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By Joybari, Asgharzadeh and Raqab

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Estimation and prediction for proportional hazard family based on a simple step-stress model with Type-II censored data

S. N. Jouybari[a], Akbar Asgharzadeh*[a], and Mohammad Z. Raqab[b]

[a]*Department of Statistics, University of Mazandaran, Babolsar, Iran*
[b]*Department of Statistics & OR, Faculty of Science, Kuwait University, Al-Shadadiyya, Kuwait*

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The accelerated life testing is the key methodology of assessing product reliability rapidly. This type of life testing is more efficient with low cost than the classical reliability testing. For this, estimating of the underlying model and predicting the future failure times are issues deserve the attention and follow-up. In this paper, a simple step-stress testing experiment is considered when the lifetime data comes from a proportional hazard family under Type-II censoring. We discuss frequentist and Bayes estimators of the underlying model parameters. Prediction of unobserved or censored lifetimes is also tackled here, and frequentist and Bayesian predictors are developed. An algorithm is presented to generate ordered lifetime data from the proportional hazard family under the simple step-stress accelerated lifetime testing. Two numerical examples are also provided to illustrate the estimation and prediction methods presented in this paper. Finally, a Monte Carlo simulation experiment is performed to evaluate the performance of the various estimation and prediction methods developed in this paper. The results show

*Corresponding authors: a.asgharzadeh@umz.ac.ir

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that the Bayesain estimation and prediction under the informative prior perform better than the ones obtained based on frequentist methods. Also, the maximum likelihood method does not work well for predicting future failure times.

keywords: Accelerated life testing, Bayes methods, Frequentist methods, Monte Carlo simulation, Prediction, Proportional hazard rate family.

1 Introduction

Due to advanced technology, competitive markets, and consumer demand, most products are highly reliable as these products may work properly for years or even decades. In this spirit, if these products are exposed to life-testing experiment under normal levels, the experiment may take a long time and it is almost impossible to get enough information about the failure times and reliability of these products. In addition, the life-testing experiment of these products may take a long time under censoring schemes. Alternatively, the experimenters tend to expose these products to accelerated life tests (ALTs) so that their failure times can be observed sooner. In ALTs, the products are exposed to higher stresses than normal conditions. These stresses may be temperature, voltage, pressure, vibration, and so on. The information obtained from such ALTs is later used to estimate the failure times and reliability of the products under normal conditions, see Nelson (1982).

One of the most common types of ALTs is the step-stress ALT (SSALT). In this test, the components are constantly exposed to higher than normal stress levels. Hence, a random sample of n identical units (components) are placed on a life test under an initial stress level s_1. Then, the stress levels are increased to $s_2, \ldots, s_{m+1}$ at fixed times $\tau_1, \tau_2, \ldots, \tau_m$ respectively. Therefore, the stress in the sample increases until all products fail or the test is terminated under a censoring scheme. In this paper, we consider a simple SSALT with only two stress levels. This model has been widely used in the literature, e.g., Miller and Nelson (1983), Bai et al. (1989) and DeGroot and Goel (1979).
For modeling structure, we also consider the popular proportional hazard rate (PHR) model for the common lifetime distribution of the units. That is, we assume that the common cumulative distribution function (CDF) of the units is

$$F(t) = 1 - [\bar{F}_0(t)]^{\theta}, \tag{1}$$

where $\theta > 0$ is an unknown parameter and $\bar{F}_0(t) = 1 - F_0(t)$ is the baseline survival function with support $[0, \infty)$ being completely known and but it does not include the parameter θ. The probability density function (PDF) is given by

$$f(t) = \theta f_0(t)[\bar{F}_0(t)]^{\theta-1},$$

where $f_0(t)$ is the PDF of $F_0(.)$. Its hazard rate function (HRF) is given by

$$h(t) = \frac{f(t)}{\bar{F}_0(t)} = \frac{\theta f_0(t)}{\bar{F}_0(t)} = \theta h_0(t),$$

where $h_0(t)$ is the baseline HRF. Several well-known lifetime distributions belong to this model, see for example, Marshall and Olkin (2007). It has received a considerable attention for modeling failure time data since it is flexible to accommodate monotonic and non-monotonic failure rate models. For inferences involving the PHR model, one may refer to Asgharzadeh and Valiollahi (2009), Asgharzadeh and Valiollahi (2010), Basirat et al. (2015), Chaturvedi et al. (2019) and Basiri and Asgharzadeh (2021).
The following commonly used lifetime distributions are members of the PRH model:

(i) **Exponential distribution**. Taking $\bar{F}_0(t) = e^{-t}$, we have $F_0(t) = 1 - e^{-\theta t}$, which is equivalent to assuming that the lifetime distribution of the items is an exponential distribution with the rate parameter θ.

(ii) **Rayleigh distribution**. Taking $\bar{F}_0(t) = e^{-t^2}$, we have $F(t) = 1 - e^{-\theta t^2}$. This is equivalent to assuming that the lifetime distribution of the items is a Rayleigh distribution with the parameter θ.

(iii) **Pareto distribution**. Considering $\bar{F}_0(t) = (1+t)^{-1}$, we have $F(t) = 1-(1+t)^{-\theta}$, which is equivalent to assuming that the lifetime distribution of the items is a Pareto distribution with shape parameter θ.

Moreover, we can also consider two-parameter PHR model. For example, the Weibull $WE(\lambda, \theta)$ and Burr Type XII, $BUR(\beta, \theta)$ distributions are members of PHR family of distributions with corresponding baseline cdfs $\bar{F}_0(t) = e^{-\lambda t}$ and $\bar{F}(t) = (1 + t^{\beta})^{-1}$.

Inferences for a simple step-stress test under Type-II censoring have been discussed in the literature. Balakrishnan et al. (2007) considered point and interval estimation for a simple SSALT test under Type-II censoring exponential lifetimes. They derived the maximum likelihood estimators (MLEs) of the exponential parameters using a cumulative exposure model and proposed several confidence intervals for the model parameters. Basak and Balakrishnan (2018) discussed prediction problem for exponential step-stress test with Type-II censoring assuming the cumulative exposure model and presented several point and interval predictors of future lifetimes. Xiong (1998) discussed the statistical inference for a simple step-stress model under Type-II censoring when the lifetimes of the items is distributed exponentially. Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring was presented by Balakrishnan and Han (2008). Xiong and Milliken (2002) provided some prediction limits for a step-stress model in accelerated life testing. Basak and Balakrishnan (2017) and Basak and Balakrishnan (2018) considered the problem of predicting the failure times of censored items for a simple step-stress model from exponential distribution under progressive Type-II censoring and Type-II right censoring, respectively. Prakash (2018) obtained the Bayes estimators under the first-failure progressive censoring scheme based on constant-stress partially ALT when the lifetime distribution is Gompertz distribution. Dey and Nassar (2020) considered different classical methods of estimation under constant stress ALT for the exponentiated Lindley distribution. The meta-analysis of Type-II censored step-stress experiment under the two parameter Weibull distribution was discussed by Samanta and Kundu (2021). Most recently, Amleh and Raqab (2021) discussed the prediction of censored Weibull lifetimes in a simple step-stress plan under Khamis-Higgens model.

In this paper, our goal is to consider statistical inferential problems including the estimation and prediction for a simple step-stress test under Type-II censoring when the test units belongs to the PHR family. The model parameters are estimated using the maximum likelihood and Bayesian methods. Further, prediction problem of the unobserved or censored lifetimes is tackled. An algorithm for generating the simple step-stress ordered lifetimes is also proposed. The main difference of our work with the other existing works is that we have considered a more general cumulative exposure model which has not been considered before. The exponential cumulative exposure model is considered as a special case of this general model. Moreover our methods are also quite general involving frequentist and Bayesian methods.

The paper is organized as follows. In Section 2, we first discuss the model description and likelihood function. In Section 3, we derive the maximum Likelihood and Bayes estimates (BEs) of the model parameters. In Section 4, the prediction of censored lifetimes is considered, and different prediction methods including the maximum Likelihood, best unbiased, conditional median and Baysian methods are developed. In Section 5, an algorithm to generate the simple step-stress ordered lifetimes is proposed. Two numerical examples are also presented to illustrate the procedures proposed and a Monte Carlo simulation is performed to assess the methods of estimation and prediction developed here.

