and radiation treatment is given by Bekker et al. (2000). The data are: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

The second data about soil fertility influence are studied by Murthy et al. (2004) for 128 plants. The data are: 0.22, 0.17, 0.11, 0.10, 0.15, 0.06, 0.05, 0.07, 0.12, 0.09, 0.23, 0.25, 0.23, 0.24, 0.20, 0.08, 0.11, 0.12, 0.10, 0.06, 0.20, 0.17, 0.20, 0.11, 0.16, 0.09, 0.10, 0.12, 0.12, 0.10, 0.09, 0.17, 0.19, 0.21, 0.18, 0.26, 0.19, 0.17, 0.18, 0.20, 0.24, 0.19, 0.21, 0.22, 0.17, 0.08, 0.08, 0.06, 0.09, 0.22, 0.23, 0.22, 0.19, 0.27, 0.16, 0.28, 0.11, 0.10, 0.20, 0.12, 0.15, 0.08, 0.12, 0.09, 0.14, 0.07, 0.09, 0.05, 0.06, 0.11, 0.16, 0.20, 0.25, 0.16, 0.13, 0.11, 0.11, 0.11, 0.08, 0.22, 0.11, 0.13, 0.12, 0.15, 0.12, 0.11, 0.11, 0.15, 0.10, 0.15, 0.17, 0.14, 0.12, 0.18, 0.14, 0.18, 0.13, 0.12, 0.14, 0.09, 0.10, 0.13, 0.09, 0.11, 0.11, 0.14, 0.07, 0.07, 0.19, 0.17, 0.18, 0.16, 0.19, 0.15, 0.07, 0.09, 0.17, 0.10, 0.08, 0.15, 0.21, 0.16, 0.08, 0.10, 0.06, 0.08, 0.12, 0.13.

The third data about failure times of 20 mechanical components are given in Murthy et al. (2004). The data are: 0.485, 0.067, 0.125, 0.076, 0.084, 0.160, 0.086, 0.098, 0.089, 0.114, 0.098, 0.115, 0.085, 0.114, 0.121, 0.131, 0.081, 0.068, 0.085, 0.149.

The KwT-W distribution and its competing distributions including the Kumaraswamy–Weibull (KW) (Cordeiro et al., 2010), Topp–Leone Weibull (TLW) (Aryal et al., 2017), beta-Weibull (BW) (Lee et al., 2007) and Kumaraswamy–Pareto (KP) (Bourguignon et al., 2012) are fitted using the three data sets.

The maximized log-likelihood ($-2\widehat{\ell}$) and other information criterion (IC) are calculated to compare the fitted distributions. These measures include the Akaike IC (AIC), consistent-AIC (CAIC), Hannan–Quinn IC (HQIC) and Bayesian IC (BIC). Additionally, other goodness-of-fit (GOF) statistics such as Cramér–von Mises (W^*), Anderson–Darling (A^*), K-S statistics and the corresponding K-S p-value are also computed.

Tables 6–8 report the information criteria, goodness-of-fit measures, estimated parameters using the ML method, and the standard errors (SEs) (in parentheses) of the KwT-W distribution and its competing distributions for the three data sets, respectively. The figures in the three tables show that the KwT-W distribution provides better fit for the analyzed data as compared to rival models. For visual comparisons, the estimated pdfs and cdfs for the KwT-W and other distributions are displayed in Figures 4–6 for the three data sets. In summary, the plots support the estimated measures in the tables, showing the superior fit of the KwT-W distribution over its competing models.

7 Conclusions

A new flexible class of continuous distributions, called the Kumaraswamy tan-G (KwT-G) family is proposed. Several properties of the KwT-G family are derived. Four sub-models of the KwT-G family are introduced. Eight estimation methods for the KwT-exponential (KwT-E) distribution are explored. It is noted from the simulation results that the maximum likelihood is the best estimation approach for the KwT-E parameters. The utility of the KwT-Weibull (KwT-W) distribution is illustrated by fitting it to three real data sets. The KwT-W distribution provides better fit as compared to the Kumaraswamy–Weibull, Topp–Leone Weibull, beta-Weibull, and Kumaraswamy–Pareto distributions.

The perspectives of the current work are many, including the construction of quantile

Table 6: Findings of chemotherapy and radiation data for the KwT-W distribution and its competing distributions.