2 Model description and likelihood function

Let us assume that the component lifetimes at stress levels s_1 and s_2 follow the PHR model in (1) with the baseline HRF $h_0(t)$ and parameters θ_1 and θ_2, respectively. To connect the distribution functions at the two stress levels, we assume that the HRF, when stress changes at the pre-fixed time τ, is of the form

$$h(t) = \begin{cases} \theta_1 h_0(t) & 0 < t \leq \tau \\ \theta_2 h_0(t) & \tau < t < \infty. \end{cases}$$

So, the cumulative HRF becomes

$$H(t) = \begin{cases} \int_0^t \theta_1 h_0(u)du = \theta_1 H_0(t) & 0 < t \leq \tau \\ \int_0^\tau \theta_1 h_0(u)du + \int_\tau^t \theta_2 h_0(u)du = \theta_1 H_0(\tau) + \theta_2[H_0(t) - H_0(\tau)] & \tau < t < \infty, \end{cases}$$

where $H_0(t) = \int_0^t h_0(u)du$ is the baseline cumulative HRF. Hence, the CDF of the lifetime is

$$G(t) = 1 - e^{-H(t)} = \begin{cases} G_1(t) = 1 - e^{-\theta_1 H_0(t)} & 0 < t \leq \tau \\ G_2(t) = 1 - e^{-\theta_1 H_0(\tau) - \theta_2[H_0(t) - H_0(\tau)]} & \tau < t < \infty, \end{cases} \tag{2}$$

and the corresponding PDF is

$$g(t) = \begin{cases} g_1(t) = \theta_1 h_0(t) e^{-\theta_1 H_0(t)} & 0 < t \leq \tau \\ g_2(t) = \theta_2 h_0(t) e^{-\theta_2 H_0(t) + (\theta_2 - \theta_1) H_0(\tau)} & \tau < t < \infty. \end{cases} \tag{3}$$

Note that for the exponential case, we have $H_0(t) = t$ and $h_0(t) = 1$. In this case, the exponential cumulative exposure model with the CDF

$$G(t) = \begin{cases} 1 - e^{-\theta_1 t} & 0 < t \leq \tau \\ 1 - e^{-\theta_2 t + (\theta_2 - \theta_1)\tau} & \tau < t < \infty, \end{cases}$$

and PDF

$$g(t) = \begin{cases} \theta_1 e^{-\theta_1 t} & 0 < t \leq \tau \\ \theta_2 e^{-\theta_2 t + (\theta_2 - \theta_1)\tau} & \tau < t < \infty, \end{cases}$$

is obtained as a special case.

Assume a sample of n identical components are placed on the simple SSALT at an initial stress level of s_1 and the stress level is changed to s_2 at the pre-fixed time τ. Under Type-II censoring, the experiment ends as soon as r-th failure ($1 \leq r \leq n$) is observed.

The advantage of Type-II censoring in accelerated life-testing experiment is that it saves the experiment time. It also saves the budget and keeps some products (or components) under experiment. If $r = n$, we observe a complete sample in the simple SSALT and therefore all the products will be lost. For the simple SSALT under Type-II censoring, suppose we observed the ordered failure times

$$\mathbf{t} = \{t_{1:n} < \cdots < t_{n_1:n} < \tau < t_{n_1+1:n} < \cdots < t_{r:n}\},$$

from

$$\mathbf{T} = \{T_{1:n} < \cdots < T_{n_1:n} < \tau < T_{n_1+1:n} < \cdots < T_{r:n}\},$$

as the r Type-II right censored order statistics from a population with PDF $g(t)$ in (3). Here n_1 is the number of failures that occurred before τ, and $r - n_1$ is the number of failures that occurred after τ. Throughout this paper, we will only consider the main case when $r > n_1$. In this case, there will be observations at both stress levels s_1 and s_2 (for the case $r = n_1$, there will be only observations at the stress level s_1 and the stress level will not reach s_2). So, the likelihood function of the observed failure times is

$$L(\theta_1, \theta_2|\mathbf{t}) = c \prod_{i=1}^{n_1} g_1(t_{i:n}) \prod_{i=n_1+1}^{r} g_2(t_{i:n}) \; [1 - G_2(t_{r:n})]^{n-r},$$

where c is a constant factor. From $g_1(.)$ and $g_2(.)$ in (3), the likelihood function is obtained as

$$\begin{aligned} L(\theta_1, \theta_2|\mathbf{t}) &= c \; \theta_1^{n_1} \; \theta_2^{r-n_1} \prod_{i=1}^{r} h_0(t_{i:n}) \; e^{-\theta_1 \left[\sum_{i=1}^{n_1} H_0(t_{i:n}) + (n-n_1) H_0(\tau) \right]} \\ &\times \quad e^{-\theta_2 \left[\sum_{i=n_1+1}^{r} \left(H_0(t_{i:n}) - H_0(\tau) \right) + (n-r) \left(H_0(t_{r:n}) - H_0(\tau) \right) \right]} . \end{aligned} \quad (4)$$

3 Estimation methods

In this section, we derive the MLEs and BEs of the model parameters of θ_1 and θ_2.

3.1 Maximum likelihood estimation

The log-likelihood function without considering constant factor is given by

$$\begin{aligned} \log L(\theta_1, \theta_2|\mathbf{t}) &= n_1 \log \theta_1 + (r - n_1) \log \theta_2 - \theta_1 \left(\sum_{i=1}^{n_1} H_0(t_{i:n}) + (n - n_1) H_0(\tau) \right) \\ &- \theta_2 \left(\sum_{i=n_1+1}^{r} \big[H_0(t_{i:n}) - H_0(\tau) \big] + (n - r) \big[H_0(t_{r:n}) - H_0(\tau) \big] \right). \end{aligned} \quad (5)$$

It follows from (5) that the likelihood equations are given by

$$\frac{\partial \log L(\theta_1, \theta_2|\mathbf{t})}{\partial \theta_1} = \frac{n_1}{\theta_1} - \sum_{i=1}^{n_1} H_0(t_{i:n}) - (n-n_1)H_0(\tau) = 0,$$

and

$$\frac{\partial \log L(\theta_1, \theta_2|\mathbf{t})}{\partial \theta_2} = \frac{(r-n_1)}{\theta_2} - \sum_{i=n_1+1}^{r} [H_0(t_{i:n}) - H_0(\tau)] - (n-r)[H_0(t_{r:n}) - H_0(\tau)] = 0.$$

The MLEs of θ_1 and θ_2 are immediately, obtained as

$$\hat{\theta}_{1ML} = \frac{n_1}{\sum_{i=1}^{n_1} H_0(t_{i:n}) + (n-n_1)H_0(\tau)}, \tag{6}$$

and

$$\hat{\theta}_{2ML} = \frac{r-n_1}{\sum_{i=n_1+1}^{r} [H_0(t_{i:n}) - H_0(\tau)] + (n-r)[H_0(t_{r:n}) - H_0(\tau)]}. \tag{7}$$

Example 1: (i) (Exponential distribution). For the exponential distribution with the baseline cumulative HRF $H_0(t) = t$, the MLEs are obtained to be

$$\hat{\theta}_{1ML} = \frac{n_1}{\sum_{i=1}^{n_1} t_{i:n} + (n-n_1)\tau},$$

and

$$\hat{\theta}_{2ML} = \frac{r-n_1}{\sum_{i=n_1+1}^{r} (t_{i:n} - \tau) + (n-r)(t_{r:n} - \tau)}.$$

(ii) (Rayleigh distribution). For the Rayleigh distribution, the baseline cumulative HRF is $H_0(t) = t^2$ and the MLEs are given by

$$\hat{\theta}_{1MLE} = \frac{n_1}{\sum_{i=1}^{n_1} t_{i:n}^2 + (n-n_1)\tau^2},$$

and

$$\hat{\theta}_{2MLE} = \frac{r-n_1}{\sum_{i=n_1+1}^{r} (t_{i:n}^2 - \tau^2) + (n-r)(t_{r:n}^2 - \tau^2)}.$$

(iii) (Pareto distribution). For the Pareto distribution, the baseline cumulative HRF is $H_0(t) = \log(1+t)$ and the MLEs are given by

$$\hat{\theta}_{1ML} = \frac{n_1}{\sum\limits_{i=1}^{n_1} \log(1+t_{i:n}) + (n-n_1)\log(1+\tau)},$$

and

$$\hat{\theta}_{2ML} = \frac{r-n_1}{\sum\limits_{i=n_1+1}^{r} \log\left(\frac{1+t_{i:n}}{1+\tau}\right) + (n-r)\log\left(\frac{1+t_{r:n}}{1+\tau}\right)}.$$

3.2 Bayesian Method

Here, we formulate the posterior density of the parameters θ_1 and θ_2 based on Type-II censored sample and then obtain the corresponding BEs of these unknown parameters under a squared error loss (SEL) function, with respect to independent $\Gamma(\alpha_1, \beta_1)$ and $\Gamma(\alpha_2, \beta_2)$ priors for θ_1 and θ_2 with PDFs as

$$\begin{aligned} p_1(\theta_1) &\propto {\theta_1}^{\alpha_1-1} e^{-\theta_1\beta_1}, \quad \alpha_1 > 0,\ \beta_1 > 0, \\ p_2(\theta_2) &\propto {\theta_2}^{\alpha_2-1} e^{-\theta_2\beta_2}, \quad \alpha_2 > 0,\ \beta_2 > 0, \end{aligned}$$

respectively. Therefore, the joint prior distribution for θ_1 and θ_2 is

$$p(\theta_1, \theta_2) = p_1(\theta_1)\ p_2(\theta_2) \propto {\theta_1}^{\alpha_1-1} e^{-\theta_1\beta_1}\ {\theta_2}^{\alpha_2-1} e^{-\theta_2\beta_2}, \tag{8}$$

and the posterior PDF of θ_1 and θ_2 given the data can be obtained using (4) and (8) as

$$\begin{aligned} p(\theta_1, \theta_2|\mathbf{t}) &\propto L(\theta_1, \theta_2|\mathbf{t}) p(\theta_1, \theta_2) \\ &\propto \prod_{i=1}^{r} h_0(t_{i:n})\ \theta_1^{n_1+\alpha_1-1}\ \theta_2^{r-n_1+\alpha_2-1} \\ &\times\ e^{-\theta_1\left(\sum\limits_{i=1}^{n_1} H_0(t_{i:n}) + (n-n_1)H_0(\tau) + \beta_1\right)} \\ &\times\ e^{-\theta_2\left(\sum\limits_{i=n_1+1}^{r} [H_0(t_{i:n}) - H_0(\tau)] + (n-r)[H_0(t_{r:n}) - H_0(\tau)] + \beta_2\right)}. \end{aligned} \tag{9}$$

Clearly, the posterior PDF of θ_1 and θ_2 can rewritten as

$$p(\theta_1, \theta_2|\mathbf{t}) \propto D_1(\theta_1) D_2(\theta_2),$$

where $D_1(\theta_1)$ and $D_2(\theta_2)$ are, respectively, the PDFs of

$$\Gamma\left(n_1 + \alpha_1, \sum_{i=1}^{n_1} H_0(t_{i:n}) + (n-n_1)H_0(\tau) + \beta_1\right),$$

and

$$\Gamma\left(r-n_1+\alpha_2, \sum_{i=n_1+1}^{r}[H_0(t_{i:n})-H_0(\tau)]+(n-r)[H_0(t_{r:n})-H_0(\tau)]+\beta_2\right),$$

distributions. Under a SEL function, the BE is the posterior mean. Therefore, the BEs of θ_1 and θ_2 under SEL function are given by

$$\hat{\theta}_{1BS}=\frac{n_1+\alpha_1}{\sum_{i=1}^{n_1}H_0(t_{i:n})+(n-n_1)H_0(\tau)+\beta_1}, \tag{10}$$

and

$$\hat{\theta}_{2BS}=\frac{r-n_1+\alpha_2}{\sum_{i=n_1+1}^{r}[H_0(t_{i:n})-H_0(\tau)]+(n-r)[H_0(t_{r:n})-H_0(\tau)]+\beta_2}. \tag{11}$$

In the case that the information prior is not available, one may assume improper gamma priors by considering $\alpha_1=\alpha_2=\beta_1=\beta_2=0$, the BEs coincide with the MLEs.

Example 2: (i) (Exponential distribution). For the exponential distribution with the baseline cumulative HRF $H_0(t)=t$, we derive the BEs as

$$\hat{\theta}_{1BS}=\frac{n_1+\alpha_1}{\sum_{i=1}^{n_1}t_{i:n}+(n-n_1)\tau+\beta_1},$$

and

$$\hat{\theta}_{2BS}=\frac{r-n_1+\alpha_2}{\sum_{i=n_1+1}^{r}(t_{i:n}-\tau)+(n-r)(t_{r:n}-\tau)+\beta_2}.$$

(ii) (Rayleigh distribution). For the Rayleigh distribution, we have $H_0(t)=t^2$ and the BEs are

$$\hat{\theta}_{1BS}=\frac{n_1+\alpha_1}{\sum_{i=1}^{n_1}t_{i:n}^2+(n-n_1)\tau^2+\beta_1},$$

and

$$\hat{\theta}_{2BS}=\frac{r-n_1+\alpha_2}{\sum_{i=n_1+1}^{r}(t_{i:n}^2-\tau^2)+(n-r)(t_{r:n}^2-\tau^2)+\beta_2}.$$

(iii) (Pareto distribution). For the Pareto distribution, the baseline cumulative HRF is $H_0(t)=\log(1+t)$ and the BEs are

$$\hat{\theta}_{1BS}=\frac{n_1+\alpha_1}{\sum_{i=1}^{n_1}\log(1+t_{i:n})+(n-n_1)\log(1+\tau)+\beta_1}, \tag{12}$$

and

$$\hat{\theta}_{2BS} = \frac{r - n_1 + \alpha_2}{\sum_{i=n_1+1}^{r} \log\left(\frac{1+t_{i:n}}{1+\tau}\right) + (n-r)\log\left(\frac{1+t_{r:n}}{1+\tau}\right) + \beta_2}. \tag{13}$$

4 Prediction Methods

The prediction of future censored observations based on the observed failure times is a fundamental problem and it is widely used in survival, medical and engineering studies. In engineering studies, one may be interested in predicting the time at which the current system might fail in order to have resources available for future purposes. Interested readers may refer to Kaminsky and Nelson (1998), Gulati and Padgett (2003), and Raqab et al. (2018) for details on these developments. In our set-up, let us observe the Type-II censored sample of the form:

$$\mathbf{t} = \{t_{1:n} < \cdots < t_{n_1:n} < \tau < t_{n_1+1:n} < \cdots < t_{r:n}\},$$

taken from $\mathbf{T} = (T_{1:n}, \cdots, T_{n_1:n}, T_{n_1+1:n}, \cdots, T_{r:n})$ with $\{T_{1:n} < \cdots < T_{n_1:n} < \tau < T_{n_1+1:n} < \cdots < T_{r:n}\}$ as the r Type-II right censored order statistics from a population with PDF $g(t)$ in (3). The aim is to predict the future failure time $Y = T_{r+j:n} (j = 1, 2, \ldots, n-r)$ based on $\mathbf{T} = \mathbf{t}$. Again, we will only consider the main case of $r > n_1$. Due to the Markovian property of Type-II right-censored order statistics, the conditional pdf of Y given $\mathbf{T} = \mathbf{t}$ is just the conditional PDF of Y given $T_{r:n} = t_{r:n}$. That is,

$$g_{Y|\mathbf{T}}(y|\mathbf{t}) = g_{Y|T_{r:n}}(y|t_{r:n}).$$

It follows that the density of Y given $\mathbf{T} = \mathbf{t}$ is the same as the density of the jth order statistic of a sample of size $(n-r)$ from the population with the right truncated PDF $g_2(y)/(1 - G_2(t_{r:n})),\ y > t_{r:n}$. Hence, the conditional PDF of Y given $\mathbf{T} = \mathbf{t}$ is

$$\begin{aligned} g_{Y|\mathbf{T}}(y|\mathbf{t}) = g_{Y|T_{r:n}}(y|t_{r:n}) = j\binom{n-r}{j} g_2(y) \left[G_2(y) - G_2(t_{r:n})\right]^{j-1} \left[1 - G_2(y)\right]^{n-r-j} \\ \times \left[1 - G_2(t_{r:n})\right]^{-(n-r)}, \quad y > t_{r:n}. \end{aligned} \tag{14}$$

4.1 Maximum likelihood predictor

Here we consider the predictive likelihood function (PLF) of the future observation $Y = T_{r+j:n}$ and the parameters θ_1 and θ_2 having observed $T = t$. The maximum likelihood predictor (MLP) of Y is derived by maximizing the PLF with respect to Y,

θ_1 and θ_2 simultaneously (Kaminsky and Rodhin (1985)). The estimators of θ_1 and θ_2 obtained here are called the predictive maximum likelihood estimators (PMLEs). The PLF of Y, θ_1 and θ_2 is

$$\begin{aligned} L^* = L^*(y, \theta_1, \theta_2; \boldsymbol{t}) &= g_{Y|\boldsymbol{T}}\ (y \mid \boldsymbol{t}, \theta_1, \theta_2)\ L(\theta_1, \theta_2 \mid \boldsymbol{t}) \\ &= g_{Y|T_{r:n}}(y \mid t_{r:n}; \theta_1, \theta_2)\ L(\theta_1, \theta_2 \mid \boldsymbol{t}) \\ &= g_{Y|T_{r:n}}(y \mid t_{r:n})\ L(\theta_1, \theta_2 \mid \boldsymbol{t}), \end{aligned} \tag{15}$$

where $L(\theta_1, \theta_2 \mid \boldsymbol{t})$ is the likelihood function of the observed failure time $\boldsymbol{t}$. Therefore, the PLF without considering the constant factor is given by

$$L^* = \prod_{i=1}^{n_1} g_1(t_{i:n}) \prod_{i=n_1+1}^{r} g_2(t_{i:n}) g_2(y) [1 - G_2(y)]^{n-r-j} [G_2(y) - G_2(t_{r:n})]^{j-1}. \tag{16}$$

From (2) and (3), the PLF without considering the constant factor is obtained to be

$$\begin{aligned} L^* &= c\ \theta_1^{n_1}\ \theta_2^{r-n_1+1} h_0(y) \prod_{i=1}^{r} h_0(t_{i:n})\ e^{-\theta_1 \left[\sum\limits_{i=1}^{n_1} H_0(t_{i:n}) + (n-n_1) H_0(\tau)\right]} \\ &\times\ e^{-\theta_2 \left[\sum\limits_{i=n_1+1}^{r} H_0(t_{i:n}) - (n-n_1) H_0(\tau) + (n-r-j+1) H_0(y) + (j-1) H_0(t_{r:n})\right]} \\ &\times\ \left[1 - e^{\theta_2 [H_0(y) - H_0(t_{r:n})]}\right]^{j-1}. \end{aligned} \tag{17}$$

The logarithm of the PLF (log PLF) is obtained as

$$\begin{aligned} \log L^* &= n_1 \log \theta_1 + (r - n_1 + 1) \log \theta_2 + \log[h_0(y)] + \sum_{i=1}^{r} \log[h_0(t_{i:n})] \\ &- \theta_1 \left[\sum_{i=1}^{n_1} H_0(t_{i:n}) + (n - n_1) H_0(\tau)\right] + (j-1) \log[1 - e^{-\theta_2 [H_0(y) - H_0(t_{r:n})]}] \\ &- \theta_2 \left[\sum_{i=n_1+1}^{r} H_0(t_{i:n}) - (n - n_1) H_0(\tau) + (n - r - j + 1) H_0(y) + (j-1) H_0(t_{r:n})\right]. \end{aligned}$$