Model	$-2\hat{\ell}$	AIC	BIC	$HQIC$	$CAIC$	W^*	A^*	K-S	P-Value	Estimates(SEs)
KwT-W	56.75	121.51	128.74	124.20	122.51	0.38	0.309	0.062	0.990	$\hat{\alpha}$=0.6554(0.3178) $\hat{\beta}$=1.1917(0.4174) $\hat{a}$=0.3283(0.4463) $\hat{b}$=0.3219(0.2121)
KW	57.35	122.71	129.94	125.41	1223.71	0.049	0.358	0.085	0.871	$\hat{\alpha}$=3.0873(0.1667) $\hat{\beta}$=0.1065(0.0167) $\hat{a}$=8.5023(0.1176) $\hat{b}$=0.8686(0.0627)
TLW	58.04	122.08	127.50	124.10	122.67	0.069	0.470	0.100	0.717	$\hat{\alpha}$=1.6172(1.9848) $\hat{a}$=0.5630(0.5588) $\hat{b}$=0.8060(0.5261)
BW	57.42	122.84	130.07	125.54	123.84	0.051	0.369	0.088	0.846	$\hat{\alpha}$=2.3120(1.0564) $\hat{\beta}$=0.1026(0.0171) $\hat{a}$=8.7439(0.0651) $\hat{b}$=0.8791(0.0560)
KP	58.01	124.01	131.24	126.70	125.01	0.089	0.597	0.111	0.588	$\hat{\alpha}$=3.4253(7.6108) $\hat{\beta}$=0.9650(0.2211) $\hat{a}$=4.0718(14.6023) $\hat{b}$=22.7727(89.8592)

Table 7: Findings of soil fertility data for the KwT-W distribution and its competing distributions.

Model	$-2\hat{\ell}$	AIC	BIC	$HQIC$	$CAIC$	W^*	A^*	K-S	P-Value	Estimates(SEs)
KwT-W	-200.41	-392.82	-381.41	-388.18	-392.49	0.065	0.395	0.069	0.563	$\hat{\alpha}$=2.9580(0.6718) $\hat{\beta}$=0.4451(0.3312) $\hat{a}$=0.4657(0.2027) $\hat{b}$=0.1295(0.0136)
KW	-196.53	-385.06	-373.65	-380.42	-384.73	0.141	0.802	0.092	0.227	$\hat{\alpha}$=5.8728(63.4739) $\hat{\beta}$=2.7748(51.9770) $\hat{a}$=11.1770(190.7604) $\hat{b}$=1.0043(8.4329)
TLW	-196.71	-387.43	-378.88	-383.96	-387.24	0.137	0.782	0.091	0.243	$\hat{\alpha}$=3.5772(2.8179) $\hat{a}$=18.4507(11.0670) $\hat{b}$=1.5193(0.5116)
BW	-196.92	-385.84	-374.43	-381.20	-385.51	0.133	0.759	0.090	0.247	$\hat{\alpha}$=6.2974(9.0385) $\hat{\beta}$=11.2307(46.2660) $\hat{a}$=3.4925(12.7307) $\hat{b}$=1.0335(0.7764)
KP	-196.05	-384.11	-372.70	-379.48	-383.79	0.173	0.958	0.109	0.094	$\hat{\alpha}$=3.8497(13.5241) $\hat{\beta}$=4.1960(0.8922) $\hat{a}$=0.3528(0.2249) $\hat{b}$=0.0424(0.0246)

Table 8: Findings of failure times data for the KwT-W distribution and its competing distributions.

Model	$-2\widehat{\ell}$	AIC	BIC	$HQIC$	$CAIC$	W^*	A^*	K-S	P-Value	Estimates(SEs)
KwT-W	-39.10	-70.20	-66.22	-69.42	-67.53	0.049	0.383	0.122	0.923	$\hat{\alpha}$=0.3880(0.0772) $\hat{\beta}$=3.9818(0.6147) $\hat{a}$=0.0043(0.0032) $\hat{b}$=0.8285(0.0772)
KW	-36.10	-65.21	-61.23	-64.44	-62.55	0.087	0.683	0.145	0.788	$\hat{\alpha}$=313.6882(165.0667) $\hat{\beta}$=0.5116(0.1921) $\hat{a}$=33.9116(8.8877) $\hat{b}$=0.7056(0.1005)
TLW	-35.98	-65.97	-62.99	-65.39	-64.47	0.101	0.777	0.135	0.855	$\hat{\alpha}$=154.5442(214.2022) $\hat{a}$=9.9339(2.1245) $\hat{b}$=0.5637(0.1275)
BW	-36.57	-65.14	-61.15	-64.36	-62.47	0.088	0.689	0.145	0.788	$\hat{\alpha}$=297.0212(364.1742) $\hat{\beta}$=0.4733(0.1408) $\hat{a}$=36.6273(4.0336) $\hat{b}$=0.7198(0.0954)
KP	-28.74	-49.48	-45.50	-48.71	-46.82	0.311	2.012	0.237	0.210	$\hat{\alpha}$=4.9861(10.7667) $\hat{\beta}$=3.2392(1.0229) $\hat{a}$=0.2398(0.1291) $\hat{b}$=0.0212(0.0222)