By differentiating the log PLF with respect to y and θ_2, we derive the predictive likelihood equations (PLEs) as

$$\begin{aligned} \frac{\partial \log L^*}{\partial y} &= \frac{h_0'(y)}{h_0(y)} - \theta_2 (n - r - j + 1) h_0(y) \\ &+ \theta_2 (j-1) h_0(y) \frac{e^{-\theta_2 [H_0(y) - H_0(t_{r:n})]}}{1 - e^{-\theta_2 [H_0(y) - H_0(t_{r:n})]}} = 0, \end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\partial \log L^*}{\partial \theta_2} &= \frac{r-n_1+1}{\theta_2} - \sum_{i=n_1+1}^{r} H_0(t_{i:n}) + (n-n_1)H_0(\tau)\\ &- (n-r-j+1)H_0(y) - (j-1)H_0(t_{r:n})\\ &+ (j-1)[H_0(y)-H_0(t_{r:n})]\frac{e^{-\theta_2[H_0(y)-H_0(t_{r:n})]}}{1-e^{-\theta_2[H_0(y)-H_0(t_{r:n})]}} = 0. \qquad (19)\end{aligned}$$

It is important to point out that $\frac{\partial \log L^*}{\partial \theta_1}$ does not depend on y and therefore it is no longer necessary for obtaining the MLP of Y. By solving (18) and (19) with respect to y and θ_2 simultaneously, we can compute the MLP of Y, $\widehat{Y}_{MLP}$, and PMLE of $\theta_2, \widehat{\theta}_{2PMLE}$.

Example 3: (i) (Exponential case). For the case of exponential distribution, we have $H_0(t) = t, h_0(t) = 1$ and $h_0'(t) = 0$. In this case, the PLEs are given by

$$\frac{\partial \log L^*}{\partial y} = -\theta_2(n-r-j+1) + \theta_2(j-1)\frac{e^{-\theta_2(y-t_{r:n})}}{1-e^{-\theta_2(y-t_{r:n})}} = 0,$$

and

$$\begin{aligned}\frac{\partial L^*}{\partial \theta_2} = \frac{(r-n_1+1)}{\theta_2} - \sum_{i=n_1+1}^{r} t_{i:n} + (n-n_1)\tau - (n-r-j+1)y\\ - (j-1)t_{r:n} + (j-1)(y-t_{r:n})\frac{e^{-\theta_2(y-t_{r:n})}}{1-e^{-\theta_2(y-t_{r:n})}} = 0.\end{aligned}$$

By solving these PLEs, we derive the MLP of Y as

$$\widehat{Y}_{MLP} = t_{r:n} + \frac{\log(\frac{n-r}{n-r-j+1})}{\widehat{\theta}_{2PMLE}}, \qquad (20)$$

where

$$\widehat{\theta}_{2PMLE} = \frac{r-n_1+1}{\sum_{i=n_1+1}^{r}(t_{i:n}-\tau) + (n-r)(t_{r:n}-\tau)}, \qquad (21)$$

is the PMLE of θ_2.

(ii) (Rayleigh case). For the case of Rayleigh distribution, $H_0(t) = t^2, h_0(t) = 2t$ and $h_0'(t) = 2$ and the PLEs can be written as

$$\frac{\partial \log L^*}{\partial y} = \frac{1}{y} - 2(n-r-j+1)\theta_2\ y + 2(j-1)\theta_2\ y\frac{e^{-\theta_2(y^2-t_{r:n}^2)}}{1-e^{-\theta_2(y^2-t_{r:n}^2)}} = 0,$$

and

$$\frac{\partial \log L^*}{\partial \theta_2} = \frac{(r-n_1+1)}{\theta_2} - \sum_{i=n_1+1}^{r} y_{i:n}^2 + (n-n_1)\tau^2 - (n-r-j+1)y^2$$
$$- (j-1)t_{r:n}^2 + (j-1)(y^2 - t_{r:n}^2)\frac{e^{-\theta_2(y^2-t_{r:n}^2)}}{1-e^{-\theta_2(y^2-t_{r:n}^2)}} = 0.$$

In this case, closed-form expressions for the MLP of Y and the PMLE of θ_2 are not available, and they must be obtained numerically by solving the above PLEs.

(iii) (Pareto case). For the case of Pareto distribution, $H_0(t) = \log(1+t), h_0(t) = (1+t)^{-1}$ and $h_0'(t) = -(1+t)^{-2}$. The PLEs reduce to

$$\begin{aligned}\frac{\partial \log L^*}{\partial y} &= -(1+y)^{-1} - \theta_2(n-r-j+1)(1+y)^{-1} \\ &+ \theta_2(j-1)(1+y)^{-1}\frac{(1+y)^{-\theta_2}}{(1+t_{r:n})^{-\theta_2} - (1+y)^{-\theta_2}} = 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L^*}{\partial \theta_2} &= \frac{r-n_1+1}{\theta_2} - \sum_{i=n_1+1}^{r} \log(1+t_{i:n}) + (n-n_1)\log(1+\tau) \\ &- (n-r-j+1)\log(1+y) - (j-1)\log(1+t_{r:n}) \\ &+ (j-1)\log\left(\frac{1+y}{1+t_{r:n}}\right)\frac{(1+y)^{-\theta_2}}{(1+t_{r:n})^{-\theta_2} - (1+y)^{-\theta_2}} = 0.\end{aligned}$$

Hence again, the MLP of Y and the PMLE of θ_2 must be obtained by solving these PLEs numerically.

4.2 Conditional median predictor

Raqab and Nagaraja (1995) used the median of the conditional distribution of $Y = T_{r+j:n}$ given $T_{r:n} = t_{r:n}$ as a predictor of $T_{r+j:n}$ and called it the conditional median predictor (CMP). The CMP of Y, denoted by Y_{CMP}, is such that

$$P(Y \le Y_{CMP} \mid T_{r:n} = t_{r:n}) = P(Y \ge Y_{CMP} \mid T_{r:n} = t_{r:n}).$$

From the conditional distribution of Y given $T_{r:n} = t_{r:n}$ in (14), it follows that the conditional distribution of

$$U = \frac{1-G_2(Y)}{1-G_2(T_{r:n})} = e^{-\theta_2[H_0(Y)-H_0(T_{r:n})]},$$

given $T_{r:n} = t_{r:n}$ is a beta distribution with parameters $(n-r-j+1)$ and j (denoted as $Beta(n-r-j+1, j)$). Using the relation

$$P(Y \le Y_{CMP} \mid T_{r:n} = t_{r:n}) = P\left(\frac{1-G_2(Y)}{1-G_2(T_{r:n})} \ge \frac{1-G_2(Y_{CMP})}{1-G_2(T_{r:n})} \mid T_{r:n} = t_{r:n}\right),$$

we readily have

$$\frac{1-G_2(Y_{CMP})}{1-G_2(t_{r:n})}=e^{-\theta_2[H_0(Y_{CMP})-H_0(t_{r:n})]}=Med(U),$$

where $Med(U)$ is the median of U . So, we find the CMP of Y as

$$Y_{CMP}=H_0{}^{-1}\left(H_0(t_{r:n})-\frac{1}{\theta_2}\log[Med(U)]\right).$$

When θ_2 is not known, it can be replaced by its MLE $\hat{\theta_2}$ and we can obtain an approximate CMP of Y.

Example 4: (i) (Exponential case). For the case of exponential distribution, $H_0(t)=t$ and $H_0{}^{-1}(t)=t$. In this case, the approximate CMP of Y is

$$Y_{CMP}=t_{r:n}-\frac{1}{\theta_2}\log[Med(U)].$$

(ii) (Rayleigh case). For the case of Rayleigh distribution, $H_0(t)=t^2$ and $H_0{}^{-1}(t)=\sqrt{t}$ and the approximate CMP is obtained to be

$$Y_{CMP}=\sqrt{t_{r:n}^2-\frac{1}{\theta_2}\log[Med(U)]}.$$

(iii) (Pareto distribution). For the Pareto distribution, $H_0(t)=\log(1+t)$ and $H_0{}^{-1}(t)=e^t-1$ and the approximate CMP is found to be

$$Y_{CMP}=(1+t_{r:n})[Med(U)]^{-\frac{1}{\theta_2}}-1\ .$$

4.3 Best unbiased predictor

A predictor $\hat{Y}$ is a best unbiased predictor (BUP) of $Y=T_{r+j:n}$ if its mean prediction error is zero and its prediction variance is less than or equal to the prediction variance of any other unbiased predictor of Y. The mean of the conditional distribution of Y given $T_{r:n}=t_{r:n}$ is the BUP of Y. Therefore, the BUP of Y takes the form:

$$\hat{Y}_{BUP}=E(Y|T_{r:n}=t_{r:n})=\int_{t_{r:n}}^{\infty}y\ g_{Y|T_{r:n}}(y|t_{r:n})dy.$$

From the conditional distribution of Y given $T_{r:n}=t_{r:n}$, the BUP is

$$\begin{aligned}\hat{Y}_{BUP} &= \int_{t_{r:n}}^{\infty}y\ g_{Y|T_{r:n}}(y|t_{r:n})dy\\ &= \int_{t_{r:n}}^{\infty}y\ j\binom{n-r}{j}\left(1-e^{\theta_2[H_0(y)-H_0(t_{r:n})]}\right)^{j-1}\left(e^{-\theta_2[H_0(y)-H_0(t_{r:n})]}\right)^{n-r-j}\\ &\times\ \theta_2\ h_0(y)e^{-\theta_2[H_0(y)-H_0(t_{r:n})]}\ dy.\end{aligned}$$

By applying the transformation, $u = e^{-\theta_2[H_0(y)-H_0(t_{r:n})]}$, we can rewrite the BUP of Y as follows.

$$\begin{aligned} \hat{Y}_{BUP} &= \int_0^1 H_0{}^{-1}\left(-\frac{1}{\theta_2}\log(u) + H_0(t_{r:n})\right)\frac{u^{n-r-j}(1-u)^{j-1}}{B(n-r-j+1,j)}du \\ &= E_U\left(H_0{}^{-1}\left(-\frac{1}{\theta_2}\log(U) + H_0(t_{r:n})\right)\right), \end{aligned} \tag{22}$$

where $E_U(.)$ is the expectation under $U \sim Beta(n-r-j+1, j)$ and $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, for $a, b \geq 0$. When θ_2 is unknown, it can be replaced by its MLE $\hat{\theta_2}$ and we obtain an approximate BUP of Y.

Example 5: (i) (Exponential case). For the case of exponential distribution, $H_0(t) = t$ and $H_0{}^{-1}(t) = t$. In this case, the BUP is

$$\hat{Y}_{BUP} = -\frac{1}{\theta_2}E_U\left[\log(U)\right] + t_{r:n}$$

(ii) (Rayleigh case). For the case of Rayleigh distribution,$H_0(t) = t^2$ and $H_0{}^{-1}(t) = \sqrt{t}$, the BUP is

$$\hat{Y}_{BUP} = E_U\left(\sqrt{t_{r:n}^2 - \frac{1}{\theta_2}\log(U)}\right).$$

(iii) (Pareto distribution). For the Pareto distribution, $H_0(t) = \log(1+t)$ and $H_0{}^{-1}(t) = e^t - 1$, the BUP is

$$\hat{Y}_{BUP} = (1 + t_{r:n})E_U(U^{-\frac{1}{\theta_2}}) - 1 .$$

4.4 Baysian prediction

In this section, our aim is to predict $Y = T_{r+j:n} (j = 1, 2, \ldots, n-r)$ based on the observed data, $\boldsymbol{t} = (t_{1:n}, t_{2:n}, \ldots, t_{r:n})$ using a Baysian approach. The Bayes predictive density function of $Y = T_{r+j:n}$ given $\boldsymbol{t}$ is

$$g^*(y \mid \boldsymbol{t}) = \int_0^\infty \int_0^\infty g(y \mid \boldsymbol{t}; \theta_1, \theta_2)\ p(\theta_1, \theta_2 \mid \boldsymbol{t})\ d\theta_1\ d\theta_2. \tag{23}$$

Under the assumption of independent gamma priors $\Gamma(\alpha_1, \beta_1)$ and $\Gamma(\alpha_2, \beta_2)$ for θ_1 and θ_2, the posterior PDF of θ_1 and θ_2 can be rewritten as

$$p(\theta_1, \theta_2|t) \propto D_1(\theta_1)D_2(\theta_2),$$

where $D_1(\theta_1)$ and $D_2(\theta_2)$ are the updated gamma PDFs of θ_1 and θ_2, respectively. Therefore,

$$g^*(y \mid \mathbf{t}) \propto \int_0^\infty \int_0^\infty g(y \mid \mathbf{t}; \theta_1, \theta_2)\ D_1(\theta_1)\ D_2(\theta_2)\ d\theta_1\ d\theta_2. \tag{24}$$

The Bayesian predictor (BP) of Y under a SEL is

$$\hat{Y}_{BP} = \int_{t_{r:n}}^{\infty} y\ g^*(y \mid \mathbf{t})dy. \tag{25}$$

For exponential, Rayleigh and Pareto distributions, the BPs are obtained by substituting $h_0(t) = 1, H_0(t) = t,\ h_0(t) = 2t, H_0(t) = t^2$ and $h_0(t) = (1+t)^{-1}, H_0(t) = \log(1+t)$, respectively. Details for computing the BPs are given in Algorithm 1.

Algorithm 1: Algorithm for computing the BPs

Step 1: Generate θ_1 from $\Gamma\left(n_1 + \alpha_1, \sum_{i=1}^{n_1} H_0(t_{i:n}) + (n-n_1)H_0(\tau) + \beta_1\right)$;

Step 2: Generate θ_2 from

$$\Gamma\left(r - n_1 + \alpha_2, \sum_{i=n_1+1}^{r} H_0(t_{i:n}) + (n-r)H_0(t_{r:n}) - (n-n_1)H_0(\tau) + \beta_2\right);$$

Step 3: Repeat Steps 1 and 2, M times and obtain $(\theta_{11}, \theta_{21}), \cdots, (\theta_{1M}, \theta_{2M})$;

Step 4: Using (24), compute the simulation consistent estimator of $g^*(y \mid \mathbf{t})$ as

$$\hat{g^*}(y \mid \mathbf{t}) = \frac{1}{M}\sum_{i=1}^{M} g(y \mid \mathbf{t}; \theta_{1i}, \theta_{2i}); \tag{26}$$

Step 5: Now, the BP of $Y = T_{r+j:n}$ under SEL can be approximated as

$$\begin{aligned} \hat{Y}_{BP} &= \int_{t_{r:n}}^{\infty} y \left(\sum_{i=1}^{M} \frac{1}{M} g(y \mid \mathbf{t}; \theta_{1i}, \theta_{2i})\right) dy = \frac{1}{M}\sum_{i=1}^{M}\left(\int_{t_{r:n}}^{\infty} y\ g(y \mid \mathbf{t}; \theta_{1i}, \theta_{2i})dy\right) \\ &= \frac{1}{M}\sum_{i=1}^{M} I(t_{r:n}; \theta_{1i}, \theta_{2i}), \end{aligned} \tag{27}$$

where

$$\begin{aligned} I(t_{r:n}; \theta_1, \theta_2) &= \int_{t_{r:n}}^{\infty} y \left[j\binom{n-r}{j} \theta_2 h_0(y) e^{-\theta_2[H_0(y) - H_0(t_{r:n})]} \right. \\ &\times \left. e^{-\theta_2(n-r-j)[H_0(y)-H_0(t_{r:n})]} \left(1 - e^{-\theta_2[H_0(y)-H_0(t_{r:n})]}\right)^{j-1} \right] dy, \end{aligned}$$

and using the binomial expansion

$$\left(1-e^{-\theta_2[H_0(y)-H_0(t_{r:n})]}\right)^{j-1}=\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^k e^{-\theta_2(j-1-k)[H_0(y)-H_0(t_{r:n})]},$$

it can be written as

$$\begin{aligned} I(t_{r:n},\theta_1,\theta_2) &= j\binom{n-r}{j}\theta_2\sum_{k=0}^{j-1}\binom{j-1}{k}(-1)^{j-k-1} \\ &\times \int_{t_{r:n}}^{\infty} y\ h_0(y)e^{-\theta_2(n-r-k)[H_0(y)-H_0(t_{r:n})]}\ dy. \end{aligned}$$

5 Numerical examples and comparative study

Here we conduct the analyses of two numerical examples with fitting model belonging to PHR family and perform a simulation study to examine the performance of the estimators and predictors developed in the previous sections. All the computations are performed using R Software (R x64 4.0.3) and the R codes can be obtained upon request from the authors.

Let's first explain how we can generate a random sample from the PHR family under the simple stress-stress model. For given $(n,\tau,\theta_1,\theta_2)$, Algorithm 2 is used to generate the ordered lifetimes, $t_{1:n},t_{2:n},\ldots,t_{n:n}$, from the PHR family under the simple SSALT model with the CDF (2).

Algorithm 2: Algorithm for generating ordered lifetimes from PHR distributions

Step 1: Generate u from the uniform distribution $\mathrm{U}(0,1)$;

Step 2: If $u<1-e^{-\theta_1 H_0(\tau)}$, set

$$t={H_0}^{-1}\left[-\frac{1}{\theta_1}\log(1-u)\right];$$

Step 3: If $u\ge 1-e^{-\theta_1 H_0(\tau)}$, set

$$t={H_0}^{-1}\left[-\frac{1}{\theta_2}\log(1-u)+\frac{\theta_2-\theta_1}{\theta_2}H_0(\tau)\right]; \tag{28}$$

Step 4: Repeat Steps $1-3$, n times to generate $t_1,t_2,\ldots,t_n$;

Step 5: Sort $t_1,t_2,\ldots,t_n$ in ascending order to obtain $t_{1:n},t_{2:n},\ldots,t_{n:n}$.

5.1 Numerical examples

In this subsection, the proposed methods are illustrated by two numerical examples. For both data sets, we compute the MLEs, BEs as well as point predictors including frequentist and Bayes predictors.

Example 1 (Rayleigh case). Here we consider a data set following Rayleigh distribution. In the following steps, we obtain different estimates and different predicted values as described in Sections 3 and 4.

(i) For given values of $(\alpha_1, \beta_1) = (2, 1)$ and $(\alpha_2, \beta_2) = (1.5, 2.5)$, we generated $\theta_1 = 0.887$ and $\theta_2 = 0.162$ from the gamma prior distributions $\Gamma(2, 1)$ and $\Gamma(1.5, 2.5)$, respectively.