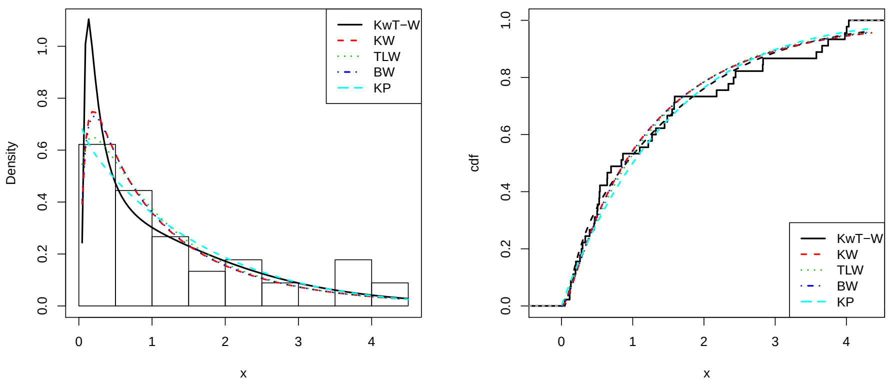


Figure 4: Estimated pdfs and cdfs for the KwT-W and other distributions for chemotherapy and radiation data.

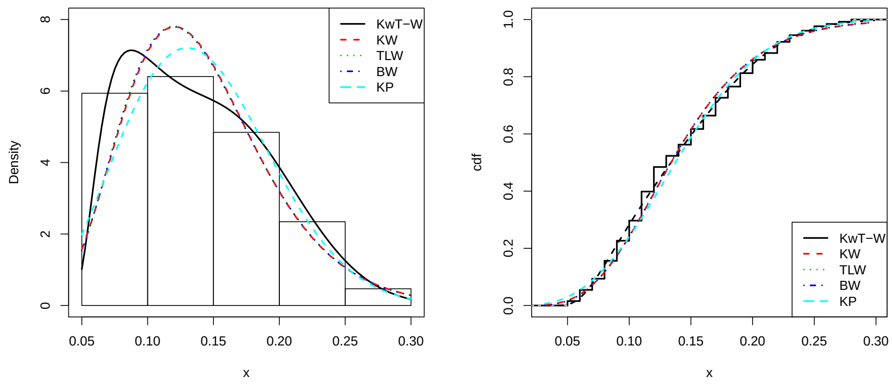


Figure 5: Estimated pdfs and cdfs for the KwT-W and other distributions for soil fertility data.

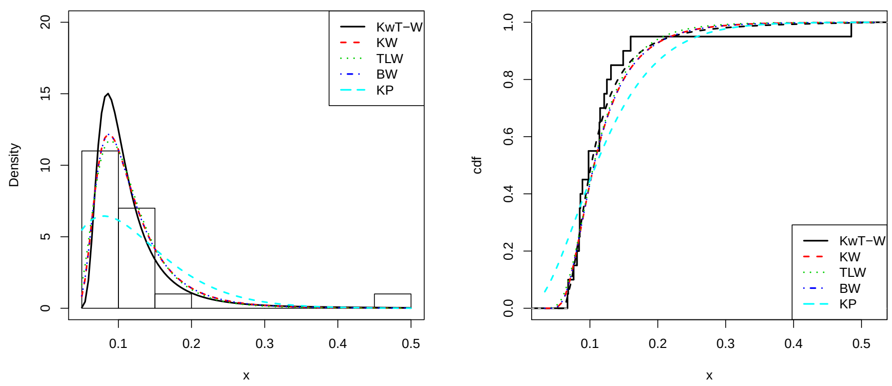


Figure 6: Estimated pdfs and cdfs for the KwT-W and other distributions for failure times data.

regression models by exploiting the flexibility of the KwT-E and KwT-W distributions, the applications of introduced lifetime sub-models of the KwT-G family, and the construction of the bivariate KwT-G family.

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