(ii) Using Algorithm 2 with $H_0(t) = t^2$ and based on the values $\theta_1 = 0.887$ and $\theta_2 = 0.162$ obtained from step (i), we then generated a random sample of size $n = 30$ from the Rayleigh model under the step-stress setting $\theta_1 = 0.887, \theta_2 = 0.162$ with $\tau = 0.5$. The generated data are presented in Table 1.

Table 1: Step-stress simulated data from Rayleigh model.

Stress level	Times to failure				
$\theta_1 = 0.887$	0.361	0.417	0.443	0.496		
$\theta_2 = 0.162$	0.765	0.940	1.013	1.016	1.037	1.103
	1.239	1.332	1.421	1.601	1.653	1.709
	1.855	1.884	1.988	1.991	2.252	2.368
	2.487	2.568	2.584	2.745	3.370	3.752
	4.521	6.137				

(iii) By taking $r = 25$, the corresponding Type-II censored data $t_{1:30}, t_{2:30}, \ldots, t_{25:30}$ are:

0.361	0.417	0.443	0.496	0.765	0.940	1.013
1.016	1.037	1.103	1.239	1.332	1.421	1.601
1.653	1.709	1.855	1.884	1.988	1.991	2.252
2.368	2.487	2.568	2.584			

(iv) Based on these data, we computed the MLEs and BEs of θ_1 and θ_2. The MLEs and BEs are computed to be $\hat{\theta}_{1ML} = 0.551$, $\hat{\theta}_{2ML} = 0.230$ and $\hat{\theta}_{1BS} = 0.727$, $\hat{\theta}_{2BS} = 0.239$.

(v) Based on the observed failure times, the predicted values of the censored lifetime $T_{25+j:30}$ $(j = 1, 2, \ldots, 5)$ are computed and presented in Table 2.

Table 2: Predicted values of $T_{25+j:30}$ $(j = 1, 2, 3, 4, 5)$.

	Actual observation	MLP	CMP	BUP	Bayesian
$T_{26:30}$	2.745	2.58	2.69	2.74	2.75
$T_{27:30}$	3.370	2.76	2.88	2.92	2.94
$T_{28:30}$	3.752	2.99	3.11	3.16	3.18
$T_{29:30}$	4.521	3.28	3.42	3.47	3.51
$T_{30:30}$	6.137	3.74	3.94	4.02	4.07

From Table 2, it is easily checked that the different predicted values obtained are quite close to the actual observations to be predicted.

Example 2 (Exponential case). Xiong (1998) presented a simulated exponential failure times data under the simple step-stress model. A random sample from the exponential model under the step-stress setting $\theta_1 = e^{-2.5}, \theta_2 = e^{-1.5}$ and $\tau = 5$, with $n = 20$ and $r = 16$ is generated. The simulated data are given in Table 3.

Table 3: The simulated data reported in Xiong (1998).

Stress levels	Times to failure				
$\theta_1 = e^{-2.5}$	2.01	3.60	4.12	4.34		
$\theta_2 = e^{-1.5}$	5.04	5.94	6.68	7.09	7.17	7.49
	7.60	8.23	8.24	8.25	8.69	12.05

Here $n_1 = 4$ and $n_2 = r - n_1 = 12$. For this simulated step-stress data set, the MLEs of θ_1 and θ_2 are computed to be $\hat{\theta}_{1ML} = 0.0420$ and $\hat{\theta}_{2ML} = 0.197$. For computing BEs, since no information is available on prior distributions, we use the improper gamma priors. Under the improper gamma priors, the BEs of θ_1 and θ_2 are obtained as $\hat{\theta}_{1BS} = 0.0420$ and $\hat{\theta}_{2BS} = 0.197$. Clearly, these estimates coincide with the MLEs values. The predicted values of the censored lifetime, $T_{r+j:n} = T_{16+j:20}$ $(j = 1, 2, 3, 4)$ are presented in Table 4.

Table 4: Predicted values of $T_{16+j:20}$ $(j = 1, 2, 3, 4)$.

	MLP	CMP	BUP	BP
$T_{17:20}$	12.05	12.92	13.31	13.41
$T_{18:20}$	13.39	14.51	14.99	15.28
$T_{19:20}$	15.28	16.86	17.52	18.02
$T_{20:20}$	18.51	21.34	22.58	23.55

5.2 Comparative study

Here, a simulation study is mainly conducted to compare the performance of MLEs and BEs of the model parameters as well as classical and Bayes predictors of future failure times, discussed in previous sections. In this simulation study, the exponential and Rayleigh distributions are considered as two special cases of the PHR family of distributions. For different n, r and τ, we have randomly generated 1000 samples of the Type-II censored lifetimes, $T_{1:n}, T_{2:n}, \ldots, T_{r:n}$, from the exponential and Rayleigh distributions in a simple SSALT model with the CDF (2). The MLEs and BEs of θ_1 and θ_2 are then computed. For BEs, we have used two following priors:

1. P1: $\alpha_i = \beta_i = 0.0001, \quad i = 1, 2$
2. P2: $\alpha_i = 1, \beta_i = 4, \quad i = 1, 2.$

In fact, the prior P2 is more informative than the prior P1, which is almost a noninformative prior. Table 5 shows the simulated biases and MSEs of the MLEs and BEs for exponential distribution. The biases and MSEs are computed as follows. Suppose $\widehat{\theta}_i$ is the estimate of θ obtained in $i-th$ iteration of simulation, where $i = 1, \ldots, N = 1000$, then the bias and MSE of $\widehat{\theta}$ are defined as

$$Bias(\widehat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i - \theta),$$

and

$$MSE(\widehat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i - \theta)^2.$$

The results for Rayleigh distribution are presented in Table 6. From Tables 5 and 6, we see that the BEs under Prior P2 perform better than the MLEs and the BEs under the prior P1. As expected, the results for MLEs and BEs under Prior P1 are approximately, the same in terms of biases and MSEs. Also in most considered cases, as τ decreases, the biases and MSEs of estimators of θ_2 decrease, but the biases and MSEs of estimators of θ_1 increase. This is explained by the fact that if τ decreases, the sample size under stress s_1 (i.e., n_1) decreases and vice versa, and then the sample size under stress s_2 (i.e., $n - r_1$) increases.

Based on the step-stress failure data $T_{1:n}, T_{2:n}, \ldots, T_{r:n}$, we have also computed different predicted values including MLP, BUP, CMP and BP for the jth future failure time $Y = T_{r+j:n}$, $(j = 1, 2, \ldots, n - r)$. We then compared the performance of these predicted values in terms of their biases and mean square prediction errors (MSPEs) over 1000 replications. Table 7 presents the results for the exponential distribution. The results for the Rayleigh distribution are given in Table 8. From Tables 7 and 8, it can be observed for both models that the BUP and BPs have the lowest MSPE values, which ensure good performance of BUP and BPs. The MLP does not work well because it provides the largest bias and MSPE. Comparing Priors P1 and P2, the BPs under Prior P2 perform better than the BPs under Prior P1. Further, for fixed sample size n, the MSPEs decrease when r (the number of observed failure times) increases for both distributions. On the other hand, the MSPEs increase for exponential and Rayleigh distributions when the value of τ gets to be increased. Furthermore, it is clearly noticed that for fixed r and n, as j gets large, the MSEs of $Y = T_{r+j}$ tend to be increased. In fact, it is explained as follows. As j increases, the variation of the variable to be predicted, $Y = T_{r+j:n}$ tends to be high and consequently the MSEs are getting large.

Finally, the results presented in this paper can be generalized in different directions. One possible work is to extend our results to 3 or more stress levels. Another possible work is to extend the results to other censoring schemes such as progressive and hybrid censoring schemes. But the generalization of the results is not simple because the likelihood function and the predictive likelihood function have complicated forms in these cases. These extensions are in progress and will be reported later.

Table 5: The biases and MSEs of the MLEs and BEs for exponential distribution with $\theta_1 = e^{-1.0}$ and $\theta_2 = e^{-2.0}$.

		θ_1			θ_2		
(n, τ, r)		MLE	Bayes		MLE	Bayes	
			P1	P2		P1	P2
$(70, 3.5, 64)$	Bias	0.0035	0.0035	0.0002	0.0863	0.0863	0.0856
	MSE	0.0027	0.0027	0.0025	0.0158	0.0158	0.0133
$(70, 3, 64)$	Bias	0.0046	0.0046	0.0007	0.0621	0.0621	0.0631
	MSE	0.0029	0.0029	0.0027	0.0072	0.0072	0.0070
$(70, 3.5, 67)$	Bias	0.0029	0.0029	0.0006	0.0378	0.0379	0.0398
	MSE	0.0028	0.0028	0.0026	0.0043	0.0043	0.0039
$(70, 3, 67)$	Bias	0.0045	0.0045	0.0004	0.0363	0.0363	0.0385
	MSE	0.0035	0.0035	0.0032	0.0036	0.0036	0.0036
$(40, 2.5, 34)$	Bias	0.0069	0.0069	0.0006	0.1262	0.1262	0.1172
	MSE	0.0059	0.0059	0.0051	0.1382	0.1382	0.0282
$(40, 2, 34)$	Bias	0.0050	0.0051	0.0034	0.0851	0.0851	0.0847
	MSE	0.0067	0.0067	0.0057	0.0147	0.0147	0.0127
$(40, 2.5, 37)$	Bias	0.0066	0.0066	0.0009	0.0490	0.0490	0.0050
	MSE	0.0060	0.0060	0.0051	0.0067	0.0067	0.0060
$(40, 2, 37)$	Bias	0.0058	0.0058	0.0027	0.0379	0.0380	0.0390
	MSE	0.0002	0.0067	0.0067	0.0057	0.0006	0.0006

Table 6: The biases and MSEs of the MLEs and BEs for Rayleigh distribution with $\theta_1 = e^{-.15}$ and $\theta_2 = e^{-1.8}$.

		θ_1			θ_2		
(n, τ, r)		MLE	Bayes		MLE	Bayes	
			P1	P2		P1	P2
$(70, 0.5, 64)$	Bias	0.0086	0.0086	0.0118	0.0322	0.0322	0.0320
	MSE	0.0568	0.0568	0.0487	0.0024	0.0024	0.0023
$(70, 1, 64)$	Bias	0.0066	0.0066	0.0042	0.0715	0.0716	0.0692
	MSE	0.0189	0.0189	0.0171	0.0094	0.0094	0.0086
$(70, 0.5, 67)$	Bias	0.0076	0.0076	0.0018	0.0170	0.0170	0.0171
	MSE	0.0562	0.0562	0.0486	0.0014	0.0014	0.0013
$(70, 1, 67)$	Bias	0.0099	0.0099	0.0068	0.0124	0.0124	0.0133
	MSE	0.0563	0.0563	0.0481	0.0007	0.0007	0.0007
$(30, 0.5, 24)$	Bias	0.0102	0.0102	0.0048	0.0673	0.0673	0.0674
	MSE	0.1340	0.1339	0.0994	0.0080	0.0080	0.0077
$(30, 1, 24)$	Bias	0.0219	0.0219	0.0092	0.4160	0.4043	0.2346
	MSE	0.0461	0.0461	0.0358	0.7496	0.7564	0.0470
$(30, 0.5, 27)$	Bias	0.0147	0.0147	0.0019	0.0455	0.0455	0.0441
	MSE	0.1325	0.1325	0.0975	0.0059	0.0059	0.0053
$(30, 1, 27)$	Bias	0.0163	0.0163	0.0097	0.0848	0.0848	0.0806
	MSE	0.0455	0.0455	0.0364	0.0196	0.0196	0.0145

Table 7: The biases and MSPEs of different point predictors under the exponential model.

						$\theta_1 = e^{-1.0}$	$\theta_2 = e^{-2.0}$		
n	τ	r			MLP	CMP	BUP	Bayes	
								P1	P2
40	2.5	34	$T_{35:40}$	Bias	1.195	0.680	0.425	0.167	0.091
				MSPE	2.879	2.654	1.467	1.423	1.367
			$T_{36:40}$	Bias	1.365	1.179	0.920	0.419	0.233
				MSPE	5.637	5.202	4.838	4.782	4.291
			$T_{37:40}$	Bias	2.328	2.544	0.251	0.550	0.302
				MSPE	14.045	14.923	13.717	10.912	10.294
			$T_{38:40}$	Bias	2.938	3.670	3.279	0.422	0.070
				MSPE	26.615	31.244	28.737	21.592	19.935
			$T_{39:40}$	Bias	3.830	5.133	4.596	1.002	0.225
				MSPE	51.604	61.001	56.572	47.290	41.937
			$T_{40:40}$	Bias	6.463	7.729	7.492	2.567	1.022
				MSPE	160.976	185.661	177.551	177.042	150.671
		37	$T_{38:40}$	Bias	2.441	1.363	0.860	0.272	0.341
				MSPE	13.005	9.827	9.920	9.093	8.892
			$T_{39:40}$	Bias	3.726	2.425	1.627	0.240	0.110
				MSPE	34.612	27.344	24.781	23.892	23.781
			$T_{40:40}$	Bias	5.029	3.780	2.273	1.781	1.070
				MSPE	107.861	96.884	84.315	94.426	85.906
70	3.5	64	$T_{65:70}$	Bias	1.980	0.701	0.443	0.065	0.008
				MSPE	3.173	2.012	1.736	1.629	1.597
			$T_{66:70}$	Bias	1.183	1.193	0.805	0.211	0.094
				MSPE	6.180	5.459	4.923	4.920	4.472
			$T_{67:70}$	Bias	1.923	0.917	1.603	0.229	0.033
				MSPE	11.520	10.763	9.653	8.044	7.687
			$T_{68:70}$	Bias	3.142	3.560	3.117	0.484	0.145
				MSPE	34.512	36.468	34.060	30.738	29.297
			$T_{69:70}$	Bias	3.357	4.282	3.715	0.822	0.302
				MSPE	48.131	54.287	50.511	45.048	41.741
			$T_{70:70}$	Bias	5.728	6.812	5.652	1.844	0.853
				MSPE	113.619	127.672	115.968	95.635	90.128
		67	$T_{68:70}$	Bias	2.586	1.224	0.620	0.375	0.299
				MSPE	14.053	8.965	7.977	7.861	7.812
			$T_{69:70}$	Bias	3.164	1.50	0.657	0.789	0.563
				MSPE	25.429	18.131	16.308	16.910	16.421
			$T_{70:70}$	Bias	6.680	4.596	3.056	0.387	0.072
				MSPE	133.960	113.199	104.038	103.2357	100.226

Table 7 (Continued).

n	τ	r			MLP	CMP	BUP	Bayes	
								P1	P2
40	2	34	$T_{35:40}$	Bias	1.114	0.551	0.301	0.190	0.137
				MSPE	2.425	1.529	1.449	1.432	1.398
			$T_{36:40}$	Bias	1.222	0.653	0.401	0.096	0.042
				MSPE	8.198	7.252	6.493	5.270	5.241
			$T_{37:40}$	Bias	1.717	1.766	1.381	0.724	0.474
				MSPE	13.320	13.468	12.473	12.878	11.999
			$T_{38:40}$	Bias	2.256	2.714	2.281	0.746	0.406
				MSPE	23.650	25.804	24.139	22.796	21.493
			$T_{39:40}$	Bias	3.208	4.060	3.437	1.036	0.551
				MSPE	34.566	39.510	35.591	33.029	30.222
			$T_{40:40}$	Bias	6.519	7.590	6.449	0.650	0.093
				MSPE	126.702	142.451	129.787	104.817	99.146
		37	$T_{38:40}$	Bias	2.616	1.183	0.549	0.053	0.033
				MSPE	13.291	7.871	6.865	6.646	6.615
			$T_{39:40}$	Bias	3.718	2.2291	1.467	0.371	0.201
				MSPE	29.289	21.1415	18.401	17.260	17.091
			$T_{40:40}$	Bias	5.544	3.369	1.785	1.263	0.825
				MSPE	101.452	85.783	80.5127	83.683	80.961
70	3	64	$T_{65:70}$	Bias	1.167	0.542	0.266	0.147	0.105
				MSPE	2.994	1.914	1.696	1.687	1.662
			$T_{66:70}$	Bias	1.689	1.274	0.944	0.056	0.149
				MSPE	8.432	7.350	6.731	6.170	6.092
			$T_{67:70}$	Bias	1.586	1.418	1.061	0.279	0.161
				MSPE	7.685	7.315	6.587	6.069	5.858
			$T_{68:70}$	Bias	2.664	2.802	2.349	0.248	0.090
				MSPE	25.965	26.740	24.834	21.420	21.031
			$T_{69:70}$	Bias	2.944	3.363	2.708	0.821	0.471
				MSPE	38.547	40.601	37.271	35.471	33.924
			$T_{70:70}$	Bias	4.899	5.390	4.159	1.796	1.235
				MSPE	107.468	112.496	103.551	103.308	98.798
		67	$T_{68:70}$	Bias	2.538	1.001	0.321	0.146	0.079
				MSPE	11.306	5.963	5.190	5.264	5.196
			$T_{69:70}$	Bias	3.391	1.738	0.808	0.366	0.193
				MSPE	32.446	24.761	22.960	23.113	22.856
			$T_{70:70}$	Bias	6.185	3.725	2.059	0.443	0.086
				MSPE	112.446	90.156	82.145	79.812	79.122

Table 8: The biases and MSPEs of different predictors under the Rayleigh model.

					$\theta_1 = e^{-1.5}$	$\theta_2 = e^{-1.8}$			
n	τ	r			MLP	CMP	BUP	Bayes	
								P1	P2
30	0.5	24	$T_{25:30}$	Bias	0.155	0.065	0.029	0.007	0.006
				MSPE	0.055	0.033	0.028	0.029	0.028
			$T_{26:30}$	Bias	0.174	0.125	0.087	0.010	0.003
				MSPE	0.133	0.120	0.113	0.107	0.107
			$T_{27:30}$	Bias	1.137	0.138	0.098	0.065	0.053
				MSPE	0.096	0.096	0.088	0.0869	0.083
			$T_{28:30}$	Bias	0.314	0.358	0.317	0.111	0.100
				MSPE	0.274	0.303	0.277	0.196	0.195
			$T_{29:30}$	Bias	0.330	0.428	0.377	0.087	0.079
				MSPE	0.457	0.528	0.490	0.367	0.362
			$T_{30:30}$	Bias	0.392	0.525	0.445	0.010	0.019
				MSPE	0.790	0.907	0.835	0.668	0.657
		27	$T_{28:30}$	Bias	0.281	0.104	0.035	0.009	0.004
				MSPE	0.149	0.083	0.075	0.075	0.074
			$T_{29:30}$	Bias	0.268	0.126	0.050	0.047	0.037
				MSPE	0.216	0.162	0.150	0.152	0.150
			$T_{30:30}$	Bias	0.420	0.277	0.172	0.014	0.003
				MSPE	0.651	0.559	0.519	0.506	0.501
70	0.5	64	$T_{65:70}$	Bias	0.135	0.050	0.015	0.009	0.007
				MSPE	0.034	0.019	0.017	0.016	0.016
			$T_{66:70}$	Bias	0.138	0.069	0.032	0.002	0.001
				MSPE	0.052	0.038	0.035	0.034	0.034
			$T_{67:70}$	Bias	0.135	0.079	0.040	0.015	0.012
				MSPE	0.096	0.084	0.079	0.078	0.078
			$T_{68:70}$	Bias	0.164	0.120	0.076	0.003	0.001
				MSPE	0.124	0.112	0.104	0.101	0.100
			$T_{69:70}$	Bias	0.179	0.141	0.088	0.025	0.020
				MSPE	0.215	0.204	0.193	0.189	0.189
			$T_{70:70}$	Bias	0.377	0.313	0.227	0.060	0.060
				MSPE	0.499	0.456	0.411	0.369	0.368
		67	$T_{68:70}$	Bias	0.252	0.095	0.032	0.018	0.017
				MSPE	0.113	0.058	0.050	0.0499	0.049
			$T_{69:70}$	Bias	0.251	0.094	0.019	0.017	0.013
				MSPE	0.194	0.141	0.133	0.134	0.133
			$T_{70:70}$	Bias	0.338	0.147	0.043	0.025	0.019
				MSPE	0.462	0.374	0.357	0.357	0.355

Table 8 (Continued).

n	τ	r			MLP	CMP	BUP	Bayes	
								P1	P2
30	1	24	$T_{25:30}$	Bias	0.155	0.065	0.029	0.007	0.006
				MSPE	0.054	0.034	0.031	0.032	0.031
			$T_{26:30}$	Bias	0.174	0.125	0.087	0.010	0.003
				MSPE	0.133	0.120	0.113	0.107	0.106
			$T_{27:30}$	Bias	0.137	0.138	0.098	0.065	0.053
				MSPE	0.096	0.096	0.088	0.086	0.083
			$T_{28:30}$	Bias	0.216	0.466	0.431	0.196	0.143
				MSPE	0.277	0.444	0.417	0.348	0.312
			$T_{29:30}$	Bias	0.202	0.293	0.243	0.064	0.045
				MSPE	0.400	0.443	0.420	0.379	0.369
			$T_{30:30}$	Bias	0.478	0.994	0.928	0.164	0.100
				MSPE	1.585	2.184	2.073	1.907	1.631
		27	$T_{28:30}$	Bias	0.316	0.141	0.075	0.033	0.018
				MSPE	0.169	0.089	0.076	0.073	0.070
			$T_{29:30}$	Bias	0.259	0.141	0.067	0.029	0.021
				MSPE	0.249	0.190	0.175	0.174	0.172
			$T_{30:30}$	Bias	0.487	0.442	0.347	0.046	0.004
				MSPE	0.960	0.938	0.873	0.764	0.738
70	1	64	$T_{65:70}$	Bias	0.148	0.060	0.023	0.007	0.004
				MSPE	0.055	0.037	0.035	0.035	0.035
			$T_{66:70}$	Bias	0.175	0.120	0.083	0.014	0.009
				MSPE	0.090	0.075	0.067	0.061	0.061
			$T_{67:70}$	Bias	0.202	0.184	0.146	0.020	0.021
				MSPE	0.120	0.115	0.103	0.082	0.081
			$T_{68:70}$	Bias	0.259	0.274	0.233	0.068	0.058
				MSPE	0.201	0.210	0.190	0.142	0.140
			$T_{69:70}$	Bias	0.363	0.412	0.362	0.118	0.133
				MSPE	0.379	0.414	0.376	0.272	0.272
			$T_{70:70}$	Bias	0.362	0.422	0.338	0.013	0.009
				MSPE	0.585	0.634	0.573	0.465	0.461
		67	$T_{68:70}$	Bias	0.235	0.070	0.026	0.027	0.024
				MSPE	0.117	0.069	0.065	0.066	0.065
			$T_{69:70}$	Bias	0.244	0.089	0.018	0.020	0.017
				MSPE	0.180	0.130	0.124	0.123	0.123
			$T_{70:70}$	Bias	0.369	0.205	0.099	0.049	0.034
				MSPE	0.599	0.511	0.483	0.486	0.482

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References

Amleh, M. and Raqab, M. (2021). Prediction of censored Weibull lifetimes in a simple step-stress plan with khamis-higgins model. *Statistics, Optimization & Information Computing*, 10(3):658–677.

Asgharzadeh, A. and Valiollahi, R. (2009). Inference for the proportional hazards family under progressive Type II censoring. *Journal of the Iranian Statistical Society*, (1-2):35–53.

Asgharzadeh, A. and Valiollahi, R. (2010). Prediction intervals for proportional hazard rate models based on progressively Type II censored samples. *Communications for Statistical Applications and Methods*, 17(1):99–106.

Bai, D. S., Kim, M., and Lee, S. (1989). Optimum simple step-stress accelerated life tests with censoring. *IEEE Transactions on Reliability*, 38(5):528–532.

Balakrishnan, N. and Han, D. (2008). Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type II censoring. *Journal of Statistical Planning and Inference*, 138(12):4172–4186.

Balakrishnan, N., Kundu, D., Ng, K. T., and Kannan, N. (2007). Point and interval estimation for a simple step-stress model with Type II censoring. *Journal of Quality Technology*, 39(1):35–47.

Basak, I. and Balakrishnan, N. (2017). Prediction of censored exponential lifetimes in a simple step-stress model under progressive Type II censoring. *Computational Statistics*, 32(4):1665–1687.

Basak, I. and Balakrishnan, N. (2018). A note on the prediction of censored exponential lifetimes in a simple step-stress model with Type II censoring. *Calcutta Statistical Association Bulletin*, 70(1):57–73.

Basirat, M., Baratpour, S., and Ahmadi, J. (2015). Statistical inferences for stress–strength in the proportional hazard models based on progressive Type II censored samples. *Journal of Statistical Computation and Simulation*, 85(3):431–449.

Basiri, E. and Asgharzadeh, A. (2021). Optimal random sample size in progressively

Type II censoring based on cost constraint for the proportional hazards family. *Journal of Statistical Computation and Simulation*, 91(15):3154–3169.

Chaturvedi, A., Pathak, A., and Kumar, N. (2019). Statistical inferences for the reliability functions in the proportional hazard rate models based on progressive Type II right censoring. *Journal of Statistical Computation and Simulation*, 89(12):2187–2217.

DeGroot, M. H. and Goel, P. K. (1979). Bayesian estimation and optimal designs in partially accelerated life testing. *Naval Research Logistics Quarterly*, 26(2):223–235.

Dey, S. and Nassar, M. (2020). Classical methods of estimation on constant stress accelerated life tests under exponentiated Lindley distribution. *Journal of Applied Statistics*, 47(6):975–996.

Gulati, S. and Padgett, W. J. (2003). *Parametric and nonparametric inference from record-breaking data*. Lecture Notes in Statistics, 172, Springer-Verlag, New York.

Kaminsky and Rodhin (1985). Maximum likelihood prediction. *Annals of the Institute of Statistical Mathematics*, 37(Part A):507–517.

Kaminsky, K. S. and Nelson, P. I. (1998). Prediction on order statistics. *In N. Balakrishnan and C. R. Rao, eds.,Handbook of Statistics*, 17:431–450.

Marshall, A. W. and Olkin, I. (2007). *Life distributions*. Springer, New York.

Miller, R. and Nelson, W. (1983). Optimum simple step-stress plans for accelerated life testing. *IEEE Transactions on Reliability*, 32(1):59–65.

Nelson, W. (1982). *Lifetime data analysis*. Wiley, New York.

Prakash, G. (2018). Bayes estimation under different censoring patterns on constant-stress partially accelerated life test. *Austrian Journal of Statistics*, 47:36–50.

Raqab, M., Bdair, O., Madi, M., and Alqallaf, F. (2018). Prediction of the remaining testing time for the generalized Pareto progressive censoring samples with applications to extreme hydrology events. *Journal of Statistical Theory and Practice*, 12(2):165–187.

Raqab, M. Z. and Nagaraja, H. N. (1995). On some predictors of future order statistics. *Metron*, 53(12):185–204.

Samanta, D. and Kundu, D. (2021). Meta-analysis of a step-stress experiment under Weibull distribution. *Journal of Statistical Computation and Simulation*, 91(9):1867–1889.

Xiong, C. (1998). Inferences on a simple step-stress model with Type II censored exponential data. *IEEE Transactions on Reliability*, 47(2):142–146.

Xiong, C. and Milliken, G. A. (2002). Prediction for exponential lifetimes based on step-stress testing. *Communications in Statistics-Simulation and Computation*, 31(4):539–556.